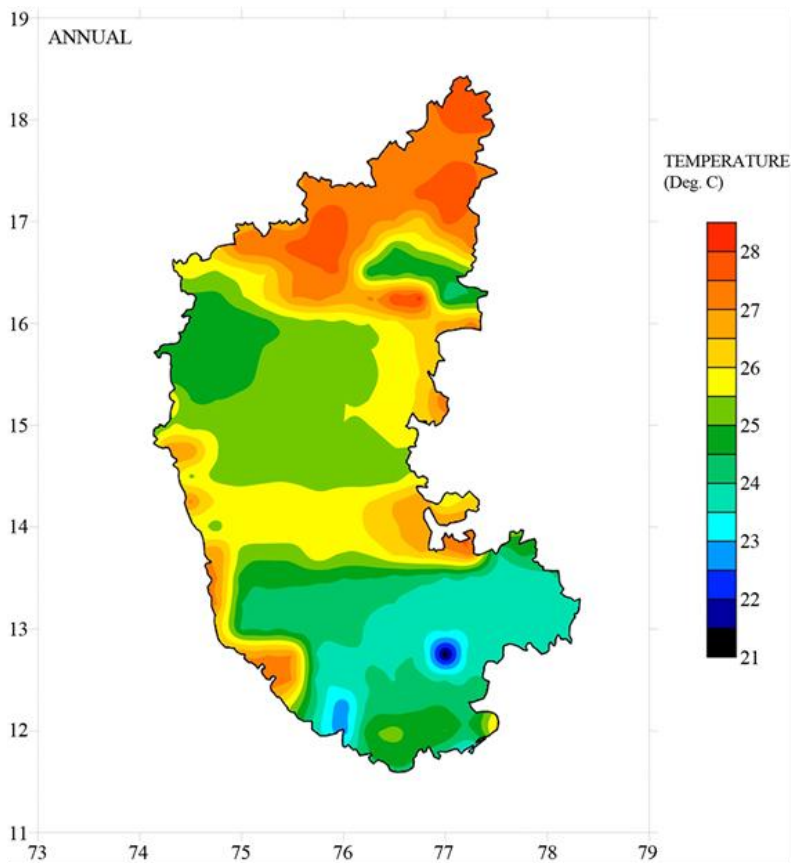
Vector Calculus

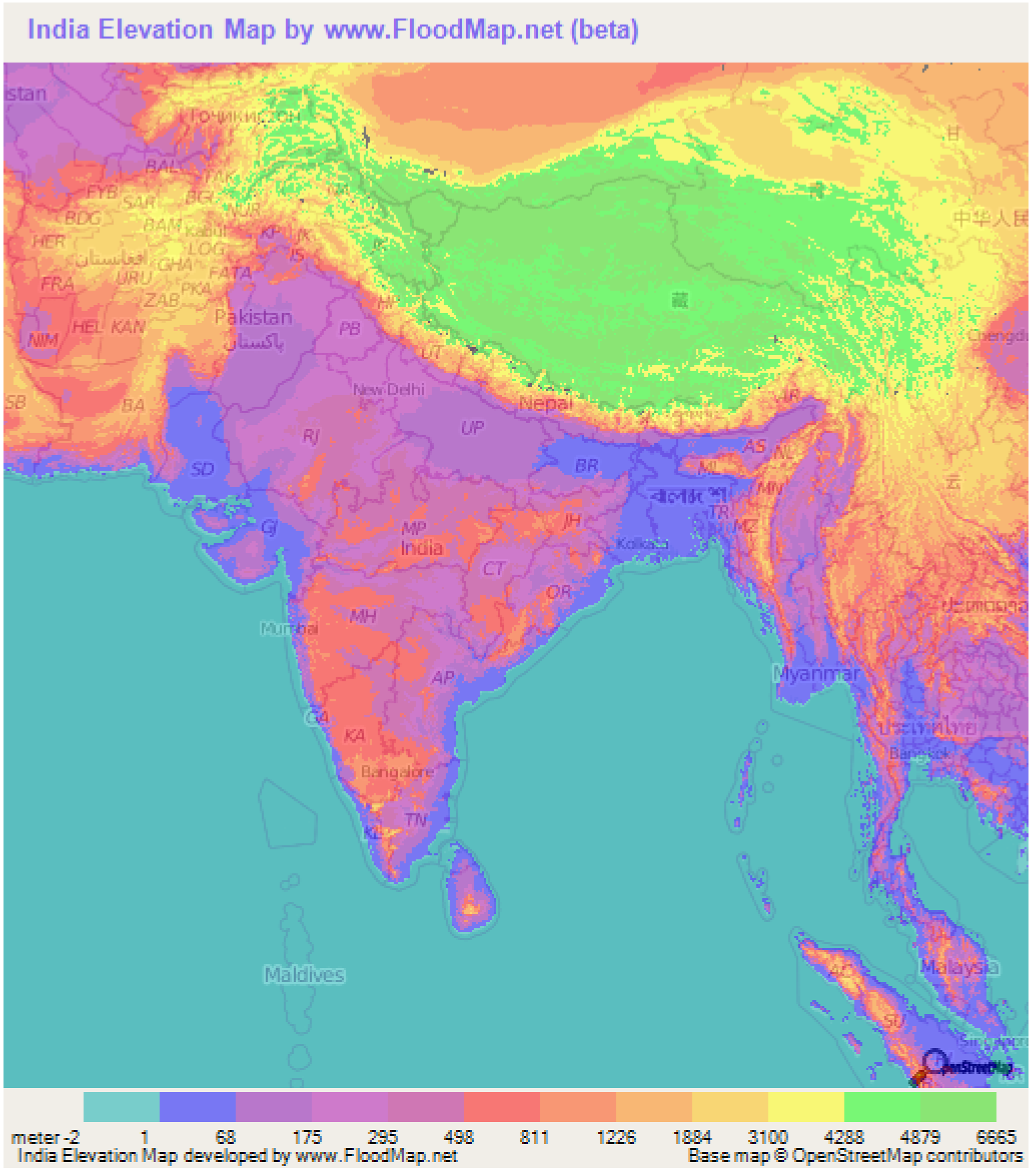
In this part of the presentation, we will learn what is known as multivariable calculus. Prerequisites are calculus of functions of one variable, vector algebra and partial differentiation. We borrow the Physics terminology for vectors, which mean that they have magnitude and direction. Examples include velocity, force and the like. We also use the word scalar, which (for this course) means real numbers. Examples include temperature, potential energy and the like. The stage for performing our calculus would be a region of 3-dimensional space we live in. You are all familiar with the Cartesian co-ordinate system and the unit vectors $\hat{\imath}, \hat{\jmath}$ and $\hat{k}$ in the canonical directions. You also need to know how to take partial differentiation of functions with more than one variable. You also need to know how to take dot and cross products of two vectors.

You must have seen a temperature map of Karnataka like the one below.



Every point in the region is labeled with the temperature of that point. Note that temperature is a scalar. This is an example of a scalar field.

You might have seen a contour map of India like the one below.

