Analysis of Determinate Structures

module - 1

Introduction and analysis of Structures Trusses

Forms of structure

Structure refers to a system of connected parts that can support loads while performing its primary functions

A structural system is composed of structural members join together by structura connections. Each structural system may be composed of one or more of the following four basic types of structures.

1. Frame structure
2. Surface structure
3. Trusses
4. cable or Arches

1. Frame structure :-

* Frames are commonly used in building structures
* Frames are composed of beams and columns that are connected together.
* Steel Frames and concrete frames are

the most commonly used Structures in buildings.

* There are two types of frames
  ① planar frame or plane frame :-
  All members are in one plane.
  ② Space frame :- All the members exist in more than one plane.

② Surface structures :-

* membranes, plates or shell type of structures with less thickness as compared to it's other dimension form the Surface Structure.

* This kind of structure exists in a single plane.
* Surface Structures may be made of rigid materials such as reinforced concrete.

③ Trusses :-

* Trusses consists of slender members arranged in a triangular pattern.
* All the members are connected together by pins which are free to rotate.
* Loads that cause the entire truss to bend are converted into axial tensile and

compressive forces in the members.

There are two types of trusses

① plane truss

Eg:- Bridges and roofs

② space truss

Eg:- stadium, transmission line tower

④ Cables and Arches :-

cables and Arch type of Structures are used to span long distances.

* cables are usually flexible and carry their loads in tension.

* The external load is usually applied vertically [not along the axis of the cable]. As a result the cable deforms with a sag.

* cables are commonly used to support bridges and building roofs.

* Cables have an advantage over beams and trusses, especially for span greater than 150 feet.

* An Arch has a reverse curvature of cable and it achieves it's strength in compression.

* An arch must be rigid in order to maintain it's shape.

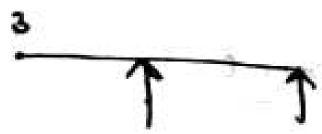
The degree of freedom of a hinged Support is equal to one since it is only allowed to or free to rotate in any direction. DOF = 1

③ Roller and Simply Supported :-

The de

The degree of freedom for roller and Simply Supported is equal to two since it is free to move in one

④ Free end :- At this point in the beam the degree of freedom is equal to free Since it is allowed to move or rotate in any direction.



Degree of inderterminacy :-

The number of equations required over an above the equations of static equilibrium for the analysis of a structure is known as the degree of indeterminacy.

It is also termed as degree of Redundancy.

For ex :-

For a beam as above the number of reactions is equal to three. Therefore, the degree of inderterminacy is equal to number of reactions minus the equations of equilibrum that is equal to number of reactions is equal to zero

$3 - 3 = 0$

②

Number of reactions = 6

Number of equations = 3

∴ The degree of inderterminacy $= 6 - 3 = 3$

Determinate and indeterminate Structures :-

* Structures are grouped into Statically determinate and Statically indeterminate Structures.

Statically determinate Structure :-

The structure which can be analised that is the Support reaction, bending moment and Shear forces can be determined with the equations of Static equilibrium only are known as Statically determinate beams.

For ex :- Simply supported beam, Cantilever beam, one end hinged and the other end roller Supported beam.

Statically indeterminate structure :-

The Structure which can not be analised that is if the Support reactions bending moment and Shear forces can not be determined with the equation of Static equilibrium are known as Statically indeterminate structure.

To analise statically indeterminate structure, one has to make use of compactability conditions and has to determine various deformations.

Linear and Non-Linear Structures

Linear Structure :- If the material has linear stress - strain relationship in a structure [only small deformation is allowed], then it is called a linear structure.

Non-Linear Structure :- If the material does not linear stress strain relationships or if the deformation is so large that the change of geometry cannot be neglected in the analysis of the structure, then that kind of structure is known as a non linear structure or System.

There are two types of Non-linear Systems.

① material non linearity :-

Due to stress - strain relationships of the material.

② **Geometric Non-Linearity** :- Due to considerable change in geometry.

Kinematicaly indeterminate Structures :-

The Kinematic indeterminacy is the minimum number of independent displacement quantities that you need to define to be able to get the complete displaced geometry of the structure.

It is also refferred to as the degree of freedom of System.

In the simply supported beam as shown above, the statical or the degree of static indeterminacy is determined using the number of reactions. Therefore Degree of Static indeterminacy is equal to number of reactions minus number of equilibrium equations.

i.e Degree of static inDeterminacy $= 3 - 3$
$= 0$

But to determine the degree of kinematic indeterminacy, the degree of freedom of the system has to be determined.

At Joint A,

DOF = 1 

At Joint B,

DOF = 2 

∴ Total DOF = 1 + 2
= 3

And in line with the definition, the degree of kinematic indeterminace =
DOF = 3

∴ Even though the beam is statically determinant, it has the kinematic indeterminacy to the degree of 3.

Analysis of trusses

A truss is a very common structure used in constructing bridges, building roofs, towers and lateral Bracings of high rise buildings.

Basic assumptions of truss analysis :-

* members are connected at their joints by smooth frictionless pin.
* All members function has two force members i.e they are subjected to either tension or compression.
* The centroidal axis of each member is straight and coincides with a line connecting the joints centres at the end of the member.
* Loads and reactions are applied to the truss at joints only.

Before analysing a truss we need to find out the statical determinacy of the structure both internally and externally.

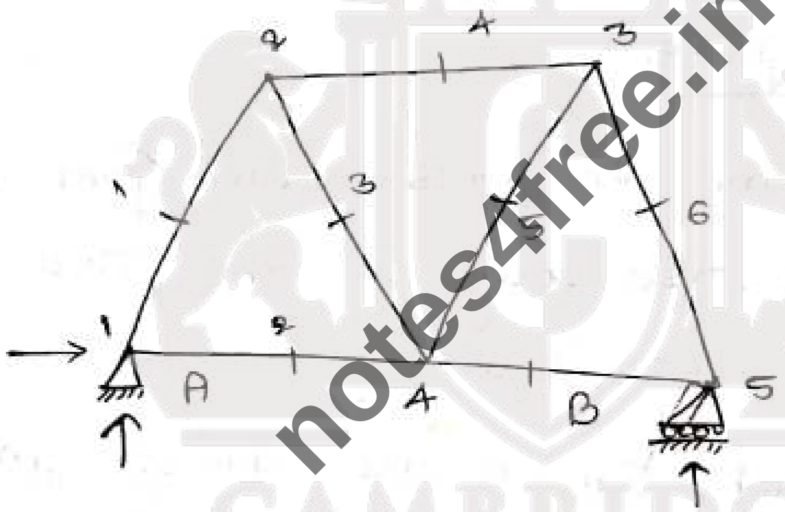
To find the static determinacy of the truss externally, we first find the number of reactions at the supports we are to be found out and use the equations of equilibrium to find out the degree of indeterminacy. To find out whether the

Structure is internally determinant, the number of members, joints and equations of equilibrium are taken into consideration.

The degree of static internal determinacy of the truss can be found out using the formula $m - 2j + r$ where

'm' is the number of members
'j' is the number of joints
'r' is the number of equilibrium equation

For example



To find static indeterminacy of the above sections.

→ Number of reactions = 3
Number of equations of equilibrium = 3

$3 - 3 = 0$

is the d

To find Static internal indeterminacy of the Structure.

Number of members = 7

Number of joints = 5

Number of reactions = 3

$m - 2j + r$

$7 - 2(5) + 3$

$= 0$

<u>method of joints</u>

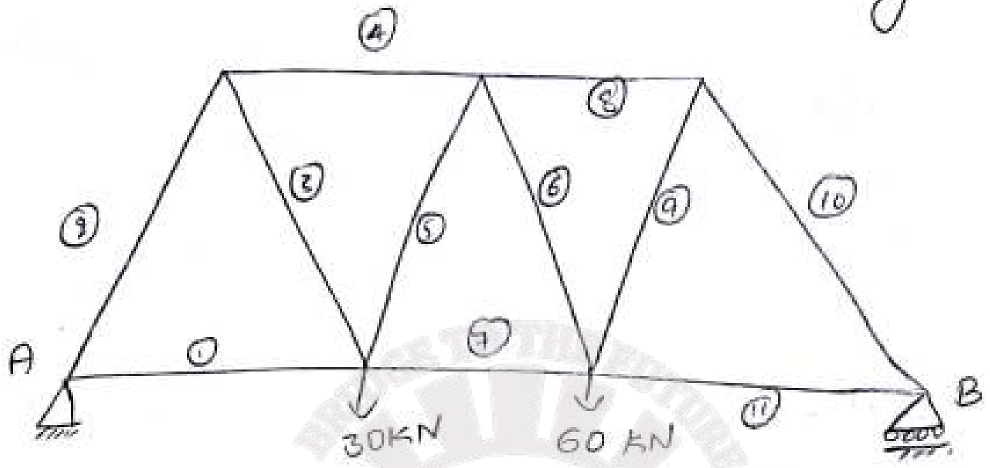
In the method of joints, each joint is isolated as a free body and the two equations of equilibrium.

$\Sigma H = 0$ and $\Sigma V = 0$ are applied and at each joint Sequencially.

At each joint their should not be more than two unknown forces. In the method of joints, the forces in the members are initially assumed to be in tension. If the force is later found as a negative value, then the member is compression.

problem :-

① Find the forces in all the members of the warren truss as shown in the figure.



Step ① - Degree of Indeterminacy

(i) External

Reactions = 3

No of equations = 3

∴ Degree of static External indeterminacy = 3 - 3 = 0

∴ The given truss is statical externally determinacy.

(ii) Internal

$m = 11$, $j = 7$, $r = 3$

$m - 2j + r$

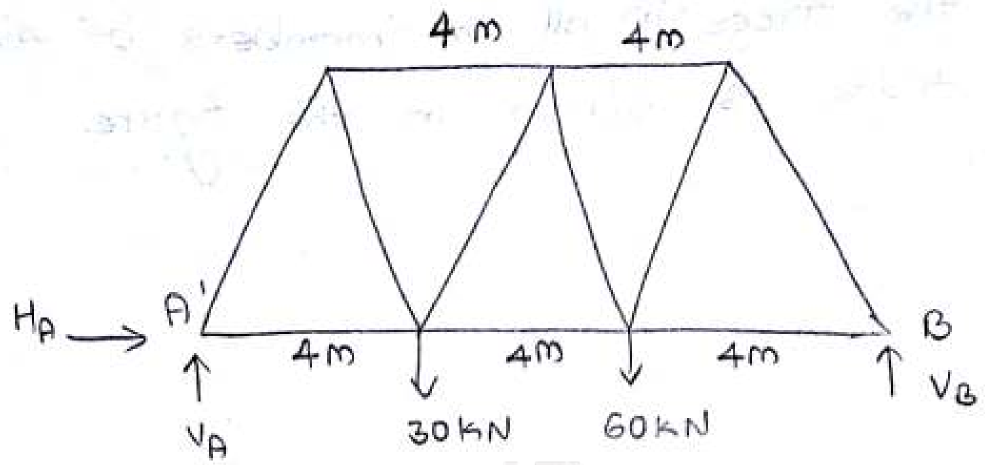
$11 - 2(7) + 3$

$11 - 14 + 3$

$= 0$

∴ The structure is internally determinant.

Step (2) :- Find the reactions at the supports.



