Laplace Transform

Introduction:

The main objective of this module is to learn new methods to solve differential equations, in particular initial value problems. Essentially this is a technique which converts differential equations to algebraic equations which are easier to solve and then interpret this solution as the solution of the original differential equation.

Definition: Laplace transform of a real function $f(t)$ is defined as

$$L[f(t)] = \int_0^\infty e^{-st} f(t)dt.$$

Here one thinks of f as a function of t which stands for time. Note that the resulting integral is a function of the variable s. To emphasize this very often we use the notation

$$L[f(t)] = F(s).$$

Since st occur as an exponent in the definition of the Laplace transform and t stands for time, we say that s is frequency. This is because physical quantities in exponent should have to be dimensionless. Also, value of f when t is negative doesn't matter. Think of f as a mapping/transformation from set of functions to set of functions. Original functions are denoted by lower case letters and their transforms by the same letters in capitals; e.g., $F(s)$ denotes the Laplace transform of $f(t)$ and $Y(s)$ denotes the Laplace transform of $y(t)$.

In general when a function is multiplied by a *standard* function and the product integrated over certain limits, one gets what is known as *integral transforms*. Laplace transform is an example of this when the *standard* function is exponential function. Fourier transform is another example you will study in this course. Integral transforms, in general, have *nice* properties which are exploited to solve differential equations.

One of the most important properties of Laplace transform is that it is a linear transformation which means for two functions f and g and constants a and b

$$L[af(t) + bg(t)] = aL[f(t)] + bL[g(t)]$$

One can compute Laplace transform of various functions from first principles using the above definition. Instead of that, here is a list of functions relevant from the point of view of this course and their Laplace transform in the form of a table. One needs to learn to manipulate them as is illustrated in the examples that follow.

$f(t)$	$L[f(t)] = F(s)$
1	$\dfrac{1}{s}$
e^{at}	$\dfrac{1}{s-a}$
$\sin at$	$\dfrac{a}{s^2+a^2}$
$\cos at$	$\dfrac{s}{s^2+a^2}$
$\sinh at$	$\dfrac{a}{s^2-a^2}$
$\cosh at$	$\dfrac{s}{s^2-a^2}$
$t^n,\ n = 1,2,3\ldots$	$\dfrac{n!}{s^{n+1}}$
$t^p,\ p > -1$	$\dfrac{\Gamma(p+1)}{s^{p+1}}$

Example 1: Find Laplace transform of $f(t) = 1 + 3t^2 - 5e^{2t} + 2e^{-10t}$.

Solution: Using linearity of Laplace transformation and then the table, we have

$$L[1+3t^2-5e^{2t}+2e^{-10t}] = L[1] + 3L[t^2] - 5L[e^{2t}] + 2L[e^{-10t}]$$

$$= \frac{1}{s} + 3\cdot\frac{2!}{s^3} - 5\cdot\frac{1}{s-2} + 2\cdot\frac{1}{s+10}.$$

Example 2: Find Laplace transform of $f(t) = 5^t - 3\cos 2t + 4\sinh 3t$.

Solution: Observe that $5^t = e^{t \log 5}$. Then using linearity of Laplace transformation and then the table, we have

$$L[5^t - 3\cos 2t + 4\sinh 3t] = L[5^t] - 3L[\cos 2t] + 4L[\sinh 3t]$$

$$= L[e^{t\log 5}] - 3L[\cos 2t] + 4[\sinh 3t]$$

$$= \frac{1}{s-\log 5} - 3\cdot\frac{s}{s^2+4} + 4\cdot\frac{3}{s^2-9}.$$

Essentially the trick is to reduce the given function to one of the elementary functions whose Laplace transform may be found in the table.

Example 3: Find Laplace transform of $\cosh^2 3t$.

Solution: By definition $\cosh 3t = \dfrac{e^{3t} + e^{=3t}}{2}$. Hence $\cosh^2 3t = \dfrac{1}{4}(e^{6t} + 2 + e^{-6t})$. But then

$$L[\cosh^2 3t] = \frac{1}{4}(L[e^{6t}] + L[2] + L[e^{-6t}])$$
$$= \frac{1}{4}\left[\frac{1}{s-6} + \frac{2}{s} + \frac{1}{s+6}\right].$$

Above trick may be used for other powers of cosh *at* and also for powers of $\sinh at = \dfrac{e^{at} - e^{-at}}{2}$.

Example 4: Find Laplace transform of $\sin^2 4t$.

Solution: Use the identity $\cos 2x = 1 - 2\sin^2 x$ with $x = 4t$. Then $\sin^2 4t = ½(1 – \cos 8t)$.

