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Engineering Mathematics-III

Module 1 :- Laplace transforms

Defn and laplace transforms of elementary functions, laplace transforms of periodic functions and unit-step-functions, problems.

Inverse laplace transforms :-

Inverse laplace transform - problems, convolution theorem to find the inverse laplace transform (without proof) and problems, solution of linear differential equations using laplace transforms.

Module 2 :- Fourier Series.

periodic functions, direchlet's condition. Fourier series of periodic functions period 2π and orbitary period. Half range fourier series. practical harmonic analysis, examples from engineering field.

Module 3 :- Fourier transforms

Infinite fourier transforms, fourier sine and cosine transforms. Inverse fourier transform, simple problems.

Difference equation and Z-transforms :-

Difference equations, basic defn, Z-transform defn, standard Z-transforms, Damping & shifting rules, initial value and final value theorems (without proof) and problems, inverse Z-transform, simple problems.

Module 4 :- Numerical solutions of Ordinary differential equation (ODE's)

Numerical solution of ODE's of first order and first degree - Taylor's series method, modified Euler's method. Runge Kutta (RK) method of fourth order, Milne's and Adam Bash forth predictor and corrector method (No derivations of formula, problems)

Module 5 :-

Numerical solution of second order of ODE calculus of variations, variation of function and functional, variational problems, Euler's equation, geo desics, hanging chain, problems.

Numerical Solutions of ODE's

Numerical methods of ODE's

Consider a differential equation of first order and first degree in the form $\frac{dy}{dx} = f(x,y)$ with the initial condition $y(x_0) = y_0$ that is $y = y_0$ at $x = x_0$.

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Taylor's series method

Consider the initial value problem $\frac{dy}{dx} = f(x,y)$ and $y(x_0) = y_0$.

Taylor's series expansion of $y(x)$ about the point x_0 in the form $\boxed{y(x) = y(x_0) + (x-x_0)y'(x_0) + \frac{(x-x_0)^2}{2!}y''(x_0) + \frac{(x-x_0)^3}{3!}y'''(x_0) + \cdots}$

Problems

1 Use Taylor series method to find "y" at $x = 0.1, 0.2, 0.3$ Considering terms upto the third degree given that $\frac{dy}{dx} = x^2+y^2$ and $y(0)=1$

Sol^n:- By data :- $\frac{dy}{dx} = x^2+y^2$, $y(0)=1$ $\quad x = 0.1$

$y(x_0)=y_0$

$x_0=0,\ y_0=1$

W.K.T

$$y(x) = y(x_0) + (x-x_0)y'(x_0) + \frac{(x-x_0)^2}{2!}y''(x_0) + \frac{(x-x_0)^3}{3!}y'''(x_0) \rightarrow ①$$

consider

$$y' = x^2+y^2 \Rightarrow y'(0) = x_0^2+y_0^2 = 0^2+1^2 = 1 \quad \boxed{y'(0)=1}$$

Differentiate w.r.t 'x'

$$y'' = 2x + 2yy' \Rightarrow y''(0) = 2x_0 + 2y_0y_0' = 2(0)+2(1)(1) = 2 \quad \boxed{y''(0)=2}$$

Again differentiate w.r.t x

$$y''' = 2 + 2[yy'' + y'y'] = 2+2[y_0y_0'' + (y_0')^2] \quad 2 + 2[(1)(2)+1] = 8$$

$$\boxed{y'''(0)=8}$$

eq^n ① becomes

$$y(x) = y(0) + (x-0)y'(0) + \frac{(x-0)^2}{2}y''(0) + \frac{(x-0)^3}{6}y'''(0)$$

$$= 1 + x(1) + \frac{x^2}{\cancel{2}}(\cancel{2}) + \frac{x^3}{\cancel{6}_3}(\cancel{8})^4$$

$$y(x) = 1 + x + x^2 + \frac{4x^3}{3} \longrightarrow ②$$

At $x=0.1$ in eqn (2)

$y(0.1) = 1+0.1+(0.1)^2+4\frac{(0.1)^3}{3}$

$y(0.1) = 1.1113.$

At $x=0.2$ in eqn (2)

$y(0.2) = 1+0.2+(0.2)^2+4\frac{(0.2)^3}{3}$

$= 1.2506.$

At $x=0.3$ in eqn (2)

$y(0.3) = 1+0.3+(0.3)^2+4\frac{(0.3)^3}{3}$

$= 1.426$

2. Find y at $x=1.02$ correct to 5 decimal places given $dy=(xy-1)dx$ and $y=2$ at $x=1$ applying Taylor Series method.

Soln :- By data :- $dy=(xy-1)dx$

$\frac{dy}{dx}=(xy-1)$, $x_0=1$ $y_0=2$ $y(x_0)=y_0$ $y(1)=2.$

WKT

$$y(x) = y(x_0)+(x-x_0)y'(x_0)+\frac{(x-x_0)^2}{2!}y''(x_0)+\frac{(x-x_0)^3}{3!}y'''(x_0) \rightarrow (1)$$

Consider

$y'=xy-1$, $y'(1)=x_0y_0-1=y'(1)=(1)(2)-1=1$ $y'(1)=1$ (or) $y_0'=1$

Differentiate w.r.t x

$y''=xy'+y$, $y''(1)=x_0y_0'+y_0 \Rightarrow y''(1)=(1)(1)+2=3$ $y''(1)=3$ (or) $y_0''=3$

Again.

$y'''=xy''+y'+y'$, $y'''(1)=x_0y_0''+2y_0' \Rightarrow (1)(3)+2(1)=5$ (or) $y'''(1)=5$

Eqn (1) becomes At $x=1.02$

$$y(1.02) = y(1) + (1.02-1)\, y'(1) + \frac{(1.02-1)^2}{2}\, y''(1) + \frac{(1.02-1)^3}{6}\, y'''(1)$$

$$y(1.02) = 2 + (0.02)(1) + (0.0002)(3) + 1.333 \times 10^{-6}\,(5)$$

$$y(1.02) = 2.0206066$$

$$= \underline{\underline{2.02061}}$$

3. From Taylor's series method y = 0.1 considering 4^{th} degree term where $y(x)$ satisfy the eqn $\frac{dy}{dx} = x - y^2$, $y(0) = 1$

Soln :- By data : $\frac{dy}{dx} = x - y^2$ $\quad y(0) = 1 \qquad y = ?$ at $x = 0.1$

$y(x_0) = y_0$

Here $x_0 = 0$ $y_0 = 1$

W.K.T

$$y(x) = y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2!}\, y''(x_0) + \frac{(x - x_0)^3}{3!}\, y'''(x_0)$$

$$+ \frac{(x - x_0)^4}{4!}\, y^{iv}(x_0) \longrightarrow (1)$$

Consider $\frac{dy}{dx} = x - y^2$

$$y' = x - y^2 \longrightarrow (2)$$

$$y'(0) = x_0 - y_0^2 = 0 - 1^2 = -1$$

Diff eqn (2) w.r.t. x.

$$y'' = 1 - 2yy'$$

$$y''(0) = 1 - 2y_0 y_0' = 1 - 2(1)(-1) = 3$$

Again diff. w.r.t x. eqn (2)

$$y''' = 0 - 2[yy'' + y'y']$$

$$= 0 - 2[yy'' + (y')^2] = 0 - 2[y_0 y_0'' + [y_0']^2]$$

$$= 0 - 2[1(3) + 1] = -8$$

Again diff w.r.t x.

$y^{iv} = -2[y\,y''' + y''\,y' + y'y'' + y'y'']$

$= -2[y\,y''' + 3y'\,y'']$

$y^{iv}(0) = -2[y_0\,y_0''' + 3y_0'\,y_0'']$

$= -2[1(-8) + 3(-1)(3)]$

$y^{iv}(0) = -2[-8-9] = 34$.

$y(0.1) = y(0) + (0.1-0)\,y'(0) + \frac{(0.1-0)^2}{2}y''(0) + \frac{(0.1-0)^3}{6}y'''(0) + \frac{(0.1-0)^4}{24}y^{iv}(0)$

$= 1 + (0.1)(-1) + 0.005(3) + 1.6\times10^{-4}(-8) + 4.16\times10^{-6}(34)$

$y(0.1) = \underline{0.9186}$

4. Using Taylor's series method $y(4.1)$ given $\frac{dy}{dx} = \frac{1}{x^2+y}$ and $y(4) = 4$.

Sol^n :- By data $\frac{dy}{dx} = \frac{1}{x^2+y}$ & $y(4) = 4$

$x_0 = 4 \quad y_0 = 4$.

$y' = \frac{1}{x^2+y}$

$y'(4) = \frac{1}{x_0^2+y_0} + \frac{1}{4^2+4} = \frac{1}{20} = 0.05$

consider $y' = \frac{1}{x^2+y}$

$y'(x^2+y) = 1$

$x^2y' + yy' = 1$

diff w.r.t x.

$x^2(y'') + y'(2x) + y(y'') + y'(y') = 1$

$y''(x^2+y) + 2xy' + (y')^2 = 0$

$$y''(x^2+y) = -2xy' - (y')^2$$

$$y'' = -\frac{2xy' + (y')^2}{x^2+y}$$

$$y''(4) = -\frac{[2[x_0 y_0' + (y_0')^2]}{x_0^2 + y_0}$$

$$= -\frac{[2(4)(0.05) + (0.05)^2]}{4^2 + 4}$$

$$= -0.020125$$

$$y(x) = y(x_0) + (x-x_0)y'(x_0) + \frac{(x-x_0)^2}{2!}y''(x_0)$$

$$y(4.1) = 4\,(4.1-4)\,0.05\,(4) + \frac{(4.1-4)^2}{2}\,\cancel{E}\,-0.0201\cancel{(4)}$$

$$y(4.1) = 4.0048$$

Using Taylor's series method to obtain power series in $(x-4)$ for the eq^n $5x\frac{dy}{dx} + y^2 - 2 = 0$, $x_0 = 4$, $y_0 = 1$ and use ~~8~~ it to find y at $x = 4.1, 4.2, 4.3$.

Sol^n: By data: $5x\frac{dy}{dx} + y^2 - 2 = 0$

$$5x\frac{dy}{dx} = 2 - y^2$$

$$\frac{dy}{dx} = \frac{2-y^2}{5x}$$

$$y' = \frac{2-y^2}{5x} \qquad y'(4) = ?$$

$$y' = \frac{2-1^2}{5(4)}$$

$$= \frac{1}{20} = 0.05$$

consider $y' = \frac{2-y^2}{5x}$.

$5xy' = 2-y^2$

diff w.r.t x.

$5[xy''+y'] = 0-2yy'$

$5xy''+5y' = -2yy'$

$\left(y'' = \frac{-2yy'-5y'}{5x}\right.$

$y'' = \frac{-[2yy' + 5y']}{5x}$.

$y''(4) = \frac{-[2(1)(0.05) + 5(0.05)]}{5(4)}$

$=)$

$5xy' + y^2 - 2 = 0$.

diff.

$(5(1)y'' + 2y' - 0 = 0)$

$5xy'' + y'(1) + 2yy'$.

$= 5(4)y'' + (0.05) + 2(1)(0.05)$

$= y'' ~~-20+(0.05)~~ +$

$y'' = \frac{0.05+2(0.05)}{5(4)}$

$= 0.055$

$y(x) = y(x_0) + (x-x_0)y'(x_0) + \frac{(x-x_0)^2}{2!}y''(x_0)$

$= y(4) + ($

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Using Taylor's series method to find y at $x = 0.1$ and 0.2 correct to 4 decimal places. in step size of 0.1 given. linear differential eqn $\frac{dy}{dx} - 2y = 3e^x$. whose solution passes through the origin.

Soln: By data: $\frac{dy}{dx} - 2y = 3e^x$.

$y(0) = 0$

$x_0 = 0, y_0 = 0$

$x = 0.1 \quad y(0.1) = ?$

$$\frac{dy}{dx} = 3e^x + 2y$$

$$y' = 3e^x + 2y \longrightarrow ①$$

$$y_0' = 3e^{x_0} + 2y_0$$

$$y_0' = 3e^0 + 2(0) = 3$$

$$\boxed{y_0' = 3}$$

eqn 1 diff w.r.t x.

$$y'' = 3e^x + 2y' \longrightarrow ②$$

$$= 3e^{x_0} + 2y_0' = 3e^0 + 2(3) = 3(1) + 6 = 9.$$

$$\boxed{y'' = 9}$$

eqn 2 diff w.r.t x.

$$y''' = 3e^x + 2y''$$

$$= 3e^{x_0} + 2y_0'' = 3e^0 + 2(9) = 3(1) + 18 = 21$$

$$\boxed{y''' = 21}$$

W.K.T

$$y(x) = y(x_0) + (x - x_0)y'(x_0) + \frac{(x-x_0)^2}{2!}y''(x_0) + \frac{(x-x_0)^3}{3!}y'''(x_0)$$

$$y(0.1) = y(0) + (0.1-0)y'(0) + \frac{(0.1-0)^2}{2}y''(0) + \frac{(0.1-0)^3}{6}y'''(0)$$

$$= 0 + (0.1)3 + \frac{0.01}{2}(9) + \frac{\cancel{1.03}\ 0.003}{6}(21)$$

$$= 0.3 + 0.005(9) + 0.063$$

$$= 0.3 + 0.$$

$$y(0.1) = y(\cancel{0.1}0) + (0.1-0)\,y'(0) + \frac{(0.1-0)^2}{2} y''(0) + \frac{(0.1-0)^3}{6} y'''(0)$$

$$= 0 + 0.1(3) + \frac{0.01}{2}\,9 + \frac{1\times 10^{-3}}{6} \times 21$$

$$= \underline{0.3 + 5\times10^{-3}(9) +}$$

$$= 0.3 + 0.0045 + 3.5\times10^{-3}$$

$$= \underline{\underline{0.3485}}$$

$$y(0.2) = y(x_0) + (x-x_0)\,y'(x_0) + \frac{(x-x_0)^2}{2!} y''(x_0) + \frac{(x-x_0)^3}{3!} y'''(x_0)$$

$$= y(0.1) + (0.2-0.1)\,y'(0.1) + \frac{(0.2-0.1)^2}{2} y''(0.1) + \frac{(0.2-0.1)^3}{6} y'''(0.1)$$

$$= 0.03484 + 0.1\,(4.0118) + 5\times10^{-3}\,(11.339) + 1.66\times10^{-4}\,(25.98)$$

$$y(0.2) = 0.8107.$$

Solve $y' = x^2 + y$ in the range $0 \le x \le 0.2$ by taking stepsize $h = 0.1$ given that $y = 10$ at $x = 0$ initially, considering terms upto 4^{th} degree.

Sol^n: By data: $y' = x^2 + y \rightarrow ①$ $h = 0.1$, $x_0 = 0$ $y_0 = 10$

$$y'(0) = x_0^2 + y_0 = 0 + 10 = 10$$

$x = x_0 + h$

$x = 0 + 0.1$

$x = 0.1$

$y(0.1) = ?$

eq^n ① Diff w.r.t x

$$y'' = 2x + y' \rightarrow ②$$

$$= 2x_0 + y'(0) = 2(0) + 10 = 10$$

$$\boxed{y''_0 = 10}$$

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again diff w.r.t x eq^n 2

$y''' = \cancel{2x}\, 2(1) + y'' \longrightarrow 3$

$= 2 + 10 = 12$

again diff w.r.t x eq^n 3

$y^{iv} = 0 + y'''$

$= 0 + 12 = 12.$

$$y(0.1) = y(x_0) + (x - x_0) y'(x_0) + \frac{(x-x_0)^2}{2!} y''(x_0) + \frac{(x-x_0)^3}{3!} y'''(x_0) + \frac{(x-x_0)^4}{4!} y^{iv}(x_0)$$

$$y(0.1) = y(0) + (0.1 - 0) y'(0) + \frac{(0.1-0)^2}{2} y''(0) + \frac{(0.10-0)^3}{6} y'''(0) + \frac{(0.1-0)^4}{24} y^{iv}(0)$$

$$= 10 + 0.1(10) + \cancel{0.05}\, 0.005 (10) + \frac{0.003}{6} (12) + \frac{0.0001}{24} 12$$

$$= 10 + 1 + 0.05 + 6 \times 10^{-3} + 5 \times 10^{-5}$$

$$= 11.056$$

$$y(0.2) = y(0) + (0.2 - 0) y'(0) + \frac{(0.2-0)^2}{2} y''(0) + \frac{(0.2-0)^3}{6} y'''(0) + \frac{(0.2-0)^4}{24} y^{iv}(0)$$

$$= 10 + (0.2)(10) + \frac{0.02}{\cancel{2}} (\cancel{10}^{5}) + \frac{0.000008}{\cancel{6}} (\cancel{12}^{2}) + \frac{0.0004}{\cancel{24}_{2}} (\cancel{12}^{1})$$

$$= (10 + 2 + 0.2 + 1.33 \times 10^{-6} \times 12 + 1.66 \times 10^{-5}) \times$$

$$= 10 + 2 + 0.1 + 1.6 \times 10^{-5} + 2 \times 10^{-4}$$

answer $= 12.2168$

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Using Taylor series method obtain the values of y at x = 0.1(0.1) 0.3 to 4 simplificant figures where y satisfy the eqn $y'' = -xy$ given that $y' = 0.5$ and $y = 1$ when $x = 0$ taking the 1st 5 terms of that Taylor series expansion.

Soln By data: $y'' = -xy \rightarrow ①$, $x_0 = 0$, $y_0 = 1$.

$y_0' = 0.5 \Rightarrow y'(0) = 0.5$

$y''(0) = -x_0 y_0 = 0.$

differentiate w.r.t x.

$y''' = -[xy' + y]$

$= -[0 + 1] = -1 \Rightarrow y'''(0) = -1$

Again diff eqn ① w.r.t x.

$y^{iv} = -[xy'' + y' + y']$

$= -[xy'' + 2y']$

$y^{iv}(0) - [x_0 y_0'' + 2y_0']$

$= - \cancel{[0 + 2(0.5)]}$

$= - [(0)(0) + (0.5)(0.5)]$

$\boxed{y^{iv}(0) = -1}$

W.K.T

$$y(x) = y(x_0) + (x - x_0)y'(x_0) + \frac{(x-x_0)^2}{2!} y''(x_0) + \frac{(x-x_0)^3}{3!} y'''(x_0) + \frac{(x-x_0)^4}{4!} y^{iv}(x_0)$$

$$y(0.1) = y(0) + (0.1 - 0)y'(0) + \frac{(0.1-0)^2}{2} y''(0) + \frac{(0.1-0)^3}{6} y'''(0) + \frac{(0.1-0)^4}{24} y^{iv}(0)$$

~~$(= 1 + (0.1)(0.5) + \frac{0.01}{2}(0) +)$~~

$$= 1 + 0.05 + 0 + (-1.66\times 10^{-4}) + (-4.166\times 10^{-6})$$

$y(0.1) = 1.05016$

At $x = 0.2$ in

$$y(0.2) = y(0) + (0.2-0)\,y'(0) + \frac{(0.2-0)^2}{2}y''(0) + \frac{(0.2-0)^3}{6}y'''(0) + y^{iv}\frac{(0.2-0)^4}{24}y^{iv}(0)$$

$$= 1 + (0.2)(0.5) + 0 + (-1.33\times10^{-3}) + (-6.66\times10^{-5})$$

$$= 1.098$$

At $x = 0.3$

$$y(0.3) = y(0) + (0.3-0)\,y'(0) + \frac{(0.3-0)^2}{2}y''(0) + \frac{(0.3-0)^3}{6}y'''(0) + \frac{(0.3-0)^4}{24}y^{iv}(0)$$

$$= 1 + (0.3)(0.5) + 0 + (-4.5\times10^{-3}) + (-3.375\times10^{-4})$$

$$= 1.145.$$

Modified Eulers Method

consider the initial value problem $\frac{dy}{dx} = f(x,y)$

$y(x_0) = y_0$ to find y at $\boxed{x_1 = x_0 + h}$

Eulers formula is given by

$$y_1^{(0)} = y_0 + hf(x_0, y_0)$$

$$y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0,y_0) + f(x_1, y_1^{(0)})\right]$$

$$y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0,y_0) + f(x_1, y_1^{(1)})\right]$$

$$y_1^{(3)} = y_0 + \frac{h}{2}\left[f(x_0,y_0) + f(x_1, y_1^{(2)})\right]$$

Problems

1. Given $\frac{dy}{dx} = 1 + \frac{y}{x}$, $y = 2$ at $x = 1$ find the approximate value of y at $x = 1.4$ by taking stepsize $h = 0.2$ applying modified Euler's method.