$\Sigma H = 0$

$\boxed{H_A = 0}$

$\Sigma V = 0$

$\Rightarrow V_A + V_B - 30 - 60 = 0$

$\boxed{V_A + V_B = 90} \longrightarrow (1)$

$\Rightarrow \Sigma M_A = 0$

$(-V_B \times 12) + (60 \times 8) + (30 \times 4) = 0$

$-12 V_B + 480 + 120$

$-12 V_B = 600$

$\boxed{V_B = 50 \text{ KN}}$

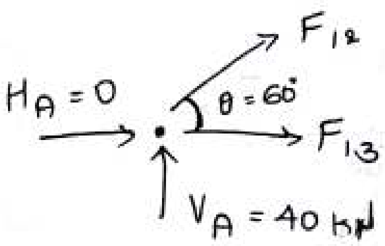
$V_A + V_B = 90$

$V_A + 50 = 90$

$\boxed{V_A = 40 \text{ KN}}$

Step(3) :- Find the forces in the members

→ Joint ①



From the figure

$$\cos\theta = \frac{2m}{4m}$$

$$= 0.5$$

$$\boxed{\theta = 60°}$$

$\Sigma H = 0$

$\Rightarrow H_A + F_{13} + F_{12}\cos 60 = 0$

$\Rightarrow F_{13} + 0.5\,F_{12} = 0$ —— ②

$\Sigma V = 0$

$\Rightarrow V_A + F_{12}\sin 60° = 0$

$\Rightarrow 40 + 0.866\,F_{12} = 0$

$\Rightarrow F_{12} = \dfrac{-40}{0.866}$

$$\boxed{F_{12} = -46.189 \text{ KN}}$$

F_{12} is a compression member

From eqn ②

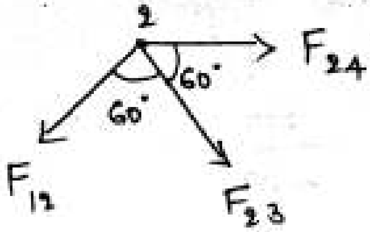
$F_{13} + 0.5\,F_{12} = 0$

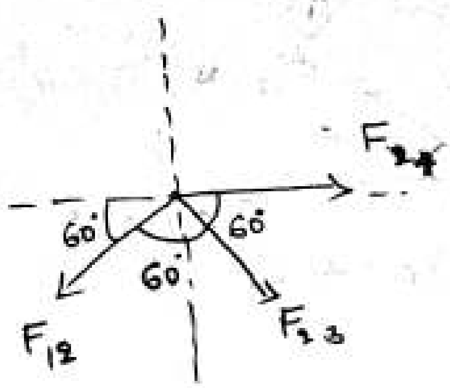
$F_{13} + 0.5(-46.189) = 0$

$F_{13} - 23.094 = 0$

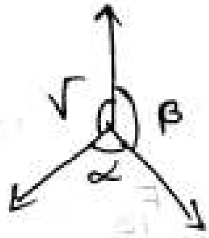
$$\boxed{F_{13} = 23.09 \text{ KN}}$$

Joint ②





Lamis theorem



$$\frac{F_{24}}{\sin 60°} = \frac{F_{12}}{\sin 60°} = \frac{F_{23}}{\sin 60°}$$

$$\frac{F_1}{\sin\alpha} = \frac{F_2}{\sin\beta} = \frac{F_3}{\sin\gamma} = K$$

$$\frac{F_{24}}{\sin 60°} = \frac{F_{23}}{\sin(180+60)} = \frac{F_{12}}{\sin 60}$$

$$\frac{F_{24}}{0.8660} = \frac{F_{23}}{-0.8660} = \frac{F_{12}}{0.8660}$$

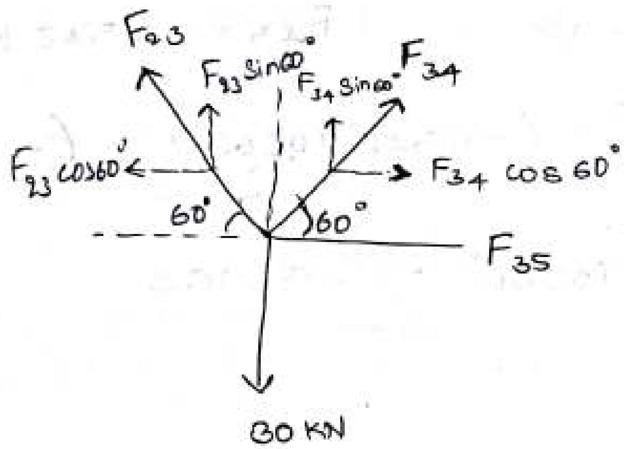
$$\frac{F_{24}}{0.8660} = \frac{-46.189}{0.8660}$$

$$\boxed{F_{24} = -46.189 \text{ KN}}$$

$$\frac{F_{23}}{-0.8660} = \frac{-46.189}{0.8660}$$

$$\boxed{F_{23} = 46.189 \text{ KN}}$$

At joint ③



$\Sigma H = 0$

$$F_{35} + F_{34}\cos 60 - F_{13} - F_{32}\cos 60° = 0$$

$$F_{35} + 0.5\,F_{34} = 46.1891$$

$\Sigma V = 0$

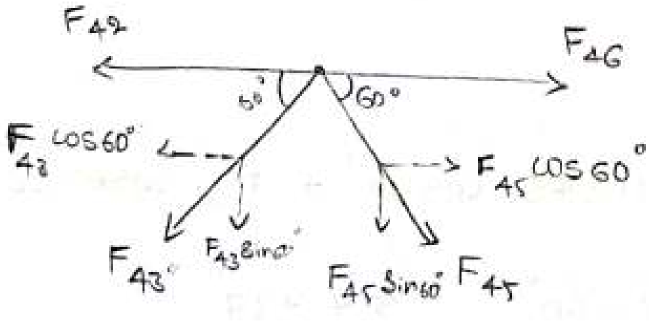
$$46.1189 \sin 60° + F_{34}\sin 60° - 30 = 0$$

$$F_{34} = -11.548 \text{ kN}$$

$$F_{35} + 0.5(-11.548) = 46.1891$$

$$\boxed{F_{35} = 51.963 \text{ kN}}$$

Joint ④



$F_{42} = -46.189\ kN$

$F_{43} = -11.548\ kN$

$$F_{46} + F_{45}\cos 60^\circ - (-11.548\cos 60^\circ) - (-46.189) = 0$$

$$F_{46} + F_{45}\cos 60^\circ = -51.963$$

$\Sigma V = 0$

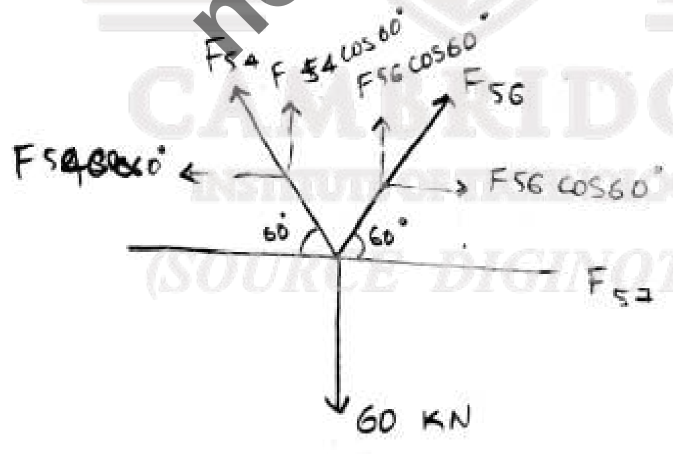
$$-(-11.548\sin 60^\circ) - (F_{45}\sin 60^\circ) = 0$$

$$F_{45} = 11.548\ kN$$

$$F_{46} + 11.548\cos 60^\circ = -51.963$$

$$\boxed{F_{46} = -57.73\ kN}$$

<u>Joint-5</u>



$F_{54} = 11.548\ kN$

$F_{53} = 51.963\ kN$

$\Sigma H = 0$

$$F_{57} - 51.963 - 11.548\cos 60^\circ + F_{56}\cos 60^\circ = 0$$

$$F_{57} + F_{56}\cos 60^\circ = 57.737$$

$\Sigma V = 0$

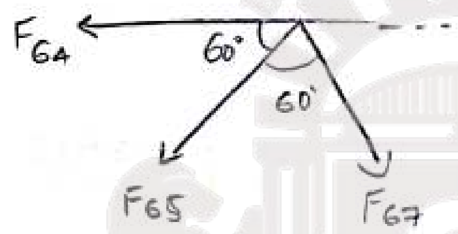
$$-60 + 11.548 \sin 60^\circ + F_{56} \sin 60^\circ = 0$$

$$\boxed{F_{56} = 57.734 \text{ kN}}$$

$$F_{57} + 57.734 \cos 60^\circ = 57.737$$

$$\boxed{F_{57} = 28.868 \text{ kN}}$$

Joint ⑥

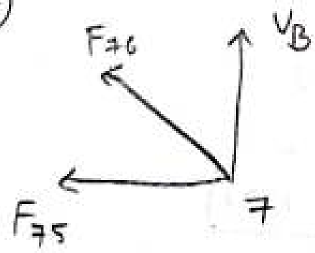


$$\frac{-57.737}{\sin 60^\circ} = \frac{57.734}{\sin(180+60^\circ)} = \frac{F_{67}}{\sin 60^\circ}$$

$$\frac{F_{67}}{\sin 60^\circ} = \frac{-57.737}{\sin 60^\circ}$$

$$\boxed{F_{67} = 57.737 \text{ kN}}$$

Joint ⑦

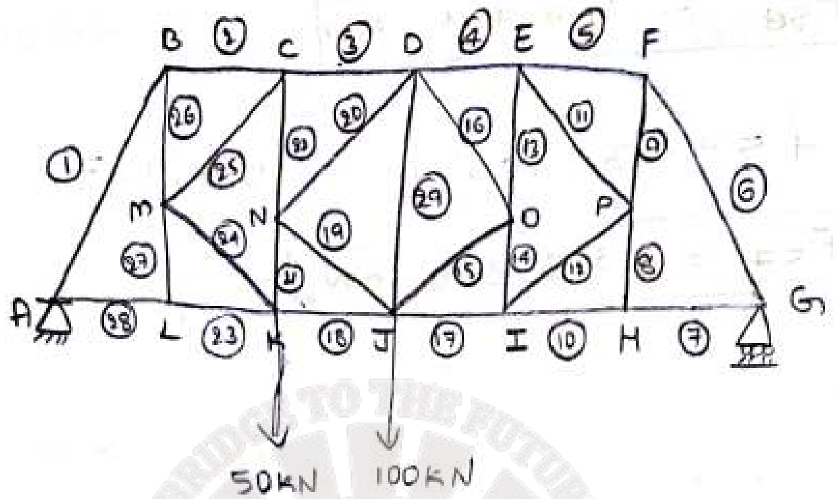


$V_B = 50$ kN

$F_{76} = -57.737$ kN

$F_{75} = 28.868$ kN

(2) Find the forces in all the members of K - Truss of shown in the figure.



Step-2

$\Sigma H = 0$

$\boxed{H_A = 0}$

$\Sigma V = 0$

$V_A + V_G - (50 + 100) = 0$

$V_A + V_G = 150 \text{ KN}$

$\Sigma M_A = 0$

$50 \times 8 + 100 \times 12 - V_G \times 24 = 0$

$\boxed{V_G = 66.667 \text{ KN}}$

$V_A + V_G = 150$

$V_A + 66.667 = 150$

$\boxed{V_A = 83.333 \text{ KN}}$

Step - 1 — Degree of Indeterminacy

(i) External

No of reactions = 3

No of equations = 3

∴ Degree of Static External indeterminacy

$= 3 - 3 = 0$

∴ The given truss is statical extremally determinacy

(ii) Internal :-

m = (No of member) = 29

No of joints j = 16   r = 3

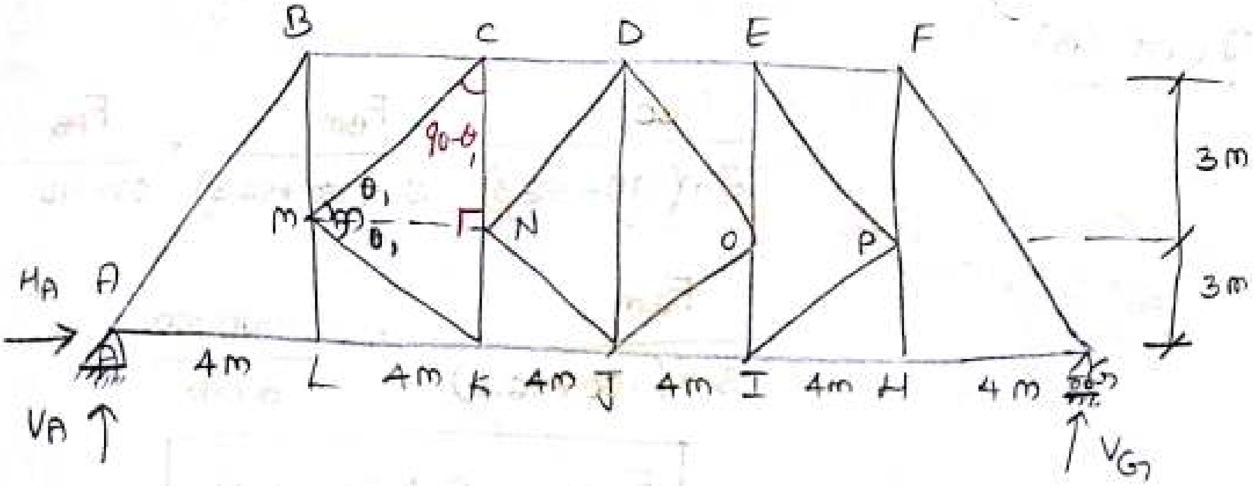
$m - 2j + r$

$29 - 2(16) + 3$

$= 0$

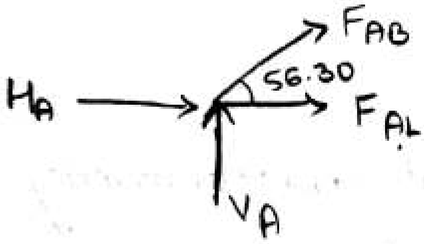
∴

∴ The structure is internally determinant.