Thus $L[\sin^2 4t] = \dfrac{1}{2}\big(L[1] - L[\cos 8t]\big) = \dfrac{1}{2}\left[\dfrac{1}{s} - \dfrac{s}{s^2 + 64}\right].$

One can use the identity $\cos 2x = 2\cos^2 x – 1$ if one wants to find Laplace transform of square of cos *x*. In analogous fashion if one has to find Laplace transform of $\sin^3 at$ or $\text{cost}^3 at$, one uses one of the identities: $\sin^3 at = ¼ (3 \sin at – \sin 3at)$ or $\cos^3 at = ¼ (3 \cos at + \cos 3at)$

Example 5: Find Laplace transform of sin 5*t* sin 3*t*

Solution: Use the identity $\sin at \cdot \sin bt = \dfrac{1}{2}[\cos(a-b)t - \cos(a+b)t]$

Thus

$$L[\sin 5t \sin 3t] = L\left[\frac{1}{2}(\cos 2t - \cos 8t)\right] = \frac{1}{2}\big(L[\cos 2t] - L[\cos 8t]\big) = \frac{1}{2}\left[\frac{s}{s^2 + 2^2} - \frac{s}{s^2 + 8^2}\right]$$

One has to use analogous formula if one has to find Laplace transformation of product of cosines or product of sine and cosine:

$$\sin at \cdot \cos bt = \frac{1}{2}[\sin(a+b)t - \sin(a-b)t]$$
$$\cos at \cdot \cos bt = \frac{1}{2}[\cos(a+b)t + \cos(a-b)t]$$

Also one may have to use these formulae repeatedly as in the case of following.

Example 6: Find Laplace transform of sin *t* sin 2*t* sin 3*t*

Solution: It is known that $\sin at \cdot \sin bt = \dfrac{1}{2}[\cos(a-b)t - \cos(a+b)t]$. Also $\sin 2x = 2 \sin x \cos x$. Hence

$$(\sin t \sin 2t)\sin 3t = \left[\frac{1}{2}(\cos t - \cos 3t)\right]\sin 3t = \frac{1}{2}[\cos t \sin 3t - \cos 3t \sin 3t]$$

$$= \frac{1}{2}\left(\cos t \sin 3t - \frac{1}{2}\sin 6t\right). = \frac{1}{2}\left[\frac{1}{2}(\sin 4t - \sin(-2t)) - \frac{1}{2}\sin 6t\right]$$

Thus

$$L[\sin t \sin 2t \sin 3t] = \frac{1}{4}L[\sin 2t + \sin 4t - \sin 6t] = \frac{1}{4}\left[\frac{2}{s^2+2^2} + \frac{4}{s^2+4^2} - \frac{6}{s^2+6^2}\right]$$

Important properties:

Listed below are some of the important properties of Laplace transformation. Proofs follow from the definition of Laplace transform, but are omitted here. Recall that $L[f(t)]$ is denoted by $F(s)$.

1. Change of scale rule: $L[f(at)] = \frac{1}{a}F\left(\frac{s}{a}\right)$
2. Shifting rule: $L[e^{at} f(t)] = F(s-a)$
3. Rule for multiplying by a monomial in t: $L[t^n f(t)] = (-1)^n \frac{d^n}{ds^n}(F(s))$

   This is most helpful when $n = 1$ in which case it takes the form $L[tf(t)] = -F'(s)$
4. Rule for division by t: $L\left[\frac{f(t)}{t}\right] = \int_s^\infty F(s)ds$
5. Laplace of derivative: $L[f'(t)] = L\left[\frac{df}{dt}\right] = sL[f(t)] - f(0) = sF(s) - f(0).$
6. Laplace of the integral: $L\left[\int_0^t f(t)dt\right] = \frac{1}{s}L[f(t)] = \frac{F(s)}{s}.$

Students are expected to become *proficient* with these properties as these will be used throughout the course. Here are a set of examples illustrating the use of these properties.

Example 7: Find Laplace transform of e^{2t} (sin $4t$ – 2cosh $3t$).

Solution: $L[e^{2t}$ (sin $4t$ – 2cosh $3t$)] = $L[e^{2t}$ (sin $4t$)] – $L[e^{2t}$ (2cosh $3t$)].

If $f(t)$= sin $4t$, then $F(s)$ = $L[\sin 4t] = \frac{4}{s^2+4^2}$. Thus by shifting rule

$L[e^{2t}$ (sin $4t$)] = $F(s-2) = \frac{4}{(s-2)^2+4^2}$. Similarly $L[e^{2t}$ (2cosh $3t$)] = 2 $L[e^{2t}$ (cosh $3t$)] =

$2\cdot\frac{s-2}{(s-2)^2-3^2}$. Thus $L[e^{2t}(\sin 4t - 2\cosh 3t) = \frac{4}{(s-2)^2+16} - 2\cdot\frac{s-2}{(s-2)^2-9}$.