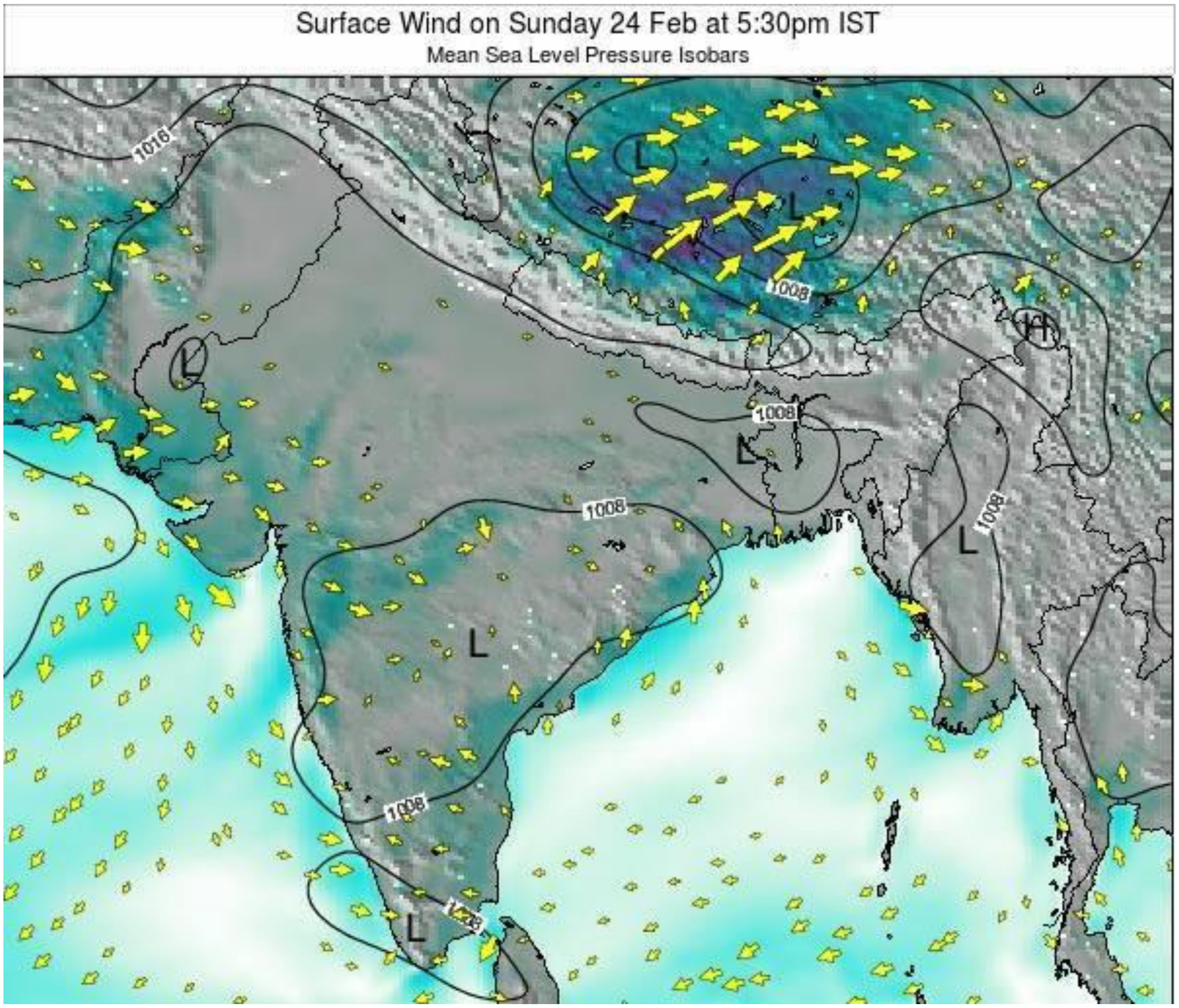


Every point in the region is labeled with the height of the point. This is another example of a scalar field.

Definition: A *scalar field* in a region is a function from the region to the scalars (real numbers)

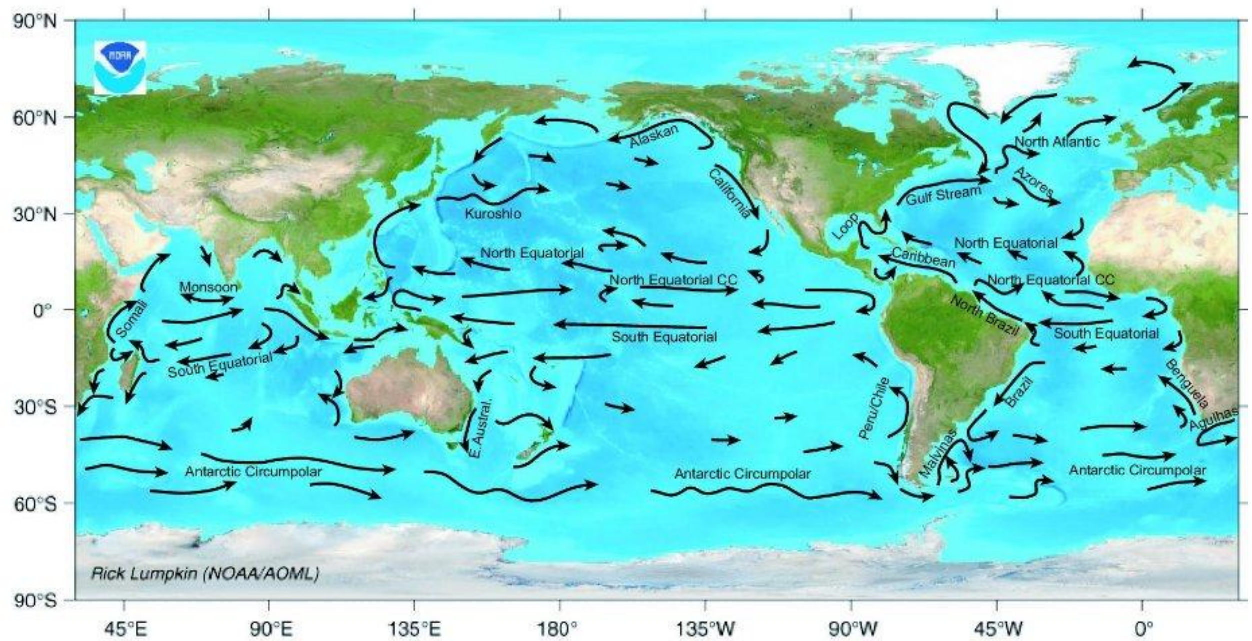
Example: The distance $\varphi(P)$ of any point from origin is a scalar field. In Cartesian coordinate system $\varphi(P) = \sqrt{(x^2 + y^2 + z^2)}$

Another important concept is vector field. If you have seen a wind map like this, you have seen a vector field.



Arrow marked at a point tells us the magnitude and direction of the movement of wind at that point. To make it precise, an arrow must be attached to every point in the region, but we do not draw all of them.

Another example is map of ocean current like this



Arrow marked at a point tells the velocity of the water particle at that point. To make it precise, an arrow must be attached to every point in the region.

Definition: A *vector field* in a region is a function from the region to set of vectors.

Example: In Cartesian coordinate system consider $\boldsymbol{V}(x,y,z) = 2x\hat{\imath} + 3y^2\hat{\jmath} - 4xz\hat{k}$. In this vector field, the point (1,0,0) has the vector $2\hat{\imath}$, (0,1,0) has $3\hat{\jmath}$ and (1,0,1) has $2\hat{\imath} - 4\hat{k}$etc. At different points one can evaluate and see the vector attached to that point.

Before we dive into the mathematics, let me bring broader stage to your attention about why we are doing this. Many different physical quantities have different values at different points in space. For example temperature in kitchen is different at different points – high near the burning gas flame and cool near an open window. Electric field around an electric charge varies – high near the charge and decreases when moved away from it. Gravitational force acting on a satellite depends on its distance from the earth. Velocity of flow of water in a stream is large when it is narrow and small where the stream is wide. In all these problems, there is a particular region of space, which is of interest for the problem at hand. May be we want to study temperature distribution of kitchen or orbit of a satellite or flow of a river. Each of these is called a field – vector field or scalar field depending on whether the physical quantity under study is a vector or scalar. For this course all fields are in 2 or 3 dimensions and differentiable as many times as we want. What this means is that there are no sudden changes and sharp turns.

The three important associated mathematical concepts are **gradient** of a scalar field, **divergence** of a vector field and **curl** of a vector field. Let us understand each of these carefully.

Suppose a marble is released from a point on an uneven surface like a hill. Of course, the marble will roll *straight down* the hill, but this *straight down* keeps changing from point to point. Essentially, it falls down in the direction in which height changes most rapidly. This direction is what we call as gradient of the hill at that point. There is nothing sacrosanct about height function. In general, let $\varphi(x, y, z)$ be a scalar field in some region. At a point P the direction in which φ changes maximum and the magnitude of the change represents gradient of φ at that point.

Definition: *Gradient* of a scalar field $\varphi(x, y, z)$ is defined by

$$\text{grad } \varphi = \frac{\partial \varphi}{\partial x}\hat{\imath} + \frac{\partial \varphi}{\partial y}\hat{\jmath} + \frac{\partial \varphi}{\partial z}\hat{k}.$$

Risking repetition I will illustrate this once more. Consider heat distribution in a region in space. This is a scalar field. We all know heat flows from hotter region to colder region. The question we want to ask is: *at any given point, how much heat is flowing and in what direction*? Answer to this question is the gradient at that point.

Observe that grad φ is a vector field.