Sol^n: By data :- $y_0 = 2$ and $x_0 = 1$

Step 1 :- $\frac{dy}{dx} = 1 + \frac{y}{x} \Rightarrow f(x,y) = 1 + y/x$

Here $x_0 = 1$, $y_0 = 2$ $h = 0.2$. $x_1 = x_0 + h$

$x_1 = 1 + 0.2$

$\boxed{x_1 = 1.2}$ $y(x_1) = ?$

$y(1.2) = ?$

$\boxed{f(x_0, y_0) = 1 + \frac{y_0}{x_0} = 1 + \frac{2}{1} = 3}$

$$y_1^{(0)} = y_0 + h f(x_0, y_0)$$
$$= 2 + 0.2(3)$$
$$= 2.6.$$

$$y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(0)})\right]$$

$$f(x_1, y_1^{(0)}) = 1 + \frac{y_1^{(0)}}{x_1} = 1 + \frac{2.6}{1.2} = 3.16.$$

$$y_1^{(1)} = 2 + \frac{0.2}{2}[3 + 3.16]$$

$$\boxed{y_1^{(1)} = 2.616}$$

$$y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(1)})\right]$$

$$\left[f(x_1, y_1^{(1)})\right] = 1 + \frac{y_1^{(1)}}{x_1} = 1 + \frac{2.616}{1.2} = 3.18$$

$$y_1^{(2)} = 2 + \frac{0.2}{2}[3 + 3.18]$$

$$\boxed{y_1^{(2)} = 2.618}$$

$$y_1^{(3)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(2)})\right]$$

$$f(x_1, y_1^{(2)}) = 1 + \frac{y_1^{(2)}}{x_1} = 1 + \frac{2.618}{1.2} = 3.1816.$$

$$y_1^{(3)} = 2 + \frac{0.2}{2}\left[3 + 3.181\right] = 2.618$$

$$\boxed{y_1^{(3)} = 2.618}$$

II stage

$x_0 = 1.2 \quad y_0 = 2.618 \quad x_1 = x_0 + h \quad h = 0.2$

$= 1.2 + 0.2$

$= 1.4$

$y(1.4) = ?$

$$f(x, y) = 1 + y/x$$

$$f(x_0, y_0) = 1 + \frac{y_0}{x_0} = 1 + \frac{2.618}{1.2} = 3.181$$

Euler's formula given by

$$y_1^{(0)} = y_0 + h\left[f(x_0, y_0)\right]$$

$$= 2.618 + 0.2\left[3.181\right]$$

$$y_1^{(0)} = 3.254$$

$$y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(0)})\right]$$

$$f(x_0, y_1^{(0)}) = 1 + \frac{y_1^{(0)}}{x_1} = 1 + \frac{3.254}{1.4} = 3.324.$$

$$y_1^{(1)} = 2.618 + \frac{0.2}{2}\left[3.181 + 3.324\right]$$

$$\boxed{y_1^{(1)} = 3.268}$$

$$y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(1)})\right]$$

$$f(x_1, y_1^{(1)}) = 1 + \frac{y_1^{(1)}}{x_1} = 1 + \frac{3.268}{1.4} = 3.334$$

$$y_1^{(2)} = 2.618 + \frac{0.2}{2}\left[3.181 + 3.334\right]$$

$$\underline{y_1^{(2)} = 3.269}$$

$$y_1^{(3)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(2)})\right]$$

$$f(x_0, y_1^{(2)}) = 1 + \frac{y_1^{(2)}}{x_1} = 1 + \frac{3.269}{1.4} = 3.336.$$

$$y_1^{(3)} = 2.618 + \frac{0.2}{2}\left[3.181 + 3.336\right]$$

$$\boxed{y(1.4) = 3.269}$$

Using modified Eulers method find y at $x = 0.2$ given $\frac{dy}{dx} = 3x + y/2$ (or) $3x + \frac{1}{2}y$. where $y(0) = 1$ taking $h = 0.1$

perform 3 iteration at each step.

By data:- $y(x_0) = y_0$ $\quad y(0) = 1$ $\quad x_0 = 0$ $\quad y_0 = 1$

$x_1 = x_0 + h$ $\quad = 0 + 0.1$ $\quad x_1 = 0.1$ $\quad y(0.1) = ?$

0
0.1
0.2

$$\frac{dy}{dx} = 3x + \frac{y}{2}$$

$$f(x, y) = 3x + \frac{y}{2}$$

$$f(x_0, y_0) = 3x_0 + \frac{y_0}{2}$$

$$= 3(0) + \frac{1}{2} = 0.5$$

$$f(x_0, y_0) = 0.5$$

Eulers formula is given by.

$$y_1^{(0)} = y_0 + h\left[f(x_0, y_0)\right]$$

$$= \cancel{0.11}\ 1 + 0.1\,[0.5]$$

$$y_1^{(0)} = \cancel{0.55}\ 1.05$$

$y_1^{(0)} = y_0 + \frac{h}{2}\left[f(x_0,y_0) + f(x_1, y_1^{(0)})\right]$

$f(x_1, y_1^{(0)}) = 3x_1 + \frac{y_1^{(0)}}{2} = 3(0.1) + \frac{1.05}{2} = 0.825$

$y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0,y_0) + f(x_1, y_1^{(1)})\right]$

$= 1 + \frac{0.1}{2}\left[0.5 + 0.825\right]$

$= 1.066.$

$y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0,y_0) + f(x_1, y_1^{(1)})\right]$

$= \cancel{1 + \frac{0.1}{2}\left[0.\right.}$

$f(x_1, y_1^{(1)}) = 3x_1 + \frac{y_1^{(1)}}{2} = 3(0.1) + \frac{1.066}{2} = 0.8333$

$y_1^{(2)} = 1 + \frac{0.1}{2}\left[0.5 + 0.8333\right]$

$y_1^{(2)} = 1.066.$

$y_1^{(3)} = y_0 + \frac{h}{2}\left[f(x_0,y_0) + f(x_1, y_1^{(2)})\right]$

$f(x_1, y_1^{(2)}) = 3x_1 + \frac{y_1^{(2)}}{2} = 3(0.1) + \frac{1.066}{2} = 0.833$

$y_1^{(3)} = 1 + \frac{0.1}{2}\left[0.5 + 0.833\right]$

$y_1^{(3)} = 1.066$

$y(0.1) = 1.066.$

II

$f(x,y) = 3x + \frac{y}{2}.$

$h = 0.1$

$x_0 = 0.1, \; y_0 = 1.066$

$f(x_0,y_0) = 3x_0 + y_0/2$

$x_1 = x_0 + h.$

$= 3(0.1) + \frac{1.066}{2}$

$= 0.1 + 0.1$

$x_1 = 0.2$

$y(0.2) = ?$

$$f(x_0, y_0) = 3x_0 + \frac{y_0}{2}$$

$$= 3(0.1) + \frac{1.066}{2}$$

$$= 0.833.$$

Eulers formula given by.

$$y_1^{(0)} = y_0 + h\,[f(x_0, y_0)]$$

$$= 1.066 + 0.1(0.833)$$

$$= \cancel{1.166 \times (0.833)}\ 0.0833$$

$$= \cancel{0.9712}\ 1.1493.$$

$$y_1^{(1)} = y_0 + \frac{h}{2}\,[f(x_0, y_0) + f(x_1, y_1^{(0)})]$$

$$\cancel{= 1.066 + \frac{0.1}{2}}$$

$$f(x_1, y_1^{(0)}) = 3x_1 + \frac{y_1^{(0)}}{2} = 3(0.2) + \frac{1.1493}{2} = 0.6 + 0.57465$$

$$= \cancel{0.874}\ 1.174.$$

$$= 1.066 + \frac{0.1}{2}[0.833 + 1.174]$$

$$= 1.066 + 0.05[2.007]$$

$$= 1.066 + 0.100$$

$$y_1^{(1)} = 1.166.$$

$$y_1^{(2)} = y_0 + \frac{h}{2}\,[f(x_0, y_0) + f(x_1, y_1^{(1)})]$$

$$f(x_1, y_1^{(1)}) = 3x_1 + \frac{y_1^{(1)}}{2} = 3(0.2) + \frac{1.166}{2} = 0.6 + 0.583$$

$$= 1.183.$$

$$= 1.066 + \frac{0.1}{2}[0.833 + 1.183]$$

$$= 1.066 + 0.05[2.016]$$

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$= 1.066 + 0.1008$

$= 1.168$

$y_1^{(3)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^{(2)})]$

$f(x_1, y_1^{(2)}) = 3x_1 + \frac{y_1^{(2)}}{2} = 3(0.2) + \cancel{[1.168 + 0.833]} \frac{1.168}{2}$

$= \cancel{0.6}(+ 2.001 = 2.601) \times \cancel{+}\ 0.6 + 0.584 = 1.184$

$= 1.066 + \frac{0.1}{2}[0.833 + 1.168]$

$= 1.066 + 0.05[2.001]$

$= 1.066 + 0.10005$

$= 1.1660$

Using modified Eulers method. find $y(0.2)$ correct to 4 decimal to 4 decimal places solving the eq^n^ $\frac{dy}{dx} = x - y^2$ $y(0) = 1$ taking $h = 0.1$

By data :- $y(0) = 1$ $x_0 = 0$ $y_0 = 1$ $x_1 = x_0 + h$ $h = 0.1$

$y(x_0) = y_0$

$= 0 + 0.1$

$x_1 = 0.1$

$\frac{dy}{dx} = x - y^2$

$f(x, y) = x - y^2$

$f(x_0, y_0) = x_0 - y_0^2$

$= 0 - 1^2$

$f(x_0, y_0) = -1$

Eulers formula.

$y_1^{(0)} = y_0 + h[f(x_0, y_0)]$

$= 1 + 0.1[-1]$

$\boxed{y_1^{(0)} = 0.9}$

$$y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(0)})\right]$$

$$f(x_1, y_1^{(0)}) = x_p - y_p^{2(0)} = 0.1 - (0.9)^2 = 0.1 - 0.81$$

$$= -0.71$$

$$= 1 + \frac{0.1}{2}\left[-1 + (-0.71)\right]$$

$$= 1 + \frac{0.1}{2}\left[-1.71\right] = 1 + 0.05\left[-1.71\right]$$

$$y_1^{(1)} = 0.9145$$

$$y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(1)})\right]$$

$$f(x_1, y_1^{(1)}) = x_1 - y_1^{2(1)} = 0.1 - (0.9145)^2 = 0.1 - 0.8363 = -0.7363$$

$$= 1 + \frac{0.1}{2}\left[-1 + (-0.7363)\right]$$

$$= 1 + \frac{0.1}{2}\left[-1.7363\right] = 1 + 0.05\left[-1.7363\right]$$

$$= 1 + (-0.0868)$$

$$= 0.9132.$$

$$y_1^{(3)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(2)})\right]$$

$$f(x_1, y_1^{(2)}) = x_1 - y_1^{2(2)} = 0.1 - (0.9132)^2 = 0.1 - 0.8339$$

$$= -0.7339.$$

$$= 1 + \frac{0.1}{2}\left[-1 + (-0.7339)\right]$$

$$= 1 + 0.05\left[-1.7339\right]$$

$$= 1 + \left[-0.0866\right]$$

$$= 0.9134.$$

II

$h=0.1 \quad y(0.2) \Rightarrow \cancel{y(x_0)=y_0} \quad x_0=0.2 \quad x_1 = x_0+h$

$= 0.2+0.1$

$= 0.3$

$f(x,y) = x - y^2$

$f(x_0,y_0) = x_0 - y_0^2 = 0.2 - 0.9134 = -0.7134.$

Eulers formula given by.

$y_1^{(0)} = y_0 + h\,[f(x_0,y_0)]$

$= 0.9134 + 0.1\,[-0.7134]$

$= 0.9134 + (-0.0713)$

$= 0.8421$

$y_1^{(1)} = y_0 + \frac{h}{2}\,[f(x_0,y_0) + f(x_1,y_1^{(0)})]$

$f(x_1,y_1^{(0)}) = x_1 - (y_1^{(0)})^2 = 0.3 - (0.8421)^2 \doteq 0.3 - 0.7091$

$= -0.4091$

$= 0.9134 + \frac{0.1}{2}\,[-0.7134 + (-0.4091)]$

$= 0.9134 + 0.05\,[-1.1225]$

$= 0.9134 + [0.0561]$

$y_1^{(1)} = 0.8573.$

$y_1^{(2)} = y_0 + \frac{h}{2}\,[f(x_0,y_0) + f(x_1,y_1^{(1)})]$

$f(x_1,y_1^{(1)}) = x_1 - (y_1^{(1)})^2 = 0.3 - 0.8573 = -0.4349.$

$= 0.9134 + \frac{0.1}{2}\,[-0.7134 + (-0.4349)]$

$= 0.9134 + 0.05\,[-1.1483]$

$= 0.9134 + [-0.0574]$

$= 0.856$

$y_1^{(3)} = y_0 + \frac{h}{2}[f(x_0,y_0) + f(x_1, y_1^{(2)})]$

$f(x_1,y_1) = x_1 - (y_1^{(2)})^2 = 0.3 - (0.856)^2 \cancel{= 0.3 - 7327}$

$= 0.3 - 0.7327$

$= -0.4327$

$= 0.9134 + \frac{0.1}{2}[-0.7134 + (-0.4327)]$.

$= 0.9134 + 0.05[-1.1461]$

$= 0.9134 + (-0.057305)$

$= 0.8564$

Using modified Eulers method find $y(20.2)$ & $y(20.4)$ given that $\frac{dy}{dx} = \log_{10}(x/y)$ where $y(20) = 5$ taking $h = 0.2$

Sol^n By data:- $h = 0.2$ $y(20) = 5$ $x_0 = 20$ 20

$y(x_0) = y_0$ $y_0 = 5$ 20.2

20.4

$f(x,y) = \log_{10}\left[\frac{x}{y}\right]$

$x_1 = x_0 + h$

$= 20 + 0.2$

$x_1 = 20.2$

$f(x_0,y_0) = \log_{10}\left[\frac{x_0}{y_0}\right]$

$= \log_{10}\left[\frac{20}{5}\right]$

$f(x_0,y_0) = \cancel{\log}\ 0.6020$

$y_1^{(0)} = y_0 + h\,[f(x_0,y_0)]$

$= 5 + 0.2(0.6020)$

$= 5.1204$

$y_1^{(1)} = y_0 + \frac{h}{2}[f(x_0,y_0) + f(x_1, y_1^{(0)})]$

$f(x_1,y_1) = \cancel{5 + \frac{0.2}{2}[5}\ \log_{10}\left[\frac{x_1}{y_1^{(0)}}\right] = \log_{10}\left[\frac{20.2}{5.1204}\right]$

$= 0.5960.$

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$= 5 + \frac{0.2}{2}\left[0.6020 + \cancel{5.1204}^{\,0.5960}\right]$

$\cancel{5+0.1[5.7224]}$ $= 5+0.1[1.198]$

$= 5+0.57224$ $= \cancel{\$}\,\underline{5.1198}.$

$= 5)\cdot X$

$$y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(1)})\right]$$

$$f(x_1, y_1^{(1)}) = \log_{10}\left[\frac{x_p}{y_1^{(7)}}\right] = \log_{10}\left(\frac{20.2}{5.1198}\right) = 0.5960$$

$$= 5 + \frac{0.2}{2}\left[0.6020 + 0.5960\right]$$

$$= 5 + 0.1[1.198]$$

$$= 5.1198.$$

$y(20.2) = 5.1198$

II^nd

$x_0 = 20.2 \quad y_0 = 5.1198 \quad h = 0.2 \quad x_1 = x_0 + h$

$= 20.2 + 0.2$

$x_1 = 20.4$

$$f(x, y) = \log_{10}\left(\frac{x}{y}\right)$$

$$f(x_0, y_0) = \log_{10}\left[\frac{x_0}{y_0}\right] = \log_{10}\left[\frac{20.2}{5.1198}\right] = 0.5960.$$

Eulers formula

$$y_1^{(0)} = y_0 + h\left[f(x_0, y_0)\right]$$

$$= 5.1198 + 0.2(0.5960).$$

$$= 5.1198 + 0.1192$$

$$y_1^{(0)} = 5.239$$

$$y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(1)})\right]$$

$$f(x_1, y_1^{(0)}) = \log_{10}\left[\frac{x_1}{y_1^{(0)}}\right] = \log_{10}\left[\frac{20.4}{5.239}\right] = 0.5903$$

$$= \cancel{5}\,5.1198 + \frac{0.2}{2}\left[0.5960 + 0.5903\right]$$

$$= 5.1198 + 0.1(1.1863)$$

$$y_1^{(1)} = 5.2384$$

$$y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(2)})\right]$$

$$f(x_0, y_1^{(1)}) = \log_{10}\left[\frac{x_1}{y_1^{(1)}}\right] = \log_{10}\left[\frac{20.4}{5.2384}\right] = 0.5904$$

$$= 5.1198 + \frac{0.2}{2}\left[0.5960 + 0.5904\right]$$

$$= 5.1198 + 0.1(1.1864)$$

$$y_1^{(2)} = \cancel{5.1198}.\ 5.2384$$

$$y(20.4) = 5.2384.$$

use modified Eulers method to solve $\frac{dy}{dx} = x + |\sqrt{y}|$ in the range $0 \le x \le 0.4$ by taking $h = 0.2$ given that $y = 1$ at $x = 0$ initially.

By data:- $\frac{dy}{dx} = x + \sqrt{y}$

$x_0 = 0 \quad y_0 = 1$

$x_0 = 0.2$

$x = 0.4$

$x_1 = x_0 + h \qquad h = 0.2$

$= 0 + 0.2$

$\boxed{x_1 = 0.2}$

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use modified Eulers ~~mode~~ method to compute $y(0.1)$ given that $\frac{dy}{dx} = x^2+y$, $y(0)=1$ by taking $h=0.05$ considering the accuracy upto 2 approximation in each step

By data: $\frac{dy}{dx} = x^2+y$

x	y
0	1
0.05	
0.1	?

$y(0)=1$
$y(x_0)=y_0$
$x_0=0 \quad y_0=1$

using Eulers predictor and corrector formulae solve $\frac{dy}{dx} = x+y$ at $x=0.2$ given that $y(0)=1$

By data:- $\frac{dy}{dx} = x+y$ $\quad y(0)=1$
$y(x_0)=y_0$
$x_0=0 \quad y_0=1$

$x_1 = 0.2$
$x_1 = x_0 + h$
$0.2 = 0 + h$
$0.2 - 0 = h$
$\boxed{h = 0.2}$

$f(x,y) = x+y$
$f(x_0, y_0) = x_0 + y_0 = 0+1 = 1$

Eulers formula is given by predictor formula.

$y_1^{(0)} = y_0 + h\,[f(x_0, y_0)]$
$= 1 + 0.2(1)$
$= 1.2$

Eulers corrector formula is given by.

$y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(0)})\right]$

$f(x_1, y_1^{(0)}) = x_1 + y_1^{(0)} = 0.2 + 1.2 = 1.4$

$= 1 + \frac{0.2}{2}[1 + 1.4]$

$= 1 + 0.1(2.4)$

$\boxed{y_1^{(1)} = 1.24}$

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$$y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0,y_0) + f(x_1, y_1^{(1)})\right]$$

$$f(x_1, y_1^{(1)}) = x_1 + y_1^{(1)} = 0.2 + 1.24 = 1.44$$

$$= 1 + \frac{0.2}{2}[1 + 1.44]$$

$$= 1 + 0.1[2.44]$$

$$\boxed{y_1^{(2)} = 1.244.}$$

$$y_1^{(3)} = y_0 + \frac{h}{2}\left[f(x_0,y_0) + f(x_1, y_1^{(2)})\right]$$

$$f(x_1, y_1^{(2)}) = x_1 + y_1^{(2)} = 0.2 + 1.244 = 1.444$$

$$= 1 + \frac{0.2}{2}[1 + 1.444]$$

$$= 1 + 0.1(2.444)$$

$$\boxed{y_1^{(3)} = 1.244.}$$

$$\boxed{y(0.2) = 1.244}$$

Using ~~usino~~ Eulers predictor & corrector formula computer $y(1.1)$ correct to ~~4~~ 5 decimal places given that $\frac{dy}{dx} + \frac{y}{x} = 1/x^2$ and $y = 1$ at $x = 1$. Also find the analytical Sol^n.