Example 8: Find Laplace transform of t sin $5t$.

Solution: Clearly rule 3 above should be helpful here. If $f(t) =$ sin 5t, then $F(s) =$ $L[\sin 5t] = \frac{5}{s^2+5^2}$. Thus by rule 3 above, $L[t\sin 5t] = -\frac{d}{ds}\left(\frac{5}{s^2+5^2}\right) = \frac{5(2s)}{\left(s^2+5^2\right)^2}$.

Example 9: Find Laplace transform of $e^{-t}\cos^2 3t$.

Solution: Use the identity $\cos 2x = 2\cos^2 x - 1$ to find $L[\cos^2 3t]$. Then use shifting rule to find the required Laplace transform.

Example 10: Find Laplace transform of $e^{-t}\sin 3t \cos 2t$.

Solution: Use formula $\sin a \cos b = ½(\sin(a+b) - \sin(a-b))$ and then use shifting rule.

Example 11: Find Laplace transform of $t^5 e^{-3t}\sinh 2t$.

Solution: Expand $e^{-3t}\sinh 2t$ by using the definition $\sinh x = ½(e^x - e^{-x})$ then use shifting rule for each term.

Example 12: Find Laplace transform of $\frac{\sin 6t}{t}$.

Solution: Let $f(t) = \sin 6t$, Then $F(s) = 6/(s^2+6^2)$. Then, by rule 4 above,

$$L\left[\frac{\sin 6t}{t}\right] = \int_s^\infty L[\sin 6t]ds = \int_s^\infty \frac{6}{s^2+36}ds = \tan^{-1}\left(\frac{s}{6}\right)\Big|_s^\infty = \frac{\pi}{2} - \tan^{-1}\left(\frac{s}{6}\right) = \cot^{-1}\left(\frac{s}{6}\right).$$

Example 13: Find Laplace transform of $\frac{e^{-6t}-e^{-5t}}{t}$.

Solution: By the rule 4 above,

$$L\left[\frac{e^{-6t}-e^{-5t}}{t}\right] = \int_s^\infty L[e^{-6t}-e^{-5t}]ds = \int_s^\infty\left(\frac{1}{s+6}-\frac{1}{s+5}\right)ds = \left[\log(s+6)-\log(s+5)\right]_s^\infty$$

$$= \frac{\log(s+6)}{\log(s+5)}\Bigg|_s^\infty \qquad = \lim_{s\to\infty}\log\left(\frac{s+6}{s+5}\right) - \log\left(\frac{s+6}{s+5}\right)$$

$$= \lim_{s\to\infty}\log\left(\frac{1-\frac{6}{s}}{1+\frac{5}{s}}\right) + \log\left(\frac{s+5}{s+6}\right)$$

$$= \log 1 + \log\left(\frac{s+5}{s+6}\right) = \log\left(\frac{s+5}{s+6}\right)$$

Example 14: Find Laplace transform of $\frac{e^{-t}\sin t}{t}$.

Solution: By shifting rule, one knows that $L\left[e^{-t}\sin t\right] = \frac{1}{(s+1)^2+1}$. Now, by rule 4

$$L\left[\frac{e^{-t}\sin t}{t}\right] = \int_s^\infty L\left[e^{-t}\sin t\right]ds = \int_s^\infty \frac{1}{(s+1)^2+1}ds = \tan^{-1}(s+1)\Big|_s^\infty = \cot^{-1}(s+1)$$

Laplace transform of periodic functions: Functions whose values repeat after a certain value are very common in engineering. Such functions are called periodic function. Discussion below is about Laplace transform of periodic functions. A function $f(t)$ is said to be periodic with period T if $f(t+T) = f(t)$ for all values of t. Sine and Cosine functions are two classic examples of periodic functions and they have period equal to 2π. The following formula is easy to verify from the definition of Laplace transform.

If f is a periodic function of period T then $L[f] = \dfrac{1}{1-e^{-sT}}\displaystyle\int_0^T e^{-st} f(t)dt.$ Observe how the limits of integration has become finite now. Here are a couple of illustrations of this formula.

Example 15: Find Laplace transform of the function f defined by $f(t) = \begin{cases} 2t & 0 < t < 1 \\ 2 & 1 < t < 2 \end{cases}$ and $f(t) = f(t+2)$ for all t.