Step-3 Find the forces in the members

Joint (A)



$\tan\theta = \frac{6}{4}$

$\theta = \tan^{-1}\left(\frac{6}{4}\right)$

$\boxed{\theta = 56.30}$

$\Sigma H = 0 \Rightarrow F_{AL} + F_{AB}\cos 56.30 + H_A = 0$

$F_{AL} + F_{AB}\cos 56.30 = 0 \longrightarrow (2)$

$\Sigma V = 0 \Rightarrow V_A + F_{AB}\sin 56.30 = 0$

$83.33 + F_{AB}\sin(56.30) = 0$

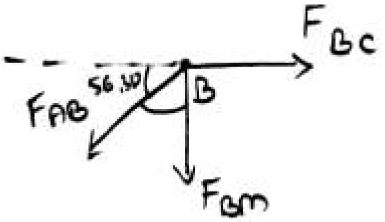
$\boxed{F_{AB} = -100.180 \text{ KN}}$

$F_{AL} + F_{AB}\cos 56.30 = 0$

$F_{AL} + -100.180(\cos 56.30) = 0$

$\boxed{F_{AL} = 55.57 \text{ KN}}$

Joint (B)



$\frac{F_{BC}}{\sin(90-56.3)} = \frac{F_{Bm}}{\sin(180+56.3)} = \frac{F_{AB}}{\sin 90°}$

$\frac{F_{Bm}}{\sin(180+56.3)} = \frac{-100.180}{\sin 90°}$

$\boxed{F_{BM} = 83.34 \text{ KN}}$

$$\frac{F_{BC}}{\sin(90-56.3)} = \frac{-100.180}{\sin 90^\circ}$$

$$\boxed{F_{BC} = -55.58 \text{ kN}}$$

At joint L

$$\frac{F_{AL}}{\sin 90} = \frac{F_{LK}}{\sin 90} = \frac{F_{Lm}}{\sin(180^\circ)}$$

$$\frac{F_{AL}}{\sin 90} = \frac{F_{LK}}{\sin 90}$$

$$\frac{F_{AL}}{\sin 90} = \frac{F_{Lm}}{\sin(180^\circ)}$$

$$\frac{55.57}{\sin 90} = \frac{F_{LK}}{\sin 90}$$

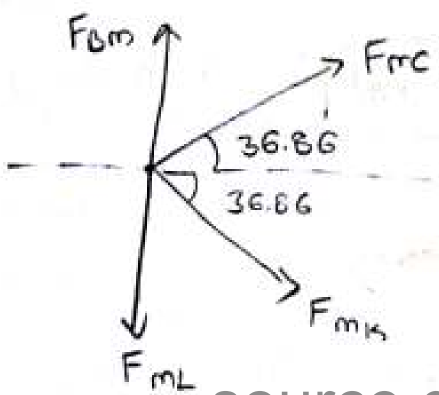
$F_{Lm} = 0$

$$\boxed{F_{LK} = 55.57 \text{ kN}}$$

$$\boxed{F_{Lm} = 0}$$

At joint m

$$\theta_1 = \tan^{-1}\left(\frac{3}{4}\right) = 36.86$$



$\Sigma H = 0 \Rightarrow$

$$F_{mc} \cos 36.86 + F_{mk} \cos 36.86 = 0 \quad \text{--- (3)}$$

$\Sigma V = 0 \Rightarrow$

$$F_{Bm} - F_{mL} + F_{mc} \sin 36.86 - F_{mk} \sin 36.86 = 0$$

$$83.34 + F_{mc} \sin 36.86 + F_{mk} \sin 36.86 = 0$$

Using eq^n (3)

$$\Rightarrow F_{mc} \cos 36.86 = -F_{mk} \cos 36.86$$

$$F_{mc} = -F_{mk} \quad \text{--- (4)}$$

Using eq^n (4) in the above equation

$$83.34 - F_{mk} \sin 36.86 - F_{mk} \sin 36.86 = 0$$

$$83.34 - 2 F_{mk} \sin 36.86 = 0$$

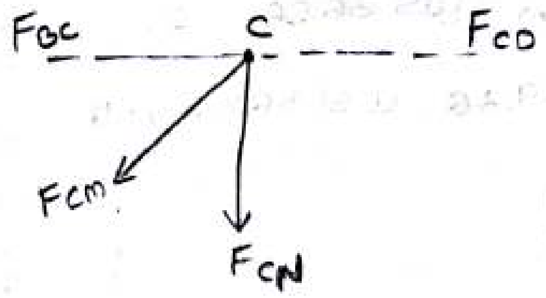
$$F_{mk} = \frac{83.34}{2 \times \sin 36.86}$$

$$\boxed{F_{mk} = 69.46 \text{ KN}}$$

$$F_{mc} = -F_{mk}$$

$$\boxed{F_{mc} = -69.46 \text{ KN}}$$

At joint ©



$\Sigma H = 0$

$F_{CD} - F_{BC} - F_{cm} \cos 36.86$

$F_{CD} + 55.58 - F_{cm} \cos 36.86$

$+55.58 + 69.48 \cos 36.86 + F_{CD} = 0$

$F_{CD} = -111.15$ KN

$\Sigma V = 0$

$-F_{CN} - F_{mc} \sin 36.87 = 0$

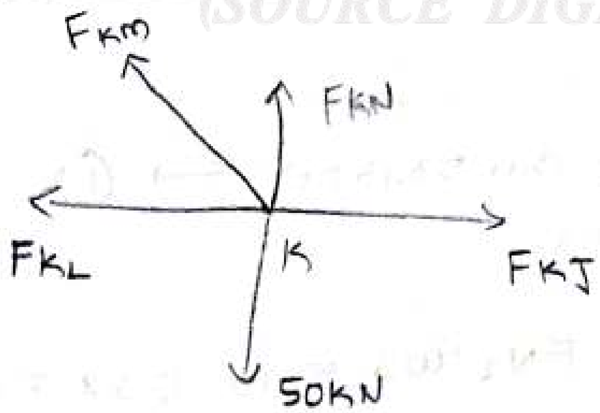
$-F_{CN} = -69.48 \times \sin 36.87$

$F_{CN} = 41.68$ KN

$F_{CN} = 41.66$

$F_{CD} = -111.06$

At Joint KL



$\Sigma H = 0$

$F_{KJ} - F_{KL} - F_{Km} \cos 36.86 = 0$

$F_{KJ} - (-55.48) - 69.46 \cos(36.86) = 0$

$$\boxed{F_{KJ} = 110.68 \text{ kN}}$$

$\Sigma V = 0$

$\Rightarrow F_{KN} + F_{Km} \sin 36.86 - 50 = 0$

$F_{KN} + 69.46 \sin 36.86 - 50 = 0$

$$\boxed{F_{KN} = 8.32 \text{ kN}}$$

At Joint (N)

$\Sigma H = 0$

$F_{ND} \sin 53.13 + F_{NJ} \sin 53.13 = 0 \longrightarrow (1)$

$\Sigma V = 0$

$41.68 + F_{ND} \cos 53.13 - F_{NJ} \cos 53.13 - 8.32 = 0$

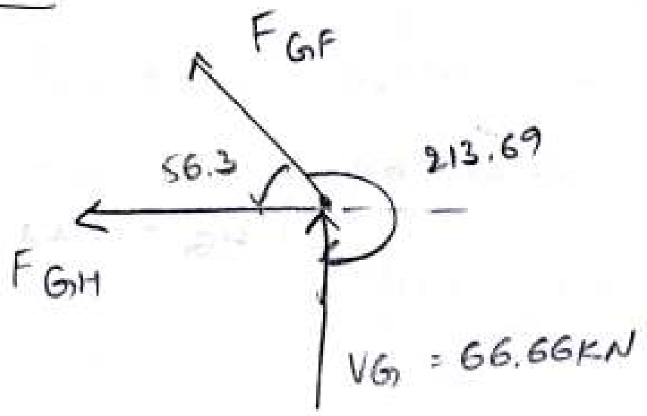
$F_{ND} \cos 53.13 - F_{NJ} \cos 53.13 = -33.36 \quad — (2)$

Solving (1) & (2)

$F_{ND} = -27.799$ kN

$F_{NJ} = 27.799$ kN

Joint (G):



$$\frac{66.667}{\sin(56.30)} = \frac{F_{GF}}{\sin 90^\circ} = \frac{F_{GH}}{\sin(180+33.7)}$$

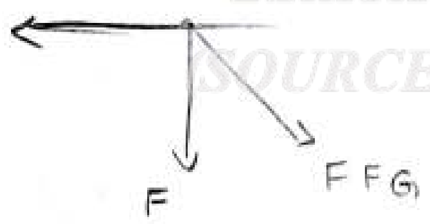
$$\frac{F_{GF}}{\sin 90^\circ} = \frac{66.667}{\sin(56.30)}$$

$F_{GF} = 80.13$ KN

$$\frac{F_{GH}}{\sin(180+33.7)} = \frac{66.667}{\sin(56.30)}$$

$F_{GH} = -44.46$ KN

Joint (F):

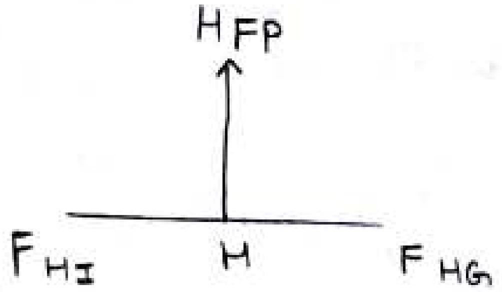


$$\frac{F_{FE}}{\sin(33.7)} = \frac{80.13}{\sin(90^\circ)} = \frac{F_{FP}}{\sin(180+56.3)}$$

$F_{FE} = 44.45$ KN

$F_{FP} = -66.66$ KN

At Joint H :-



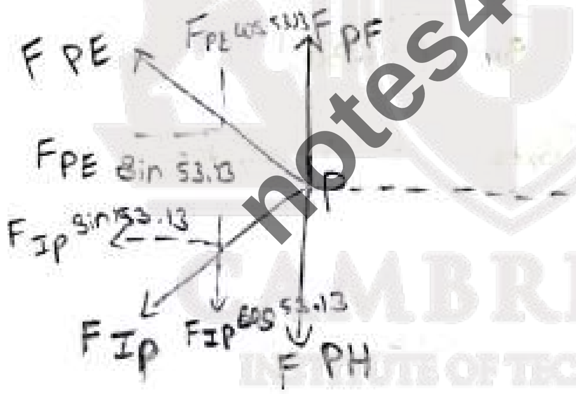
$F_{HG} = -44.46$ kN

$$\frac{-44.46}{\sin(90°)} = \frac{F_{HP}}{\sin(180°)} = \frac{F_{HI}}{\sin(90°)}$$

$F_{HP} = 0$

$F_{HI} = -44.46$ kN

At Joint P :-



$\Sigma H = 0$

$-F_{DE} \sin 53.13 - F_{P} \sin 53.13 = 0$

$\Sigma V = 0$

$F_{FP} - \cancel{F_{PH}}^{0} \; F_{PE} \cos 53.13 - F_{PR} \cos 53.13 = 0$

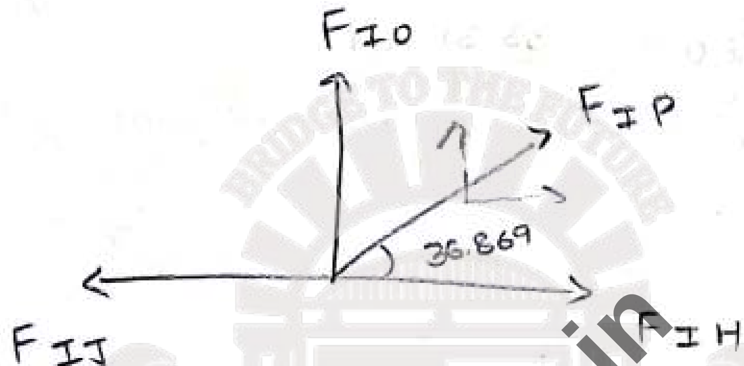
$-66.66 + F_{PE} \cos 53.13 - F_{PR} \cos 53.13 = 0$