Example: Let $\varphi(x, y, z) = xy + yz + zx$. Find grad φ at $(1,2, -1)$

By definition, grad $\varphi = \frac{\partial \varphi}{\partial x}\hat{\imath} + \frac{\partial \varphi}{\partial y}\hat{\jmath} + \frac{\partial \varphi}{\partial z}\hat{k}$

$\frac{\partial \varphi}{\partial x} = (y + z), \frac{\partial \varphi}{\partial y} = (x + z)$ and $\frac{\partial \varphi}{\partial z} = (y + x)$. Thus grad $\varphi = (y + z)\hat{\imath} + (x + z)\hat{\jmath} + (y + x)\hat{k}$.
At $(1, 2, -1)$ this become $\hat{\imath} + 3\hat{k}$.

In addition to gradient being denoted by grad φ it is also denoted by $\nabla\varphi$. Due to historical and computational reasons following notational conveniences are developed.

Definition: Let $\boldsymbol{\nabla} = \frac{\partial}{\partial x}\hat{\boldsymbol{\imath}} + \frac{\partial}{\partial y}\hat{\boldsymbol{\jmath}} + \frac{\partial}{\partial z}\hat{\boldsymbol{k}}$

Observe that this $\boldsymbol{\nabla}$ (pronounced nabla) is an operator and not a *true* vector as $\frac{\partial}{\partial x}$etc are not scalars. The result is a scalar only when $\boldsymbol{\nabla}$ is *operated* or *applied* on some real valued function. $\boldsymbol{\nabla}$ is called an operator because when it is applied on a scalar function, it results in a vector. We shall clarify this with one more example.

Example: Find the gradient of $\varphi(x, y, z) = x^2yz$.grad $\varphi = \boldsymbol{\nabla}\, \varphi = \frac{\partial \varphi}{\partial x}\hat{\imath} + \frac{\partial \varphi}{\partial y}\hat{\jmath} + \frac{\partial \varphi}{\partial z}\hat{k}$

$$= \frac{\partial}{\partial x}(x^2yz)\hat{\imath} + \frac{\partial}{\partial y}(x^2yz)\hat{\jmath} + \frac{\partial}{\partial z}(x^2yz)\hat{k}$$

$$= 2xyz\,\hat{\imath} + x^2z\,\hat{\jmath} + x^2y\,\hat{k}$$

Clearly, **∇** φ is a vector field.

I would like to bring it to your notice that **∇** is an operator. It has no meaning on its own; much like *d/dx* has no meaning unless you specify a function *f* in *x* and operate *d/dx* on *f* to obtain *df/dx* which is the derivative of *f* with respect to *x*. In fact, there are many properties of **∇** much like *d/dx*, like linearity, Leibnitz rule, quotient rule which are beyond the scope of this course.

To summarize, gradient of a scalar field is a vector field which denotes the greatest change in scalar field. i.e., at every point, gradient of a scalar field denotes the vector which is the magnitude and direction of the greatest change in the scalar function. If φ is the scalar field,

grad φ denoted by **∇** φ $= \frac{\partial \varphi}{\partial x}\hat{\imath} + \frac{\partial \varphi}{\partial y}\hat{\jmath} + \frac{\partial \varphi}{\partial z}\hat{k}$

So, remember to find gradient of a (scalar) function you differentiate with respect to *x*, put it as *x* –component by writing $\hat{\imath}$ after that and similarly for *y* and *z* components.

Now let us move onto capturing change in a vector field. Imagine a widening river. Think of a small (imaginary) sphere fixed at a point. It is easy to convince that total velocity of water particles coming into the sphere is less than the total velocity of water particles coming out of the sphere. i.e., there is net flow of water *out* of the sphere. In some sense water is becoming less *compressed*, it is spreading out or it is diverging. One says such a vector field has positive divergence. Similarly one has negative divergence when water flow is constricted. Of course one can have 0 divergence scenario too. Essentially divergence captures the difference in outflow to inflow of vectors at a point. Clearly net outflow is a scalar and thus we come to associate a scalar to every point. Once we do this at all point we get a scalar field.

Another physical situation where divergence comes in handy is magnets. Magnetic field near north pole has positive divergence whereas near south pole it has negative divergence.

Mathematical expression for divergence is as follows.

Definition: Let $\boldsymbol{V}(x, y, z) = V_1(x, y, z)\hat{\imath} + V_2(x, y, z)\hat{\jmath} + V_3(x, y, z)\hat{k}$ be a vector field in a region. Then *divergence* of *V* is defined as

$$\text{div}\,\boldsymbol{V} = \frac{\partial}{\partial x}V_1(x, y, z) + \frac{\partial}{\partial y}V_2(x, y, z) + \frac{\partial}{\partial z}V_3(x, y, z).$$

Observe that this expression gives a scalar which is the sum of change in *x*-component of ***V*** with respect to *x*, change in *y*-component of ***V*** with respect to *y* and change in *z*-component of ***V*** with respect to *z*. This is precisely the total change in ***V*** at a point.

In our previous notation this corresponds to the dot product of

$\nabla = \frac{\partial}{\partial x}\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath} + \frac{\partial}{\partial z}\hat{k}$ and $V = V_1\hat{\imath} + V_2\hat{\jmath} + V_3\hat{k}$. Hence

$$\text{div } \boldsymbol{V} = \boldsymbol{\nabla}.\boldsymbol{V} = \left(\frac{\partial}{\partial x}\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath} + \frac{\partial}{\partial z}\hat{k}\right).(V_1\hat{\imath} + V_2\hat{\jmath} + V_3\hat{k}) = \frac{\partial V_1}{\partial x} + \frac{\partial V_2}{\partial y} + \frac{\partial V_3}{\partial z}$$

Example: Let $\boldsymbol{V}(x, y, z) = x^2 z\hat{\imath} + y^2 x\hat{\jmath} + y^2 x\hat{k}$ be a vector field. Its divergence is given by div $\boldsymbol{V} = \frac{\partial V_1}{\partial x} + \frac{\partial V_2}{\partial y} + \frac{\partial V_3}{\partial z}$ where $V_1 = x^2 z$,$V_2 = y^2 x$ and $V_3 = y^2 x$.

Hence $\frac{\partial V_1}{\partial x} = 2xz, \frac{\partial V_2}{\partial y} = 2yz$ and $\frac{\partial V_3}{\partial z} = 2xy$. Then div $\boldsymbol{V} = \frac{\partial V_1}{\partial x} + \frac{\partial V_2}{\partial y} + \frac{\partial V_3}{\partial z} = 2(xz + yz + xy)$.

Example: Find divergence of gradient of $\varphi(x, y, z) = x^3 + y^3 + z^3 - 3xyz$.

Note that here a scalar function φ is given and first we have to find its gradient. We know that

$$\text{grad } \varphi = \boldsymbol{\nabla}\varphi = \frac{\partial \varphi}{\partial x}\hat{\imath} + \frac{\partial \varphi}{\partial y}\hat{\jmath} + \frac{\partial \varphi}{\partial z}\hat{k}$$

$$= \frac{\partial}{\partial x}(x^3 + y^3 + z^3 - 3xyz)\hat{\imath} + \frac{\partial}{\partial y}(x^3 + y^3 + z^3 - 3xyz)\hat{\jmath} + \frac{\partial}{\partial z}(x^3 + y^3 + z^3 - 3xyz)\hat{k}$$

$$= 3(x^2 - yz)\,\hat{\imath} + 3(y^2 - xz)\,\hat{\jmath} + 3(z^2 - xy)\,\hat{k}$$

Now we have to find div $\boldsymbol{V}$ where

$$\boldsymbol{V} = 3(x^2 - yz)\,\hat{\imath} + 3(y^2 - xz)\,\hat{\jmath} + 3(z^2 - xy)\,\hat{k}$$

div $V = \nabla.V=$

$$\left(\frac{\partial}{\partial x}\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath} + \frac{\partial}{\partial z}\hat{k}\right).(3(x^2 - yz)\hat{\imath} + 3(y^2 - xz)\hat{\jmath} + 3(z^2 - xy)\hat{k})$$

$$= \frac{\partial}{\partial x}3(x^2 - yz) + \frac{\partial}{\partial y}3(y^2 - xz) + \frac{\partial}{\partial z}3(z^2 - xy) = 3(2x) + 3(2y) + 3(2z)$$

$$= 6(x + y + z)$$

Here is terminology. A vector field is said to be **solenoidal** if its divergence is identically zero. This means that total outflow of the field is equal to the total inflow at every point. Trivial example is that of a constant vector field. Another example is the magnetic field in the region of perpendicular bisector of a bar magnet. Terminology comes from the fact that when an electric current passes through a solenoid, the magnetic field in the interior of the coil has no divergence.