$x_0 = 1, y_0 = 1$

$y(1.1) = ?$

$x_1 = 1.1$

By data:-

$$\frac{dy}{dx} + \frac{y}{x} = \frac{1}{x^2}$$

$$\frac{dy}{dx} = \frac{1}{x^2} - \frac{y}{x}$$

$$\frac{dy}{dx} = \frac{1 - xy}{x^2}$$

$$f(x,y) = \frac{1-xy}{x^2}$$

$x_1 = x_0 + h$

$1.1 = 1 + h$

$1.1 - 1 = h$

$\boxed{0.1 = h}$

$$f(x_0, y_0) = \frac{1 - x_0 y_0}{x_0^2} = \frac{1 - 1 \times 1}{1^2} = 0$$

Eulers formula.

$$\cancel{y_0}\ \cancel{f(x}, y_1^{(0)} = y_0 + h\,[f(x_0, y_0)]$$

$$= 1 + 0.1\,(0)$$

$$y_1^{(0)} = 1$$

$$y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(0)})\right]$$

$$f(x_1, y_1) = \frac{1 - x_1 y_1^{(0)}}{x_1^2} = \frac{1 - (1.1)(1)}{(1.1)^2} = -0.08264$$

$$= 1 + \frac{0.1}{2}\left[0 + \cancel{(-0.1)}\,(-0.8264)\right]$$

$$= \cancel{1 + 0.05}\ \cancel{0.99868}\ 0.99586.$$

$$y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(1)})\right]$$

$$f(x_1, y_1^{(1)}) = \frac{1 - x_1 y_1^{(1)}}{x_1^2} = \frac{1 - (1.1)(0.99586)}{(1.1)^2}$$

$$= \frac{1 - 1.095446}{1.21} = \frac{-0.095446}{1.21} = -0.0788$$

$$= 1 + \frac{0.1}{2}\left[0 + (-0.0788)\right]$$

$$= 1 + 0.05\,(-0.0788)$$

$$= 1 + (-3.94 \times 10^{-3})$$

$$= 0.99605$$

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$y_1^{(3)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(2)})\right]$

$f(x_1, y_1^{(2)}) = \frac{1 - x_1 y_1^{(2)}}{x_1^2} = \frac{1 - (1.1)(0.99605)}{(1.1)^2}$

$= -0.07905$

$= 1 + \frac{0.1}{2}\left[0 + (-0.07905)\right]$

$= 1 + 0.05(-0.07905)$

$y_1^{(3)} = 0.99604$

$\boxed{y(1.1) = 0.99604}$

Analytical solution.

$\frac{dy}{dx} + \frac{y}{x} = \frac{1}{x^2}$ is of the form $\frac{dy}{dx} + Py = Q$ where $P = \frac{1}{x}$

and $\left(Q = \frac{1}{x^2} \quad \frac{dy}{dx} + \left(\frac{1}{x}\right)y = \frac{1}{x^2}\right)$ whose solution is given by

$y \cdot e^{\int P\,dx} = \int Q\, e^{\int P\,dx}\, dx + c.$

$y \cdot e^{\int \frac{1}{x}dx} = \int \frac{1}{x^2} e^{\int \frac{1}{x}dx} dx + c$

$y \cdot e^{\log x} = \int \frac{1}{x^2} \cdot e^{\log x} \cdot dx$

$y \cdot x = \int \frac{1}{x^2} \cdot x\, dx$

$yx = \int \frac{1}{x} \cdot dx$

$yx = \log x + c.$

using $y(1) = 1$

$(1)(1) = \log(1) + 1$

$1 = 1 = c$

(2) $\Rightarrow yx = \log x + 1$

$$y = \frac{\log x + 1}{x}$$

$x = 1.1$

$$y = \frac{\log(1.1) + 1}{1.1}$$

$$\boxed{y = 0.9957}$$

VVIMP.

Runge-Kutta method of IVth Order

Consider the initial value problem $\frac{dy}{dx} = f(x, y)$, $y(x_0) = y_0$.

We have to first compute K_1, K_2, K_3, K_4 by the following formula.

$$K_1 = \cancel{y_0} h\, f(x_0, y_0)$$

$$K_2 = h\, f\left(x_0 + \frac{h}{2}, y_0 + \frac{K_1}{2}\right)$$

$$K_3 = h f\left(x_0 + \frac{h}{2}, y_0 + \frac{K_2}{2}\right)$$

$$K_4 = h f\left(x_0 + h, y_0 + K_3\right)$$

The required $\boxed{y(x_0 + h) = y_0 + \frac{1}{6}\left[K_1 + 2K_2 + 2K_3 + K_4\right]}$

Problems

1. Given $\frac{dy}{dx} = 3x + \frac{y}{2}$, $y(0) = 1$ compute $y(0.2)$ by taking $h = 0.2$ using R-K method of IVth order

$y(0) = 1$ $y(x_0) = y_0$

By data: $\frac{dy}{dx} = 3x + \frac{y}{2}$ $y_0 = 1$ $x_0 = 0$, $h = 0.2$

$$f(x, y) = 3x + \frac{y}{2}.$$

$$f(x_0, y_0) = 3x_0 + \frac{y_0}{2} = 3(0) + \frac{1}{2} = 0.5$$

$$K_1 = hf(x_0, y_0)$$

$$\boxed{K_1 = 0.2(0.5) = 0.1}$$

$$K_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{h}{2}\right)$$

$$K_2 = 0.2f\left(0 + \frac{0.2}{2}, 1 + \frac{0.1}{2}\right)$$

$$K_2 = 0.2f(0.1, 1.05)$$

$$f(0.1, 1.05) = 3(0.1) + \frac{1.05}{2} = 0.825$$

$$K_2 = 0.2(0.825)$$

$$\boxed{K_2 = 0.165}$$

$$K_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{K_2}{2}\right)$$

$$K_3 = 0.2f\left(0 + \frac{0.2}{2}, 1 + \frac{0.165}{2}\right)$$

$$K_3 = 0.2f(0.1, 1.0825)$$

$$f(0.1, 1.0825) = 3(0.1) + \frac{1.0825}{2} = 0.84125$$

$$K_3 = 0.2(0.84125)$$

$$\boxed{K_3 = 0.1682}$$

$$K_4 = hf(x_0 + h, y_0 + K_3)$$

$$= 0.2f(0 + 0.2, 1 + 0.1682)$$

$$= 0.2f(0.2, 1.1682)$$

$$\cancel{= 0.2f}\ f(0.2, 1.1682) = 3(0.2) + \frac{1.1682}{2} = 1.184$$

$$K_4 = 0.2(1.184)$$

$$\boxed{K_4 = 0.2368}$$

$y(x_0+h) = y_0 + \frac{1}{6}[K_1 + 2K_2 + 2K_3 + K_4]$

$y(0+0.2) = 1 + \frac{1}{6}[0.1 + 2(0.165) + 2(0.1682) + 0.2368]$

$y(0.2) = 1.1672$

2. Use 4^{th} order R-K method to solve $(x+y)\frac{dy}{dx} = 1$, $y(0.4) = 1$ at $x = 0.5$ correct to 4 decimal places.

By data:- $\frac{dy}{dx} = \frac{1}{(x+y)}$ $y(0.4) = 1$ & $x_0 = 0.4$ $y_0 = 1$

$x = x_0 + h$

$0.5 = 0.4 + h$

$0.5 - 0.4 = h$

$0.1 = h$.

$f(x,y) = \frac{1}{(x+y)}$

$f(x_0,y_0) = \frac{1}{(x_0+y_0)} = \frac{1}{(0.4+1)} = 0.7142$

$K_1 = hf(x_0,y_0)$

$= 0.1(0.7142)$

$\boxed{K_1 = 0.07142.}$

$K_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{K_1}{2}\right)$

$= 0.1 f\left(0.4 + \frac{0.1}{2}, 1 + \cancel{\frac{0.1}{2}}\right) \frac{0.01742}{2})$

$= (0.1) f(0.45, 1.035)$

$f(0.45, 1.035) = \frac{1}{(0.45+1.035)} = \frac{1}{1.485} = \cancel{0.6666}\ 0.6734$

$= 0.1 \times 0.6666$

$\cancel{K_2 = 0.06734}$ $\boxed{K_2 = 0.06734}$

$$K_3 = hf\left(x_0+\frac{h}{2},\ y_0+\frac{K_2}{2}\right)$$

$$=0.1f\left(0.4+\frac{0.1}{2},\ 1+\frac{0.0673}{2}\right)$$

$$=0.1f(0.45,\ 1.0336)$$

$$f(0.45, 1.0336) = \frac{1}{0.45+1.0336} = \frac{1}{1.4836} = 0.6740$$

$$=0.1\times 0.674$$

$$\boxed{K_3 = 0.0674}$$

$$K_4 = hf(x_0+h,\ y_0+K_3)$$

$$=0.1f(0.4+0.1,\ 1+0.0674)$$

$$=0.1f(0.5,\ 1.0674)$$

$$f(0.5, 1.0674) = \frac{1}{0.5+1.0674} = 0.6379$$

$$=0.1\times 0.6379$$

$$\boxed{K_4 = 0.06379}$$

$$y(x_0+h) = y_0+\frac{1}{6}[K_1+2K_2+2K_3+K_4]$$

$$y(0.4+0.1) = 1+\frac{1}{6}[0.07142+2(0.06734)+2(0.0674)+0.0637$$

$$y(0.5) = 1+\frac{1}{6}[0.07142+0.13468+0.1348+0.06379]$$

$$y(0.5) = 1+0.166[0.40469]$$

$$\boxed{y(0.5) = \cancel{1.} 1.0671}$$

Euler's Assignment.

1. Use modified Eulers method to solve $\frac{dy}{dx} = x + |\sqrt{y}|$ in the range $0 \le x \le 0.4$ by taking $h = 0.2$ given that $y = 1$ at $x = 0$ initially.

Sol^n: By data :- $\frac{dy}{dx} = x + \sqrt{y}$ $\quad y_0 = 1 \quad x_0 = 0 \quad x_1 = x_0 + h \quad h = 0.2$

$x = 0.2 \qquad = 0 + 0.2$

$x = 0.4 \qquad \boxed{x_1 = 0.2}$

$$f(x, y) = x + \sqrt{y}$$

$$f(x_0, y_0) = x_0 + \sqrt{y_0} = 0 + \sqrt{1} = 1$$

$$y_1^{(0)} = y_0 + h[f(x_0, y_0)]$$
$$= 1 + 0.2(1)$$
$$= 1.2$$

$$y_1^{(1)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^{(0)})]$$
$$(= 1 + \frac{0.2}{2}[1 + 1.2]$$
$$= 1.22)^{\times}$$

$$f(x_1, y_1^{(0)}) = x_1 + \sqrt{y_1^{(0)}} = 0.2 + \sqrt{1.2} = 1.0954$$

$$= 1 + \frac{0.2}{2}[1 + 1.0954]$$
$$= 1 + 0.1(2.0954)$$
$$= 1.209$$

$$y_1^{(2)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^{(1)})]$$

$$f(x_1, y_1^{(1)}) = x_1 + \sqrt{y_1^{(1)}} = 0.2 + \sqrt{1.209} = 1.2995$$

$$= 1 + \frac{0.2}{2}[1 + 1.2995]$$
$$= 1 + 0.1(2.2995)$$
$$= 1.229$$

$y_1^{(3)} = y_0 + \frac{h}{2}\left[f(x_0,y_0) + f(x_1, y_1^{(2)})\right]$

$f(x_1, y_1^{(2)}) = x_1 + \sqrt{y_1^{(2)}} = 0.2 + \sqrt{1.299} = 1.339$

$= \cancel{0.}1 + \frac{0.2}{2}\left[1 + 1.339\right]$

$= 1 + 0.1(2.339)$

$y_1^{(3)} = 1.2339$

$\underline{\text{II}}^{nd}$

$x_0 = 0.2 \quad y_0 = 1.2339 \quad h = 0.2 \quad x_1 = x_0 + h$

$= 0.2 + 0.2$

$x_1 = 0.4$

$f(x,y) = x + \sqrt{y}$

$f(x_0,y_0) = x_0 + \sqrt{y_0} = 0.2 + \sqrt{1.2339} = 1.3108$

$y_1^{(0)} = y_0 + h\left[f(x_0,y_0)\right]$

$= 1.2339 + 0.2[1.3108]$

$= 1.49606.$

$y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0,y_0) + f(x_1,y_1^{(0)})\right]$

$f(x_1, y_1^{(0)}) = x_1 + \sqrt{y_1^{(0)}} = 0.4 + 1.49606 = \cancel{1.89606}\ 1.223$

$= 1.2339 + \frac{0.2}{2}\left[1.3108 + \overset{1.223}{\cancel{1.89606}}\right]$

$= 1.2339 + 0.1\ \cancel{[3.20686]}\ [2.5338]$

$= \cancel{1.554}\ 1.4872$

$y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0,y_0) + f(x_1, y_1^{(1)})\right]$

$f(x_1, y_1^{(1)}) = x_1 + \sqrt{y_1^{(1)}} = 0.4 + \sqrt{1.554} = 1.646$

$= \cancel{0}1.2339 + \frac{0.2}{2}\left[1.3108 + 1.646\right]$

$= 1.2339 + 0.1(2.9568)$

$= 1.529$

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$y_1^{(2)} = y_0 + \frac{h}{2}[f(x_0,y_0) + f(x_1, y_1^{(1)})]$

$f(x_1, y_1^{(1)}) = x_1 + \sqrt{y_1^{(1)}} = 0.4 + \sqrt{1.4872} = 1.619$

$= 1.2339 + \frac{0.2}{2}[1.3108 + 1.619]$

$y_1^{(2)} = 1.526$.

$y_1^{(3)} = y_0 + \frac{h}{2}[f(x_0,y_0) + f(x_1, y_1^{(2)})]$

$f(x_1, y_1^{(2)}) = x_1 + \sqrt{y_1^{(2)}} = 0.4 + \sqrt{1.526} = 1.635$

$= 1.2339 + \frac{0.2}{2}[1.3108 + 1.635]$

$y_1^{(3)} = 1.528$.

$y_1(0.4) = 1.528$.

Use modified Eulers method to compute $y(0.1)$ given that $\frac{dy}{dx} = x^2 + y$, $y(0) = 1$ by taking $h = 0.05$ considering the accuracy upto 2 approximation in each step.

By data: $\frac{dy}{dx} = x^2 + y$ $y(0) = 1$

$y(x_0) = y_0$

$x_0 = 0$ $y_0 = 1$

x	y
0	1
0.05	?
0.1	?

$h = 0.05$.

$x_1 = x_0 + h$

$= 0 + 0.05$

$\boxed{x_1 = 0.05}$

$f(x, y) = x^2 + y$

$\boxed{f(x_0 + y_0) = 0^2 + 1 = 1}$

$y_1^{(0)} = y_0 + h[f(x_0, y_0)]$

$= 1 + 0.05[1]$

$y_1^{(0)} = 1.05$

$y_1^{(1)} = y_0 + \frac{h}{2}[f(x_0,y_0) + f(x_1, y_1^{(0)})]$

$f(x_1, y_1^{(0)}) = x_1^2 + y_1^{(0)} = (0.05)^2 + 1.05 = 1.0525$

$= 1 + \frac{0.05}{2}[1 + 1.0525]$

$= 1 + 0.025[0.0513]$

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$\boxed{y_1^{(1)} = 1.0513}$

$y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(1)})\right]$

$f(x_1, y_1^{(1)}) = x_1^2 + y_1^{(1)} = (0.05)^2 + 1.0513 = 1.0538$

$= 1 + \frac{0.05}{2}\left[1 + 1.0538\right]$

$= 1 + 0.025\left[2.0538\right]$

$\boxed{y_1^{(2)} = 1.0513}$

$y_1^{(3)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(2)})\right]$

$f(x_1, y_1^{(2)}) = x_1^2 + y_1^{(2)} = (0.05)^2 + 1.0513 = 1.0538$

$= 1 + \frac{0.05}{2}\left[1 + 1.0538\right]$

$= 1 + 0.025\left[2.0538\right]$

$\boxed{y_1^{(3)} = 1.0513}$

$\boxed{y(0.05) = 1.0513}$

II^{nd} $x_0 = 0.05$ $h = 0.05$ $y_0 = 1.0513$ $x_1 = x_0 + h$ $h = 0.05$

$= 0.05 + 0.05$

$x_1 = 0.1$

$\frac{dy}{dx} = x^2 + y$

$f(x, y) = x^2 + y$

$f(x_0, y_0) = x_0^2 + y_0 = (0.05)^2 + 1.0513 = 1.0538$

$y_1^{(0)} = y_0 + h\left[f(x_0, y_0)\right]$

$= 1.0513 + 0.05(1.0538)$

$\boxed{y_1^{(1)} = 1.103}$

$y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(0)})\right]$

$f(x_1, y_1^{(0)}) = x_1^2 + y_1^{(0)} = (0.05)^2 + 1.103 = 1.1055$

$= 1.0513 + \frac{0.05}{2}\left[1.0538 + 1.1055\right]$

$= 1.0513 + \frac{0.05}{2}[2.1593]$

$\boxed{y_1^{(1)} = 1.105}$

$y_1^{(2)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^{(1)})]$

$f(x_1, y_1^{(1)}) = x_1^2 + y_1^{(1)} = (0.05)^2 + 1.105 = 1.1075$

$= 1.0513 + \frac{0.05}{2}[1.0538 + 1.1075]$

$\boxed{y_1^{(2)} = 1.105}$

$y_1^{(3)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^{(2)})]$

$f(x_1, y_1^{(2)}) = x_1^2 + y_1^{(2)} = (0.05)^2 + 1.105 = 1.1075$

$= 1.0513 + \frac{0.05}{2}[1.0538 + 1.1075]$

$\boxed{y_1^{(3)} = 1.105}$

$\boxed{y(0.1) = 1.105}$

R-K method. continuation.