$F_{PE} \cos 53.13 - F_{PR} \cos 53.13 = 66.66$

$F_{PE} = 55.54$ kN

$F_{PR} = -55.54$ kN

At Joint I :-



$\Sigma H = 0$

$-44.46 - F_{JR} - 55.54 \sin 53.13 = 0$

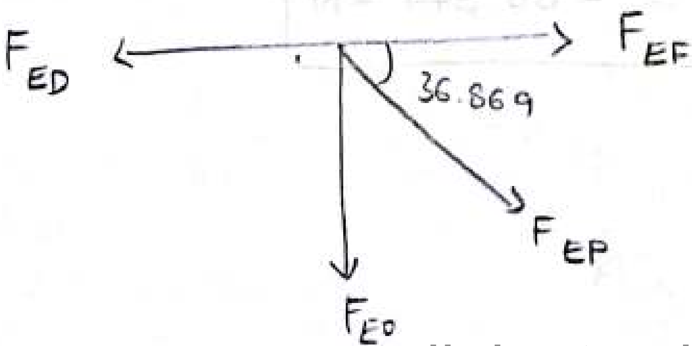
$F_{JI} = -88.89$ kN

$\Sigma V = 0$

$F_{IO} + (-55.54 \cos 53.13) = 0$

$F_{IO} = 33.32$ kN

Joint E :-



$\Sigma H = 0$

$$44.45 + 55.54 \cos 36.86 - F_{DE} = 0$$

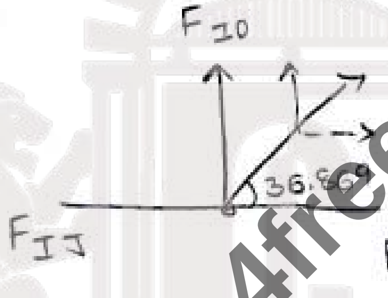
$$F_{DE} = 88.88 \text{ kN}$$

$\Sigma V = 0$

$$-F_{EO} - 55.54 \sin 36.86 = 0$$

$$F_{EO} = -33.31 \text{ kN}$$

Joint - I



$F_{IP} = -55.55$ kN

$F_{IH} = -44.439$ kN

$\Sigma V = 0$

$$F_{IO} - 55.55 \sin(36.869) = 0$$

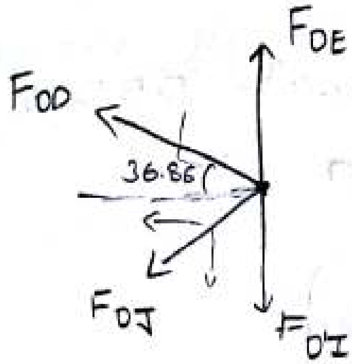
$$\boxed{F_{IO} = 33.329 \text{ kN}}$$

$\Sigma H = 0$

$$-F_{IJ} - 55.55 \cos(36.869) - 44.439 = 0$$

$$\boxed{F_{IJ} = -88.879 \text{ kN}}$$

Joint - O



$F_{OE} = -33.329$ KN

$F_{OI} = 33.329$ KN

$\Sigma H = 0$

$$-F_{OD}\cos(36.869) - F_{OJ}\cos(36.869) = 0$$

$$F_{OD} + F_{OJ} = 0 \rightarrow (b)$$

$\Sigma V = 0$

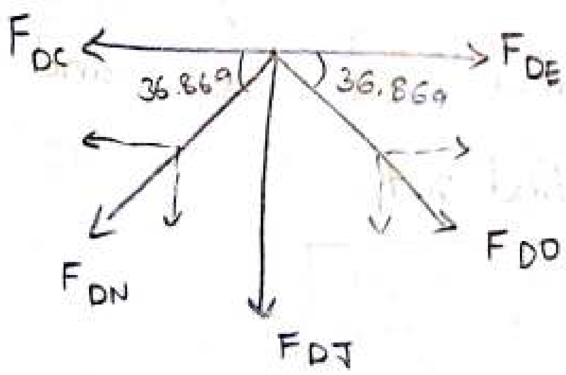
$$-33.329 - 33.329 + F_{OD}\sin(36.869) - F_{OJ}\sin(36.869) = 0$$

$$F_{OD} - F_{OJ} = +11.09 \rightarrow (a)$$

$$\boxed{F_{OD} = 55.545 \text{ KN}}$$

$$\boxed{F_{OJ} = -55.545 \text{ KN}}$$

Joint - D



$F_{DE} = 88.879$ KN

$F_{DC} = 111.70$ KN

$F_{DO} = 55.545$ KN

$F_{DN} = -27.148$ KN

$\Sigma V = 0$

$$-F_{DJ} + 27.148 \sin(36.869) - 55.545 \sin(36.869) = 0$$

$$\boxed{F_{DJ} = -17.037 \text{ KN}}$$

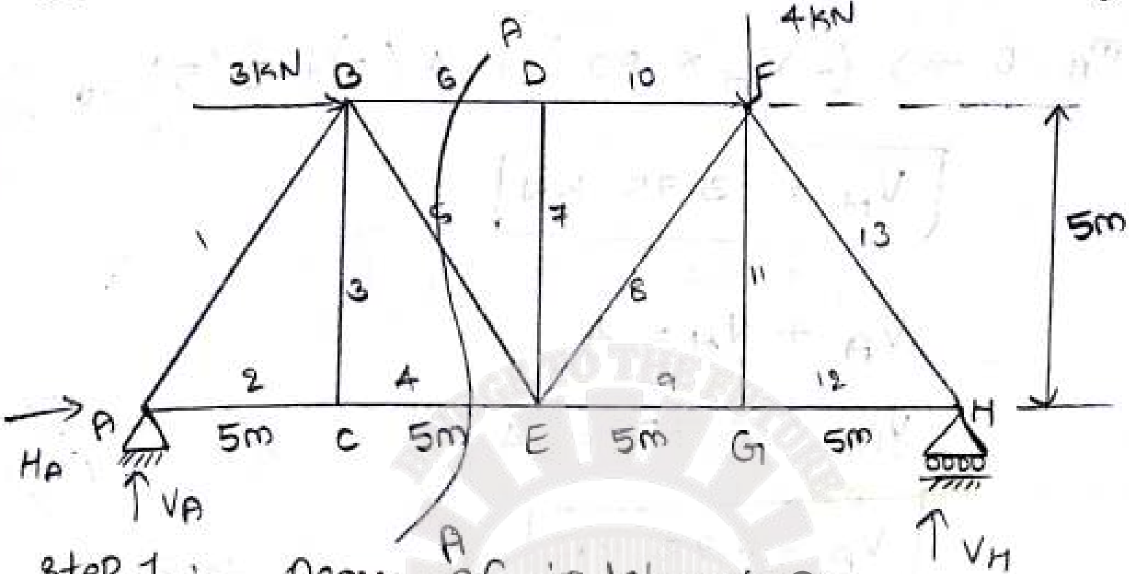
Method of Sections

Unlike the method of joints, the method of Section is based on the concept that each Section of the truss will be in equilibrium when cut into two different Sections. This method is Specifically helpful in problems where only two or ~~three~~ forces are to be find out in the entire truss.

First an optimum Section is taken to Solve the problem and then the equilibrium equations are used to Solve the problem. Here all three equations of equilibrium. $\Sigma H = 0$, $\Sigma V = 0$, $\Sigma M = 0$ are used and member forces are find out.

problem

① Find the forces in the member BD, BE, CE of the truss as shown in the figure.



Step 1 :- Degree of indeterminacy

① External

Reactions = 3

Equations = 3

3 − 3 = 0

② Internal

m = 13

j = 8

r = 3

$m - 2j + r$

$13 - 2(8) + 3$

$= 0$

Step ② :- Find the reactions

$\Sigma H = 0 \Rightarrow \boxed{H_A + 3 = 0}$

$\boxed{H_A = -3 \text{ KN}}$

$\Sigma V = 0 \Rightarrow V_A + V_H - 4 = 0$

$V_A + V_H = 4$ —— ①

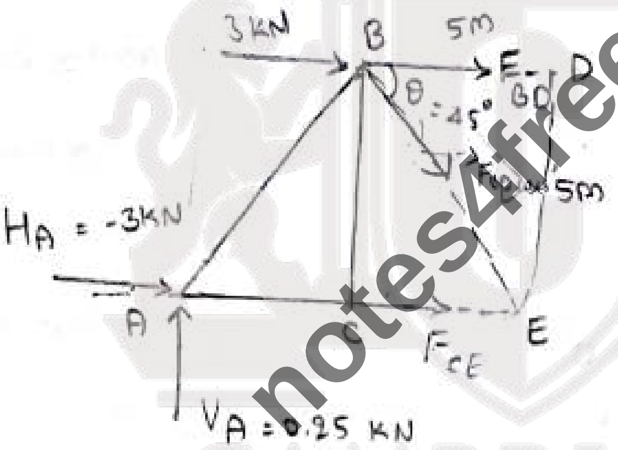
$\Sigma m_A = 0 \Rightarrow (-V_H \times 20) + 4(15) + 3(5) = 0$

$\boxed{V_H = 3.75 \text{ KN}}$

$V_A + V_H = 4$

$V_A + 3.75 = 4$

$\boxed{V_A = 0.25 \text{ KN}}$



$\tan\theta = \frac{5}{5}$

$\boxed{\theta = 45°}$

$\Sigma H = 0 \Rightarrow H_A + 3 + F_{CE} + F_{BD} + F_{BE}\cos 45° = 0$

$F_{CE} + F_{BD} + F_{BE}\cos 45° = 0$ —— ②

$\Sigma V = 0$

$\Rightarrow V_A - F_{BE}\sin 45 = 0$

$\Rightarrow 0.25 - F_{BE}\sin 45 = 0$

$\boxed{F_{BE} = 0.35 \text{ KN}}$

$\Sigma M_B = 0$

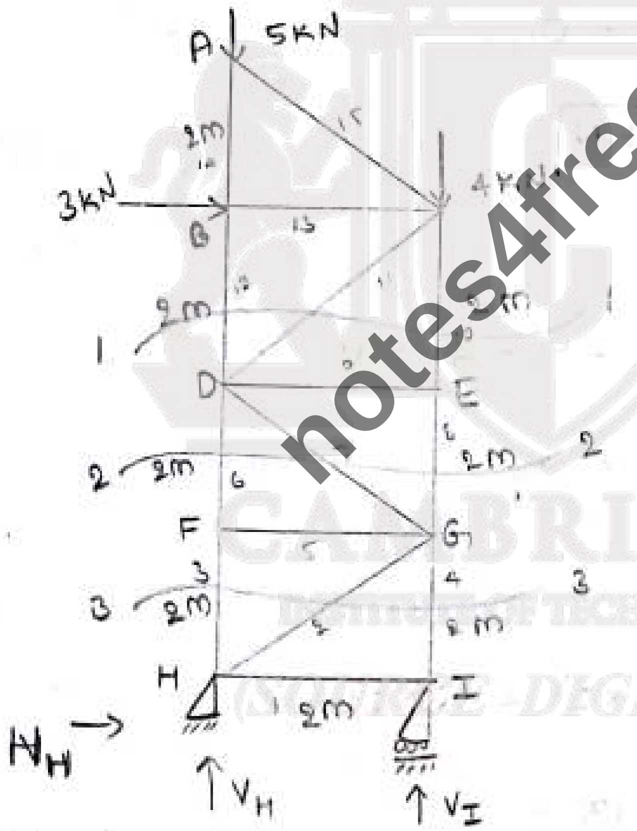
$\Rightarrow \; -(-3 \times 5) + (0.25 \times 5) - (F_{CE} \times 5) = 0$

$$\boxed{F_{CE} = 3.25 \text{ kN}}$$

sub in (2)

$$\Rightarrow \boxed{F_{BD} = -3.49 \text{ kN}}$$

(2) Find the forces in the members CE, DF and GI.