Example: Show that $\boldsymbol{V}(x,y,z) = 3y^4z^2\hat{\imath} + 4x^3z^2\hat{\jmath} + 3x^2y^2\hat{k}$ is solenoidal.

div $\boldsymbol{V} = \nabla.\boldsymbol{V}= \left(\frac{\partial}{\partial x}\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath} + \frac{\partial}{\partial z}\hat{k}\right).\left(3y^4z^2\hat{\imath} + 4x^3z^2\hat{\jmath} + 3x^2y^2\hat{k}\right)$

$$= \frac{\partial}{\partial x}(3y^4z^2) + \frac{\partial}{\partial y}(4x^3z^2) + \frac{\partial}{\partial z}3(x^2y^2) = 0 + 0 + 0 = 0.$$

Hence, $\boldsymbol{V}$ is solenoidal.

Example: Find the value of a so that $\boldsymbol{V}(x,y,z) = (x+3y)\hat{\imath} + (y-2z)\hat{\jmath} + (x-az)\hat{k}$ is solenoidal.

Again div $\boldsymbol{V} = \nabla.\boldsymbol{V}=\left(\frac{\partial}{\partial x}\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath} + \frac{\partial}{\partial z}\hat{k}\right).\left((x+3y)\hat{\imath} + (y-2z)\hat{\jmath} + (x-az)\hat{k}\right)$

$= \frac{\partial}{\partial x}(x+3y) + \frac{\partial}{\partial y}(y-2z) + \frac{\partial}{\partial z}(x-az) = 1 + 1 - a = 2 - a.$

If $\boldsymbol{V}$ has to be solenoidal, div $\boldsymbol{V}$ = 0 and hence $2 - a$ = 0 which means a = 2.

Now let us move onto the concept of directional derivative. Let $\varphi(x,y,z)$ be scalar field in a region. Let P be a point in that region. Moving in different directions from P, φ could change by different amounts. Change in φ in the direction of $\boldsymbol{v} = v_1\hat{\imath} + v_2\hat{\jmath} + v_3\hat{k}$ is called directional derivative of φ with respect to $\boldsymbol{v}$. It is denoted by

$$D_v(\varphi) = \lim_{s\to 0}\frac{\varphi(Q) - \varphi(P)}{s}$$

where Q is a variable point on the ray C in the direction of v.

Definition: Let P be a point in a region where the scalar field $\varphi(x,y,z)$ is defined. For any vector v, *directional derivative* of $\varphi(x,y,z)$ in the direction of v is defined to be $D_v(\varphi) = \text{grad}(\varphi).\hat{v}$ where $\hat{v}$ is the unit vector in the direction of v.

An example should clarify this

Example: Find the directional derivative of $\varphi(x,y,z) = 2xy + 3y^2 - z^3$ at the point P(1, −1,2) in the direction of the vector $\boldsymbol{v} = \hat{\imath} - \hat{\jmath} + \hat{k}$.

First we find that gradφ=$\frac{\partial\phi}{\partial x}\hat{\imath} + \frac{\partial\phi}{\partial y}\hat{\jmath} + \frac{\partial\phi}{\partial z}\hat{k} = 2y\hat{\imath} + 6y\hat{\jmath} + (-3z^2)\hat{k}$. This evaluated at (1, −1,2) gives grad $\varphi|_{(1,-1,2)} = -2\hat{\imath} - 6y\hat{\jmath} - 12\hat{k}$.

The unit vector in the direction of ***v*** is $\hat{\boldsymbol{v}} = \frac{v}{\sqrt{3}} = \frac{\hat{\imath}-\hat{\jmath}+\hat{k}}{\sqrt{3}}$.

Thus $D_v(\varphi) = \text{grad}(\varphi).\hat{\boldsymbol{v}} = (-2\hat{\imath} - 6y\hat{\jmath} - 12\hat{k}).\left(\frac{\hat{\imath}-\hat{\jmath}+\hat{k}}{\sqrt{3}}\right) = \frac{-2+6-12}{\sqrt{3}} = -\frac{8}{\sqrt{3}}$.

So, anytime you want to find the directional derivative of a scalar field in the direction of *v*, just take the dot product of grad (φ) and $\hat{\boldsymbol{v}}$.

Another geometric property of gradient we need to understand is that whenever it is non-zero, it is normal to a particularly important surface. Let me explain this in detail. Let $\varphi(x, y, z)$be a scalar field. Consider *S* = {(*x*,*y*,*z*) | $\varphi(x, y, z)$= *c*}. This means *S* is the set of all points (x, y, z)in the region where φ will take value *c*. In case of height function you may think of this as all points with same height. In case of temperature distribution, you may think of this as isotherms. One can show that such points form a surface. Gradient (if it is non-zero) at any point on *S* will be the normal to the surface S at that point. Proof of this is beyond the scope of this talk, but we will illustrate this using a couple of examples.

Example: Find normal to the surface $x^2 + y^2 + z^2 = 4$ at the point (2, 0, 0).

Those of you who have understood your analytical geometry well will see that the given surface is a sphere centered at origin with radius 2. At (2, 0, 0) the normal to the surface is the vector (1, 0, 0). We will see how it is possible to conclude this geometric insight using gradient computations.

Take $\varphi(x, y, z) = x^2 + y^2 + z^2 - 4$

grad φ =∇ φ =$\frac{\partial\varphi}{\partial x}\hat{\imath} + \frac{\partial\varphi}{\partial y}\hat{\jmath} + \frac{\partial\varphi}{\partial z}\hat{k}$

$= \frac{\partial}{\partial x}(x^2 + y^2 + z^2 - 4)\hat{\imath} + \frac{\partial}{\partial y}(x^2 + y^2 + z^2 - 4)\hat{\jmath} + \frac{\partial}{\partial z}(x^2 + y^2 + z^2 - 4)\hat{k}$

$= 2x\,\hat{\imath} + 2y\,\hat{\jmath} + 2z\,\hat{k}$

At (2,0,0), grad φ =4$\hat{\imath}$

Required normal is the unit vector in the direction of grad φ which is $\frac{4\hat{\imath}}{4} = \hat{\imath} = (1,0,0)$ as expected.

Example: Find the unit normal vector of $z^2 = 4(x^2 + y^2)$ at (1, 0, 2).

Let $\varphi(x, y, z) = z^2 - 4(x^2 + y^2)$

grad φ =∇ φ =$\frac{\partial\varphi}{\partial x}\hat{\imath} + \frac{\partial\varphi}{\partial y}\hat{\jmath} + \frac{\partial\varphi}{\partial z}\hat{k}$

$= \frac{\partial}{\partial x}(z^2 - 4(x^2 + y^2))\hat{\imath} + \frac{\partial}{\partial y}(z^2 - 4(x^2 + y^2))\hat{\jmath} + \frac{\partial}{\partial z}(z^2 - 4(x^2 + y^2))\hat{k}$

$= (-8x)\hat{\imath} + (-8y)\hat{\jmath} + (2z)\ \hat{k}$

At (1,0,2), grad φ $= -\ 8\hat{\imath} + 4\hat{k}$.

Unit vector in the direction of grad φ =$\frac{1}{\sqrt{(64+16)}}\left(-8\hat{\imath} + 4\hat{k}\right) = \frac{1}{\sqrt{5}}\left(-2\hat{\imath} + \hat{k}\right)$

Example: Find the angle between the surfaces $x \log z = y^2 - 1$ and $x^2 y = 2 - z$ at the point (1,1,1).

First note that (1,1,1) is indeed a point on both the surfaces by substituting *x* = *y* = *z* = 1 in both the surfaces. Next recall that angle between the surfaces at a point of their intersection is defined to be the angle between the normals to the surfaces at the point of intersection. Thus we need to find the normals to both the surfaces at (1,1,1).