3. Using R-K method of 4^{th} order find $y(0.2)$ for the equation $\frac{dy}{dx} = \frac{y-x}{y+x}$, $y(0) = 1$ taking $h = 0.2$

Soln By data: $\frac{dy}{dx} = \frac{y-x}{y+x}$ $y(0) = 1$ $h = 0.2$

$x_0 = 0$ $y_0 = 1$

$f(x, y) = \frac{y-x}{y+x}$

$f(x_0, y_0) = \frac{1-0}{1+1} = 1$

$K_1 = hf(x_0, y_0)$

$= 0.2 \times 1$

$\boxed{K_1 = 0.2}$

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$K_2 = hf(x_0 + \frac{h}{2}, y_0 + \frac{K_1}{2})$

$= 0.2f(0 + \frac{0.2}{2}, 1 + \frac{0.2}{2})$

$= 0.2f(0.1, 1.1)$

$f(x_0, y_0) = \frac{y-x}{y+x} = \frac{1.1-0.1}{1.1+0.1} = \frac{1}{1.2} = 0.833$

$= 0.2 \times 0.833$

$\boxed{K_2 = 0.166.}$

$K_3 = hf(x_0 + \frac{h}{2}, y_0 + \frac{K_2}{2})$

$= 0.2f(0 + \frac{0.2}{2}, 1 + \frac{0.166}{2})$

$= 0.2(0.1, 1.083)$

$f(x_0, y) = \frac{y-x}{y+x} = \frac{1.083-0.1}{1.083+0.1} = \frac{0.983}{1.183} = 0.830$

$= 0.2 \times 0.830 = 0.166$

$\boxed{K_3 = 0.166}$

$K_4 = hf(x_0 + h, y_0 + K_3)$

$= 0.2f(0 + 0.2, 1 + 0.166)$

$= 0.2(\overset{x}{0.2}, \overset{y}{1.166})$

$f(x, y) = \frac{y-x}{y+x} = \frac{1.166 - 0.2}{1.166 + 0.2} = \frac{0.966}{1.366} = \cancel{2.332}\ 0.707$

$= \cancel{0.2 \times 2.332}\ 0.2 \times 0.707$

$\boxed{K_4 = 0.1414}$

$y(x_0 + h) = y_0 + \frac{1}{6}[K_1 + 2K_2 + 2K_3 + K_4]$

$y(0+0.2) = 1 + \frac{1}{6}[0.2 + 2(0.166) + 2(0.166) + 0.1414]$

$\boxed{y(0.2) = 1.167}$

7. Use 4^{th} order R-K method to find y at $x=0.1$ given that $\frac{dy}{dx} = 3e^x + 2y$, $y(0)=0$. and $h=0.1$

By data:- $\frac{dy}{dx} = 3e^x + 2y$ $y(0)=0$ $h=0.1$

$x_0=0$ $y_0=0$

$$f(x,y) = 3e^x + 2y$$
$$= 3e^0 + 2(0)$$
$$= 3.$$

$$K_1 = hf(x_0, y_0)$$
$$= 0.1 \times 3$$
$$\boxed{K_1 = 0.3}$$

$$K_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{K_1}{2}\right)$$
$$= 0.1 f\left(0 + \frac{0.1}{2}, 0 + \frac{0.3}{2}\right)$$
$$= 0.1 f(0.05, 0.15)$$

$f(x_0, y_0) = 3e^x + 2y = 3e^{0.05} + 2(0.15)$

$= 1.05 + 0.3 = 3.45$

$$= 0.1 \times 3.45$$
$$\boxed{K_2 = 0.345}$$

$$K_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{0.345}{2}\right)$$
$$= 0.1 f\left(0 + \frac{0.1}{2}, 0 + 0.1725\right)$$
$$= f(0.05, 0.1725)$$

$$f(x,y) = 3e^{0.05} + 2(0.172)$$

$= \cancel{1.05} + 3.15 + 0.344 = 3.494$

$$K_3 = 0.1 \times 3.494$$
$$\boxed{K_3 = 0.349}$$

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$K_4 = hf(x_0+h, y_0+K_3)$

$= 0.1f(0+0.1, 0+0.349)$

$= 0.1f(0.1, 0.349]$

$f(x,y) = 3e^{0.1} + 2\times 0.349 = 3.31+0.698 =$

~~$=0.1\times$~~ $\boxed{K_4 = 0.401}$

$y(x_0+h) = y_0 + \frac{1}{6}[K_1+2K_2+2K_3+K_4]$

$y(0+0.1) = 0+\frac{1}{6}[0.3+2[0.345]+2[0.349]+0.401]$

$\boxed{y(0.1) = 0.348}$

5 Use 4^{th} order R-K method to compute $y(1.1)$ given that $\frac{dy}{dx} = xy^{1/3}$, $y(1)=1$

By data: $\frac{dy}{dx} = xy^{1/3}$

$y(1) = 1 = x_0 = 1$

$x_0 = y_0 \quad y_0 = 1$

$f(x_0, y_0) = x_0 y_0^{1/3}$

$x_1 = x_0+h$

$1.1 = 1+h$

$1.1-1 = h$

$\boxed{0.1 = h}$

$= 1\times 1^{(1/3)} = 1$

$K_1 = hf(x_0, y_0)$

$= 0.1(1) = 0.1$

$K_2 = hf\left(x_0+\frac{h}{2}, y_0+\frac{K_1}{2}\right)$

$= 0.1f\left(1+\frac{0.1}{2}, 1+\frac{0.1}{2}\right)$

$= 0.1f(1.05, 1.05)$

$= 0.1[(1.05)(1.05)^{1/3}]$

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$= 0.1 [(1.05)(1.016)]$

$\boxed{K_2 = 0.106}$

$K_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{K_2}{2}\right)$

$= 0.1 f\left(1 + \frac{0.1}{2}, 1 + \frac{0.106}{2}\right)$

$= 0.1 f(1.053, 1.053)$

$= 0.1 \left[(1.05)(1.053)^{1/3}\right]$

$\boxed{K_3 = 0.106}$

$K_4 = hf(x_0 + h, y_0 + K_3)$

$= 0.1 f(1 + 0.1, 1 + 0.106)$

$= 0.1 (1.1, 1.106)$

$= 0.1 \left[(1.1)(1.106)^{1/3}\right]$

$\boxed{K_4 = 0.113.}$

$y(x_0 + h) = y_0 + \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4)$

$y(1 + 0.1) = 1 + \frac{1}{6}[0.1 + 2(0.106) + 2(0.106) + 0.113)]$

$\boxed{y(1.1) = 1.106}$

6. Using R-K method of 4^{th} order solve $\frac{dy}{dx} + y = 2x$ and $x = 1.1$ given that $y = 3$ at $x = 1$ initially

By data $\frac{dy}{dx} + y = 2x$ $\quad y_0 = 3 \quad x_0 = 1 \quad x_1 = x_0 + h$

$1.1 = 1 + h$

$\frac{dy}{dx} = 2x - y$

$1.1 - 1 = h$

$\boxed{0.1 = h}$

7 Using R-K method of 4^{th} order find $y(0.2)$ for the eqn

$\frac{dy}{dx} = \frac{y-x}{y+x}$, $y(0)=1$ and taking $h=0.1$

By data: $\frac{dy}{dx} = \frac{y-x}{y+x}$ $y(0)=1$ $x_0=0$ $h=0.1$

$y(x_0)=y_0$ $y_0=1$

0

x_1 0.1

0.2

$$f(x_0 y_0) = \frac{y_0-x_0}{y_0+x_0} = \frac{1-0}{1+0} = 1$$

$K_1 = hf(x_0, y_0)$

$\boxed{K_1 = 0.1 \times 1 = 0.1}$

$K_2 = hf(x_0+\frac{h}{2}, y_0+\frac{K_1}{2})$

$= 0.1 f(0+\frac{0.1}{2}, 1+\frac{0.1}{2})$

$= 0.1 f(\overset{x}{0.05}, \overset{y}{1.05})$

$$0.1 \times f(x,y) = \frac{y-x}{y+x} = \frac{1.05-0.05}{1.05+0.05} = \frac{1}{1.1} = 0.909$$

$= 0.1 \times 0.909$

$\boxed{K_2 = 0.0909}$

$K_3 = hf(x_0+\frac{h}{2}, y_0+\frac{K_2}{2})$

$= 0.1 f(0+\frac{0.1}{2}, 1+\frac{0.090}{2})$

$= 0.1 f(0.05, 1.045)$

$$\frac{y-x}{y+x} = \frac{1.045-0.05}{1.045+0.05} = \frac{0.995}{1.095} = 0.908$$

$= 0.1 \times 0.908$

$\boxed{K_3 = 0.090}$

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$K_4 = hf(x_0+h, y_0+K_3)$

$= 0.1 f(0+0.1, 1+0.090)$

$= 0.1(\overset{x}{0.1}, \overset{y}{1.09})$

$$\frac{y-x}{y+x} = \frac{1.09-0.1}{1.09+0.1} = \frac{0.99}{1.19} = 0.8$$

$= 0.1 \times 0.8$

$\boxed{K_4 = 0.08}$

$y(x_0+h) = y_0 + \frac{1}{6}[K_1 + 2K_2 + 2K_3 + K_4]$

$y(0+0.1) = 1 + \frac{1}{6}[0.1 + 2(0.090) + 2(0.0909) + 0.08]$

$\boxed{y(0.1) = 1.09}$

II^{nd} Stage

~~Using RK~~ $y(0.1) = 1.09$ $x_0 = 0.1$, $y_0 = 1.09$ $h = 0.1$

$x = x_0 + h$

$= 0.1 + 0.1$

$x = 0.2$

$$f(x_0, y_0) = \frac{y_0 - x_0}{y_0 + x_0} = \frac{1.09-0.1}{1.09+0.1} = \frac{0.99}{1.19} = 0.831$$

$K_1 = hf(x_0, y_0)$

$= 0.1 \times 0.831$

$\boxed{K_1 = 0.0831}$

$K_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{K_1}{2}\right)$

$= 0.1f\left(0.1 + \frac{0.1}{2}, 1.09 + \frac{0.0831}{2}\right)$

$= 0.1f(\overset{x}{0.15}, \overset{y}{1.131})$

$$= 0.1\left[\frac{1.131-0.15}{1.131+0.15}\right] = 0.1\left[\frac{0.981}{1.281}\right] =$$

$\boxed{K_2 = 0.076}$

$K_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{K_2}{2}\right)$

$= 0.1 f\left(0.1 + \frac{0.1}{2}, 1.09 + \frac{0.076}{2}\right)$

$= 0.1 f(\overset{x}{0.15}, \overset{y}{1.128})$

$= 0.1\left[\frac{1.128 - 0.15}{1.128 + 0.15}\right] = 0.1\left[\frac{0.978}{1.278}\right]$

$\boxed{K_3 = 0.076}$

$K_4 = hf(x_0 + h, y_0 + K_3)$

$= 0.1 f(0.1 + 0.1, 1.09 + 0.076)$

$= 0.1 f(0.2, 1.166)$

$= 0.1\left[\frac{1.166 - 0.2}{1.166 + 0.2}\right] = 0.1\left[\frac{0.966}{1.366}\right]$

$\boxed{K_4 = 0.070}$

$y(x_0 + h) = y_0 + \frac{1}{6}\left[K_1 + 2K_2 + 2K_3 + K_4\right]$

$y(0.1 + 0.1) = 1.09 + 0.166\left[0.0831 + 2(0.076) + 2(0.076) + 0.070\right]$

$y(0.2) = 1.09 + 0.166\left[0.0831 + 0.152 + 0.152 + 0.070\right]$

$y(0.2) = 1.09 + 0.166\left[0.4571\right]$

$\boxed{y(0.2) = 1.165}$

8 Solve $(y^2 - x^2) \cdot dx = y^2 + x^2 \cdot dy$ for $x = 0(0.2)0.4$ given that $y = 1$ and $x = 0$ initially by applying R-K method of order 4

By data : $(y^2 - x^2) \cdot dx = (y^2 + x^2) \cdot dy$ $\qquad h = 0.2$

$\frac{dy}{dx} = \frac{y^2 - x^2}{y^2 + x^2}$ $\qquad x = 0(0.2)0.4$

$y_0 = 1 \quad x_0 = 0$

$f(x_0 y_0) = \frac{1^2 - 0^2}{1^2 + 0^2} = \frac{1}{1} = 1$

$x = x_0 + h$

$= 0 + 0.2$

$\boxed{x = 0.2}$

$K_1 = hf(x_0, y_0)$

$= 0.2 \times 1$

$\boxed{K_1 = 0.2}$

$$K_2 = hf\left(x_0 + \frac{h}{2},\ y_0 + \frac{K_1}{2}\right)$$

$$= 0.2\, f\left(0 + \frac{0.2}{2},\ 1 + \frac{0.2}{2}\right)$$

$$= 0.2 f(\overset{x}{0.1},\ \overset{y}{1.1})$$

$$f(x,y) = \frac{y^2 - x^2}{y^2 + x^2} = \frac{(1.1)^2 - (0.1)^2}{(1.1)^2 + (0.1)^2} = \frac{1.21 - 0.01}{1.21 + 0.01} = \frac{1.2}{1.22} = 0.983$$

$$= 0.2 \times 0.983$$

$$\boxed{K_2 = 0.196}$$

$$K_3 = hf\left(x_0 + \frac{h}{2},\ y_0 + \frac{K_2}{2}\right)$$

$$= 0.2\, f\left(0 + \frac{0.2}{2},\ 1 + \frac{0.196}{2}\right)$$

$$= 0.2 f(0.1,\ 1.098)$$

$$= 0.2\left[\frac{(1.098)^2 - (0.1)^2}{(1.098)^2 + (0.1)^2}\right] = \left(\frac{1.205 - 0.01}{1.205 + 0.01}\right) = \left[\frac{1.195}{1.215}\right] = 0.983$$

$$= 0.2 \times 0.983$$

$$\boxed{K_3 = 0.196}$$

$$K_4 = hf(x_0 + h,\ y_0 + K_3)$$

$$= 0.2\, f(0 + 0.2,\ 1 + 0.196)$$

$$= 0.2 f(\overset{x}{0.2},\ \overset{y}{1.196})$$

$$= 0.2\left[\frac{(1.196)^2 - (0.2)^2}{(1.196)^2 - (0.2)^2}\right] = \left(\frac{1.430 - 0.04}{1.430 + 0.04}\right) = \left(\frac{1.39}{1.47}\right) = 0.945$$

$$= 0.2 \times 0.945$$

$$\boxed{K_4 = 0.189}$$

$$y(x_0 + h) = y_0 + \frac{1}{6}\left[K_1 + 2K_2 + 2K_3 + K_4\right]$$

$$y(0 + 0.2) = 1 + \frac{1}{6}\left[0.2 + 2(0.196) + 2(0.196) + 0.945\right]$$

$$= 1 + \frac{1}{6}\left[0.2 + 0.392 + 0.392 + 0.945\right]$$

$$= 1 + \frac{1}{6}(1.929)$$

$$\boxed{y(0.2) = 1.3215}$$

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6. Using R-K method of fourth order solve $\frac{dy}{dx}+y=2x$ and $x=1.1$ given that $y=3$ at $x=1$ initially.

Soln By data: $\frac{dy}{dx}+y=2x$ $\quad x_0=1 \quad y_0=3$ $\quad x=x_0+h$

$1.1=1+h$

$1.1-1=h$

$\boxed{0.1=h}$

$\frac{dy}{dx}=2x-y$

$f(x,y)=2x-y$

$\boxed{f(x_0,y_0)=2x_0-y_0=2(1)-3=-1}$

$K_1=hf(x_0,y_0)$

$K_1=\cancel{x_0}\ 0.1(-1)$

$\boxed{K_1=-0.1}$

$K_2=hf\left(x_0+\frac{h}{2},\ y_0+\frac{K_1}{2}\right)$

$=0.1f\left(1+\frac{0.1}{2},\ 3+\frac{(-0.1)}{2}\right)$

$=0.1f(\overset{x}{1.05},\ \overset{y}{2.95})$

$f(x_0,y_0)=2(1.05)-(2.95)=-0.85$

$=0.1\times-0.85$

$\boxed{K_2=-0.085}$

$K_3=hf\left(x_0+\frac{h}{2},\ y_0+\frac{K_2}{2}\right)$

$=0.1f\left(1+\frac{0.1}{2},\ 3+\frac{-0.085}{2}\right)$

$=0.1f(1.05,\ 2.95)$

$f(x_0,y_0)=2(1.05)-(2.95)=-0.85$

$=0.1\times-0.85$

$\boxed{K_3=-0.085}$

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$K_4 = hf(x_0 + h, y_0 + K_3)$

$= 0.1 f(1 + 0.1, 3 + (-0.085))$

$= 0.1 f(\overset{x}{1.1}, \overset{y}{2.915})$

$f(x,y) = 2(1.1) - 2.915 = -0.715$

$= 0.1 \times -0.715$

$\boxed{K_4 = -0.0715}$

$y(x_0+h) = y_0 + \frac{1}{6}[K_1 + 2K_2 + 2K_3 + K_4]$

$y(1+0.1) = \cancel{3} + \frac{1}{\cancel{6}2}[(-0.1) + 2(-0.085) + 2(-0.085) + (-0.0715)]$

$y(1.1) = 0.5[(-0.1) + (-0.17) + (-0.17) + (-0.715)]$

$\boxed{y(1.1) = 0.5775}$

Neumerical predictor & Corrector method.

1. Milne's method

2. Adam's - Bashforth method

Milne's predictor and corrector formulae

$y_4^{(P)} = y_0 + \frac{4h}{3}(2y_1' - y_2' + 2y_3')$

$p \rightarrow$ predictor formula

$y_4^{(C)} = y_2 + \frac{h}{3}[y_2' + 4y_3' + y_4'] \rightarrow$ corrector formula.

Adam's Bashforth predictor and corrector formula

$$y_4^{(P)} = y_3 + \frac{h}{24}\left[55y_3' - 59y_2' + 37y_1' - 9y_0'\right]$$

predictor formula.

$$y_4^{(c)} = y_3 + \frac{h}{24}\left[9y_4' + 19y_3' - 5y_2' + y_1'\right] \rightarrow \text{corrector}$$

Problems

1. Given that $\frac{dy}{dx} = x - y^2$ and the data $y(0)=0$, $y(0.2)=0.02$ $y(0.4)=0.0795$, $y(0.6)=0.1762$ compute y at $x=0.8$ applying Milne's method.

Sol^n: By data :

x	y	$y' = x - y^2$
$x_0 = 0$	$y_0 = 0$	$y_0' = x_0 - y_0^2 = 0$
$x_1 = 0.2$	$y_1 = 0.02$	$y_1' = x_1 - y_1^2 = 0.2 - (0.02)^2 = 0.1996$
$x_2 = 0.4$	$y_2 = 0.0795$	$y_2' = x_2 - y_2^2 = 0.4 - (0.0795)^2 = 0.3936$
$x_3 = 0.6$	$y_3 = 0.1762$	$y_3' = x_3 - y_3^2 = 0.6 - (0.1762)^2 = 0.5689$
$x_4 = 0.8$	$y_4 = ?$	$y_4' = ?$

W.K.T Milne's corrector and predictor formulae

$$y_4^{(P)} = y_0 + \frac{4h}{3}\left[2y_1' - y_2' + 2y_3'\right]$$

$$= 0 + \frac{4(0.2)}{3}\left[2(0.1996) - (0.3936) + 2(0.5689)\right]$$

$$\boxed{y_4^{(P)} = 0.3049}$$

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$$\boxed{y_4^{(P)'} = x_4 - [y_4^{(P)}]^2 = 0.8 - (0.3049)^2 = 0.7070.}$$

Apply corrector formula

$$y_4^{(C)} = y_2 + \frac{h}{3}[y_2' + 4y_3' + y_4']$$

$$= 0.0795 + \frac{0.2}{3}[0.3936 + 4[0.5689] + 0.7070]$$

$$\boxed{y_4^{(C)} = 0.30458 \simeq 0.3046.}$$

Again apply corrector

$$y_4^{(C)'} = x_4 - (y_4^{(1)})^2 = 0.8 - (0.30458)^2 = 0.7072$$

$$y_4^{(C)} = y_2 + \frac{h}{3}[y_2' + 4y_3' + y_4']$$

$$= 0.0795 + \frac{0.2}{3}[0.3936 + 4[0.5089] + 0.7072]$$

$$y_4^{(C)} = 0.30459 \simeq 0.3046$$

$$\boxed{y(0.8) = 0.3046.}$$

2. Apply Milne's method to compute $y(1.4)$ correct to 4 decimal places. given $\frac{dy}{dx} = x^2 + \frac{y}{2}$ and following the data $y(1) = 2$; $y(1.1) = 2.2156$ $y(1.2) = 2.4649$ $y(1.3) = 2.7514$

Sol^n By data:

x	y	$y' = x^2 + y/2$
$x_0 = 1$	$y_0 = 2$	$y_0' = 1^2 + 2/2 = 1 + 1 = 2$
$x_1 = 1.1$	$y_1 = 2.2156$	$y_1' = (1.1)^2 + \frac{2.2156}{2} = 2.3175$
$x_2 = 1.2$	$y_2 = 2.4649$	$y_2' = (1.2)^2 + \frac{2.4649}{2} = 2.6754.$
$x_3 = 1.3$	$y_3 = 2.7514$	$y_3' = (1.3)^2 + \frac{2.7514}{2} = 3.0657$
$x_4 = 1.4$	$y_4 = ?$	

W.K.T Milnes P & C formula.

$$y_4^{(P)} = y_0 + \frac{4h}{3}[2y_1' - y_2' + 2y_3']$$

$$= 2 + \frac{4(0.1)}{3}[2(2.3175) - 2.6754 + 2(3.0657)]$$

$$\boxed{y_4^{(P)} = 3.0797}$$

$$\boxed{y_4' = x^2 + y/2 = x_4^2 + \frac{y_4^{(P)}}{2} = (1.4)^2 + \frac{3.0797}{2} = 3.499}$$

Apply Corrector formula.

$$y_4^{(c)} = y_2 + \frac{h}{3}[y_2' + 4y_3' + y_4']$$

$$= 2.4649 + \frac{0.1}{3}[2.6754 + 4[3.0657] + 3.499]$$

$$\boxed{y_4^{(c)} = 3.0793}$$

Again apply corrector formula

$$y_4' = x_4^2 + \frac{y_4^{(c)}}{2} = (1.4)^2 + \frac{3.0793}{2} = 3.4996$$

$$y_4^{(c)} = y_2 + \frac{h}{3}[y_2' + 4y_3' + y_4']$$

$$= 2.4649 + \frac{0.1}{3}[2.6754 + 4[3.0657] + 3.4996]$$

$$\boxed{y_4^{(c)} = 3.0793}$$

$$\boxed{y_4(1.4) = 3.0793}$$

4. If $\frac{dy}{dx} = 2e^x - y$, $y(0) = 2$, $y(0.1) = 2.010$, $y(0.2) = 2.040$, $y(0.3) = 2.090$ find $y(0.4)$ correct to 4 decimal places by using Milne's predictor & corrector method.