Step-1 Degree of indeterminacy

Reactions = 3

Equations = 3

3 - 3 = 0

Internal

$m = 15$ $\qquad m - 2j + r$

$j = 9$ $\qquad 15 - 2(9) + 3$

$r = 3$ $\qquad = 0$

Step-2

Find the reactions

$\Sigma H = 0$

$\Rightarrow H_H + 3 = 0$

$\boxed{H_H = -3 \text{ KN}}$

$\Sigma V = 0$

$\Rightarrow V_H + V_I - 5 - 4 = 0$

$\boxed{V_H + V_I = 9_{KN}}$

$\Sigma m_H = 0$

$(-V_I \times 2) + (4 \times 2) + (3 \times 6) = 0$

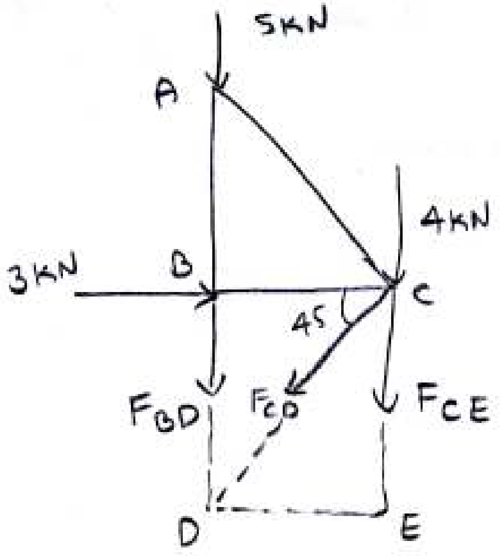
$-V_I = -13$

$\boxed{V_I = 13 \text{ KN}}$

$V_H + V_I = 9$

$V_H + 13 = 9$

$\boxed{V_H = -4 \text{ KN}}$



$\Sigma H = 0$

$\Rightarrow \quad 3 - F_{CD}\cos 45 = 0$

$F_{CD} = 4.24 \text{ KN}$

$\Sigma V = 0 \Rightarrow$

$-5 - 4 - F_{CD}\sin 45 - F_{BD} - F_{CE} = 0 \quad — (2)$

$-5 - 4 - 4.24\sin 45 - F_{BD} - F_{CE} = 0$

$-F_{BD} - F_{CE} - 11.99 = 0$

$F_{BD} + F_{CE} = -11.99 \text{ KN}$

$\Sigma M_c = 0$

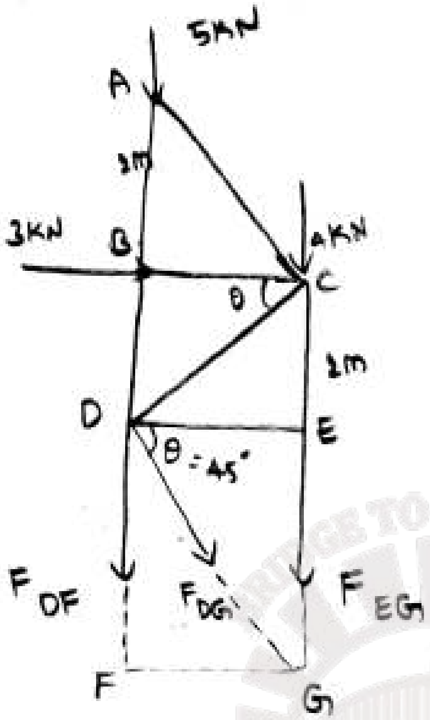
$-F_{BD} \times 2 - 5 \times 2 = 0$

$F_{BD} = -5 \text{ KN}$

$-5 + F_{CE} = -11.99$

$\boxed{F_{CE} = -6.99 \text{ KN}}$

Section - ②



$\Sigma H = 0$

$3 - F_{DG} \cos 45 = 0$

$\boxed{F_{DG} = 4.24 \text{ kN}}$

$\Sigma V = 0$

$-5 - 4 - F_{DG} \sin 45 - F_{DF} - F_{EG} = 0$

$-5 - 4 - 4.24 \sin 45 - F_{DF} - F_{EG}$

$- F_{DF} - F_{EG} = 6$

$F_{DF} + F_{EG} = -6$

$\Sigma M_D = 0$

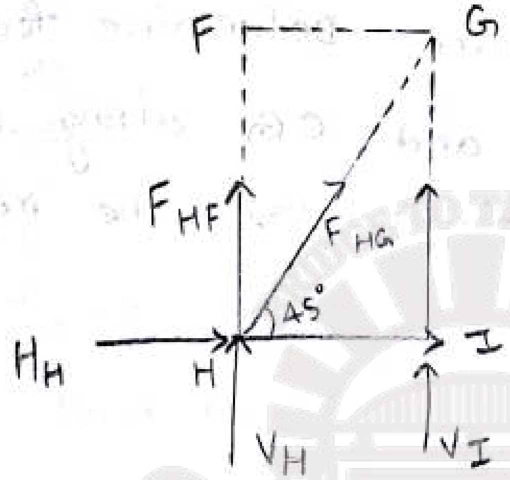
$3 \times 2 + 4 \times 2 + F_{EG} \times 2 = 0$

$\boxed{F_{EG} = -7 \text{ kN}}$

$F_{DF} - 7 = 6$

$\boxed{F_{DF} = 13\ KN}$

Section 3-3 :-



$V_I = 13\ KN$

$V_H = -4\ KN$

$H_H = -3\ KN$

$\Sigma H = 0$

$-3 + F_{HG} \cos 45° = 0$

$\boxed{F_{HG} = 4.24\ KN}$

$\Sigma V = 0$

$-4 + 13 + F_{HF} + F_{GI} + F_{HG} \sin 45° = 0$

$-4 + 13 + F_{HF} + F_{GI} + 4.24 \sin 45° = 0$

$F_{HF} + F_{GI} = -11.99$

$\Sigma M_H = 0$

$-13 \times 2 - F_{GI} \times 2 = 0$

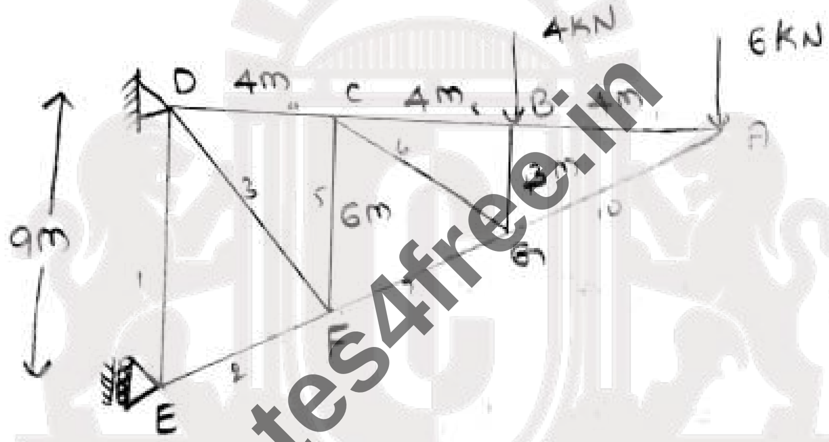
$F_{GI} \times 2 = -26$

$\boxed{F_{GI} = -13\ KN}$

$F_{HF} - 13 = -11.99$

$$\boxed{F_{HF} = 1.01 \text{ KN}}$$

③ A pin joint truss is loaded and supported as shown in the figure. Determine forces in members BC, GF, and CG along with the nature of the forces. Use the method of Sections.



<u>Step-1</u> Degree of indeterminacy

Reactions = 3

Equations = 3

3 - 3 = 0

<u>Internal</u>

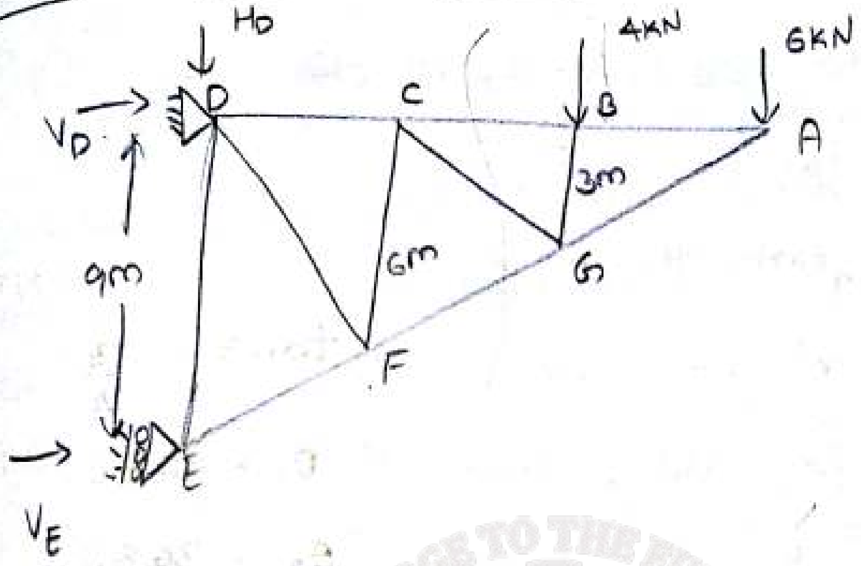
$m = 11$ $\quad m - (2j) + r$

$j = 7$ $\quad 11 - 2(7) + 3$

$r = 3$ $\quad = 0$

So it is internally determinate

Step-2 Find the reactions



$\Sigma_H = 0$

$V_D + V_E = 0$

$\Sigma_V = 0$

$-H_D - 4 - 6 = 0$

$-H_D - 10 = 0$

$\boxed{H_D = -10 \text{ KN}}$

$\Sigma_m = 0$

$4 \times 8 + 6 \times 12 - V_E \times 9 = 0$

$\boxed{V_E = 11.55 \text{ KN}}$

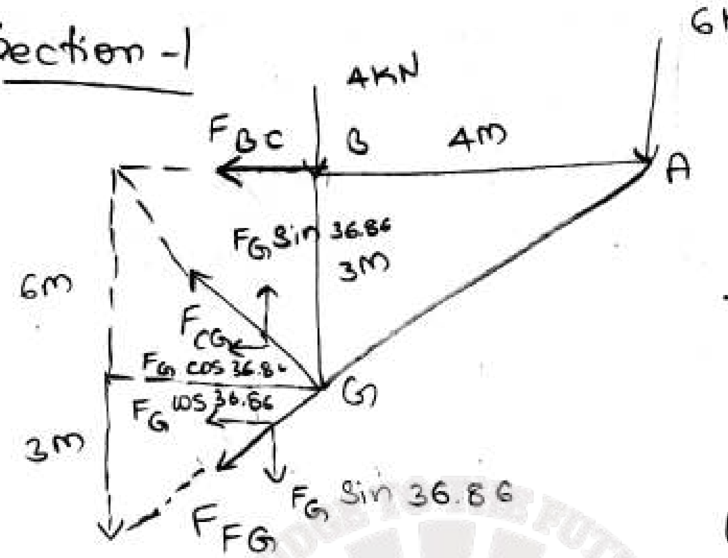
$V_D + V_E = 0$

$V_D + 11.55 = 0$

$\boxed{V_D = -11.55 \text{ KN}}$

Step - ③

Section - 1



$\tan\theta = \frac{3}{4}$

$\theta_1 = 36.86$

$\theta_2 = 36.86$

$\Sigma H = 0$

$-F_{CG} \cos 36.86 - F_{FG} \cos 36.86 - F_{BC} = 0$

$\Sigma V = 0$

$-6 - 4 + F_{CG} \sin 36.86 - F_{FG} \sin 36.86$

$F_{CG} \sin 36.86 - F_{FG} \sin 36.86 = 10 \rightarrow ①$

$\Sigma M_G = 0$

$(6 \times 4) - F_{BC} \times 3) = 0$

$\boxed{F_{BC} = 8 \text{ KN}}$

$-F_{CG} \cos 3686 - F_{FG} \cos 36.86 = 8 \rightarrow ②$

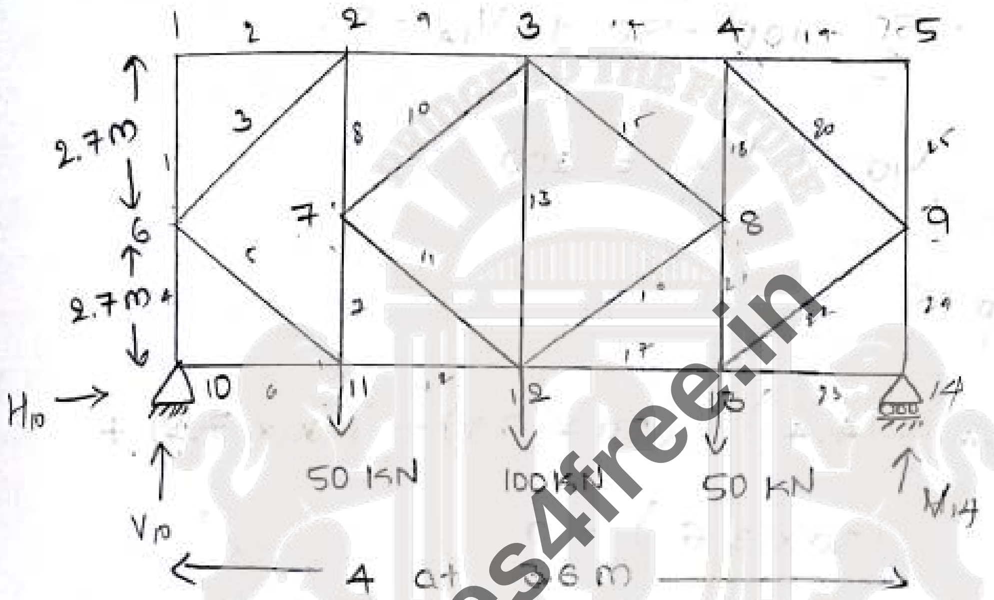
Solving ① and ②

$\boxed{F_{CG} = 3.33 \text{ KN}}$

$\boxed{F_{FG} = -13.33 \text{ KN}}$

∴ The forces in members BC, CG and GF are 8, 3.33, -13.33 KN respectively.

(4) Using method of sections determine the bar forces, indicated in the members of the truss as shown in the figure.



2-3
7-3
7-12
11-12
2-7
7-11

<u>Step-1</u> Degree of indeterminacy

Reactions = 3

Equations = 3

3 - 3 = 0

Internal

m = 25

j = 10

r = 3

$m - 2j + r$

$25 - (2 \times 14) + 3$

$= 0$

∴ So it is internally determinant.