Let $\varphi(x, y, z) = x \log z - y^2 + 1$ and $\psi(x, y, z) = x^2 y - 2 + z$

grad φ =∇ φ =$\frac{\partial\varphi}{\partial x}\hat{\imath} + \frac{\partial\varphi}{\partial y}\hat{\jmath} + \frac{\partial\varphi}{\partial z}\hat{k}$

$= \frac{\partial}{\partial x}(x \log z - y^2 + 1)\hat{\imath} + \frac{\partial}{\partial y}(x \log z - y^2 + 1)\hat{\jmath} + \frac{\partial}{\partial z}(x \log z - y^2 + 1)\hat{k}$

$= (\log z)\hat{\imath} + (-2y)\hat{\jmath} + \left(\frac{x}{z}\right)\hat{k}$

$u =$ grad φ at (1,1,1) is $-2\hat{\jmath} + \hat{k}$.

grad ψ =$\nabla\psi$ =$\frac{\partial\psi}{\partial x}\hat{\imath} + \frac{\partial\psi}{\partial y}\hat{\jmath} + \frac{\partial\psi}{\partial z}\hat{k}$

$= \frac{\partial}{\partial x}(x^2 y - 2 + z)\hat{\imath} + \frac{\partial}{\partial y}(x^2 y - 2 + z)\hat{\jmath} + \frac{\partial}{\partial z}(x^2 y - 2 + z)\hat{k}$

$= 2xy\hat{\imath} + (x^2)\hat{\jmath} + (1)\hat{k}$

$v =$ grad ψ at (1,1,1) is $2\hat{\imath} + \hat{\jmath} + \hat{k}$.

We know that $u.v = |u|.|v| \cos\theta$ where θ is the angle between u and v.

From the above $u.v = -2 + 1 = -1$, $|u| = \sqrt{5}$ and $v = \sqrt{6}$.

Thus $\theta = \cos^{-1}\left(-\frac{1}{30}\right)$.

Now we turn our attention to curl of a vector field.

Definition: Let $\boldsymbol{V}(x, y, z) = V_1(x, y, z)\hat{\imath} + V_2(x, y, z)\hat{\jmath} + V_3(x, y, z)\hat{k}$ be a vector field in some region. ThenCurl $\boldsymbol{V} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_1 & V_2 & V_3 \end{vmatrix}$

$$= \left(\frac{\partial V_3}{\partial y} - \frac{\partial V_2}{\partial z}\right)\hat{\imath} + \left(\frac{\partial V_1}{\partial z} - \frac{\partial V_3}{\partial x}\right)\hat{\jmath} + \left(\frac{\partial V_2}{\partial x} - \frac{\partial V_1}{\partial y}\right)\hat{k}$$

What exactly this unnerving, but algebraically symmetric definition capture physically? Assume the above vector field represents flow of water in some region of space. Imagine a tiny sphere whose center is fixed at a point. Of course, if the sphere is not fixed, it would flow away with the flow. So imagine it is *fixed* to a point by an imaginary pin. Because of push and pulls of the flow, this sphere would rotate along some axis passing through the center of the sphere. The speed of rotation and the axis of rotation is what is captured by Curl. Proof of this beyond the scope of this presentation, but keep in mind when you are dealing with curl. Computation of curl is straightforward from definition.

Example: If $\boldsymbol{V}$ is a constant vector field, meaning if V_1, V_2 and V_3 are all some constants (real numbers), then clearly Curl $V = 0$

Example: Compute curl of $\boldsymbol{V}(x,y,z) = z\hat{\imath} + x\hat{\jmath} + y\hat{k}$.

$$\text{Curl } V = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_1 & V_2 & V_3 \end{vmatrix} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z & x & y \end{vmatrix} = \left(\frac{\partial y}{\partial y} - \frac{\partial x}{\partial z}\right)\hat{\imath} + \left(\frac{\partial z}{\partial z} - \frac{\partial y}{\partial x}\right)\hat{\jmath} + \left(\frac{\partial x}{\partial x} - \frac{\partial z}{\partial y}\right)\hat{k}$$

$$= \hat{\imath} + \hat{\jmath} + \hat{k}.$$

Example: Compute curl of $\boldsymbol{V}(x, y, z) = xyz^2\hat{\imath} + xy^2z\hat{\jmath} + x^2yz\hat{k}$.

$$\text{Curl } \boldsymbol{V} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_1 & V_2 & V_3 \end{vmatrix} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xyz^2 & xy^2z & x^2yz \end{vmatrix}$$

$$= \left(\frac{\partial}{\partial y}(x^2yz) - \frac{\partial}{\partial z}(xy^2z)\right)\hat{\imath} + \left(\frac{\partial}{\partial z}(xyz^2) - \frac{\partial}{\partial x}(x^2yz)\right)\hat{\jmath}$$

$$+ \left(\frac{\partial}{\partial x}(xy^2z) - \frac{\partial}{\partial y}(xyz^2)\right)\hat{k}$$

$$= (x^2z - xy^2)\hat{\imath} + (2xyz - 2xyz)\hat{\jmath} + (y^2z - xz^2)\hat{k}$$

$$= (x^2z - xy^2)\hat{\imath} + (y^2z - xz^2)\hat{k}$$

Here is an obvious terminology. If the curl of a vector field is identically 0 (ie 0 everywhere), then such a vector field is called an *irrotational* vector field.

Example: $\boldsymbol{V}(x, y, z) = (4xy - z^3)\hat{\imath} + 2x^2\hat{\jmath} - 3xz^2\hat{k}$.

$$\text{Curl } V = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_1 & V_2 & V_3 \end{vmatrix} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 4xy - z^3 & 2x^2 & -3xz^2 \end{vmatrix}$$

$$= \left(\frac{\partial}{\partial y}(-3xz^2) - \frac{\partial}{\partial z}(2x^2)\right)\hat{\imath} + \left(\frac{\partial}{\partial z}(4xy - z^3) - \frac{\partial}{\partial x}(-3xz^2)\right)\hat{\jmath}$$

$$+ \left(\frac{\partial}{\partial x}(2x^2) - \frac{\partial}{\partial y}(4xy - z^3)\right)\hat{k}$$

$$= (0 - 0)\hat{\imath} + (-3z^2 - (-3z^2))\hat{\jmath} + (4x - 4x)\hat{k} = 0$$

Example: Show that the vector field $\mathbf{r}/r^3$ is both solenoidal and irrotational**.**

Whenever you see **r** in this course, remember it stands for the position vector defined by $\mathbf{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$. And $r = \sqrt{x^2 + y^2 + z^2}$. Thus, the given vector field is

$$\boldsymbol{V}(x, y, x) = \frac{x}{\left(\sqrt{x^2+y^2+z^2}\right)^3}\hat{\imath} + \frac{y}{\left(\sqrt{x^2+y^2+z^2}\right)^3}\hat{\jmath} + \frac{z}{\left(\sqrt{x^2+y^2+z^2}\right)^3}\hat{k}$$

$$= \frac{x}{(x^2+y^2+z^2)^{3/2}}\hat{\imath} + \frac{y}{(x^2+y^2+z^2)^{3/2}}\hat{\jmath} + \frac{z}{(x^2+y^2+z^2)^{3/2}}\hat{k}$$

div $\boldsymbol{V} = \boldsymbol{\nabla}.\boldsymbol{V}$

$$= \left(\frac{\partial}{\partial x}\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath} + \frac{\partial}{\partial z}\hat{k}\right).\left(\frac{x}{(x^2+y^2+z^2)^{\frac{3}{2}}}\hat{\imath} + \frac{y}{(x^2+y^2+z^2)^{\frac{3}{2}}}\hat{\jmath} + \frac{z}{(x^2+y^2+z^2)^{\frac{3}{2}}}\hat{k}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{x}{(x^2+y^2+z^2)^{\frac{3}{2}}}\right) + \frac{\partial}{\partial y}\left(\frac{x}{(x^2+y^2+z^2)^{\frac{3}{2}}}\right) + \frac{\partial}{\partial z}\left(\frac{x}{(x^2+y^2+z^2)^{\frac{3}{2}}}\right) =$$

$$\frac{(x^2+y^2+z^2)^{\frac{3}{2}}.\frac{\partial}{\partial x}(x) - x.\frac{3}{2}(x^2+y^2+z^2)^{\frac{1}{2}}.2x}{(x^2+y^2+z^2)^3} + \frac{(x^2+y^2+z^2)^{\frac{3}{2}}.\frac{\partial}{\partial y}(y) - y.\frac{3}{2}(x^2+y^2+z^2)^{\frac{1}{2}}.2y}{(x^2+y^2+z^2)^3} +$$

$$\frac{(x^2+y^2+z^2)^{\frac{3}{2}}.\frac{\partial}{\partial z}(z) - z.\frac{3}{2}(x^2+y^2+z^2)^{\frac{1}{2}}.2z}{(x^2+y^2+z^2)^3} = \frac{(x^2+y^2+z^2)^{\frac{1}{2}}}{(x^2+y^2+z^2)^3}\left[\left(\left((x^2+y^2+z^2) - 3x^2\right) + \left((x^2+y^2+z^2) - 3y^2\right) + \left((x^2+y^2+z^2) - 3z^2\right)\right)\right]$$