Adam's

Q.1 Given that $\frac{dy}{dx} = x - y^2$, ~~and~~ $y(0) = 0$, $y(0.2) = 0.02$, ~~y(0.4)~~ $= 0.$ $y(0.4) = 0.0795$, $y(0.6) = 0.1762$ compute y at $x = 0.8$ by applying Adam's Bash-forth method.

x	y	$y' = x - y^2$
$x_0 = 0$	$y_0 = 0$	$y_0' = 0 - 0^2 = 0$
$x_1 = 0.2$	$y_1 = 0.02$	$y_1' = 0.2 - (0.02)^2 = 0.1996$
$x_2 = 0.4$	$y_2 = 0.0795$	$y_2' = 0.4 - (0.0795)^2 = 0.3936$
$x_3 = 0.6$	$y_3 = 0.1762$	$y_3' = 0.6 - (0.1762)^2 = 0.5689$
$x_4 = 0.8$	$y_4 = ?$	$y_4' = ?$

W.K.T Adam's Bash forth P & C formulae.

$$y_4^{(P)} = y_3 + \frac{h}{24}\left[55y_3' - 59y_2' + 37y_1' - 9y_0'\right]$$

$$= 0.1762 + \frac{0.2}{24}\left[55(0.5689) - 59(0.3936) + 37(0.1996) - 9(0)\right]$$

$$= 0.1762 + 8.33\times10^{-3}\left[31.2895 - 23.2224 + 7.3852 - 0\right]$$

$$= 0.1762 + 8.33\times10^{-3}(15.4523)$$

$$= 7.5614) \times$$

$$= 0.3049.$$

Now $y_4' = x_4 - (y_4^{(P)})^2 = 0.8 - [0.3049]^2 = 0.7070.$

Apply corrector formula.

$$y_4^{(c)} = y_3 + \frac{h}{24}[9y_4' + 19y_3' - 5y_2' + y_1']$$

$$= 0.1762 + \frac{0.2}{24}[9(0.7070) + 19(0.5689) - 5(0.3936) + 0.1996]$$

$y_4^{(c)} = 0.3045.$

$y_4' = x_4 - (y_4^{c})^2 = 0.8 - (0.3045)^2 = 0.7072.$

Again apply corrector formula.

$$y_4^{(c)} = y_3 + \frac{h}{24}[9y_4' + 19y_3' - 5y_2' + y_1']$$

$$= 0.1762 + \frac{0.2}{24}[9(0.7072) + 19(0.5689) - 5(0.3936) + 0.1996]$$

$y_4^{(c)} = 0.3045$

$y(0.8) = 0.3045.$

2 Apply Adam's Bash forth method to solve the eq^n

$(y^2+1)dy - x^2 \cdot dx = 0.$

x	0.	~~2.~~ 0.25	0.5	0.75
y	1	1.0026	1.0206	1.0679

find y at $x=1$

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By data: $(y^2+1)dy - x^2dx = 0$

$(y^2+1)\,dy = x^2dx.$

$(y^2+1)\dfrac{dy}{dx} = x^2$ $\qquad h = 0.25$

$\dfrac{dy}{dx} = \dfrac{x^2}{y^2+1}$

x	y	$y' = \dfrac{x^2}{y^2+1}$
$x_0 = 0$	y_0 1	$y_0' = \dfrac{0^2}{1^2+1} = 0$
$x_1 = 0.25$	y_1 1.0026	$y_1' = \dfrac{(0.25)^2}{(1.0026)^2+1} = 0.0311$
$x_2 = 0.5$	y_2 1.0206	$y_2' = \dfrac{(0.5)^2}{(1.0206)^2+1} = 0.122$
$x_3 = 0.75$	y_3 1.0679	$y_3' = \dfrac{(0.75)^2}{(1.0679)^2+1} = 0.2628.$
x_4		

W.K.T

$$y_4^{(P)} = y_3 + \frac{h}{24}\left[55y_3' - 59y_2' + 37y_1' - 9y_0'\right]$$

$$= 1.0679 + \frac{\overset{0.0104}{0.25}}{24}\left[55(0.2628) - 59(0.122) + 37(0.0311) - 9(0)\right]$$

$$\boxed{y_4^{(P)} = 1.1554}$$

$$y_4' = \frac{x_4^2}{(y_4^{P})^2+1} = \frac{1^2}{(1.1554)^2+1} = 0.428$$

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Apply corrector formula.

$$y_4^{(c)} = y_3 + \frac{h}{24}\left[9y_4' + 19y_3' - 5y_2' + y_1'\right]$$

$$= 1.0679 + \frac{0.25}{24}\left[9(0.428) + 19(0.2628) - 5(0.122) + 0.0311\right]$$

$$\boxed{y_4^{(c)} = 1.154}$$

Again

$$\boxed{y_4' = \frac{x_4^2}{(y_4^{(c)})^2 + 1} = \frac{1^2}{(1.154)^2 + 1} = 0.428.}$$

Again apply corrector formula.

$$y_4^{(c)} = y_3 + \frac{h}{24}\left[9y_4' + 19y_3' - 5y_2' + y_1'\right]$$

$$= 1.0679 + \frac{0.25}{24}\left[9(\overset{0.428}{\cancel{1.154}}) + 19(0.2628) - 5(0.122) + 0.0311\right]$$

$$\boxed{y_4^{(c)} = 1.154}$$

$$\boxed{y(1) = 1.154}$$

3. Solve the differential eqn $y' + y + xy^2 = 0$ with the initial values of y: $y_0 = 1$, $y_1 = 0.9008$, $y_2 = 0.8066$, $y_3 = 0.722$ corresponding to the values of x: $x_0 = 0$, ~~y~~ $x_1 = 0.1$, $x_2 = 0.2$, $x_3 = 0.3$ by computing the value of y corresponding to $x = 0.4$ apply Adam's Bash-forth predictor and corrector formula.

By data: $y' + y + xy^2 = 0$

$y' = -x^2 - y$

$y' = -[x^2 + y]$

4. If $\frac{dy}{dx} = 2e^x - y$, $y(0) = 2$, $y(0.1) = 2.010$, $y(0.2) = 2.040$ and $y(0.3) = 2.090$ find $y(0.4)$ correct to 4 decimal places by using Adam's Bash-forth P & C formula.

MODULE - 5

Numerical Solution of Second Order ODE

The given 2^{nd} order ODE where two initial conditions will reduce to two first order simultaneous ODE's. Let

Let $y'' = g(x, y, y')$ with the initial conditions $y(x_0) = y_0$ and $y'(x_0) = y_0'$ with the given 2^{nd} order ODE

Let $y' = \frac{dy}{dx} = z$

$$y'' = \frac{d^2y}{dx^2} = \frac{dz}{dx}.$$

The given second order differential equation assumes the form $\frac{dz}{dx} = g(x, y, z)$ with the conditions $y(x_0) = y_0$ and $z(x_0) = z_0$ where $\underline{y_0'}$ is denoted by $\underline{z_0}$.

<u>Two first order simultaneous ODE</u>

$\frac{dy}{dx} = z$ and $\frac{dz}{dx} = g(x, y, z)$ with $y(x_0) = y_0$ and $z(x_0) = z_0$.

Taking $f(x, y, z) = z$ we know have the following Systems of equations for solving $\underline{\frac{dy}{dx} = f(x, y, z)}$ and

$\frac{dz}{dx} = g(x, y, z)$ where $y(x_0) = y_0$ and $z(x_0) = z_0$

Runge-Kutta method of ~~2nd~~ 4th order

$K_1 = hf(x_0, y_0, z_0)$, $l_1 = hg(x_0, y_0, z_0)$

$K_2 = hf\left[x_0 + \frac{h}{2}, y_0 + \frac{K_1}{2}, z_0 + \frac{l_1}{2}\right]$

$l_2 = hg\left[x_0 + \frac{h}{2}, y_0 + \frac{K_1}{2}, z_0 + \frac{l_1}{2}\right]$

$K_3 = hf\left[x_0 + \frac{h}{2}, y_0 + \frac{K_2}{2}, z_0 + \frac{l_2}{2}\right]$

$l_3 = hg\left[x_0 + \frac{h}{2}, y_0 + \frac{K_2}{2}, z_0 + \frac{l_2}{2}\right]$

$K_4 = hf\left[x_0 + h, y_0 + K_3, z_0 + l_3\right]$

$l_4 = hg\left[x_0 + h, y_0 + K_3, z_0 + l_3\right]$

The required $y(x_0 + h) = y_0 + \frac{1}{6}\left[K_1 + 2K_2 + 2K_3 + K_4\right]$

and $y'(x_0 + h) = z(x_0 + h) = z_0 + \frac{1}{6}\left[l_1 + 2l_2 + 2l_3 + l_4\right]$

Problems

Imp

1. Given $\frac{d^2y}{dx^2} - x^2\frac{dy}{dx} - 2xy = 1$, $y(0) = 1$, $y'(0) = 0$. Evaluate $y(0.1)$ using R-K method of 4^{th} order.

By data: $\frac{d^2y}{dx^2} - x^2\frac{dy}{dx} - 2xy = 1 \longrightarrow (1)$

$y(0) = 1$ $\quad y'(0) = 0$

$x_0 = 0 \quad y_0 = 1 \quad y_0' = z_0 = 0$

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put $\frac{dy}{dx} = z \quad f(x,y,z) = z \rightarrow (2)$

Diff. w.r.t x.

$$\frac{d^2y}{dx^2} = \frac{dz}{dx}$$

eq^n (1) becomes

$$\frac{dz}{dx} - x^2z - 2xy = 1 \qquad g(x,y,z) = 1 + x^2z + 2xy \rightarrow (3)$$

$$\frac{dz}{dx} = 1 + x^2z + 2xy \leftarrow (3)$$

$$K_1 = hf(x_0, y_0, z_0)$$

$$= 0.1 f(0, 1, 0)$$

$$f(x_0, y_0, z_0) = z_0$$

$$K_1 = 0.1 (0)$$

$$K_1 = 0$$

$$l_1 = hg(x_0, y_0, z_0)$$

$$= 0.1\, g(0, 1, 0)$$

$$g(x_0, y_0, z_0) = 1 + x^2z + 2xy$$

$$= 1 + 0 + 0 = 1$$

$$l_1 = 0.1 \times 1 = 0.1$$

$$K_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{K_1}{2}, z_0 + \frac{l_1}{2}\right)$$

$$= 0.1 f\left(0 + \frac{0.1}{2}, 1 + \frac{0}{2}, 0 + \frac{0.1}{2}\right)$$

$$= 0.1 f(0.05, 1, 0.05)$$

$$K_2 = 0.1 \times 0.05$$

$$K_2 = 0.005$$

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$l_2 = hg\left[x_0+\frac{h}{2}, y_0+\frac{k_1}{2}, z_0+\frac{l_1}{2}\right]$

$= 0.1g\left[0+\frac{0.1}{2}, 1+\frac{0}{2}, 0+\frac{0.1}{2}\right]$

$= 0.1g(0.05, 1, 0.05)$

$l_2 = 0.1\left(1+(0.05)^2(0.05)+2(0.05)(1)\right]$

$l_2 = 0.1100$

$K_3 = hf\left[x_0+\frac{h}{2}, y_0+\frac{K_2}{2}, z_0+\frac{l_2}{2}\right]$

$= 0.1f\left[0+\frac{0.1}{2}, 1+\frac{0.005}{2}, 0+\frac{0.1100}{2}\right]$

$= 0.1f\left[0.05, 1.0025, 0.055\right]$

$K_3 = 0.1(0.055)$

$K_3 = 0.0055.$

$l_3 = hg\left[x_0+\frac{h}{2}, y_0+\frac{K_2}{2}, z_0+\frac{l_2}{2}\right]$

$= 0.1g\left[0+\frac{0.1}{2}, 1+\frac{0.005}{2}, 0+\frac{0.1100}{2}\right]$

$= 0.1g\left[\overset{x}{0.05}, \overset{y}{1.0025}, \overset{z}{0.055}\right]$

$l_3 = g\,0.1\left[1+(0.05)^2(0.055)+2(0.05)(1.0025)\right]$

$l_3 = 0.1102$

$K_4 = hf\left[x_0+h, y_0+K_3, z_0+l_3\right]$

$= 0.1f\left[0+0.1, 1+0.0055, 0+0.1102\right]$

$= 0.1f\left[0.1, 1.0055, 0.1102\right]$

$K_4 = 0.1[0.1102] = 0.01102$

$l_4 = hg(x_0+h,\ y_0+k_3,\ z_0+l_3)$

$= 0.1g(0+0.1,\ 1+0.0055,\ 0+0.1102)$

$= 0.1g[0.1,\ 1.0055,\ 0.1102]$

$= 0.1[1+(0.1)^2(0.1102)+2(0.1)(1.0055)]$

$l_4 = 0.1202$

$y(x_0+h) = y_0 + \frac{1}{6}[K_1+2K_2+2K_3+K_4]$

$y(0+0.1) = 1 + \frac{1}{6}[0+2(0.005)+2(0.0055)+0.01102]$

$\boxed{y(0.1) = 1.0053}$

Milne's assignment

3. The following table gives the solution of $5xy'+y^2-2=0$ find the value of y at $x=4.5$ using Milne's P & C formula. use the corrector formula twice.

x	4	4.1	4.2	4.3	4.4
y	1	1.0049	1.0097	1.00143	1.0187

By data: $5xy'+y^2-2=0$

$5xy' = y^2-2$

$y' = \frac{2-y^2}{5x}$

$y(4.5) = ?$

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We require y_5, the equalent form of these formulae.

$$y_5^{(P)} = y_1 + \frac{4h}{3}\left[2y_2' - y_3' + 2y_4'\right]$$

$$y_5^{(C)} = y_3 + \frac{h}{2}\left[y_3' + 4y_4' + y_5'\right]$$

x y

~~$x_0=4$~~ ~~$y_0=1$~~

x	y	$y' = \frac{2-y^2}{5x}$
$x_0=4$	$y_0=1$	$y_0' = \frac{2-(1)^2}{5(4)} = \frac{2-1}{20} = \frac{1}{20} = 0.05$
$x_1=4.1$	$y_1=1.0049$	$y_1' = \frac{2-(1.0049)^2}{5(4.1)} = 0.0483$
4.2	1.0097	$y_2' = \frac{2-(1.0097)^2}{5(4.2)} = 0.0466$
4.3	1.0143	$y_3' = \frac{2-(1.0143)^2}{5(4.3)} = 0.0451$
4.4	1.0187	$y_4' = \frac{2-(1.0187)^2}{5(4.4)} = 0.0437$
4.5	?	$y_5' = ?$

W.K.T Milne's P & C formula.

$$y_5^{(P)} = y_1 + 4\frac{h}{3}\left[2y_2' - y_3' + 2y_4'\right]$$

$$= 1.0049 + 4\frac{(0.1)}{3}\left[2(0.0466) - 0.0451 + 2(0.0437)\right]$$

$$\boxed{y_5^{(P)} = 1.0229}$$

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$$y_5^{\prime} = \frac{2-y_5^2}{5x_5} = \frac{2-(1.0229)^2}{5(4.5)} = 0.0423$$

Apply corrector formula

$$y_5^{(c)} = y_3 + \frac{h}{2}\left[y_3' + 4y_4' + y_5'\right]$$

$$= 1.0143 + \frac{0.1}{2}\left[0.0451 + 4(0.0437) + 0.0423\right]$$

$$\boxed{y_5^{(c)} = 1.0274}$$

Again apply corrector formulae.

$$y_5^{(c)} = y_3 + \frac{h}{2}\left[y_3' + 4y_4' + y_5'\right]$$

$$y_5' = \frac{2-y_5^2}{5x} = \frac{2-(1.0274)^2}{5(4.5)} = 0.0419$$

$$= 1.0143 + \frac{0.1}{2}\left[0.0451 + 4(0.0437) + 0.0419\right]$$

$$\boxed{y_5^{(c)} = 1.027}$$

$$\boxed{y(4.5) = 1.027}$$

4. If $\frac{dy}{dx} = 2e^x - y$, $y(0)=2$, $y(0.1)=2.010$, $y(0.2)=2.040$, $y(0.3)=2.090$ find $y(0.4)$ correct to 4 decimal places by using Milne's predictor and corrector method.

By data: $\frac{dy}{dx} = 2e^{x} - y$ $\quad h = 0.1$

x	y	$y' = 2e^{x} - y$
$x_0 = 0$	$y_0 = 2$	$y_0' = 2e^{x_0} - y_0 = 2e^{(0)} - 2 = 0$
$x_1 = 0.1$	$y_1 = 2.010$	$y_1' = 2e^{(0.1)} - 2.010 = 0.200$
$x_2 = 0.2$	$y_2 = 2.040$	$y_2' = 2e^{(0.2)} - 2.040 = 0.402$
$x_3 = 0.3$	$y_3 = 2.090$	$y_3' = 2e^{(0.3)} - 2.090 = 0.609$
$x_4 = 0.4$	$y_4 = ?$	

W.K.T Milne's P 4 C formula.

$$y_4^{(P)} = y_0 + 4\frac{h}{3}\left[2y_1' - y_2' + 2y_3'\right]$$

$$= 2 + 4\,\overset{0.1333}{\frac{(0.1)}{3}}\left[2(0.200) - 0.402 + 2(0.609)\right]$$

$$y_4^{(P)} = 2.1620$$

$$y_4' = 2e^{x_4} - y_4^{(P)} = 2e^{(0.4)} - 2.1620 = 0.8216$$

Apply corrector formula.

$$y_4^{(C)} = y_2 + \frac{h}{3}\left[y_2' + 4y_3' + y_4'\right]$$

$$= 2.040 + \overset{0.0333}{\frac{0.1}{3}}\left[0.402 + 4(0.609) + 0.8216\right]$$

$$y_4^{(C)} = 2.1618.$$

Again apply corrector formula.

$$y_4^{(c)\prime} = 2e^{x} - y = 2e^{(0.4)} - 2.1618 = 0.8218$$

$$y_4^{(c)} = y_2 + \frac{h}{3}\left[y_2' + 4y_3' + y_4'\right]$$

$$\cancel{0.4027} = 2.040 + \frac{0.1}{3}\left[0.402 + 4(0.609) + 0.8218\right]$$

$$y_4^{(c)} = 2.1618$$

$$y(0.4) = 2.1618$$

Adam's Assignment

3. Solve the differential equation $y' + y + xy^2 = 0$ with the initial values of y: $y_0 = 1$, $y_1 = 0.9008$, $y_2 = 0.8066$, $y_3 = 0.722$ corresponding to the values of x: $x_0 = 0$, $x_1 = 0.1$, $x_2 = 0.2$, $x_3 = 0.3$ by computing the value of y at $x = 0.4$ apply Adam's Bash-forth P&C formula

By data: $y' + y + xy^2 = 0$

$$\cancel{y'+y}\ y' = -x^2 - y$$

$$y' = -[x^2 + y]$$

	x	y	$y' = -[x^2+y]$
x_0	0.	y_0 $\cancel{0}$1	$y_0' = -[x_0^2 + y_0] = -[0^2 + \phi] = \cancel{0}1$
	0.1	y_1 0.9008	$y_1' = -[(0.1)^2 + 0.9008] = -0.9108$
	0.2	y_2 0.8066	$y_2' = -[(0.2)^2 + 0.8066] = -0.8466$
	0.3	y_3 0.722	$y_3' = -[(0.3)^2 + 0.722] = -0.812$
	0.4	?	