Step-2 Find the reactions

$\Sigma_H = 0$

$\boxed{H_{10} = 0}$

$\Sigma_V = 0$

$V_{10} - 50 - 100 - 50 + V_{14} = 0$

$V_{10} + V_{14} = 200$

$\Sigma m_{10} = 0$

$(-V_{14} \times 14.4) + (50 \times 10.8) + (100 \times 7.2) + (50 \times 3.6) = 0$

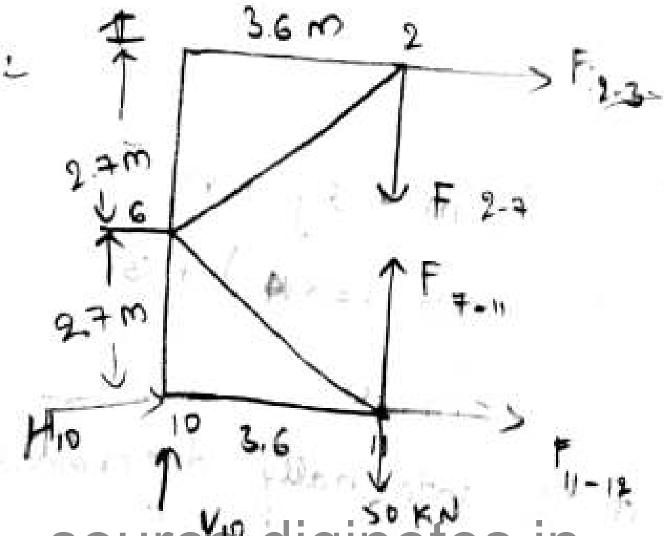
$\boxed{V_{14} = 100 \text{ KN}}$

$V_{10} + V_{14} = 200$

$V_{10} + 100 = 200$

$\boxed{V_{10} = 100 \text{ KN}}$

Step-3 :



$\Sigma H = 0$

$H_{10} + F_{11-12} + F_{2-3} = 0$

$F_{11-12} + F_{2-3} = 0 \longrightarrow (1)$

$\Sigma V = 0$

$V_{10} - 50 - F_{2-7} + F_{11-12} = 0$

$100 - 50 - F_{2-7} + F_{11-12} = 0$

$50 - F_{2-7} + F_{11-7} = 0$

$-F_{2-7} + F_{7-11} = -50 \longrightarrow (2)$

$\Sigma M_{11} = 0$

$F_{23} \times 5.4 + V_{10} \times 3.6 = 0$

$F_{23} \times 5.4 + 100 \times 3.6 = 0$

$\boxed{F_{23} = -66.66 \text{ KN}}$

$\Sigma M_{12} = 0$

$-F_{11-12} \times 5.4 - 0 \times 5.4 \times 100 \times 3.6 = 0$

$\boxed{F_{11-12} = 66.66 \text{ KN}}$

$\Sigma M_{10} = 0$

$+ 50 \times 3.6 - 66.66 \times 5.4 - F_{11-7} \times 3.6$

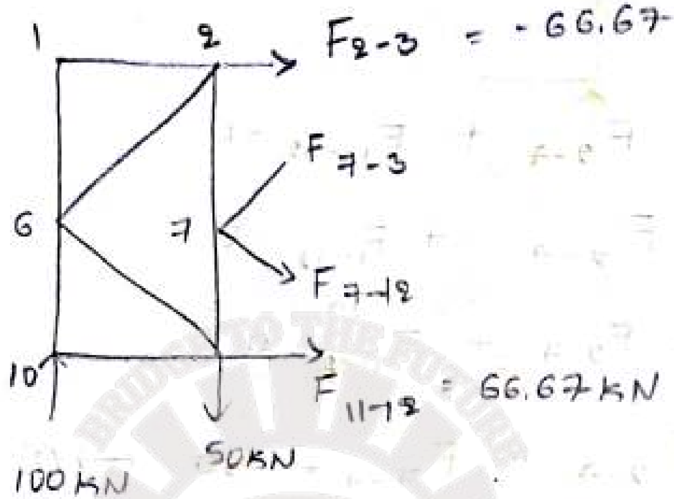
$+ F_{2-7} \times 3.6 = 0$

$180 - 359.964 - F_{11-7} \times 3.6 + F_{2-7} \times 3.6 = 0$

$F_{2-7} \times 3.6 - F_{7-11} \times 3.6 = 179.964 \longrightarrow (3)$

$$-F_{2-7} + F_{7-11} = -50.$$

Section B-B



$\Sigma H = 0$

$$\Rightarrow F_{7-3} \cos 36.86 + F_{7-12} \cos 36.86 + 66.67 - 66.67 = 0$$

$$\Rightarrow F_{7-3} + F_{7-12} = 0 \Rightarrow F_{7-3} - F_{7-12} \rightarrow (3)$$

$\Sigma V = 0$

$$\Rightarrow 100 - 50 + F_{7-3} \sin 36.86 - F_{7-12} \sin 36.86 = 0$$

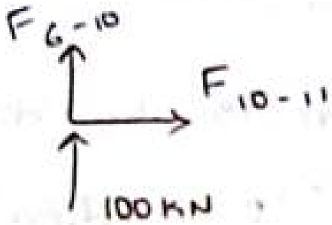
$$F_{7-3} \sin 36.86 - F_{7-12} \sin 36.86 = -50 \text{ -(4)}$$

$$\boxed{F_{7-3} = -41.67 \text{ kN}}$$

$$\boxed{F_{7-12} = 41.67 \text{ kN}}$$

Considering Joint (7)

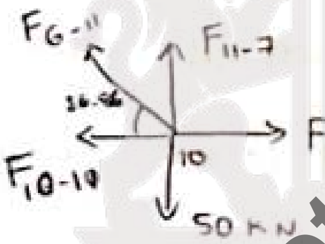
Considering Joint (10)



$\Sigma H = 0 \Rightarrow F_{10-11} = 0$

$\Sigma V = 0 \Rightarrow \boxed{F_{6\cdot10} = -100}$

Joint - (11)



$\Sigma H = 0$

$\Rightarrow -F_{6-11} \cos 36.86 + F_{11-12} = 0$

$\Rightarrow F_{6-11} = \frac{+66.67}{\cos 36.86}$

$F_{6-11} = +83.34$ KN

$\Sigma V = 0$

$\Rightarrow -50 + F_{11-7} + F_{6-11} \sin 36.86 = 0$

$F_{11-7} = 99.99 \approx 100$ KN

$\boxed{F_{8-7} = +50 \text{ KN}}$

23/02/2017

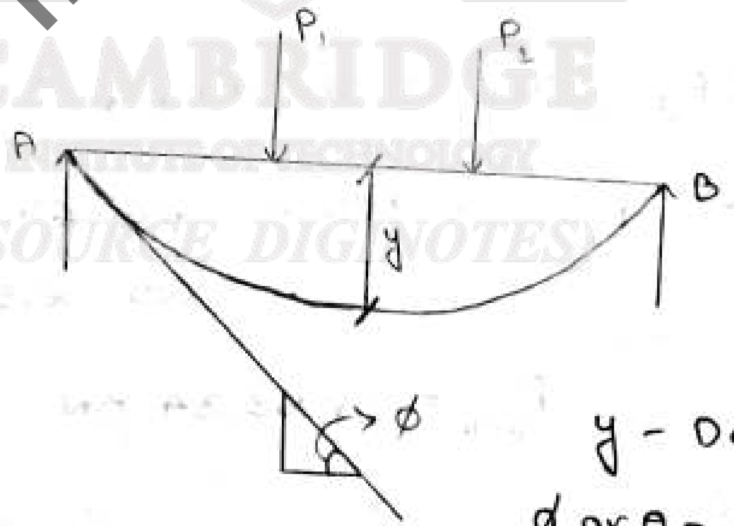
module - 2

DEFLECTION OF BEAMS

Deflection :- Deflection is the distance through which a point on a longitudinal axis displaces in a direction transverse to the longitudinal axis.

Deflection curve :- A beam subjected to transverse loads undergoes deflection. The deflected shape of longitudinal axis of the beam is called as Deflection curve.

Deflection curve is also known as elastic curve.



y - Deflection

ϕ or θ - Slope

Slope :-

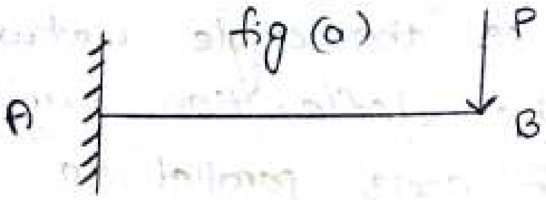
As shown in the figure above the simple supported beam is considered with two point loading and the deflection curve is plotted.

The slope 'ϕ' or 'θ' of the deflected beam at a point is the angle between the tangent to the deflection curve at that point and an axis parallel to the longitudinal axis

Curvature :- When a loads are applied to a beam its longitudinal axis is deformed in to a curve. The resulting strains and stresses in the beam are directly related to the curvature of the deflection curve. Simply put curvature is a measure of how sharply a beam is bend. If the load on a beam is small the beam will be nearly straight, the radius of curvature will be very large and the curvature will be very small. If the load is increased the amount of bending will increase that is the radius of curvature will becomes smaller and the curvature will become larger.

To further understand the concept of curvature, consider a cantilever beam subjected to a load P acting at the free end of the beam as shown in the figure.

fig (a)



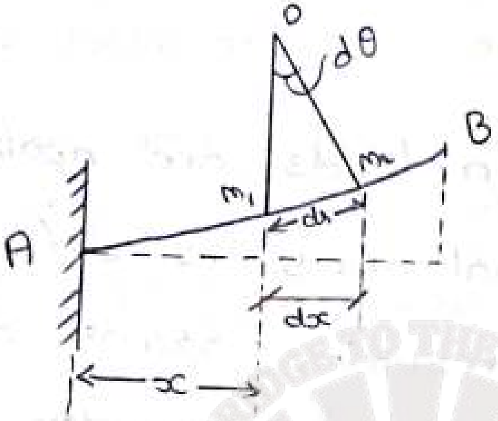


fig (b)

The deflection curve of this beam is shown in fig (b). we identify two points m_1 and m_2 on the deflection curve. point m_1 is selected at an obtirary distance 'x' from the 'y' axis and point m_2 is located a distance 'ds' further along the curve. At each of these points we draw a line normal to the tangent to the deflection curve that is normal to the curve itself. These normals intersect at a point 'O' which is known as the centre of curvature of the deflection curve. The distance m_1O from the curve to the centre of curvature is called as radius of curvature [R] and the curvature 'K' is defined as the reciprocal of the radius of curvature. Thus $\boxed{K \propto \frac{1}{R}}$

From the geometry of the triangle om_1, om_2 we obtain $R\,d\theta = ds$ in which $d\theta$ is the infinitesimal angle between the normals and 'ds' is the infinitesimal distance along the curve between the points m_1 and m_2

$$\Rightarrow \quad R\,d\theta = ds$$

$$\Rightarrow \quad R = \frac{ds}{d\theta}$$

$$K = \frac{1}{R}$$

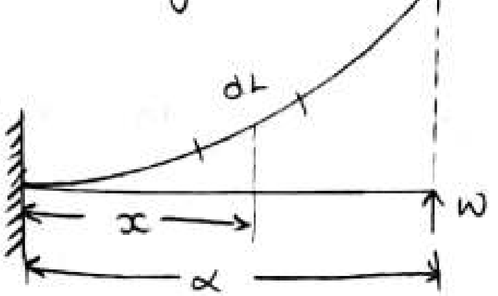
$$\boxed{K = \frac{d\theta}{ds}}$$

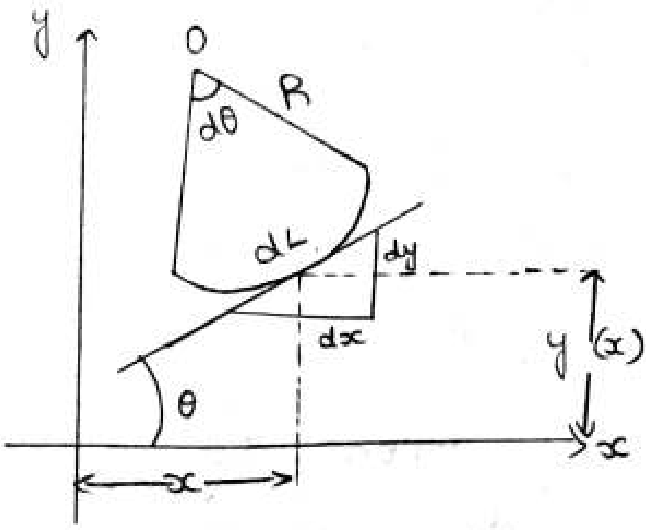
27/02/2017

MOMENT - CURVATURE RELATIONSHIP

$dL = ld$

fg (a)





fg (b)

dL

The general equation for the deflection of beam is based on the fact that the moment varies as the function of position along the length of the beam. Consider a cantilever beam as shown in the figure whose neutral axis coincides with 'x' axis and is subjetected to a point load at it's free end. With the application of load the neutral axis is bent into a curve known as the deflection curve and the deflection at any point along the length of the cantilever can be written as $y(x)$. Consider a small portion 'dL' of the deflected curve at a distance x from the fixed end as shown in the figure. The radius of curvature of the deflection curve is 'R' and the curve makes an angle '$d\theta$' at the centre of the curvature. Draw a tangent at 'x' to get the slope of the curve.