= 0

Since the divergence of $\boldsymbol{V}$ is zero, it is solenoidal.

$$\text{Curl } \boldsymbol{V} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_1 & V_2 & V_3 \end{vmatrix} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{x}{(x^2+y^2+z^2)^{3/2}} & \frac{y}{(x^2+y^2+z^2)^{3/2}} & \frac{z}{(x^2+y^2+z^2)^{3/2}} \end{vmatrix}$$

$$= \left(\frac{\partial}{\partial y}\left(\frac{z}{(x^2+y^2+z^2)^{3/2}}\right) - \frac{\partial}{\partial z}\left(\frac{y}{(x^2+y^2+z^2)^{3/2}}\right)\right)\hat{\imath}$$

$$+ \left(\frac{\partial}{\partial z}\left(\frac{x}{(x^2+y^2+z^2)^{3/2}}\right) - \frac{\partial}{\partial x}\left(\frac{z}{(x^2+y^2+z^2)^{3/2}}\right)\right)\hat{\jmath}$$

$$+ \left(\frac{\partial}{\partial x}\left(\frac{y}{(x^2+y^2+z^2)^{3/2}}\right) - \frac{\partial}{\partial y}\left(\frac{x}{(x^2+y^2+z^2)^{3/2}}\right)\right)\hat{k}$$

$$= \left(\frac{0 - z\frac{\partial}{\partial y}(x^2+y^2+z^2)^{3/2}}{(x^2+y^2+z^2)^3} - \frac{0 - y\frac{\partial}{\partial z}(x^2+y^2+z^2)^{3/2}}{(x^2+y^2+z^2)^3}\right)\hat{\imath}$$

$$+ \left(\frac{0 - x\frac{\partial}{\partial z}(x^2+y^2+z^2)^{3/2}}{(x^2+y^2+z^2)^3} - \frac{0 - z\frac{\partial}{\partial x}(x^2+y^2+z^2)^{3/2}}{(x^2+y^2+z^2)^3}\right)\hat{\jmath}$$

$$+ \left(\frac{0 - y\frac{\partial}{\partial x}(x^2+y^2+z^2)^{3/2}}{(x^2+y^2+z^2)^3} - \frac{0 - x\frac{\partial}{\partial y}(x^2+y^2+z^2)^{3/2}}{(x^2+y^2+z^2)^3}\right)\hat{k}$$

$$= \left(\frac{-3zy}{(x^2+y^2+z^2)^{\frac{5}{2}}} - \frac{-3zy}{(x^2+y^2+z^2)^{\frac{5}{2}}}\right)\hat{\imath} + \left(\frac{-3xz}{(x^2+y^2+z^2)^{\frac{5}{2}}} - \frac{-3xz}{(x^2+y^2+z^2)^{\frac{5}{2}}}\right)\hat{\jmath}$$

$$+ \left(\frac{-3xy}{(x^2+y^2+z^2)^{\frac{5}{2}}} - \frac{-3xy}{(x^2+y^2+z^2)^{\frac{5}{2}}}\right)\hat{k} = 0$$

Since Curl $\boldsymbol{V}$ is zero, $\boldsymbol{V}$ is irrotational.

We now look at integration over vector fields. Let $\boldsymbol{F}(x,y,z) = F_1(x,y,z)\hat{\imath} + F_2(x,y,z)\hat{\jmath} + F_3(x,y,z)\hat{k}$ be a vector field in some region of space. Let C be a path in the region. We want to make sense of integral of $\boldsymbol{F}(x,y,z)$ along the path C. Naively we want to sum up values of $\boldsymbol{F}$ along C. Rephrasing this amounts to asking sum of components of $\boldsymbol{F}$ along C. What this means is: Take component of $\boldsymbol{F}$ along tangent to C and sum them all. To take tangent to C at any point we take a parameterization of C. Let $\boldsymbol{r}(x,y,z) = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$ be the position vector of any point on C. Then tangent vector to C at (x,y,z) would be $\boldsymbol{dr} = dx\hat{\imath} + dy\hat{\jmath} + dz\hat{k}$. Taking component of $\boldsymbol{F}$ along C means taking dot product of $\boldsymbol{F}$ and $\boldsymbol{dr}$. Thus we have the following definition.

Definition: Let $\boldsymbol{F}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}) = F_1(x,y,z)\hat{\imath} + F_2(x,y,z)\hat{\jmath} + F_3(x,y,z)\hat{k}$ be a vector field in some region of space. Let *C* be a path in the region. Then *line integral* of $\boldsymbol{F}$ over *C* is defined by

$$\int_C \boldsymbol{F}.\boldsymbol{dr} = \int_C \left(F_1(x,y,z)\hat{\imath} + F_2(x,y,z)\hat{\jmath} + F_3(x,y,z)\hat{k}\right).(dx\hat{\imath} + dy\hat{\jmath} + dz\hat{k})$$

$$= \int_C (F_1(x,y,z)dx + F_2(x,y,z)dy + F_3(x,y,z)dz)$$

To give a more physical intuition, if $\boldsymbol{F}(x,y,z)$ denotes a fluid flow in some region and *C* is a path in that region, the above integral gives us a measure of total component of $\boldsymbol{F}$along *C*.

Example: Consider the constant vector field $\boldsymbol{F}(x,y,z) = \hat{\imath}$. Physically this means that flow is in the *x*-direction and magnitude of vector at every point is 1. Let *C* denote the straight line from (0,0,0) to (1,0,0). Then $\boldsymbol{r}(x,y,z) = x\hat{\imath}$ and hence $\boldsymbol{dr} = dx\hat{\imath}$. Therefore

$\int_C F.dr = \int_C \hat{\imath}.dx\hat{\imath} = \int_{x=0}^{x=1} dx = x|_0^1 = 1 - 0 = 1.$

Example: Consider the constant 2 dimensional vector field $\boldsymbol{F}(\boldsymbol{x},\boldsymbol{y}) = \hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}}$. Let us find the integral of this along the path $\boldsymbol{y} = \boldsymbol{x}$ from (0,0) to (1,1). In this we note that $\boldsymbol{z}$ and hence $\boldsymbol{dz}$ are both zero. Also, since $\boldsymbol{y} = \boldsymbol{x}$, we have $\boldsymbol{dy} = \boldsymbol{dx}$. Further, as one moves from (0,0) to (1,0), $\boldsymbol{x}$ changes from 0 to 1.Thus

$$\int_C \boldsymbol{F}.\boldsymbol{dr} = \int_{(0,0)}^{(1,1)} (\hat{\imath} + \hat{\jmath}).(dx\hat{\imath} + dy\hat{\jmath}) = \int_{(0,0)}^{(1,1)} (dx + dy) = \int_{x=0}^{x=1} (dx + dx) = 2.$$

For the same vector field, let us integral along the path from (0,0) to (1,1) along $\boldsymbol{y} = \boldsymbol{x}^2$. In this case $\boldsymbol{dy} = \boldsymbol{2xdx}$ and as one moves from (0,0) to (1,0), $\boldsymbol{x}$ changes from 0 to 1. Thus

$$\int_C \boldsymbol{F}.\boldsymbol{dr} = \int_{(0,0)}^{(1,1)} (\hat{\imath} + \hat{\jmath}).(dx\hat{\imath} + dy\hat{\jmath}) = \int_{(0,0)}^{(1,1)} (dx + dy) = \int_{x=0}^{x=1} (dx + 2xdx) = (x + x^2)|_0^1 = 2.$$

Observe that the value of the integral is same in either of the paths.

Example: Let $\boldsymbol{F}(x,y) = 3xy\hat{\imath} - y^2\hat{\jmath}$. Find $\int_{(0,0)}^{(1,2)} \boldsymbol{F}.\boldsymbol{dr}$ along the curve *C* given by $y = 2x^2$.