Solution: Given function is a periodic function with period 2. Hence by the above formula $L[f]$

$$L[f] = \frac{1}{1-e^{-sT}}\int_0^T e^{-st} f(t)dt = \frac{1}{1-e^{-2s}}\int_0^2 e^{-st} f(t)dt = \frac{1}{1-e^{-2s}}\left\{\int_0^1 e^{-st}(2t)dt + \int_1^2 e^{-st}(2)dt\right\}$$

Integrating the first integral by parts one has

$$L[f] = \frac{1}{1-e^{-2s}}\left(\frac{1}{s^2} + \frac{e^{-s}}{s^2} - \frac{e^{-2s}}{s}\right)$$

Example 16: A square wave function $f(t)$ with period a is given by $f(t) = \begin{cases} E & 0 \le t < a/2 \\ -E & a/2 \le t < a. \end{cases}$

Show that $L[f(t)] = (E/s)\tanh(as/4)$.

Solution: The given function is a periodic function with period a. Hence by the above formula

$$L[f] = \frac{1}{1-e^{-sT}}\int_0^T e^{-st} f(t)dt = \frac{1}{1-e^{-as}}\int_0^a e^{-st} f(t)dt = \frac{1}{1-e^{-as}}\left\{\int_0^{a/2} e^{-st}(E)dt + \int_{a/2}^a e^{-st}(-E)dt\right\}$$

$$= \frac{E}{1-e^{-as}}\left\{\int_{t=0}^{a/2} e^{-st}dt - \int_{t=a/2}^{a} e^{-st}dt\right\} = \frac{E}{1-e^{-as}}\left\{\left.\frac{e^{-st}}{-s}\right|_{t=0}^{a/2} - \left.\frac{e^{-st}}{-s}\right|_{t=a/2}^{a}\right\} = \frac{E}{s(1-e^{-as})}\left\{-e^{-sa/2} + 1 + e^{-sa} - e^{-sa/2}\right\}$$

$$= \frac{E(1-e^{-sa/2})^2}{s(1+e^{sa/2})(1-e^{-sa/2})^2} = \frac{E}{s}\cdot\frac{(1-e^{-sa/2})}{(1+e^{-sa/2})} = \tanh(as/4)$$

Step function and its Laplace transform: The following is a function which shows up often in engineering: $H(t-a)=\begin{cases}0 & t\le a\\ 1 & t>a\end{cases}$

This is named after the British electrical engineer Oliver Heaviside. In literature it is also named unit step function. Best way to understand this is to draw a graph of this function. This is a discontinuous function and clearly it is like a switch which comes *on* at time $t = a$. Also draw the graphs of $f(t)H(t-a)$ and $f(t-a)\,H(t-a)$ to understand the role of step function better. In these examples, take f to be t or sin t or cos t. The important result from the point of view of this course is $L\left[f(t-a)H(t-a)\right]=e^{-as}L[f(t)]$.

Example 17: Find Laplace transform of $e^{(t-1)}\,H(t-1)$.

Solution: Taking $f(t) = e^t$ and applying the formula directly tells us that

$$L[e^{(t-1)}H(t-1)]=e^{-s}L[e^t]=\frac{e^{-s}}{s-1}.$$

Example 18: Find Laplace transform of $t^2H(t-2)$.

Solution: Write t^2 in terms of $(t-2)^2$ as follows: $t^2 = (t-2+2)^2 = (t-2)^2 + 4(t-2) + 4$. Calling $f(t) = t^2 + 4\,t + 4$, we are expected to find $L[f(t-2)H(t-2)]$. This follows from the formula and hence $L[t^2H(t-2)] = L[f(t-2)H(t-2)] = e^{-2s}\,L[f(t)] = e^{-2s}\,L[t^2 + 4\,t + 4]$

$$=e^{-2s}\left(\frac{2!}{s^3}+4\cdot\frac{1}{s^2}+4\cdot\frac{1}{s}\right).$$

In general the trick is to write $f(t)H(t-a)$ as $g(t-a)H(t-a)$ by rewriting $f(t) = f(t-a+a)$ and using the definition of f. If the given function has a discontinuity at a, following trick is useful to find its Laplace transform. Let $f(t)=\begin{cases}f_1(t) & \text{for} \quad t\le a\\ f_2(t) & \text{for} \quad t>a\end{cases}$

Then $\quad f(t)=f_1(t)+\{f_2(t)-f_1(t)\}H(t-a).$

Example 19: Find Laplace transform of $\quad f(t)=\begin{cases}\cos t & for \quad 0<t\le\pi\\ \sin t & for \quad t>\pi\end{cases}$