From fig (b) we know that

$$dL = R\,d\theta$$

where dL = Length of the arc

$$\frac{1}{R} = \frac{d\theta}{dL} = K \quad \text{——— } ①$$

Length of the arc can also be given by

$$dL = \sqrt{dx^2 + dy^2}$$

$$dL = dx\sqrt{1+\left(\frac{dy}{dx}\right)^2} \longrightarrow ②$$

we know that the slope of the curve at

$$x = \tan\theta = \frac{dy}{dx}$$

Differentiating the slope equation w.r.t 'x' we get

$$\sec^2\theta \cdot \frac{d\theta}{dx} = \frac{d^2y}{dx^2}$$

$$d\theta = \frac{d^2y}{dx^2} \cdot \frac{1}{\sec^2\theta} \cdot dx$$

But we know that $\sec^2\theta = 1 + \tan^2\theta$,

Substituting the same

$$d\theta = \frac{d^2y}{dx^2} \cdot \frac{1}{1+\tan^2\theta} dx$$

$$\Rightarrow d\theta = \frac{d^2y}{dx^2} \cdot \frac{1}{1+\left(\frac{dy}{dx}\right)^2} \cdot dx$$

From eqn ① $d\theta = \frac{dL}{R}$

$$\therefore \frac{dL}{R} = \frac{d^2y}{dx^2} \cdot \frac{1}{\left[1+\left(\frac{dy}{dx}\right)^2\right]} dx$$

From eq^n ②

w.k.T

$$dL = dx\sqrt{1+\left(\frac{dy}{dx}\right)^2}$$

$$\therefore \quad \frac{1}{R} = \frac{dy/dx^2}{\left[1+\left(dy/dx\right)^2\right]^{3/2}}$$

But here for the length of the arc 'dL'

$\theta = \frac{dy}{dx}$ is small and $\left(\frac{dy}{dx}\right)^2$ is very small.

Hence we get, $\frac{1}{R} = \frac{d^2y}{dx^2}$

we know that, From bending equation

$$\frac{M}{I} = \frac{f}{y} = \frac{E}{R} \quad \text{(or)}$$

$$\frac{1}{R} = \frac{M}{EI}$$

$$\therefore \quad \boxed{\frac{1}{R} = \frac{M}{EI} = \frac{d^2y}{dx^2}} \quad \text{or}$$

$$\boxed{M = \frac{d^2y}{dx^2} \times EI}$$

This is known as the moment - curvature relationship.

NOTE:-

* The moment - curvature relationship is the governing differential equation of the second order for the deflected curve.

* The equation also indicates that the curvature $\frac{1}{R}$ of the deflected curve is the rate of change of slope.

* The curvature of the deflected curve at a point is directly proportional to the corresponding moment and is inderectly proportional to the flexural rigidity 'EI' over unit length of the beam.

IMPORTANT RELATIONS TO SOLVE PROBLEMS

The relations involving moment, shear force and load intensity may be obtained from the moment curvature relationship as shown below.

Slope $= \theta = \frac{dy}{dx}$

moment $= m = EI \frac{d^2y}{dx^2}$

Shear force = $SF = \frac{dm}{dx}$

$$\boxed{SF = EI \frac{d^3y}{dx^3}}$$

Load Intensity = $w = \frac{d(SF)}{dx}$

$$\boxed{w = EI. \frac{d^4y}{dx^4}}$$

Assumptions :-

* The following assumptions have been made while deriving the equation for moment - curvature relationship.

① Deflections and slopes are small.

② The beam is very long when compared to its cross sectional dimensions.

③ Deflection due to shear force is negligable when compared to deflection due to moment.

④ The values of young's modulus (E) and moment of Inertia (I) remain constant for a given interval along the length of a beam.

Double integration method :-

we can find slope and deflection at a given location on the length of the beam by integrating corresponding moment curvature relation successively.

i.e moment curvature relation is

$$EI \frac{d^2y}{dx^2} = M$$

when integrated once w.r to 'x', we get the slope equation.

By integrating the slope equation w.r to 'x', we get the deflection equation. Hence this method is also known as successive integration method.

Boundary conditions :-

The values of both slope and deflection or any one of them at the supports will be zero based on the nature of support provided.

The following particulars are important for obtaining boundary conditions

① A beam does not undergo any deflection at its supports irrespective of the type of support provided. i.e $y=0$ at supports.

② In case of hinged or simply supported beam, the beam will be free to rotate at its support.

A tangent to a deflected curve at the hinged support has an inclination of '0'. w.r to x-axis of the beam as shown in the figure.

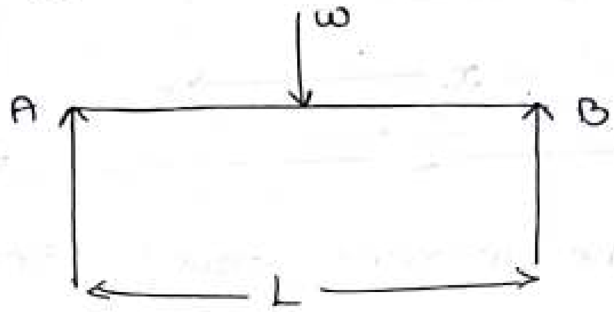
Therefore slope has a finite value at the supports of a hinged beam or a simply supported beam.

③ A beam is not free to rotate at its fixed support i.e slope $\frac{dy}{dx}$ will be equal to 'zero' at fixed support.

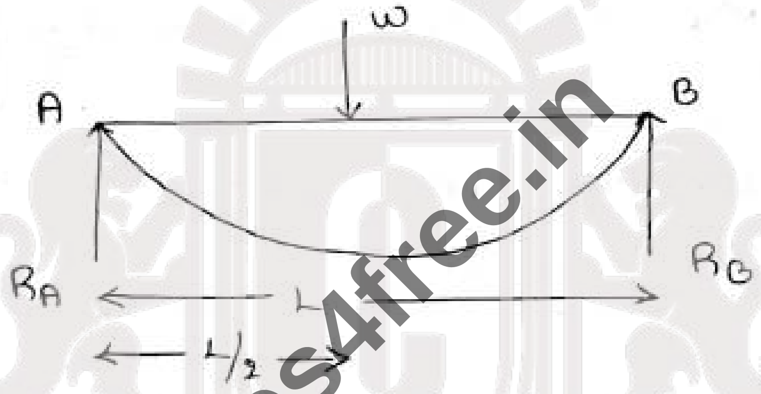
④ The deflection of a beam will be maximum at a point on its longitudinal axis where the slope changes its sign.

Deflection and slopes of different beams:

① Simply supported beam with point load at its centre.



Step 1 :- Find the reactions



$\Sigma H = 0$

$\Sigma V = 0$

$R_A + R_B = W$

$\Sigma M_A = 0$

$W \times L/2 - R_B \times L = 0$

$R_B \cancel{L} = W \times \frac{\cancel{L}}{2}$

$$\boxed{R_B = \frac{W}{2}}$$

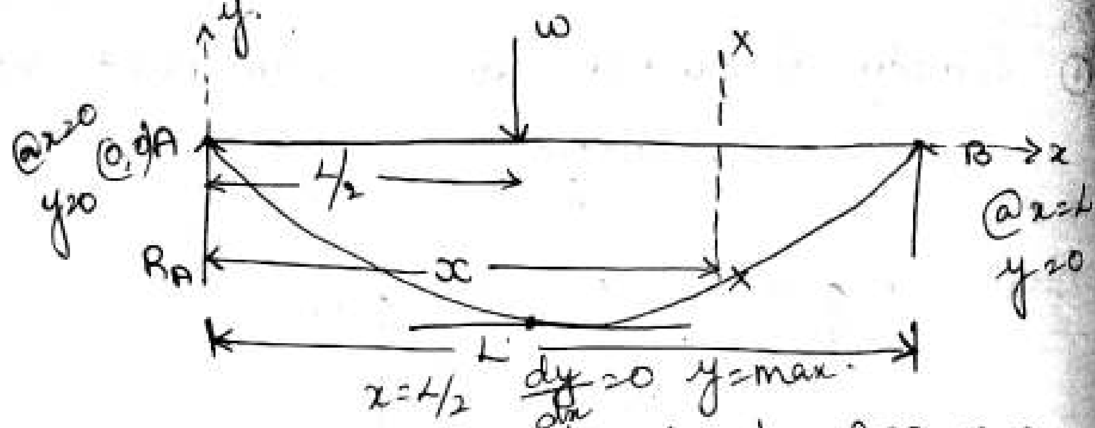
$R_A + R_B = W$

$R_A + \frac{W}{2} = W$

$R_A = W - \frac{W}{2}$

$$\boxed{R_A = \frac{W}{2}}$$

Step - 2 :- Consider a Section $x-x$



Step - 3 :- Find the moment about sec $x-x$

$$m_x = R_A \times x - w \times \left(x - \frac{L}{2}\right)$$

$$\boxed{m_x = \frac{w}{2}x - wx + \frac{wL}{2}}$$

Step 4 :- Consider the moment curvature equation

i.e

$$EI \cdot \frac{d^2y}{dx^2} = M_x = \frac{wx}{2} - wx + \frac{wL}{2} \longrightarrow (1)$$

Step 5 :- Integrate eqⁿ (1) w.r to 'x'

$$\int EI \frac{d^2y}{dx^2} dx = \int \frac{wx}{2} dx - \int wx\,dx + \int \frac{wL}{2}$$

$$EI\left(\frac{dy}{dx}\right) = \frac{wx^2}{4} - \frac{wx^2}{2} + \frac{wLx}{2} + C_1 \longrightarrow (2)$$

Equation (2) is the slope equation

Step 6 :- Integrate eq^n (2) wr to 'x'

$\theta = \frac{dy}{dx}$

$$\int EI\left(\frac{dy}{dx}\right)dx = \int \frac{wx^2}{4}dx - \int \frac{wx^2}{2}dx + \int \frac{wLx}{2}dx + \int c_1\,dx$$

$$EI.y = \frac{wx^3}{12} - \frac{wx^3}{6} + \frac{wLx^2}{4} + c_1x + c_2 \rightarrow (3)$$

<u>Step 7</u> :- Apply boundary conditions to find c_1 and c_2

(i) at $x = \frac{L}{2}$, $\frac{dy}{dx} = 0$

Substituting $\frac{dy}{dx} = 0$ in eq^n (2)

$$0 = \frac{w\left(\frac{L}{2}\right)^2}{4} - \frac{w\left(\frac{L}{2}\right)^2}{2} + \frac{wL\left(\frac{L}{2}\right)}{2} + c_1$$

$$c_1 = -\frac{wL^2}{16} + \frac{wL^2}{8} - \frac{wL^2}{4}$$

$$\boxed{c_1 = -\frac{3wL^2}{16}}$$

(ii) at $x = 0$, $y = 0$

Substituting $y = 0$ in eq^n (3)

$$0 = 0 - 0 + 0 + 0 + c_2$$

$$\therefore \boxed{c_2 = 0}$$

Step 8 :- Substitute the values of C_1 and C_2 in equation (2) & (3)

eqn (2) $\Rightarrow$

$$EI \cdot \frac{dy}{dx} = \frac{wx^2}{4} - \frac{wx^2}{2} + \frac{wLx}{2} - \frac{3wL^2}{16} \rightarrow (4)$$

Similarly,

eqn (3) $\Rightarrow$

$$EIy = \frac{wx^3}{12} - \frac{wx^3}{6} + \frac{wLx^2}{4} - \frac{3wL^2x}{16} \quad (5)$$

Step 9 :- To find maximum slope and deflection

(i) maximum slope is at $x = 0$

Substituting $x=0$ in eqn (4)

$$EI \frac{dy}{dx} = -\frac{3wL^2}{16}$$

(or)

$$\boxed{\left(\frac{dy}{dx}\right)_{(A \text{ or } B)} = \frac{-3wL^2}{16EI}}$$

(ii) maximum deflection is at $x = \frac{L}{2}$

Substituting $x = \frac{L}{2}$ in eqn (5)

$$EIy = \frac{w\left(\frac{L}{2}\right)^3}{12} - \frac{w\left(\frac{L}{2}\right)^3}{6} + \frac{wL\left(\frac{L}{2}\right)^2}{4} - \frac{3wL^2\left(\frac{L}{2}\right)}{16}$$