Here $\boldsymbol{dy = 4xdx}$ and as one moves from (0,0) to (1,2), $\boldsymbol{x}$ changes from 0 to 1. Thus

$$\int_C \boldsymbol{F.dr} = \int_{(0,0)}^{(1,2)} (3xy\hat{\imath} - y^2\hat{\jmath}).(dx\hat{\imath} + dy\hat{\jmath})$$

$$= \int_{(0,0)}^{(1,2)} (3xydx - y^2dy) = \int_{(0,0)}^{(1,2)} (3x(2x^2)dx - (2x^2)^2(4xdx))$$

$$= \int_{x=0}^{x=1} (6x^3 - 16x^5)dx = -\frac{7}{6}.$$

Often we need to consider paths such that the starting and ending points of the path coincide and we end up with a *simple loop*. In this case the integral over the entire loop is denoted by the symbol$\oint \boldsymbol{F.dr}$ to emphasize that we are considering a closed loop as against open ended path.

Example: Find$\oint \boldsymbol{F.dr}$along the whole circle $\boldsymbol{x^2 + y^2 = a^2}$ when $\boldsymbol{F(x,y) = \sin y\, \hat{\imath} + x(1 + \cos y)\hat{\jmath}}$.

Parameterization for the circle is $x = a\cos t$ and $y = a \sin t$. To get the whole circle, *t* need to vary from 0 to 2π. Also $dx = -a\sin t\ \ dt$ and $dy = a\cos t\ \ dt$. Thus

$$\oint \boldsymbol{F.dr} = \oint (\sin y\,\hat{\imath} + x(1 + \cos y)\hat{\jmath}).(dx\hat{\imath} + dy\hat{\jmath})$$

$$= \oint (\sin y\, dx + x(1 + \cos y)dy) = \oint d(x \sin y) + xdy$$

$$= \int_{t=0}^{2\pi} d(a\cos t \sin(a \sin t)) + a\ \cos t\,(a\cos t)dt$$

$$= 0 + \int_{t=0}^{2\pi} a^2(\cos t)^2\, dt = \frac{a^2}{2}\int_0^{2\pi} (1 + \cos 2t) = \frac{a^2}{2}\left[t + \frac{\sin 2t}{2}\right]|_0^{2\pi} = \pi a^2.$$

Example: Find the work done in moving a particle in the force field $F(x, y, z) = 3x^2\hat{\imath} + (2xz - y)\hat{\jmath} + z\hat{k}$ along (a) C_1: a straight line from (0,0,0) to (2,1,3) and (b) C_2: the curve defined by $x^2 = 4y, 3x^3 = 8z$ from *x* = 0 to *x* = 2.

Recall that work done in moving a particle along a curve *C* is the integral$\int \boldsymbol{F.dr}$ along the curve *C*.

For part (a), parameterization is got by $\frac{x}{2} = \frac{y}{1} = \frac{z}{3} = t$. Then $x = 2t, y = t$ and $z = 3t$ which implies $dx = 2dt, dy = dt$ and $dz = 3dt$. When *x* varies from 0 to 2, *t* varies from 0 to 1. Thus along C_1

$$\int \boldsymbol{F}.\boldsymbol{dr} = \int (3x^2\hat{\imath} + (2xz - y)\hat{\jmath} + z\hat{k}).(dx\hat{\imath} + dy\hat{\jmath}) = \int 3x^2dx + (2xz - y)dy + zdz$$

$$= \int_{t=0}^{t=1} 3(2t)^2 2dt + (2(2t)(3t) - t)\,dt + (3t)3dt$$

$$= \int_{t=0}^{t=1} (36t^2 + 8t)dt = 12t^3 + 4t^2|_0^1 = 16$$

For part (b) parameterization is got by putting $x = t, y = \frac{t^2}{4}$ and $z = \frac{3t^3}{8}$. Then $dx = dt, dy = \frac{tdt}{2}$ and $dz = \frac{9}{8}t^2dt$ and *t* varies from 0 to 2. Thus along C_2

$$\int \boldsymbol{F}.\boldsymbol{dr} = \int (3x^2\hat{\imath} + (2xz - y)\hat{\jmath} + z\hat{k}).(dx\hat{\imath} + dy\hat{\jmath}) = \int 3x^2dx + (2xz - y)dy + zdz$$

$$= \int_{t=0}^{t=1} 3(t)^2dt + \left(2(t)\left(\frac{3t^3}{8}\right) - \frac{t^2}{4}\right)\frac{tdt}{2} + \left(\frac{3t^3}{8}\right)\frac{9}{8}t^2dt = 16$$

Again, we note that in either of the paths the work done is the same.

Example: Compute $\oint ydx - xdy$ along *C* where *C* is the triangle whose vertices are (1,0), (0,1) and $(-1,0)$.

Since *C* is piecewise smooth, we have the following result.

$\oint ydx - xdy = \int_{(-1,0)}^{(1,0)} (ydx - xdy) + \int_{(1,0)}^{(0,1)} (ydx - xdy) + \int_{(0,1)}^{(-1,0)} (ydx - xdy)$ where each path is the straight line joining appropriate starting and end points.

Observe that for the 1st integral in RHS, *y* =0 and *dy* = 0. Thus the 1st integral is zero.

For the second integral, parameterization is got by writing the equation of the line joining (1,0) and (0,1) which is $y = 1 - x$. Putting $x = t$, we have $y = 1 - t$ and hence $dx = dt$ and $dy = -dt$. Also t varies from 1 to 0. Thus

$$\int_{(1,0)}^{(0,1)} (ydx - xdy) = \int_{t=1}^{t=0} (1 - t)dt - t(-dt)) = \int_1^0 dt = -1$$

For the third integral, parameterization is got by writing the equation of the line joining (0,1) and $(-1,0)$which is $y = 1 + x$. Putting $x = t$, we have $y = 1 + t$ and hence $dx = dt$ and $dy = dt$. Also t varies from 0 to -1. Thus

$$\int_{(0,1)}^{(-1,0)} (ydx - xdy) = \int_{t=0}^{t=-1} (1+t)dt - t(dt)) = \int_0^{-1} dt = -1$$

Thus

$$\oint ydx - xdy = \int_{(-1,0)}^{(1,0)} (ydx - xdy) + \int_{(1,0)}^{(0,1)} (ydx - xdy) + \int_{(0,1)}^{(-1,0)} (ydx - xdy) = 0 - 1 - 1 = -2$$

Now let us turn our attention towards understanding the three of the fundamental theorems of vector calculus. In fact these are the generalizations of the fundamental theorem of Calculus. Fundamental theorem of calculus of one variable says$\int_a^b df = f(b) - f(a)$. What this says is the definite integral of differential of a function between two values can be expressed in terms of the values of the function at the two values which are the boundary points of the interval over which the integration is being carried out. Below we make analogous statements for multivariable case. We start with Green's theorem, a generalization to the 2 dimensional case.

Green's theorem: Let $\boldsymbol{F}(x,y) = F_1(x,y)\hat{\imath} + F_2(x,y)\hat{\jmath}$ be a *smooth* vector field in a region R. Let C be a simple closed curve (*loop*) in R enclosing a region D. Then

$$\iint_D \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dA = \oint_C \boldsymbol{F}.\boldsymbol{dr}.$$

Note that LHS is a surface integral and RHS is a line integral and C is the boundary of D. It is instructive to stare at this formula for some time and convince yourself that it is indeed a generalization of the fundamental theorem of calculus of one variable. Usefulness of this theorem stems from the fact that it equates a surface integral to a line integral; it may happen in some situations that it is computationally simpler to carry out one rather than the other. For physical interpretation of this visit www.mathinsight.org

Example: Evaluate $\oint_C \boldsymbol{F}.\boldsymbol{dr}$ where $\boldsymbol{F}(x,y) = y^2\hat{\imath} + 3xy\hat{\jmath}$ and C is the counterclockwise oriented boundary of the upper-half unit disk D.