Solution: Observe that $f(t) = \cos t + \{\sin t - \cos t\}H(t-\pi)$. One needs to write sin t and cos t in terms of $(t-\pi)$ so as to be able to apply the formula, Clearly, $\sin t = -\sin(t-\pi)$ and $\cos t = -\cos(t-\pi)$. Then

$$\begin{aligned}L[f(t)]&=L[\cos t]+L[-\sin(t-\pi)H(t-\pi)]+L[\cos(t-\pi)H(t-\pi)]\\ &=L[\cos t]+e^{-\pi s}L[\sin t]+e^{-\pi s}L[\cos t]\\ &=\frac{s}{s^2+1}+e^{-\pi s}\cdot\frac{1}{s^2+1}+e^{-\pi s}\cdot\frac{s}{s^2+1}.\end{aligned}$$

The above trick may be used when there are more points of discontinuity in the function. Let f be a function with two points of discontinuity; a and b.

$$f(t) = \begin{cases} f_1(t) & \text{for} \quad 0 < t \le a \\ f_2(t) & \text{for} \quad a < t \le b \\ f_3(t) & \text{for} \quad t > b \end{cases}$$

Then rewrite $f(t) = f_1(t) + \{f_2(t) - f_1(t)\}H(t-a) + \{f_3(t) - f_1(t)\}H(t-b)$ and then follow the above procedure.

Inverse Laplace Transform(ILT): As pointed out earlier, Laplace transform is best visualized as a mapping from set of functions to set of functions. It is natural to ask if this mapping has an inverse; meaning given a function, is there funtion whose Laplace is the given one? This question is not answered in this generality in this course, but suffices to know that Laplace transform does have an inverse for the class of functions dealt in this course. More to the point one says $f(t)$ is inverse Laplace transform of $F(s)$ if $F(s)$ is the Laplace transform of $f(t)$. Mathematically, if $L[f(t)] = F(s)$, we say $L^{-1}[F(s)] = f(t)$. Similar to the table of Laplace transform, one can have a table of inverse Laplace transform which is the same table as that of Laplace transform, bur read the other way. An important property of ILT that will be exploited is linearity. This follows from linearity of Laplace transfrom. This means $L^{-1}[aF(s) + bG(s)] = aL^{-1}[F] + bL^{-1}[G]$.

Example 20: Find ILT of $(3s^3 - 5s^2 - 2)/s^4$.

Solution: Let $F(s) = (3s^3 - 5s^2 - 2)/s^4$.

$L^{-1}[F(s)]=L^{-1}[(3s^3-5s^2-2)/s^4]= L^{-1}[3/s - 5/s^2 - 2/s^4]$. Then by linearity,

$L^{-1}[F(s)] = 3L^{-1}(1/s) - 5L^{-1}(1/s^2) - 2L^{-1}(1/s^4) = 3(1) - 5t/2! - 2t^3/3!$

Example 21: Find ILT of $5s/(s^2+36)$

Solution: Recall that $L[\cos at] = s/(s^2+a^2)$ and hence $L^{-1}[s/(s^2+a^2)] = \cos at$. Thus by linearity $L^{-1}[5s/(s^2+36)] = 5\,L^{-1}[s/(s^2+6^2)] = 5\cos 6t$.

Very often one has to break up the given function using partial fractions as illustrated by next two examples,

Example 22: Find ILT of $k/(s+a)(s+b)$.

Solution: Write

$$\frac{1}{(s+a)(s+b)} = \frac{A}{(s+a)} + \frac{B}{(s+b)} \Rightarrow 1 = A(s+b) + B(s+a)$$

Putting $s = -b$, we get $B = 1/(a-b)$ and putting $s = -a$, we get $A = 1/(b-a)$. Thus

$$\frac{1}{(s+a)(s+b)} = \frac{1}{b-a}\left[\frac{1}{(s+a)} - \frac{1}{(s+b)}\right].$$

Hence

$$L^{-1}\left[\frac{k}{(s+a)(s+b)}\right] = \frac{k}{b-a}\left[L^{-1}\left[\frac{1}{(s+a)}\right] - L^{-1}\left[\frac{1}{(s+b)}\right]\right] = \frac{k}{b-a}\left[e^{-at} - e^{-bt}\right]$$

Example 23: Find ILT of $\dfrac{s^2+2s-4}{(s^2+9)(s-5)}$.

Solution: Write $\dfrac{s^2+2s-4}{(s^2+9)(s-5)} = \dfrac{A}{s-5} + \dfrac{Bs+C}{s^2+9}$.