W.K.T Adam's Bash-forth formula.

$$y_4^{(P)} = y_3 + \frac{h}{24}\left[55y_3' - 59y_2' + 37y_1' - 9y_0'\right]$$

$$= 0.722 + \frac{\overset{4.16\times10^{-3}}{0.1}}{24}\left[55(-0.812) - 59(-0.8466) + 37(-0.9108) - 9(-1)\right]$$

$$y_4^{(P)} = 0.6412$$

$$y_4' = -[x^2+y] = -[(0.4)^2 + 0.6412] = -0.8012.$$

Apply corrector formula

$$y_4^{(C)} = y_3 + \frac{h}{24}\left[9y_4' + 19y_3' - 5y_2' + y'\right]$$

$$= 0.722 + \frac{0.1}{24}\left[9(-0.8012) + 19(-0.812) - 5(-0.8466) + (-0.9108)\right]$$

$$y_4^{(C)} = 0.6416.$$

$$y_4' = -[x^2+y] = -[(0.4)^2 + 0.6416] = -0.8016$$

Again apply corrector formula.

$$y_4^{(C)} = y_3 + \frac{h}{24}\left[9y_4' + 19y_3' - 5y_2' + y'\right]$$

$$= 0.722 + \frac{0.1}{24}\left[9(-0.8016) + 19(-0.812) - 5(-0.8466) + (-0.9108)\right]$$

$$y_4^{(C)} = 0.6416$$

4 If $\frac{dy}{dx} = 2e^x - y$, $y(0) = 2$, $y(0.1) = 2.010$, $y(0.2) = 2.040$ and $y(0.3) = 2.090$ find $y(0.4)$ correct to 4 decimal places by using Adam's Bash-forth P&C formula

By data : $\frac{dy}{dx} = 2e^{x} - y$ $\qquad h = 0.1$

$y' = 2e^{x} - y$

x	y	$y' = 2e^{x} - y$
$x_0 = 0$	$y_0 = 2$	$y_0' = 2e^{(0)} - 2 = 0$
$x_1 = 0.1$	$y_1 = 2.010$	$y_1' = 2e^{(0.1)} - 2.010 = 0.2003$
$x_2 = 0.2$	$y_2 = 2.040$	$y_2' = 2e^{(0.2)} - 2.040 = 0.4028$
$x_3 = 0.3$	$y_3 = 2.090$	$y_3' = 2e^{(0.3)} - 2.090 = 0.6097$
$x_4 = 0.4$	$y_4 = ?$	

W.K.T Adam's Bash-forth

$$y_4^{(P)} = y_3 + \frac{h}{24}\left[55y_3' - 59y_2' + 37y_1' - 9y_0'\right]$$

$$= 2.090 + \frac{0.1}{24}\left[55(0.6097) - 59(0.4028) + 37(0.2003) - 9(0)\right]$$

$$y_4^{(P)} = 2.1614$$

$$y_4' = 2e^{x} - y = 2e^{(0.4)} - 2.1614 = 0.8222$$

Apply corrector formula.

$$y_4^{(C)} = y_3 + \frac{h}{24}\left[9y_4' + 19y_3' - 5y_2' + y_1'\right]$$

$$= 2.090 + \frac{0.1}{24}\left[9(0.8222) + 19(0.6097) - 5(0.4028) + 0.2003\right]$$

$$= 2.1614$$

$$y_4' = 2e^{x} - y = 2e^{(0.4)} - 2.1614 = 0.8222$$

Again apply corrector formula.

$$y_4^{(c)} = y_3 + \frac{h}{24}\left[9y_4' + 19y_3' - 5y_2' + y_1'\right]$$

$$= 2.090 + \frac{0.1}{24}\left[9(0.822) + 19(0.6097) - 5(0.4028) + 0.2003\right]$$

$$y_4^{(c)} = 2.1614$$

$$y(0.4) = 2.1614$$

Module-5 R-K method continuation

By R-K method $\frac{d^2y}{dx^2} = x\left[\frac{dy}{dx}\right]^2 - y^2$ for $x = 0.2$ correct to 4 decimal places, using the initial conditions $y=1$ and $y'=0$ when $x=0$

By data: $\frac{d^2y}{dx^2} = x\left[\frac{dy}{dx}\right]^2 - y^2 \longrightarrow 1$

$x = 0.2 \quad x_0 = 0 \quad y_0 = 1 \quad z_0 = 0$

$x = x_0 + h$

$0.2 = 0 + h$

$\boxed{h = 0.2}$

put $\frac{dy}{dx} = z \rightarrow f(x,y,z) = z \quad — 2$

Diff w.r.t x.

$$\frac{d^2y}{dx^2} = \frac{dz}{dx}$$

These values substitute in eqn 1

$$\frac{dz}{dx} = xz^2 - y^2$$

$$g(x,y,z) = xz^2 - y^2 \quad — 3$$

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$K_1 = hf(x_0, y_0, z_0)$

$= 0.2\, f(0, 1, 0)$

$= 0.2 \times 0$

$K_1 = 0.$

$l_1 = hg(x_0, y_0, z_0)$

$= hg(0, 1, 0)$

$= (0.2)[0(0)^2 - 1^2]$

$l_1 = -0.2$

$K_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{K_1}{2}, z_0 + \frac{l_1}{2}\right)$

$= 0.2 f\left[0 + \frac{0.2}{2}, 1 + \frac{0}{2}, \cancel{z_0}\, 0 + \frac{(-0.2)}{2}\right]$

$= 0.2 f(0.1, 1, -0.1)$

$K_2 = 0.2(-0.1) = -0.02$

$l_2 = hg\left(x_0 + \frac{h}{2}, y_0 + \frac{K_1}{2}, z_0 + \frac{l_1}{2}\right)$

$= 0.2\, g\left(0 + \frac{0.2}{2}, 1 + \frac{0}{2}, 0 + \frac{(-0.2)}{2}\right)$

$= 0.2 g(0.1, 1, -0.1)$

$= 0.2[0.1(-0.1)^2 - 1^2]$

$= -0.1998$

$K_3 = hf\left(x_0 + \frac{h}{2}, y_0, \frac{K_2}{2}, z_0 + \frac{l_2}{2}\right)$

$= 0.2 f\left[0 + \frac{0.2}{2}, 1 + \frac{(-0.02)}{2}, 0 + \frac{(-0.1998)}{2}\right]$

$= 0.2 f(0.1, 0.99, -0.099)$

$\cancel{K_3}$

$$K_3 = (0.2)(-0.099) = -0.0199$$

$$l_3 = hg\left[x_0 + \frac{h}{2},\ y_0 + \frac{K_2}{2},\ z_0 + \frac{l_2}{2}\right]$$

$$= 0.2\, g\left[0 + \frac{0.2}{2},\ 1 + \frac{(-0.02)}{2},\ 0 + \frac{(-0.1998)}{2}\right]$$

$$= 0.2\, g\left[0.1,\ 0.99,\ -0.099\right]$$

$$= 0.2\left[0.1(-0.099)^2 - (0.99)^2\right]$$

$$l_3 = -0.1958$$

$$K_4 = h\cancel{g}f\left[x_0 + h,\ y_0 + K_3,\ z_0 + l_3\right]$$

$$= 0.2\, f\left[0 + 0.2,\ \cancel{y}\,1 + (-0.0199),\ \cancel{z_0}\,0\,\cancel{f}\,(-0.1958)\right]$$

$$= 0.2\, f\left[0.2,\ 0.9801,\ -0.1958\right]$$

$$= 0.2(-0.1958) = -0.03916.$$

$$l_4 = hg\left[x_0 + h,\ y_0 + K_3,\ z_0 + l_3\right]$$

$$= 0.2\, g\left[0 + 0.2,\ 1 + (-0.0199),\ 0 + (-0.1958)\right]$$

$$= 0.2\, g(0.2,\ 0.9801,\ -0.1958)$$

$$= 0.2\left[(0.2)(-0.1958)^2 - (0.9801)^2\right]$$

$$= -0.1905$$

$y_0(x_0+h) = y_0 + \frac{1}{6}[K_1 + 2K_2 + 2K_3 + K_4]$

$y(0.2+0.2) = 1 + \frac{1}{6}[0 + 2(-0.02) + 2(-0.0199) + (-0.03916)]$

$y(0.4) = 1.166[-0.11896]$

$\therefore y(0.4) = -0.19818$

Compute $y(0.1)$ given $\frac{d^2y}{dx^2} = y^3$ and $y=10$, $\frac{dy}{dx} = 5$ at $x=0$ by RK method of 4^{th} order.

$x_0 = 0 \quad y_0 = 10 \quad \frac{dy}{dx} = z = 5 \quad \frac{d^2y}{dx^2} = y^3$ — ①

put $\frac{dy}{dx} = z \to f(x,y,z) \to$ ②

$x = x_0 + h$

$0.1 = 0 + h$

$\boxed{h = 0.1}$

Diff w.r.t x

$\frac{d^2y}{dx^2} = \frac{dz}{dx}$

These value substitute in eqn ①

$\frac{dz}{dx} = y^3$

$g(x,y,z) = y^3 \to$ ③

$K_1 = hf(x_0, y_0, z_0)$

$= 0.1 f(0, 10, 5)$

$= 0.1 \times 5$

$K_1 = 0.5$

$l_1 = hg(x,y,z)$

$= 0.1 g(0, 10, 5)$

$= 0.1 \times 10^3$

$l_1 = 100$

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$$K_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{K_1}{2}, z_0 + \frac{l_1}{2}\right)$$

$$= 0.1 f\left(0 + \frac{0.1}{2}, \overset{5}{\cancel{10}} + \frac{0.5}{2}, \overset{5}{\cancel{0}} + \frac{\overset{150}{\cancel{100}}}{2}\right)$$

$$= 0.1 f\left(0.05, \underset{10.25}{\cancel{10.5}\ \cancel{10.25}}, 55\right)$$

$$= 0.1 \times 55$$

$$K_2 = 5.5$$

$$l_2 = hg\left(x_0 + \frac{h}{2}, y_0 + \frac{K_1}{2}, z_0 + \frac{l_1}{2}\right)$$

$$= 0.1 g\left(0 + \frac{0.1}{2}, \overset{5}{\cancel{10}} + \frac{0.5}{2}, 5 + \frac{100}{2}\right)$$

$$= 0.1 g\left(0.05, \overset{10.25}{\cancel{5.5}}, 55\right)$$

$$= 0.1 \times (10.25)^3$$

$$l_2 = \cancel{16.6375}\ 107.68$$

$$K_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{K_2}{2}, z_0 + \frac{l_2}{2}\right)$$

$$= 0.1 f\left(0 + \frac{0.1}{2}, 10 + \frac{\overset{5.5}{\cancel{55}}}{2}, 5 + \frac{107.68}{2}\right)$$

$$= 0.1 f(0.05, 12.75, 58.84)$$

$$= (0.1)(58.84)$$

$$= 5.884.$$

$$l_3 = hg\left(x_0 + \frac{h}{2}, y_0 + \frac{K_2}{2}, z_0 + \frac{l_2}{2}\right)$$

$$= 0.1 g\left(0 + \frac{0.1}{2}, 10 + \frac{5.5}{2}, 5 + \frac{107.68}{2}\right)$$

$$= 0.1 g(\overset{x}{0.05}, \overset{y}{12.75}, \overset{z}{58.84})$$

$$= 0.1 \times (12.75)^3$$

$l_3 = 207.26$

$K_4 = hf(x_0+h, y_0+K_3, z_0+l_3)$

$= 0.1 f(0+0.1, 10+5.884, 5+207.26)$

$= 0.1 f(0.1, 15.884, 212.26)$

$= 0.1(212.26)$

$K_4 = 21.226$

$l_4 = hg(x_0+h, y_0+K_3, z_0+l_3)$

$= 0.1 g(0+0.1, 10+5.884, 5+207.26)$

$= 0.1 g(0.1, 15.884, 212.26)$

$= 0.1 (15.884)^3$

$l_4 = 400.755$

$y(x_0+h) = y_0 + \frac{1}{6}[K_1 + 2K_2 + 2K_3 + K_4]$

$y(0+0.1) = 10 + \frac{1}{6}[0.5 + 2(5.5) + 2(5.884) + 21.226]$

$y(0.1) = 10.166(44.494)$

$y(0.1) = 452.326$

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3 Given $y'' - xy' - y = 0$ with the initial conditions $y(0) = 1$, $y'(0) = 0$ compute $y(0.2)$ and $y'(0.2)$ using 4^{th} order R-K method.

4 Obtain the value of x and $\frac{dx}{dt}$ when $t = 0.1$ given that x satisfies the eqn $\frac{d^2x}{dt^2} = t\frac{dx}{dt} - 4x$ and $x = 3$, $\frac{dx}{dt} = 0$. when $t = 0$ initially. use fourth order R-K method.

Soln:

$$\frac{d^2x}{dt^2} = t\frac{dx}{dt} - 4x \quad \text{--- (1)}$$

put $y = \frac{dx}{dt} \Rightarrow f(t, x, y) = y.$ --- (2)

diff w.r.t "t".

$x_0 = 3, y_0 = 0, t_0 = 0$

$t = 0.1 \quad t = t_0 + h$

$0.1 = 0 + h$

$\boxed{h = 0.1}$

$$\frac{d^2x}{dt^2} = \frac{d}{dt}\left[\frac{dx}{dt}\right] = \frac{dy}{dt}$$

Substitute values in eqn (1)

$$\frac{dy}{dt} = ty - 4x \Rightarrow g(t, x, y) = ty - 4x \rightarrow (3)$$

$K_1 = hf(t_0, x_0, y_0)$

$= 0.1 f(0, 3, 0)$

$= 0.1 \times 0$

$K_1 = 0$

$l_1 = hg(t_0, x_0, y_0)$

$= 0.1 g(0, 3, 0)$

$= 0.1[0(0) - 4(3)]$

$= -1.2$

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$$K_2 = hf\left[t_0+\frac{h}{2}, x_0+\frac{K_1}{2}, y_0+\frac{l_1}{2}\right]$$

$$= 0.1\left[0+\frac{0.1}{2}, 3+\frac{0}{2}, 0+\frac{(-1.2)}{2}\right]$$

$$= 0.1(0.05, 3, -0.6)$$

$$= 0.1(-0.6)$$

$$K_2 = -0.06$$

$$l_2 = hg\left[t_0+\frac{h}{2}, x_0+\frac{K_1}{2}, y_0+\frac{l_1}{2}\right]$$

$$= 0.1\,g\left[0+\frac{0.1}{2}, 3+\frac{0}{2}, 0+\frac{(-1.2)}{2}\right]$$

$$= 0.1g(0.05, 3, -0.6)$$

$$= 0.1\left[(0.05)(-0.6) - 4(3)\right]$$

$$= -1.203.$$

$$K_3 = hf\left[t_0+\frac{h}{2}, x_0+\frac{K_2}{2}, y_0+\frac{l_2}{2}\right]$$

$$= 0.1f\left[0+\frac{0.1}{2}, 3+\frac{\overset{-0.06}{\cancel{0.06}}}{2}, 0+\frac{(-1.203)}{2}\right]$$

$$= 0.1f(0.05, \overset{2.97}{\cancel{3.03}}, -0.6015)$$

$$= (0.1)(-0.6015)$$

$$= -0.06015$$

$\cancel{K_3} = l_3 = -1.191$

$K_4 = +0.1191$

$l_4 = -1.18785$

$$l_3 = hg\left[t_0+\frac{h}{2}, x_0+\frac{K_2}{2}, y_0+\frac{l_2}{2}\right]$$

$$= 0.1g\left[0+\frac{0.1}{2}, 3+\frac{(-0.06)}{2}, 0+\frac{(-1.203)}{2}\right]$$

$$= 0.1g(\overset{t}{0.05}, \overset{2.97}{\cancel{3.03}}, -\overset{y}{0.6015})$$

$$= 0.1\left[(0.05)(-0.6015) - 4(\overset{2.97}{\cancel{3.03}})\right]$$

$$= 0.1(-0.030075 - 11.88)$$

$$= -1.19100$$

$K_4 = hf\left[\cancel{x_0}^{t_0}+h,\ x_0+k_3,\ y_0+l_3\right]$

$= 0.1 f\left[0+0.1,\ 3+\cancel{0.1}^{(-0.06015)},\ \cancel{0+0.1}\right]$ $0+(-1.191)$

$= 0.1 f\left[0.1,\ \cancel{3.1}^{2.939},\ -1.191\right]$

$= 0.1\ \cancel{(0.1)}\ (-1.191)$

$K_4 = \cancel{0.01}\ -0.1191$

$l_4 = hg\left[t_0+h,\ x_0+k_3\ y_0+l_3\right]$

$= 0.1\ g\left[0+0.1,\ 3\cancel{+0.1}+(-0.06015),\ 0+\cancel{0.1}(-1.191)\right]$

$= 0.1\ g(\overset{t}{0.1},\ \overset{x}{\cancel{3.1}}^{-0.06015},\ \overset{y}{0.1}))\times\ 0.1\ g(\overset{t}{0.1},\ \overset{x}{2.939},\ \overset{y}{-1.191})$

$= ((0.1)(0.1)(0.1) - 4(3.1))\times 0.1\left[(0.1)(-1.191) - 4(2.939)\right]$ 11.756

$= (1.2399)\times\ -1.18751$

$x(t_0+h) = x_0 + \frac{1}{6}\left[k_1 + 2k_2 + 2k_3 + k_4\right]$

$x(0+0.1) = 3 + \frac{1}{6}\left[0 + 2(-0.06) + 2(-0.06015) + 2(-0.06) + (-0.1191)\right]$

$x(\cancel{0.1}^{0.1}) = \cancel{3.166}\ [$

$= \cancel{1.5177}\ -1.13786$

$y_0(t_0+h) = 3 + \frac{1}{6}\left[k_1 + 2k_2 + 2k_3 + k_4\right]$

$y(0+0.1) = 3 + \frac{1}{6}\left[(-1.2) + 2(-1.203) + 2(-1.191) + (-1.187)\right]$

$y_0(0.1) = 3.166\ [$

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Milne's Method

Note: Method to solve $y'' = f(x, y, y')$ then $y(x_0) = y_0$ and $y'(x_0) = y_0'$

put $y' = z$ and $y'' = \frac{dz}{dx} = z'$

The given differential equation becomes $z' = f(x, y, z)$

Predictor

$$y_4^{(P)} = y_0 + \frac{4h}{3}\left[2z_1 - z_2 + 2z_3\right] \quad \text{where } y' = z$$

$$z_4^{(P)} = z_0 + \frac{4h}{3}\left[2z_1' - z_2' + 2z_3'\right]$$

Corrector

$$y_4^{(C)} = y_2 + \frac{h}{3}\left[z_2 + 4z_3 + z_4\right]$$

$$z_4^{(C)} = z_2 + \frac{h}{3}\left[z_2' + 4z_3' + z_4'\right]$$

Problems

1. Apply Milne's method to solve $\frac{d^2y}{dx^2} = 1 + \frac{dy}{dx}$ given the following table of initial values.

x	x_0: 0	x_1: 0.1	x_2: 0.2	x_3: 0.3
y	y_0: 1	y_1: 1.01103	y_2: 1.2427	y_3: 1.399
y'	z_0: 1	z_1: 1.2103	z_2: 1.4427	z_3: 1.699

Compute $y(0.4)$

Soln: $\frac{d^2y}{dx^2} = 1 + \frac{dy}{dx}$ — (1)

put $\frac{dy}{dx} = z$

$$\frac{d^2y}{dx^2} = \frac{dz}{dx} = z'$$

Eqn (1) $\Rightarrow$ $z' = 1 + z$

$$z_0' = 1 + z_0 = 1 + 1 = 2$$

$$z_1' = 1 + z_1 = 1 + 1.2103 = 2.2103$$

$$z_2' = 1 + z_2 = 1 + 1.4427 = 2.4427$$

$$z_3' = 1 + z_3 = 1 + 1.699 = 2.699$$

W.K.T predictor formula.

$$y_4^{(P)} = y_0 + \frac{4h}{3}\left[2z_1 - z_2 + 2z_3\right]$$

$$= 1 + \frac{4(0.1)}{3}\left[2(1.2103) - 1.4427 + 2(1.699)\right]$$

$$\boxed{y_4^{(P)} = 1.583}$$

$$z_4^{(P)} = z_0 + \frac{4h}{3}\left[2z_1' - z_2' + 2z_3'\right]$$

$$= 1 + \frac{4(0.1)}{3}\left[2(2.2103) - 2.4427 + 2(2.699)\right]$$

$$\boxed{z_4^{(P)} = 1.983}$$

W.K.T Milne's corrector formula.

$$y_4^{(C)} = y_2 + \frac{h}{3}\left[z_2 + 4z_3 + z_4\right]$$

$$= 1.2427 + \frac{0.1}{3}\left[1.4427 + 4(1.699) + 1.983\right]$$

$$\boxed{y_4^{(C)} = 1.5834}$$

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$$z_4^{(c)} = z_2 + \frac{h}{3}\left[z_2' + 4z_3' + z_4'\right]$$

$$z_4' = 1 + z_4^{(P)} = 1 + 1.983 = 2.983$$

$$z_4^{(c)} = 1.4427 + \frac{0.1}{3}\left[2.4427 + 4(2.699) + 2.983\right]$$

$$\boxed{z_4^{(c)} = 1.9834}$$

Again apply Milne's corrector formula.

$$y_4^{(c)} = y_2 + \frac{h}{3}\left[z_2 + 4z_3 + z_4\right].$$

$$= 1.2427 + \frac{0.1}{3}\left[1.4427 + 4(1.699) + 1.9834\right]$$

$$\boxed{y_4^{(c)} = 1.5834}$$

$$z_4^{(c)} = z_2 + \frac{h}{3}\left[z_2' + 4z_3' + z_4'\right]$$

$$z_4^{(c)'} = 1 + 1.9834 = 2.9834$$

$$z_4^{(c)} = 1.4427 + \frac{0.1}{3}\left[2.4427 + 4(2.699) + 2.983\right]$$

$$\boxed{z_4^{(c)} = 1.9834}$$

$$\boxed{y(0.4) = 1.5834}$$

2. Apply Milne's method to compute $y(0.8)$ given that $\frac{d^2y}{dx^2} = 1 - 2y\frac{dy}{dx}$ and the following table of initial values

x	x_0 0	x_1 0.2	x_2 0.4	x_3 0.6
y	y_0 0	y_1 0.02	y_2 0.0795	y_3 0.1762
$y' = z$	z_0 0	z_1 0.1996	z_2 0.3937	z_3 0.5689

find the value of y at $x = 0.8$

soln:- $h = 0.2$.