Here $F_1(x,y) = y^2$ and $F_2(x,y) = 3xy$. So $\frac{\partial F_1}{\partial y} = 2y$ and $\frac{\partial F_2}{\partial x} = 3y$. Using these in Green's theorem we have

$$\oint_C \boldsymbol{F}.\boldsymbol{dr} = \iint_D \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dA = \iint_D y\, dA = \int_{-1}^{1}[\int_0^{\sqrt{1-x^2}} y dy]\, dx = \frac{2}{3}.$$

One of the fallouts of Green's theorem is that it may be used to find areas of smooth shapes. The heart of the matter is that if we can choose $\boldsymbol{F}(x,y)$ such that $\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) = 1$, then surface integral in Green's theorem is nothing but the area of D. Then one can parameterize C and evaluate the line integral on the RHS to get the area. The following example should clarify this idea.

Example: Find the area of the disk D of radius *a* defined by $x^2 + y^2 \leq a^2$.

We first try to find a $\boldsymbol{F}(x,y)$ such that $\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) = 1$. There are many candidates for this, but we stick to $\boldsymbol{F}(x,y) = -\frac{y}{2}\hat{\imath} + \frac{x}{2}\hat{\jmath}$. Parameterization of circle is given by $x = a\cos t\,, y = a\sin t$ where *t* varies from 0 to 2π. Then $dx = -a\sin t\, dt, dy = a\cos t\, dt$. By Green's theorem

$$\iint_D \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dA = \oint_C \boldsymbol{F}.\boldsymbol{dr}.$$

$$\iint_D dA = \oint_C (-\frac{y}{2}\hat{\imath} + \frac{x}{2}\hat{\jmath}).(dx\hat{\imath} + dy\hat{\jmath}) = \oint_C \left(-\frac{y}{2}dx + \frac{x}{2}dy\right)$$

$$A = \frac{1}{2}\int_{t=0}^{2\pi}(-a\sin t)(-a\sin t)dt + (a\cos t)(a\cos t)dt = \pi a^2.$$

Example: Find the area enclosed between the parabolas $x^2 = 4y$ and $y^2 = 4x$.

Draw the two parabolas and find that the points of intersections are (0,0) and (4,4). Denote the enclosed region by D and the two parabolas by C_1 and C_2.

As before we choose $\boldsymbol{F}(x,y) = -\frac{y}{2}\hat{\imath} + \frac{x}{2}\hat{\jmath}$ so that $\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) = 1$. Parameterization for C_1 is given by $x = 4t, y = \frac{x^2}{4} = 4t^2$. So $dx = 4dt, dy = 8tdt$.

When *x* varies from 0 to 4, *t* varies from 0 to 1. Similarly for C_2 $y = 4t, x = \frac{y^2}{4} = 4t^2$. So $dx = 8tdt, dy = 4dt$. When *x* varies from 4 to 0, *t* varies from 1 to 0. Let C denote the closed curve from (0,0) to (4,4) along C_1 and from (4,4) to (0,0) along C_2. Dente the required area by *A*.

By Green's theorem we have

$$\iint_D \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dA = \oint_C \boldsymbol{F}.\boldsymbol{dr}.$$

$$\iint_D dA = \oint_C \left(-\frac{y}{2}\hat{\imath} + \frac{x}{2}\hat{\jmath}\right).(dx\hat{\imath} + dy\hat{\jmath}) = \oint_C \left(-\frac{y}{2}dx + \frac{x}{2}dy\right)$$

$$A = \frac{1}{2}\oint_C (-ydx + xdy) = \frac{1}{2}\int_{C_1} (-ydx + xdy) + \frac{1}{2}\int_{C_2} (-ydx + xdy)$$

$$\int_{C_1} (-ydx + xdy) = \int_{t=0}^{1} (-4t^2)(4dt) + (4t)(8tdt) = \frac{16}{3}$$

$$\int_{C_2} (-ydx + xdy) = \int_{t=1}^{0} (-4t)(8tdt) + (4t^2)(4dt) = \frac{16}{3}$$

Thus $A = \frac{1}{2}\left(\frac{16}{3} + \frac{16}{3}\right) = \frac{16}{3}$.

Intuitively Green's theorem can be explained as follows. Imagine a thin sheet of metal plate with uneven temperature distribution. Consider a region on the plate whose boundary is a loop. Then Green's theorem says that total amount of heat circulating in the region is equal to the total amount of heat circulating on the boundary.

Stokes' Theorem which is a three dimensional generalization says that this plate need not be in a plane, but a curved surface in 3 dimensions. This kind of intuitive explanations help in understanding concepts, but should not be taken literally as we have not developed enough mathematical machinery. So without much ado, we will state Stokes' theorem.

Stokes' Theorem: Let $\boldsymbol{F}(x,y,z) = F_1(x,y,z)\hat{\imath} + F_2(x,y,z)\hat{\jmath} + F_3(x,y,z)\hat{k}$ be a vector field in some region of space. Let *S* be a *smooth* surface bounded by a closed curve *C*. Then

$$\int_S (\text{Curl } \boldsymbol{F}).\hat{\boldsymbol{n}}\, ds = \int_C F.dr$$

Where $\hat{\boldsymbol{n}}$ is the unit normal at any point of *S*. Note that there are two unit normal at any point (one is negative of the other!). One makes a choice based on the following: Orient *C* arbitrarily. If we curl our right hand and keep the thumb in the direction of *C*, the direction in which the four fingers point would define the direction of $\hat{\boldsymbol{n}}$. Note that RHS in Stokes' theorem is a line integral and LHS is a surface integral.

Example: Evaluate $\oint_C \boldsymbol{F}.\boldsymbol{dr}$ using Stokes' theorem where $\boldsymbol{F}(x,y,z) = y\hat{\imath} + z\hat{\jmath} + x\hat{k}$ and C is the boundary of the upper half of the sphere $x^2 + y^2 + z^2 = 1$.

To apply Stokes' theorem we need to find curl of the vector field and unit normal to the surface at every point on the surface.

Curl $\boldsymbol{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & z & x \end{vmatrix} = -\hat{\imath} + \hat{\jmath} - \hat{k}$ and $\hat{\boldsymbol{n}} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$.

(Curl $\boldsymbol{F}$)$.\hat{\boldsymbol{n}} = -x + y - z$.

By Stokes' theorem $\int_C F.dr = \int_S$ (Curl $\boldsymbol{F}$)$.\hat{\boldsymbol{n}}\, ds = \int_S (-x + y - z)\, ds$.

To evaluate the last surface integral we use parameterization of unit sphere and the elemental area, which are given by $x = \cos\theta \sin\varphi, y = \sin\theta \sin\varphi, z = \cos\varphi$ and $ds = \sin\theta\, d\theta d\varphi$. For the upper half of the sphere, we have θ varying from 0 to $\frac{\pi}{2}$ and φ varying from 0 to 2π. Substituting all these in the surface integral we get its value to be equal to 0.

Finally, we now state the divergence theorem due to Gauss.

Divergence Theorem: Let **F** be a vector field in a region R. Let *S* be a closed surface bounding a volume *D*. Then $\int_D$ div $\boldsymbol{F}\, dV = \int_S \boldsymbol{F}.\hat{\boldsymbol{n}}\, dS$ where $\hat{\boldsymbol{n}}$ is the unit normal to *S* at any point.

Observe that LHS is a volume integral and RHS is a surface integral. Observe that $\boldsymbol{F}.\hat{\boldsymbol{n}}$ denotes component of $\boldsymbol{F}$ in the outward direction from *S*. Thus the integral on the RHS denotes net volumetric flow of $\boldsymbol{F}$ out of *S*. Another term physicists use for this is net flux of $\boldsymbol{F}$ through *S*. Divergence theorem says that this net flux is equal to the volume integral of divergence of $\boldsymbol{F}$ over *D*.

Example: Compute net flux out of a unit cube placed in a vector field defined by $\boldsymbol{F}(x,y,z) = 10x\hat{\imath} - 3xz\hat{\jmath} + 5z\hat{k}$.

By Gauss' theorem net flux equals $\int_D$ div $\boldsymbol{F}\, dV$. Recall Div $\boldsymbol{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$ and hence

Div $\boldsymbol{F} = \frac{\partial}{\partial x}(10x) + \frac{\partial}{\partial y}(-3xz) + \frac{\partial}{\partial z}(5z) = 10 + 0 + 5 = 15$. $\int_D$ div $\boldsymbol{F}\, dV = \int_D 15\, dV = 15\int_D dV = 15\, V = 15$.

Following are some of the good websites which could give students a fantastic learning experience. Enjoy.

https://betterexplained.com

https://www.3blue1brown.com/

https://mathinsight.org/

https://www.khanacademy.org/math/multivariable-calculus