Then $A(s^2+9)+(Bs+C)(s-5) = s^2+2s-4.$ Putting $s = 5$, one finds that $A = 31/34$. Equating powers of s^2, $B = 3/34$ and hence $C = 83/84$. Thus

$$L^{-1}\left[\frac{s^2+2s-4}{(s^2+9)(s-5)}\right] = L^{-1}\left[\frac{31}{34}\frac{1}{s-5} + \frac{1}{34}\frac{3s+83}{(s^2+9)}\right]$$

$$= \frac{31}{34}\cdot L^{-1}\left[\frac{1}{s-5}\right] + \frac{3}{34}L^{-1}\left[\frac{s}{(s^2+9)}\right] + \frac{83}{34}\cdot L^{-1}\left[\frac{s}{(s^2+9)}\right]$$

$$= \frac{31}{34}e^{5t} + \frac{3}{34}\cos 3t + \frac{83}{34}\cdot\frac{1}{3}\sin 3t.$$

One can get useful formulae to find ILT from the properties of Laplace transform. Here is the shifting property. $L[e^{at}f(t)] = F(s-a) \Leftrightarrow L^{-1}[F(s-a)] = e^{at}f(t) = e^{at}L^{-1}[F(s)]$

Example 24: Find ILT of $\dfrac{5}{s^2+4s+13}$..

Solution: Complete the square of the denominator. $L^{-1}\left[\dfrac{5}{s^2+4s+13}\right] = L^{-1}\left[\dfrac{5}{(s+2)^2+9}\right]$

$$= 5e^{-2t}L^{-1}\left[\frac{1}{s^2+3^2}\right] = 5e^{-2t}\sin 3t.$$

Example 25: Find ILT of $(4s+2)/(s-1)^2(s+2)$.

Solution: Write

$$\frac{4s+2}{(s-1)^2(s+2)} = \frac{A}{s-1} + \frac{B}{(s-1)^2} + \frac{C}{s+2} \Rightarrow A(s+2)(s-1) + B(s+2) + C(s-1)^2 = 4s+5.$$

Putting s = 1, 2 and 0 respectively one gets A=1/3, B = 3 and C = –1/3. Thus

$$L^{-1}\left[\frac{4s+5}{(s-1)^2(s+2)}\right]=L^{-1}\left[\frac{1}{3}\cdot\frac{1}{s-1}+3\cdot\frac{1}{(s-1)^2}-\frac{1}{3}\cdot\frac{1}{s+2}\right]$$

$$=\frac{1}{3}\cdot L^{-1}\left[\frac{1}{s-1}\right]+3\cdot L^{-1}\left[\frac{1}{(s-1)^2}\right]-\frac{1}{3}\cdot L^{-1}\left[\frac{1}{s+2}\right]$$

$$=\frac{1}{3}e^t+3e^tL^{-1}\left[\frac{1}{s^2}\right]-\frac{1}{3}e^{-2t}$$

$$=\frac{1}{3}e^t+3te^t-\frac{1}{3}e^{-2t}$$

From the rule $L[t^n f(t)]=(-1)^n\left(F^{(n)}(s)\right)$, one gets $L^{-1}\left[F^{(n)}(s)\right]=(-1)^n t^n f(t)$. This is particularly useful for n = 1 which can be rewritten as $L^{-1}[F'(s)]=-tf(t)=-tL^{-1}[F(s)]$.

Thus knowing ILT of F, one can deduce ILT of its derivative and viceversa.

Example 26: Find ILT of $\cot^{-1}\left[\frac{s}{a}\right]$.

Solution: Observe that it is easy to find ILT of derivative of this function.

$$F(s)=\cot^{-1}\left(\frac{s}{a}\right)\quad\Rightarrow\quad F'(s)=-\frac{a}{a^2+s^2}.$$

$$L^{-1}[F'(s)]=-L^{-1}\left[\frac{a}{a^2+s^2}\right]=-\sin at$$

$$L^{-1}\left[\cot^{-1}\left(\frac{s}{a}\right)\right]=-\frac{1}{t}(-\sin at)=\frac{\sin at}{t}.$$

Formula for Laplace of integral tells $L\left[\int_0^t f(t)dt\right]=\frac{F(s)}{s}$. From this one can conclude that $L^{-1}\left[\frac{F(s)}{s}\right]=\int_0^t f(t)dt=\int_0^t L^{-1}[F(s)]dt.$

Example 27: Find ILT of $\frac{1}{s(s^2+a^2)}$.

Solution: Observe that it is easy to find ILT of $F(s)$ = $1/(s^2+a^2)$. In fact

$$f(t)=L^{-1}[F(s)]=L^{-1}\left[\frac{1}{(s^2+a^2)}\right]=\frac{1}{a}\sin at.$$

Then

$$L^{-1}\left[\frac{1}{s(s^2+a^2)}\right]=L^{-1}\left[\frac{F(s)}{s}\right]=\int_0^t f(t)dt=\frac{1}{a^2}(1-\cos at).$$

Convolution Theorem: Convolution is a way of combining two functions. It is defined as follows: If *f* and *g* are two functions, their convolution is a function defined as

$$f * g = \int_0^t f(t-u)g(u)du.$$

For example, convolution of t and t^2 is given by $\int_0^t (t-u)u^2 du = \left(tu^3 - \frac{u^4}{4} \right)\Big|_0^t = \frac{3}{4}t^4.$

Convolution theorem for Laplace transform says $L[f(t)] \cdot L[g(t)] = L[f(t) * g(t)].$

Proof of this is not included here but verification of this for a few *easy* functions is illustrated

Example 28: Verify convolution theorem for $f(t) = t$ and $g(t) = e^t$.