$$\frac{d^2y}{dx^2} = 1 - 2y\frac{dy}{dx} \quad \text{---} \ ①$$

put $\frac{dy}{dx} = z$

eqn ① implies

$z' = 1 - 2yz$

$z_0' = 1 - 2(0)(0) = 1$

$z_1' = 1 - 2(0.02)(0.1996) = 0.9920$

$z_2' = 1 - 2(0.0795)(0.3937) = 0.9374$

$z_3' = 1 - 2(0.1762)(0.5689) = 0.7995$

W.K.T Milne's predictor formula.

$$y_4^{(P)} = y_0 + \frac{4h}{3}\left[2z_1 - z_2 + 2z_3\right]$$

$$= 0 + \frac{4(0.2)}{3}\left[2(0.1996) - 0.3937 + 2(0.5689)\right]$$

$y_4^{(P)} = 0.30488$

$$z_4^{(P)} = z_0 + \frac{4h}{3}\left[2z_1' - z_2' + 2z_3'\right]$$

$$= 0 + \frac{4(0.2)}{3}\left[2(0.9920) - 0.9374 + 2(0.7995)\right]$$

$z_4^{(P)} = 0.7054$

W.K.T Milne's corrector formula.

$$y_4^{(C)} = y_2 + \frac{h}{3}\left[z_2 + 4z_3 + z_4^{(P)}\right]$$

$$= 0.0795 + \frac{0.2}{3}\left[0.3937 + 4(0.5689) + 0.7054\right]$$

$y_4^{(C)} = 0.30448$

$$Z_4^{(c)} = Z_2 + \frac{h}{3}\left[Z_2' + 4Z_3' + Z_4'\right]$$

$$Z_4' = 1 - 2y_4^{(P)} Z_4^{(P)} = 1 - 2(0.30488)(0.7054) = 0.5698$$

$$Z_4^{(c)} = 0.3937 + \frac{0.2}{3}\left[0.9374 + 4(\overset{0.7995}{\cancel{0.5689}}) + 0.5698\right]$$

$$Z_4^{(c)} = 0.70738$$

Again apply Milne's corrector formula.

$$y_4^{(c)} = y_2 + \frac{h}{3}\left[Z_2 + 4Z_3 + Z_4\right]$$

$$= 0.0795 + \frac{0.2}{3}\left[0.3937 + 4(0.5689) + 0.70738\right]$$

$$y_4^{(c)} = 0.3046$$

$$Z_4^{(c)} = Z_2 + \frac{h}{3}\left[Z_2' + 4Z_3' + Z_4'\right]$$

$$Z_4' = 1 - 2y_4^{(c)} Z_4^{(c)} = 1 - 2(0.3046)(0.70738) = 0.5690$$

$$= 0.3937 + \frac{0.2}{3}\left[0.9374 + 4(0.7995) + (0.5690)\right]$$

$$Z_4' = 0.7073$$

$$\boxed{y(0.8) = 0.3046}$$

Obtain the solution of the eq^n $2\frac{d^2y}{dx^2} = 4x + \frac{dy}{dx}$ by computing the value of the dependent variable corresponding to the value 1.4 of the independent variable by applying method using the following data

	x_0	x_1	x_2	x_3
x	1	1.1	1.2	1.3
	y_0	y_1	y_2	y_3
y	2	2.2156	2.4649	2.7514
	z_0	z_1	z_2	z_3
y'	2	2.3178	2.6725	3.0657

By data:

$$2\frac{d^2y}{dx^2} = 4x + \frac{dy}{dx} \quad \text{—(1)}$$

put $\frac{dy}{dx} = z$

$$\frac{d^2y}{dx^2} \cdot 2z' = 4x + z$$

$$z' = \frac{4x+z}{2} = 2x + \frac{z}{2}$$

$$z_0' = 2x + \frac{z}{2} = 2(1) + \frac{2}{2} = 3$$

$$z_1' = 2x + \frac{z_1}{2} = 2(1.1) + \frac{2.3178}{2} = 3.3589$$

$$z_2' = 2x + \frac{z_2}{2} = 2(1.2) + \frac{2.6725}{2} = 3.7362$$

$$z_3' = 2x + \frac{z_3}{2} = 2(1.3) + \frac{3.0657}{2} = 4.51328$$

W.K.T Milne's predictor formula.

$$y_4^{(P)} = y_0 + \frac{4h}{3}\left[2z_1 - z_2 + 2z_3\right]$$

$$= 2 + \frac{4(0.1)}{3}\left[2(2.3178) - 2.6725 + 2(3.0657)\right]$$

$$= 3.0792$$

$$z_4^{(P)} = z_0 + \frac{4h}{3}\left[2z_1' - z_2' + 2z_3'\right]$$

$$= 2 + \frac{4(0.1)}{3}\left[2(3.3589) - 3.7362 + 2(4.1328)\right]$$

$$z_4^{(P)} = 3.4996$$

W.K.T Milne's corrector formula.

$$y_4^{(C)} = y_2 + \frac{h}{3}\left[z_2 + 4z_3 + z_4^{(P)}\right]$$

$$= 2.4649 + \frac{0.1}{3}\left[2.6725 + 4(3.0657) + 3.4996\right]$$

$$y_4^{(C)} = 3.0993$$

$$z_4^{(C)} = z_2 + \frac{h}{3}\left[z_2' + 4z_3' + z_4'\right]$$

$$z_4' = 2x_4 + \frac{z_4^{(P)}}{2} = \left[2(1.4) + \frac{3.4996}{2}\right] = 4.549.$$

$$= 2.6725 + \frac{0.1}{3}\left[3.7362 + 4(4.1328) + 4.549\right]$$

$$z_4' = 3.499$$

Again apply corrector formula.

$$y_4^{(C)} = y_2 + \frac{h}{3}\left[z_2 + 4z_3 + z_4^{(C)}\right]$$

$$= 2.4649 + \frac{0.1}{3}\left[2.6725 + 4(3.0657) + 3.499\right]$$

$$y_4^{(C)} = 3.079$$

$$z_4^{(C)} = z_2 + \frac{h}{3}\left[z_2' + 4z_3' + z_4'\right]$$

$$z_4' = 2x_4 + \frac{z_4^{(P)}}{2} = \left[2(1.4) + \frac{3.499}{2}\right] \doteq 4.549$$

$= 2.6725 + \frac{0.1}{3}\left[\overset{3.7362}{\cancel{2.6725}} + 4(\cancel{2}.01328) + 4.549\right]$

$\underline{z_4' = 3.499}$

5) Applying Milne's predictor and corrector formula compute $y(0.8)$ given that y satisfy the equation $y'' = 2yy'$ and y & y' are governed by the following values.

$y(0) = 0$, $y(0.2) = 0.2027$, $y(0.4) = 0.4228$, $y(0.6) = 0.6841$

$y'(0) = 0$, $y'(0.2) = 1.041$, $y'(0.4) = 1.179$, $y'(0.6) = 1.468$

apply corrector formula twice.

	x_0	x_1	x_2	x_3
x	0	0.2	0.4	0.6
	y_0	y_1	y_2	y_3
y	0	0.2027	0.4228	0.6841
	z_0	z_1	z_2	z_3
$y'z$	0	1.041	1.179	1.468

By data: $y'' = 2yy'$ —①

put $y' = z$.

$y'' = 2yz$

$z' = 2yz$

$z_0' = 2(0)(0) = 0$

$z_1' = 2(0.2027)(1.041) = 0.422$

$z_2' = 2(0.4228)(1.179) = 0.996$

$z_3' = 2(0.6841)(1.468) = 2.008$

W.K.T Milne's predictor

$y_4^{(p)} = y_0 + \frac{4h}{3}\left[2z_1 - z_2 + 2z_3\right]$

$= 0 + \frac{4(0.2)}{2}\left[2(1.041) - (1.179) + 2(1.468)\right] = 1.023$

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$$Z_4^{(P)} = Z_0 + \frac{4h}{3}\left[2Z_1' - Z_2' + 2Z_3'\right]$$

$$= 0 + \frac{4(0.2)}{3}\left[2(0.422) - 0.996 + 2(2.008)\right]$$

$$Z_4^{(P)} = 1.0304$$

W.K.T Milne's corrector formula.

$$y_4^{(c)} = y_2 + \frac{4h}{3}\left[Z_2 + 4Z_3 + Z_4^{(P)}\right]$$

$$= 0.4228 + \frac{4(0.2)}{3}\left[1.179 + 4(1.468) + 1.0304\right]$$

$$= 2.577$$

$$Z_4^{(c)} = Z_2 + \frac{h}{3}\left[Z_2' + 4Z_3' + Z_4'\right]$$

$$Z_4' = 2(2.577)(1.0304) = 5.310$$

$$= \cancel{1.179 + 4(1.468) + 5.310} \quad 1.179 + \frac{0.2}{3}\left[0.996 + 4(2.008) + 5.31\right]$$

$$= 2.134.$$

Again apply Milne's corrector formula.

$$y_4^{(c)} = y_2 + \frac{4h}{3}\left[Z_2 + 4Z_3 + Z_4^{(c)}\right]$$

$$= 0.4228 + \frac{4(0.2)}{3}\left[1.179 + 4(1.468) + 2.134\right]$$

$$= 2.872$$

$$Z_4^{(c)} = Z_2 + \frac{h}{3}\left[Z_2' + 4Z_3' + Z_4'\right]$$

$$Z_4' = \cdot$$

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CALCULUS OF VARIATION

20/9/19

In differential calculus we are familiar with finding extreme values (maximum and minimum values) of a ~~functionat~~ of 1 and 2 variables.

Euler's equation

$$\boxed{\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0}$$

Problems

1. Find the extremal of the functional integration from ~~x_1 to x_2 y'~~ $\int_{x_1}^{x_2} \left(y' + x^2 (y')^2\right) dx$.

Sol^n: $f(x, y, y') = (y' + x^2 (y')^2)$

W.K.T Euler's eq^n

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left[\frac{\partial f}{\partial y'}\right] = 0$$

diff w.r.t y

$$0 - \frac{d}{dx}\left[(1 + x^2 \cdot 2y')\right] = 0$$

$$\frac{d}{dx}\left[(1 + 2x^2 y')\right] = 0$$

Integrating on B.S w.r.t x

$$(1 + 2x^2 y') = K_1$$

$$2x^2 y' = K_1 - 1$$

$$y' = \frac{K_1 - 1}{2x^2}$$

$$y' = \frac{K_1}{2x^2} - \frac{1}{2x^2}$$

$$y' = \frac{K_1}{2}\left[\frac{1}{x^2}\right] - \frac{1}{2}\left[\frac{1}{x^2}\right]$$

Again integrating w.r.t x

$$y = \frac{K_1}{2}\left[\frac{-1}{x}\right] - \frac{1}{2}\left[-\frac{1}{x}\right] + C_2$$

$$y = \frac{-K_1}{2x} + \frac{1}{2x} + C_2$$

$$y = \frac{1}{2x}\left[1 - K_1\right] + C_2$$

$$y = \left(\frac{1-K_1}{2}\right)\frac{1}{x} + C_2 \qquad \text{where } \frac{1-K_1}{2} = C_1$$

$$y = \frac{C_1}{x} + C_2$$

2. Find the function y which makes the integral ~~integration~~ ~~from~~ $\int_{x_1}^{x_2} (1 + xy' + x(y')^2)\,dx$ an extremum.

Sol^n: Let $f(x,y,y') = (1 + xy' + x(y')^2)$

W.K.T Euler's Eq^n

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$$

$$(0+0+0) - \frac{d}{dx}\left(0 + x + x2y'\right) = 0$$

$$\frac{d}{dx}\left(x + 2xy'\right) = 0$$

Integrate on B.S

$x + 2xy' = K_1$

$2xy' = K_1 - x$

$y' = \frac{K_1 - x}{2x}$

$y' = \frac{K_1}{2x} - \frac{x}{2x}$

$y' = \frac{K_1}{2x} - \frac{1}{2}$

Again integrate w.r.t x

$y = \frac{K_1}{2} \log x - \frac{1}{2} x + C_2$

$y = C_1 \log x - \frac{x}{2} + C_2$. where $C_1 = \frac{K_1}{2}$

3. Find the extremal of the function $\int_{x_1}^{x_2} ((y')^2 + Ky^2)\, dx$

Sol^n: Let $f(x, y, y') = ((y')^2 + Ky^2)\, dx$.

W.K.T Euler's eq^n

tion

$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$

$(0 + 2Ky) - \frac{d}{dx}(2y') = 0$

$2Ky - 2y'' = 0 \qquad \div 2.$

$Ky - y'' = 0 \qquad \div -$

$y'' - Ky = 0 \qquad \left[y'' = \frac{d^2y}{dx^2} \quad \frac{d}{dx} = D \quad \frac{d^2}{dx^2} = D^2 \right.$

$D^2y - Ky = 0 \qquad \left. y'' = D^2 y \right]$

$(D^2 - K)y = 0$

This is a second order ODE and the solution depends on the nature of K

Case 1: let $K=0$ then $y=C_1x+C_2$

Case 2: let $K=p^2$ then $y=C_1e^{px}+C_2e^{-px}$

case 3: Let $K=-p^2$ then $y=C_1\cos px+C_2\sin px$.

4. Find the extremal of the functional $\int_{x_1}^{x_2}(y^2+(y')^2+2ye^x)\,dx$

Sol^n: let $f(x,y,y')=y^2+(y')^2+2ye^x$

WKT Euler's eq^n

$$\frac{\partial f}{\partial y}-\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)=0$$

$$=(2y+2e^x)-\frac{d}{dx}(2y')=0 \quad \div 2.$$

$$=y+e^x-\frac{d}{dx}(y')=0$$

diff. w.r.t.

$$y''=\frac{d^2y}{dx^2}$$

$$=y+e^x-y''=0.$$

$$e^x=y''-y$$

where $D=\frac{d}{dx}$

$$y''-y=e^x$$

$$D^2=\frac{d^2}{dx^2}$$

$$[D^2y-y]=e^x$$

$$[D^2-1]y=e^x.$$

$$A.E=m^2-1=0$$

$$m=\pm 1$$

$$y_c=C_1e^x+C_2e^{-x}$$

$$y_p=\frac{e^x}{D^2-1}$$ *x to numerator

. diff denominator

$= \frac{x e^x}{2D}$ where $D = 1$

$y_p = \frac{x e^x}{2}$

$y = y_c + y_p$

$= C_1 e^x + C_2 e^{-x} + \frac{x e^x}{2}$

5. Find the extremal of the functional $\int_a^b (x^2 (y')^2 + 2y^2 + 2xy) dx$

Soln Let $f(x, y, y') = x^2 (y')^2 + 2y^2 + 2xy$

W.K.T Euler's eqn

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$$

diff. w.r.t y.

$(4y + 2x) - (2x^2 y') = 0.$ (product rule.)

$(4y + 2x) - [2x^2 y'' + y' 4x] = 0$ $(\div 2)$.

$2y + x - [x^2 y'' + 2xy'] = 0.$ $(\div -)$

$2y + x - x^2 y'' - 2xy' = 0.$

$-2y - x + x^2 y'' + 2xy' = 0$

$x^2 y'' + 2xy' - 2y = x \longrightarrow (1)$

| $x^2 y'' = D(D-1) y$
| $x y' = Dy$.

put $t = \log x$, $x = e^t$, $x^2 y'' = D(D-1) y$, $xy' = Dy$

eqn (1) becomes

$[D(D-1) + 2D - 2] y = e^t$ y common.

$[D^2 - D + 2D - 2] y = e^t$

$[D^2 + D - 2] y = e^t$

$y_c = m^2 + m - 2 = 0$

$m^2 + 2m - m - 2 = 0$

$m(m+2) - 1(m+2) = 0$

$(m+2)(m-1) = 0$

$m+2 = 0, \quad m-1 = 0$

$m = -2, \quad m = 1$

$$\boxed{y_c = C_1 e^{-2t} + C_2 e^{t}}$$

$$y_p = \frac{e^t}{D^2 + D - 2}$$

mul t numerator

diff denominator

$$= \frac{t e^t}{2D + 1} \qquad D = 1$$

$$y_p = \frac{t e^t}{2(1) + 1}$$

$$\boxed{y_p = \frac{t e^t}{3}}$$

$y = y_c + y_p$

$$= C_1 e^{-2t} + C_2 e^t + t \frac{e^t}{3}$$

$$\boxed{y = C_1 e^{-2\log x} + C_2 e^{\log x} + \log x \frac{e^{\log x}}{3}}$$

6. Show that the functional $\int_{x_1}^{x_2} (y^2 + x^2 y')\,dx$ assumes extreme values on the straight line $y = x$

Sol^n: $f(x, y, y') = y^2 + x^2 y'$

W.K.T Euler's eq^n

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$$

$$2y - \frac{d}{dx}(x^2) = 0$$

$$2y - 2x = 0$$

$$\cancel{2}y = \cancel{2}x$$

$$\boxed{y = x}$$

7. Find the extremal of the functional $\int_{x_1}^{x_2} ((y')^2 - y^2 + 2y\sec x)\, dx$

Soln: $f(x, y, y') = (y')^2 - y^2 + 2y\sec x$

W.K.T Euler's eqn

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$$

$$-2y + 2\sec x - \frac{d}{dx}(2y') = 0$$

$$-2y + 2\sec x - 2y'' = 0 \quad \div -2$$

$$y - \sec x + y'' = 0$$

$$y'' + y = \sec x$$

$$(D^2y + y) = \sec x$$

$$(D^2 + 1)y = \sec x.$$

$$yc = m^2 + 1 = 0$$

$$m^2 = -1$$

$$m = \pm\sqrt{-1}$$

$$m = \pm i$$

$$yc = C_1 \cos x + C_2 \sin x$$

Let $y = A\cos x + B\sin x$ be the solution of the ODE where A & B are functions of x to be determined. If y^1 & y^2 are the solution of the homogeneous equation

$f(D)y = 0$ then we know that $A = -\int \frac{y_2 \phi(x)}{W} dx$,

$B = \int \frac{y_1 \phi(x)}{W} dx$. where $W = y_1 y_2' - y_2 y_1'$ and

$\phi(x) = \sec x$ then $y_1 = \cos x$, $y_2 = \sin x$ then $y_1' = -\sin x$

$y_2' = \cos x$. Hence $W = y_1 y_2' - y_2 y_1'$

$$W = \cos x(\cos x) + \sin x(\sin x)$$

$$W = 1$$

Now $A = -\int \frac{y_2 \phi(x)}{W} dx$

$$A = \int -\sin x . \sec x \, dx$$

$$= \int -\sin x . \frac{1}{\cos x} . dx$$

$$A = \int -\tan x . dx.$$

$$A = \log(\sec x) + C_1$$

$$B = \int \frac{y_1 \phi(x)}{W} dx$$

$$= \int \cos x . \sec x \, dx.$$

$$= \int \cancel{\cos x} . \frac{1}{\cancel{\cos x}} . dx$$

$$= \int 1 . dx$$

$$B = x + C_2$$

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then $y = A\cos x + B\sin x$

$= [\log(\sec x) + C_1]\cos x + [x + C_2]\sin x.$

8. Find the curve on which the functional $I = \int_0^1 ((y')^2 + 12xy)\,dx$ with $y(0) = 0$ and $y(1) = 1$ can be determined.

Sol^n: $f(x, y, y') = (y')^2 + 12xy$

W.K.T Euler's eq^n

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$$

$$12x - \frac{d}{dx}(2y') = 0$$

$$12x - 2y'' = 0$$

$$\cancel{12}^{6}x = \cancel{2}y''$$

$$y'' = 6x$$

integrate w.r.t x

$$y' = \frac{6x^2}{2} + C_1$$

$$y' = 3x^2 + C_1$$

Again integrating w.r.t x.

$$y = \cancel{3}\frac{x^3}{\cancel{3}} + C_1x + C_2$$

$$y = x^3 + C_1x + C_2 \quad \text{——} \; (1)$$

Given condition $y(0) = 0$ & $y(1) = 1$

$x = 0, y = 0$ and $y = 1$ $x = 1$

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Eq^n ① implies

$x = 0.$

$y = x^3 + c_1 x + c_2.$

$0 = 0 + c_1(0) + c_2$

$\boxed{c_2 = 0}$

Another condition.