Solution: $L[f(t)] = L[t] = \frac{1}{s^2}$ and $L[g(t)] = L[e^t] = \frac{1}{s-1}$

$$L[f(t)] \cdot L[g(t)] = \frac{1}{s^2(s-1)}.$$

$$f(t) * g(t) = \int_0^t f(t-u)g(u)du = \int_0^t (t-u)e^u du = -t + e^t - 1$$

$$L\left[f(t) * g(t)\right] = L\left[e^t - t - 1\right] = \frac{1}{s-1} - \frac{1}{s^2} - \frac{1}{s} = \frac{1}{s^2(s-1)}$$

and hence $L[f(t)] \cdot L[g(t)] = L[f(t) * g(t)].$

In fact the convolution theorem may be restated as

$$L^{-1}[F(s)G(s)] = f * g = \int_0^t f(t-u)g(u)du.$$

This may be used to evaluate ILT for some special cases and below is an illustration of this.

Example 29: Find ILT of $\frac{1}{s(s^2+a^2)}$.

Solution: First step is to recognize the given funtion as a product of Laplace transform of two functions. Thus let $F(s) = 1/s$ and $G(s) = 1/(s^2+a^2)$. Then $f(t) = 1$ and $g(t) = (\sin at)/a$.

$$L^{-1}\left[\frac{1}{s(s^2+a^2)}\right] = L^{-1}\left[F(s) \cdot G(s)\right] = f(t) * g(t) = \int_0^t f(t-u)g(u)du = \int_0^t \frac{\sin au}{a} du = \frac{1}{a^2}(1 - \cos at).$$

Example 30: Find ILT of $\dfrac{s^2}{(s^2+a^2)(s^2+b^2)}, \quad b \neq a.$

Solution: Let $F(s) = s/(s^2+a^2)$ and $G(s) = s/(s^2+b^2)$. Then $f(t) = \cos at$ and $g(t) = \cos bt$. Thus

$$L^{-1}\left[\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right] = L^{-1}[F(s)\cdot G(s)] = f * g = \int_0^t \cos a(t-u)\cos bu\, du$$

$$= \frac{1}{2}\int_0^t \{\cos[at+(b-a)u] + \cos[at-(b+a)u]\}du = \frac{1}{2}\left\{\frac{\sin bt - \sin at}{b-a} + \frac{\sin bt + \sin at}{b+a}\right\}$$

This may be simplified to $\dfrac{1}{(b^2-a^2)}[b\sin bt - a\sin at]$

Solving differential equations: As promised earlier, all this technique of finding Laplace transform and inverse Laplace transform will be put to use to solve diffrenetial equations. In particular observe how this method is simpler to the other methods you have studies till now. Also, historically this is the reason for popularity of Laplace transforms.

As is to be expected, behaviour of Laplace transform of derivatives of functions play an important role. For the sake of convenience reproduced below is a list of relevant properties for a function $f(t)$:

$$L[f'(t)] = sF(s) - f(0)$$
$$L[f''(t)] = s^2F(s) - sf(0) - f'(0)$$
$$L[f'''(t)] = s^3F(s) - s^2 f(0) - sf'(0) = f''(0)$$

Example 31: Solve $\dfrac{dy}{dt} + y = \sin t$, given $y(0) = 0$.

Solution: Take Laplace transform of the given differential equation.

$$L[y' + y] = L[\sin t]$$

Use linearity $\quad L[y'] + L[y] = \dfrac{1}{s^2+1}$

Use property above $\quad sY - y(0) + Y = \dfrac{1}{s^2+1}$

Solve for Y $\quad (s+1)Y = \dfrac{1}{s^2+1} \quad \Rightarrow Y = \dfrac{1}{(s+1)(s^2+1)}$

Now take inverse Laplace to get
$$y = L^{-1}[Y] = L^{-1}\left[\frac{1}{(s+1)(s^2+1)}\right]$$
$$= \frac{1}{2}L^{-1}\left[\frac{1}{s+1} - \frac{s}{s^2+1} + \frac{1}{s^2+1}\right]$$
$$= \frac{1}{2}\left[e^{-t} - \cos t + \sin t\right]$$

Example 32: Solve $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = e^x$, such that when $x = 0$, $y = 2$ and $dy/dx = -1$.

Solution: Take Laplace transform of the ODE.

Use linearity $\quad L[y''] - 2L[y'] + L[y] = L[e^x]$

Use the properties above $\quad [s^2Y - sy(0) - y'(0)] - 2[sY - y(0)] + Y = \frac{1}{s-1}$

Solve for Y $\quad Y = \frac{2s^2 - 7s + 6}{(s-1)^3} = \frac{2}{(s-1)} - \frac{3}{(s-1)^2}\frac{1}{(s-1)^3}.$

Taking inverse Laplace $\quad y = 2L^{-1}\left[\frac{1}{(s-1)}\right] - 3L^{-1}\left[\frac{1}{(s-1)^2}\right] + L^{-1}\left[\frac{1}{(s-1)^3}\right]$
$$= 2e^x - 3e^x x + \frac{1}{2}e^x x^2$$

Example 33: Solve $\frac{d^2y}{dx^2} + 9y = \cos 2x,\ y(0) = 1,\ y(\pi/2) = -1$

Solution: Let $y'(0) = a.$ Taking Laplace of the ODE

Use linearity $\quad L[y''] + 9L[y] = L[\cos 2t]$

Use the property above $\quad \left(s^2Y - sy(0) - y'(0)\right) + 9Y = \frac{s}{s^2+4}$

Solve for Y $\quad Y = \frac{a}{s^2+9} + \frac{1}{5}\cdot\frac{s}{s^2+4} + \frac{4}{5}\cdot\frac{s}{s^2+9}$

Taking inverse Laplace

$$y(x) = \frac{a}{3}\sin 3t + \frac{1}{5}\cos 2t + \frac{4}{5}\cos 3t$$

Putting $t = \pi/2$ one gets a=4/5. Substitute this in the above to get the final solution.

Example 34: Solve $ty'' + 2y' + ty = \cos t$ given that $y(0) = 1$.

Solution: Take Laplace of the ODE. Use the fact that $L[t\,f(t)] = -dF(s)/ds$

$$L[ty'' + 2y' + ty] = L[\cos t] \quad \Leftrightarrow \quad L[ty''] + L[2y'] + L[ty] = L[\cos t]$$
$$-\frac{d}{ds}[L[y'']] + 2L[y'] - \frac{d}{ds}[L[y]] = L[\cos t]$$
$$-\frac{d}{ds}\left(s^2Y - sy(0) - y'(0)\right) + 2\left(sY - y(0)\right) - \frac{d}{ds}(Y) = \frac{s}{s^2+1}$$
$$-\left(s^2\frac{d}{ds}(Y) + 2sY\right) + y(0) + 2sY - 2y(0) - \frac{d}{ds}(Y) = \frac{s}{s^2+1}$$

Take ILT of this $L^{-1}\left[\frac{dY}{ds}\right] = L^{-1}\left[-\frac{1}{s^2+1} - \frac{s}{(s^2+1)^2}\right]$

$$ty = -\sin t - \frac{1}{2} t \sin t$$

$$y = \frac{1}{2}\left(1 + \frac{1}{2}\right)\sin t$$

These notes are prepared with a view to introduce students of VTU to learn the techniques involved with Laplace transform. For further studies and clarifications feel free to contact me at seearepi@gmail.com

Appended below is a list of urls which might help you pursue this study more effectively. By no means is the following list complete or the best. It just reflects my taste.

- https://www.youtube.com/watch?v=sZ2qulI6GEk The most wonderful introduction to LT (part of MIT ocw) power series to LT
- http://tutorial.math.lamar.edu/Classes/DE/LaplaceIntro.aspx Dry, but useful
- https://web.stanford.edu/~boyd/ee102/laplace.pdf Computation of Laplace of all elementary functions (and not in your syllabus!!)
- http://www.personal.psu.edu/sxt104/class/Math251/Notes-LT1.pdf Lots of examples but uses complex numbers, but not too much
- http://www.personal.psu.edu/wxs27/250/NotesLaplace.pdf More examples, but initially a bit math oriented.
- https://faculty.atu.edu/mfinan/4243/Laplace.pdf Very comprehensive, but can be intimidating because of heavy math
- https://www.math.fsu.edu/~fusaro/EngMath/Ch5/index.htm superb step-by-step introduction

- https://lpsa.swarthmore.edu/LaplaceXform/FwdLaplace/LaplaceXform.html very well organized

Visualization videos

- https://www.youtube.com/watch?v=6MXMDrs6ZmA part of physics channel by E Khutoryansky
- https://johnflux.com/2019/02/12/laplace-transform-visualized/ animation for cubic, square wave etc
- https://www.youtube.com/watch?v=Wc46Y3SXcdo animation about how a specific function is how close to exponential of negative *st.*

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