$y(1) = 1$

$x = 1 \quad y = 1$

eq^n ① implies.

$y = x^3 + c_1 x + c_2$

$1 = 1^3 + c_1(1) + 0$

$1 - 1 = c_1$

$\boxed{c_1 = 0}$

c_1 and c_2 values are Substituted in eq^n ①

$y = x^3 + 0(x) + 0$

$\boxed{y = x^3}$

9. Solve the variational problem $\delta \int_0^1 (x + y + (y')^2)\, dx = 0$ under the conditions $y(0) = 1$ and $y(1) = 2$

Sol^n: Let $I = \int_0^1 (x + y + (y')^2)\, dx.$

$\delta I = 0$ is the equalent to Euler's equation

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$$

$f(x, y, y') = x + y + (y')^2$

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$$1 - \frac{d}{dx}(2y') = 0.$$

$$1 - 2y'' = 0$$

$$1 = 2y'' \Rightarrow y'' = \frac{1}{2}$$

Integrating w.r.t x.

$$y' = \frac{1}{2}.x + C_1$$

Again integrating w.r.t x.

$$y = \frac{1}{2}\frac{x^2}{2} + C_1 x + C_2.$$

$$y = \frac{x^2}{4} + C_1 x + C_2 \quad \text{— (1)}$$

Given condition.

$y(0) = 1$

$x = 0 \quad y = 1$

Eqⁿ (1) implies.

$$y = \frac{x^2}{4} + C_1 x + C_2$$

$$1 = \frac{0}{4} + C_1(0) + C_2$$

$$C_2 = 1$$

Another condition.

$y(1) = 2.$

$x = 1 \quad y = 2$

Eqⁿ (1) becomes.

$$2 = \frac{1^2}{4} + C_1(1) + 1$$

$$2 - 1 - \frac{1}{4} = C_1$$

$$1 - \frac{1}{4} = C_1$$

$$\frac{H-1}{4} = C_1$$

$$\boxed{C_1 = \frac{3}{4}}$$

Substitute C_1 & C_2 values in Eqn ① becomes.

$$y = \frac{x^2}{4} + \frac{3}{4}x + 1$$

10. Solve the variational problem $\delta \int_0^{\pi/2} (y^2 - (y')^2) . dx = 0$,

$y(0) = 0$, ~~and~~ $y(\pi/2) = 2$

Soln : Let $I = \int_0^{\pi/2} (y^2 - (y')^2) . dx.$

$$\delta I = 0$$

$$f(x, y, y') = y^2 - (y')^2.$$

W.K.T Euler's eqn

$$\frac{\delta f}{\delta y} - \frac{d}{dx}\left(\frac{\delta f}{\delta y'}\right) = 0$$

$$2y - \frac{d}{dx}(-2y') = 0$$

$$2y + 2y'' = 0 \quad \div 2.$$

$$y + y'' = 0$$

$$y'' + y = 0$$

$$D^2 y + y = 0$$

$$[D^2 + 1] y = 0$$

$$m^2 + 1 = 0$$

$$m^2 = -1$$

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$m = \pm\sqrt{-1}$ (or) $m = \pm i$

$y = C_1 \cos x + C_2 \sin x$. —①

By data, $y(0) = 0$, $x = 0$ & $y = 0$

Eqn ① implies.

$0 = C_1 \cos 0 + C_2 \sin 0$

$= C_1 + 0$

$\therefore C_1 = 0$

Another condition $y(\pi/2) = 2$, $x = \pi/2$, $y = 2$.

Eqn ① =>

$2 = 0 + C_2$

$C_2 = 2$

C_1 and C_2 values are substituted in eqn ①

$\boxed{y = 2\sin x}$

11. Show that $\int_{x_1}^{x_2} (y^2 (y')^2) \cdot dx$. as an extremum

VIMP

when $y(x)$ is of the form $C_1 \sqrt{x^2 + C_2}$

Soln: $f(x, y, y') = y^2 (y')^2$

W.K.T Euler's eqn

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$$

$$2y(y')^2 - \frac{d}{dx}(2y'y^2) = 0$$

$$2y(y')^2 - 2[y' 2yy' + y^2 y''] = 0$$

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$$2y(y')^2 - 4(y')^2 y - 2y^2 y'' = 0$$

$$-2y(y')^2 - 2y^2 y'' = 0 \quad \div -2$$

$$y(y')^2 + y^2 y'' = 0$$

$$\underbrace{y((y')^2 + yy'')}_{} = 0$$

$$yy'' + (y')^2 = 0$$

$$\frac{d}{dx}(yy') = 0$$

$\frac{d}{dx}(yy')$ product rule

$$yy'' + y'y' = yy'' + (y')^2$$

Integrating on B.S

$$yy' = K_1$$

$$y\frac{dy}{dx} = K_1$$

$$y\,dy = K_1\,dx$$

Integrating on B.S

$$\int y \cdot dy = \int K_1 \cdot dx.$$

$$\frac{y^2}{2} = K_1 x + K_2$$

$$y^2 = 2[K_1 x + K_2]$$

$$y = \sqrt{2K_1 x + 2K_2}$$

$$y = \sqrt{2K_1\left[x + \frac{K_2}{K_1}\right]}$$

where $C_1 = \sqrt{2K_1}$

$C_2 = \frac{K_2}{K_1}$

$$y = C_1\sqrt{x + C_2}$$

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12. Show that an extremal of $\int_{x_1}^{x_2} \left[\frac{y'}{y}\right]^2 dx$ is expressible in the form $y = \cancel{Ae}\ a e^{bx}$

Soln: Let $f(x, y, y') = \frac{(y')^2}{y^2}$

W.K.T Euler's eqn

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$$

$$\left[-\frac{2}{y^3} \cdot (y')^2\right] - \frac{d}{dx}\left[\frac{2y'}{y^2}\right] = 0$$

$$\Rightarrow \frac{-2(y')^2}{y^3} - \left[\frac{y^2 2y'' - 2y' 2yy'}{(y^2)^2}\right] = 0 \quad \text{Quotient rule.}$$

$$\Rightarrow \frac{-2(y')^2}{y^3} - \frac{\left[2y^2 y'' - 4(y')^2 y\right]}{y^4} = 0$$

$$\Rightarrow \frac{-2(y')^2}{y^3} - \frac{y\left[2yy'' - 4(y')^2\right]}{y^4} = 0$$

$$\Rightarrow \frac{-2(y')^2}{y^3} - \frac{2yy'' + 4(y')^2}{y^3} = 0.$$

$$\Rightarrow \frac{2(y')^2 - 2yy''}{y^3} = 0$$

$$\Rightarrow 2(y')^2 - 2yy'' = 0$$

$$\Rightarrow 2\left[(y')^2 - yy''\right] = 0$$

$$(y')^2 - yy'' = 0 \quad \div \text{ '}-\text{'}$$

$$yy'' - (y')^2 = 0$$

$$\frac{d}{dx}\left[\frac{y'}{y}\right] = 0$$

$$\frac{y'y' - yy''}{y^2} = 0$$

$$yy'' - (y')^2 = 0$$

$yy'' - (y')^2 = 0$ can be put in the form

$\frac{d}{dx}\left[\frac{y'}{y}\right] = 0$ Hence integration of $\int \frac{y'}{y} = \int c_1 \, dx + c_2$

$\log y = c_1 x + c_2$

$y = e^{c_1 x + c_2}$

$y = e^{c_1 x} + e^{c_2}$

Here $c_1 = b$

$e^{c_2} = a$.

$y = e^{bx} \cdot a$

$\underline{y = a e^{bx}}$

VVImp Standard variational problem

1. Prove that the shortest distance b/n two points in a plane is along the straight line joining them ~~or~~ OR prove that the Geodesics on a plane or straight line.

Sol^n Let $y = y(x)$ be a curve joining two points $P(x_1, y_1)$ and $Q(x_2, y_2)$ in the XOY plane

W.K.T the arc length between P & Q is given by

$$S = \int_{x_1}^{x_2} \frac{ds}{dx} \cdot dx = \int_{x_1}^{x_2} \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \cdot dx$$

$$I = \int_{x_1}^{x_2} \sqrt{1 + (y')^2} \cdot dx$$

W.K.T Euler's equation

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$$

$$0 - \frac{d}{dx}\left[\frac{1}{\sqrt{1+(y')^2}}\right] = 0$$

$\sqrt{x} = \frac{1}{2\sqrt{x}}$.

Quotient rule:

$$\frac{d}{dx}\left[\frac{y'}{\sqrt{1+(y')^2}}\right] = 0$$

$$\frac{\left[\sqrt{1+(y')^2}\, y'' - y' \frac{1}{\cancel{2}\sqrt{1+(y')^2}} \cancel{2} y' y''\right]}{\left(\sqrt{1+(y')^2}\right)^{\cancel{2}}} = 0$$

$$\frac{\sqrt{1+(y')^2}\, y'' - \frac{(y')^2 y''}{\sqrt{1+(y')^2}}}{1+(y')^2} = 0$$

$$\sqrt{1+(y')^2}\, y'' - \frac{y' y''}{\sqrt{1+(y')^2}} = 0$$

$$\frac{\left(1+(y')^2\right) y'' - (y')^2 y''}{\sqrt{1+(y')^2}} = 0$$

$$\left[1+(y')^2\right] y'' - (y')^2 y'' = 0$$

$$y'' + \cancel{y''(y')^2} - \cancel{(y')^2 y''} = 0$$

$$y'' = 0$$

Integrating w.r.t 'x'

$$y' = c_1$$

Again Integrating w.r.t x.

$$y = c_1 x + c_2.$$

Hence it is the straight line.

2) Find the Geodesics surface given that the arc length on the surfaces is

$$S = \int_{x_1}^{x_2} \sqrt{x(1+(y')^2)}\, dx.$$

Sol^n: $$I = \int_{x_1}^{x_2} \sqrt{x(1+(y')^2)}\,.\,dx.$$

W.K.T Euler's equation

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0.$$

$$0 - \frac{d}{dx}\left[\frac{1 \cdot x(2y')}{2\sqrt{x(1+(y')^2)}}\right] = 0.$$

$$\frac{d}{dx}\left[\frac{x y'}{\sqrt{x(1+(y')^2)}}\right] = 0$$

$$\frac{\cancel{\sqrt{x}} \cdot \sqrt{x} \cdot y'}{\cancel{\sqrt{x}}\sqrt{(1+(y')^2)}} = C$$

$$\sqrt{x}\, y' = C\sqrt{1+(y')^2}$$

S B S

$$x(y')^2 = C^2(1+(y')^2)$$

$$x(y')^2 = C^2 + C^2(y')^2$$

$$x(y')^2 - C^2(y')^2 = C_2$$

$$(y')^2[x - C^2] = C^2$$

$$(y')^2 = \frac{C^2}{x - C^2}$$

$$y' = \sqrt{\frac{C^2}{x - C^2}}$$

$$y' = \frac{c}{\sqrt{x - c^2}}$$

Integrating w.r.t x

$$y = \int \frac{c}{\sqrt{x-c^2}} \cdot dx$$

$$y = 2c\sqrt{x-c^2} + c_1$$

$$(y - c_1) = 2c\sqrt{x - c^2}$$

SBS

$$(y - c_1)^2 = 4c^2(x - c^2) \Rightarrow \therefore \text{ parabola.}$$

3) prove that catenary is the curve which rotated about a line generates a surface of minimum area

Soln: The expression for the total surface area given by $\int 2\pi y \cdot ds$ where the curve is rotated about the x-axis

$$I = \int_{x_1}^{x_2} 2\pi y \cdot \frac{ds}{dx} \cdot dx$$

$$= \int_{x_1}^{x_2} 2\pi y \sqrt{1 + (y')^2} \cdot dx.$$

Since 2π is a constant we can as we will take $f(x, y, y') = y\sqrt{1 + (y')^2}$ which is independent of x.

Therefore it is convenient to take Euler's equation in the form $f - y'\left(\frac{\partial f}{\partial y'}\right) = \text{constant}$.

$$y\sqrt{1+(y')^2} - y'\left[y \frac{1}{\cancel{2}\sqrt{1+(y')^2}} \cancel{2}y'\right] = c_1$$

$$y\sqrt{1+(y')^2} - y'\left[\frac{y(y')^2}{\sqrt{1+(y')^2}}\right] = c_1$$

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Q!
$$\frac{y\sqrt{1+(y')^2}-(y')^2 y}{\sqrt{1+(y')^2}}=c$$

So

$$\frac{y(1+(y')^2)-(y')^2 y}{\sqrt{1+(y')^2}}=c$$

$$y+\cancel{y(y')^2}-\cancel{y(y')^2}=c\sqrt{1+(y')^2}$$

$$y=c\sqrt{1+(y')^2}$$

SBS

$$y^2=c^2(1+(y')^2)$$

$$y^2=c^2+c^2(y')^2$$

$$y^2-c^2=c^2(y')^2$$

$$(y')^2=\frac{y^2-c^2}{c^2}$$

$$y'=\sqrt{\frac{y^2-c^2}{c^2}}$$

$$y'=\frac{\sqrt{y^2-c^2}}{c}$$

$$\frac{dy}{dx}=\frac{\sqrt{y^2-c^2}}{c}$$

$$\frac{dy}{\sqrt{y^2-c^2}}=\frac{dx}{c}$$

Integrating on B.S

$$\int\frac{dy}{\sqrt{y^2-c^2}}=\int\frac{dx}{c}$$

$$\cosh^{-1}\left[\frac{y}{c}\right] = \frac{x}{c} + K$$

$$\cosh^{-1}\left[\frac{y}{c}\right] = \frac{x + cK}{c} \qquad a = cK.$$

$$\frac{y}{c} = \cosh\left[\frac{x+a}{c}\right]$$

$$y = c.\cosh\left[\frac{x+a}{c}\right]$$

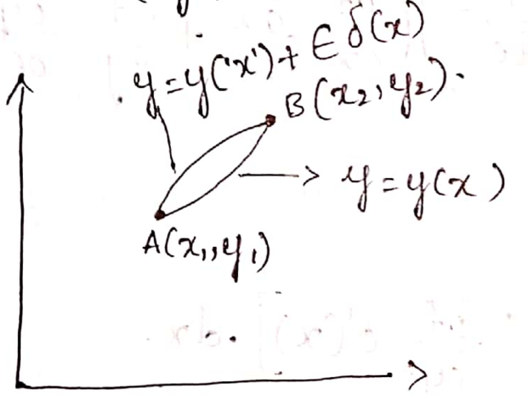
Required catenary equation.

Euler's Eq^n

2 Intervals A necessary condition for the integral $I = \int_{x_1}^{x_2} f(x, y, y').dx$

where $y(x_1) = y_1$ and $y(x_2) = y_2$ to be an extremum is

that $\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$



Let $y = y(x)$ be a eqn of curve joining the points $A(x_1, y_1)$ and $B(x_2, y_2)$ now make the variation problem $I = \int_{x_1}^{x_2} f(x, y, y').dx$ to be an extremum. Let $y = y(x) + \epsilon\delta(x) \longrightarrow$ (1) be a neighbouring curve joining the points $A(x_1, y_1)$ and $B(x_2, y_2)$ such that

$\delta(x_1)=0$ at the point A and $\delta(x_2)=0$ at the point B where ϵ is a small quantity (If $\epsilon=0$) then the neighbouring curve becomes $y=y(x)$ Now the value of I along the neighbouring curve is given by

$$\int_{x_1}^{x_2} f(x, y+\epsilon\delta, y'+\epsilon'\delta').dx$$

Now clearly I is a function of ϵ we have a necessary condition for this function to be an extremum (either maxima or minima) at $\epsilon=0$ is $\frac{dI}{d\epsilon}=0$

Now differentiate with respect to ϵ by using "Leibnitz rule" for the differentiation under integral sign then we get $\frac{dI}{d\epsilon}=\int_{x_1}\left(\frac{\delta f}{\delta x}\frac{\partial x}{\partial\epsilon}+\frac{\partial f}{\partial y}\frac{\partial y}{\partial\epsilon}+\frac{\partial f}{\delta y'}\cdot\frac{\delta y'}{\delta\epsilon}\right).dx.$

$$=\int_{x_1}^{x_2}\left[\frac{\partial f}{\delta x}(0)+\frac{\delta f}{\delta y}\partial(x)+\frac{\partial f}{\partial y'}.\delta'(x)\right].dx.$$

$$=\int_{x_1}^{x_2}\frac{\partial f}{\partial y}\partial x\,dx+\int_{x_1}^{x_2}\frac{\partial f}{\partial y'}\delta'(x).dx.$$

$$=\int_{x_1}^{x_2}\frac{\partial f}{\partial y}.\delta x.dx+\left[\frac{\partial f}{\partial y'}\right]\left[\delta'(x)-\int\int\delta(x).\frac{d}{dx}\frac{\partial f}{\partial y'}.dx\right|_{x_1}^{x_2}$$

$$= \int_{x_1} \frac{\partial f}{\partial y} \delta x . dx + \left[\frac{\partial f}{\partial y'} \delta x - \int \delta x \frac{d}{dx} \frac{\delta f}{\delta y'} . dx \right|_{x_1}^{x_2}$$

$$= \int_{x_1}^{x_2} \frac{\delta f}{\delta y} \delta x . dx + \frac{\partial f}{\partial y'} \delta(x_2) - \frac{\delta f}{\delta y'} \delta(x_1) - \int_{x_1}^{x_2} \frac{d}{dx} \left[\frac{\delta f}{\delta y'} \right] \delta x . dx)$$

$$= \int_{x_1}^{x_2} \frac{\partial f}{\delta y} \delta x . dx + 0 + 0 - \int_{x_1}^{x_2} \frac{d}{dx} \left[\frac{\delta f}{\delta y'} \right] . \delta x . dx .$$

$$0 = \int_{x_1}^{x_2} \frac{\partial f}{\delta y} - \frac{d}{dx} \left[\frac{\partial f}{\partial y'} \right] \partial x . dx .$$

$\because \delta(x)$ is an orbitrary function.

For the extremum $\frac{dI}{d\epsilon} = 0$

$$\Rightarrow \int_{x_1}^{x_2} \frac{\partial f}{\partial y} - \frac{d}{dx} \left(\frac{\partial f}{\partial y'} \right)) \delta x . dx = 0$$

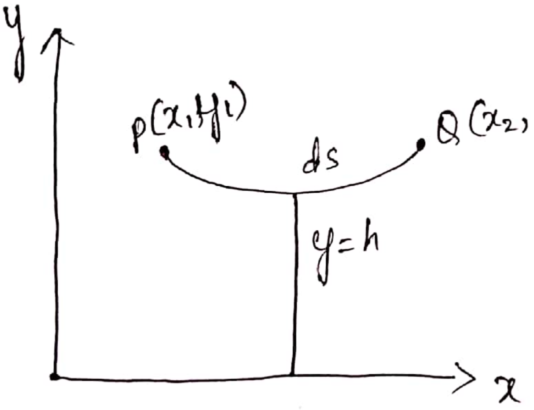
$$0 = \frac{\delta f}{\delta y} - \frac{d}{dx} \left[\frac{\delta f}{\delta y'} \right] . dx .$$

which is a required. Euler's eq^n for solving variational Problem.

Hanging chain problems

1. A heavy cable hangs freely under gravity between two fixed points. Show that the shape of the cable is a catenary.

Sol^n:



Let $P(x_1, y_1)$ and $Q(x_2, y_2)$ be the two fixed points of the hanging cable let us consider an ~~extremum~~ elementary arc length ds of the cable. let 'ρ' be the (density) of the cable so that ρ ~~into~~ ρds is the mass of element. If g is acceleration due to gravity then the potential energy of elements (mgh) is given by $(\rho ds).g.y$ where x-axis is taken as the line of reference therefore total P.E of the cable is given by

$$T = \int_P^Q (\rho ds).g.y.dx = \int_{x_1}^{x_2} \rho g y \frac{ds}{dx}.dx.$$

But, $\frac{ds}{dx} = \sqrt{1+(y')^2}$ here $f(x, y, y') = \rho g y\sqrt{1+(y')^2}$

$= \text{Constant } y\sqrt{1+(y')^2}$

It is convenient to take Euler's eqn in the form

$$f - y' \frac{\delta f}{\delta y'} = \text{constant}.$$

$$= y.\sqrt{1+(y')^2} - y' \frac{y.1}{2}$$