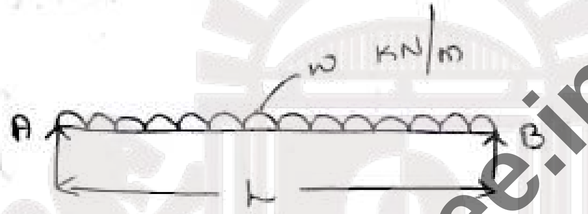
$$= \frac{wL^3}{96} - \frac{wL^3}{48} + \frac{wL^3}{16} - \frac{3wL^3}{32}$$

$$= wL^3 \left[\frac{1}{96} - \frac{1}{48} + \frac{1}{16} - \frac{3}{32}\right]$$

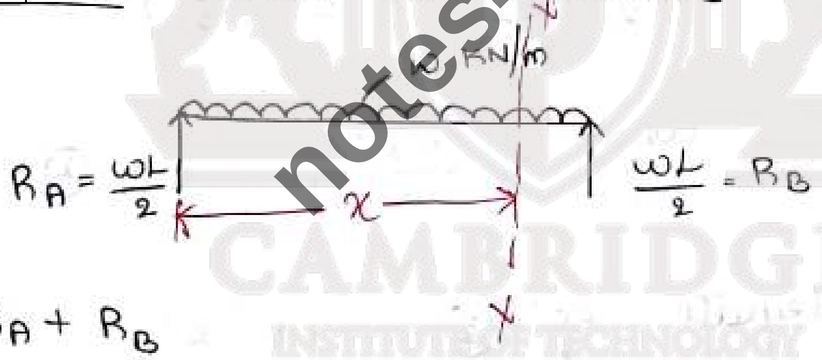
$$EIy = -\frac{wL^3}{24}$$

$$\boxed{y = -\frac{wL^3}{24EI}}$$

② Simply Supported beam with UDL :-



Step ① :- Find the Reactions



$R_A + R_B$

Step 3 :- Find moment about section x - x

$$M_x = \frac{wL}{2} \times x - w \times x \times \frac{x}{2}$$

$$M_x = \frac{wLx}{2} - \frac{wx^2}{2}$$

Step 4 :- Consider the moment curvature equation

$$EI \cdot \frac{d^2y}{dx^2} = M_x = \frac{wLx}{2} - \frac{wx^2}{2} \longrightarrow (1)$$

Step 5 :- Integrate equation (1) w.r.t x

$$\int EI \frac{d^2y}{dx^2} \cdot dx = \int \frac{wLx}{2} \cdot dx - \int \frac{wx^2}{2} dx$$

$$EI \frac{dy}{dx} = \frac{wL}{2}\left(\frac{x^2}{2}\right) - \frac{w}{2}\left(\frac{x^3}{3}\right) + C_1$$

$$EI \frac{dy}{dx} = \frac{wLx^2}{4} - \frac{wx^3}{6} + C_1 \quad \text{---} (2)$$

Step 6 :- Integrate eq^n (2) w.r.t 'x'

$$\int EI \frac{dy}{dx} \cdot dx = \int \frac{wLx^2}{4} dx - \int \frac{wx^3}{6} dx + \int C_1 dx$$

$$\Rightarrow EIy = \frac{wL}{4}\left(\frac{x^3}{3}\right) - \frac{w}{6}\left(\frac{x^4}{4}\right) + C_1 x + C_2$$

$$\Rightarrow EIy = \frac{wLx^3}{12} - \frac{wx^4}{24} + C_1 x + C_2 \quad \text{---} (3)$$

Step 7 :- Apply boundary conditions to find C_1 and C_2

@ $x = \frac{L}{2}$, $\frac{dy}{dx} = 0$

Substituting $\frac{dy}{dx} = 0$ in eq^n (2)

$$0 = \frac{wL}{4}\left(\frac{L}{2}\right)^2 - \frac{w}{6}\left(\frac{L}{2}\right)^3 + C_1$$

$$0 = \frac{wL^3}{16} - \frac{wL^3}{48} + C_1$$

$$0 = \frac{3wL^3 - wL^3}{48} + C_1$$

$$0 = \frac{+2wL^3}{48} + C_1$$

$$\boxed{C_1 = -\frac{wL^3}{24}}$$

(ii) at $x=0$, $y=0$

$$\Rightarrow \quad 0 = 0 - 0 + 0 + C_2$$

$$\boxed{C_2 = 0}$$

Step 8 :- Substitute the values of C_1 and C_2 in eq^n (2) & (3)

eq^n (2) $\Rightarrow$

$$EI\frac{dy}{dx} = \frac{wLx^2}{4} - \frac{wx^3}{6} - \frac{wL^3}{24} \longrightarrow (4)$$

Similarly
eq^n (3) $\Rightarrow$

$$EIy = \frac{wLx^3}{12} - \frac{wx^4}{24} - \frac{wL^3x}{24} \longrightarrow (5)$$

Step 9 :- To find maximum slope and deflection

(i) maximum slope is at $x=0$

Substitute $x=0$ at eq^n (4)

$$EI\frac{dy}{dx} = \frac{-wL^3}{24}$$

$$\boxed{\frac{dy}{dx} = \frac{-wL^3}{24EI}}$$

(ii) maximum deflection is at $x = \frac{L}{2}$

Substituting $x = \frac{L}{2}$ in eq^n (5)

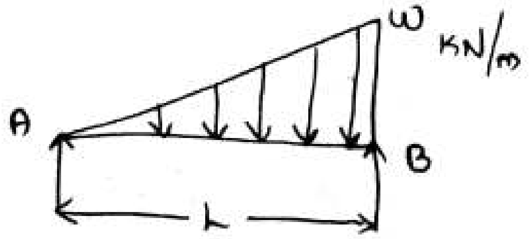
$$EIy = \frac{wL}{12}\left(\frac{L}{2}\right)^3 - \frac{w}{24}\left(\frac{L}{2}\right)^4 - \frac{wL^3}{24}\left(\frac{L}{2}\right)$$

$$= \frac{wL^4}{96} - \frac{wL^4}{384} - \frac{wL^4}{48}$$

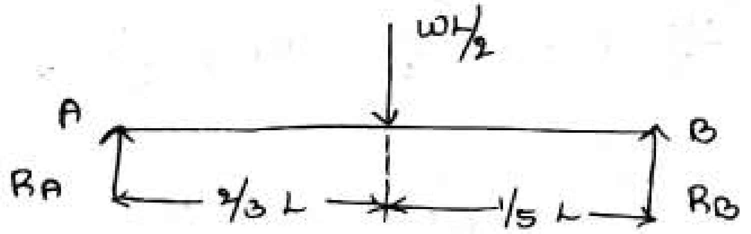
$$= \frac{-5}{384}wL^4$$

$$\boxed{y = \frac{-5wL^4}{384EI}}$$

Simply Supported beam with UVL



Step -1 : To find the reactions



$$\frac{1}{2} \times L \times w \Rightarrow \frac{wL}{2}$$

$\Sigma H = 0$

$$\Sigma V = 0 \Rightarrow R_A + R_B - \frac{wL}{2} = 0$$

$$R_A + R_B = \frac{wL}{2} \longrightarrow (1)$$

$\Sigma M_A = 0$

$$\frac{wL}{2} \times \frac{2}{3}L - R_B \times L = 0$$

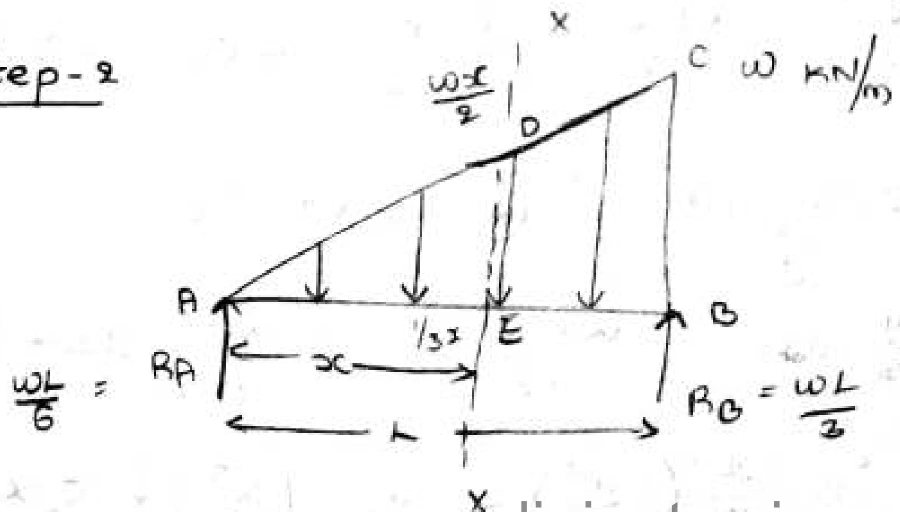
$$\Rightarrow \frac{wL^2}{3} - R_B L = 0$$

$$\boxed{\frac{wL}{3} = R_B}$$

$$R_A = \frac{wL}{2} - \frac{wL}{3}$$

$$\boxed{R_A = \frac{wL}{6}}$$

Step-2



Step-3:- Find the moment about Sec x-x

$$M_x = \frac{wL}{6} \times x - \left(\frac{1}{2} \times \frac{wx}{L} \times x\right) \times \frac{1}{3}x$$

$$= \frac{wLx}{6} - \frac{wx^3}{6L} \quad \text{——— (2)}$$

$|||^{lar}$ Δ^{le}

$$\frac{BC}{AB} = \frac{DE}{AE}$$

$$\Rightarrow \frac{w}{L} = \frac{DE}{x}$$

$$\Rightarrow \boxed{DE = \frac{wx}{L}}$$

Step 4:- Consider the moment curvature relation

$$EI\frac{d^2y}{dx^2} = M_x = \frac{wLx}{6} - \frac{wx^3}{6L} \longrightarrow (3)$$

Step 5:- Integrate eq^n (3) w.r.t 'x'

$$\int EI \frac{d^2y}{dx^2}\, dx = \int \frac{wLx}{6}\, dx - \int \frac{wx^3}{6L}\, dx$$

Slope eq^n

$$EI\left(\frac{dy}{dx}\right) = \frac{wLx^2}{12} - \frac{wx^4}{24L} + C_1 \longrightarrow (4)$$

Step 6:- Integrate eq^n (4) w.r.t 'x'

$$\int EI \frac{dy}{dx}\, dx = \int \frac{wLx^2}{12} dx - \int \frac{wx^4}{24L} dx + \int C_1 dx$$

$$\Rightarrow EIy = \frac{wL}{12}\left(\frac{x^3}{3}\right) - \frac{w}{24L}\left(\frac{x^5}{5}\right) + C_1 x + C_2$$

$$\Rightarrow EIy = \frac{wLx^3}{36} - \frac{wx^5}{120L} + c_1x + c_2 \rightarrow (5)$$

Step-7 :- Apply boundary conditions to find C_1 and C_2

(i) at $x=0$, $y=0$

$$0 = 0 - 0 + 0 + C_2$$

$$\boxed{C_2 = 0}$$

(ii) at $x=L$, $y=0$

$$EIy = \frac{wLx^3}{36} - \frac{wx^5}{120L} + C_1x + C_2$$

$$0 = \frac{wL(L^3)}{36} - \frac{w(L)^5}{120L} + C_1L$$

$$-C_1L = wL^4\left[\frac{1}{36} - \frac{1}{120}\right]$$

$$C_1 = wL^3\left[-\frac{1}{36} + \frac{1}{120}\right]$$

$$\boxed{C_1 = \frac{-7wL^3}{360}}$$

Step-8 :- Substitute the values of C_1 and C_2 in eq^n (4) and (5)

eq^n (4) $\Rightarrow$

$$EI\frac{dy}{dx} = \frac{wLx^2}{12} - \frac{wx^4}{24L} + \left(-\frac{7wL^3}{360}\right) \rightarrow (6)$$

eq^n (5) $\Rightarrow$

$$EIy = \frac{wLx^3}{36} - \frac{wx^5}{120L} + \left(-\frac{7wL^3}{360}\right) \rightarrow (7)$$

<u>step (9)</u> :- To find maximum slope and deflecti

(i)

$$\frac{dy}{dx} = 0 \quad , y = max$$

$$0 = \frac{wLx^2}{12} - \frac{wx^4}{24L} - \frac{7wL^3}{360}$$

$$0 = \frac{Lx^2}{12} - \frac{x^4}{24L} - \frac{7L^3}{360}$$

$$\Rightarrow \frac{Lx^2}{12} = \frac{x^4}{24L} + \frac{7L^3}{360}$$

$$\frac{Lx^2}{12} = \frac{15x^4 + 7L^4}{360L}$$

$$\Rightarrow \frac{360L^2x^2}{12} = 15x^4 + 7L^4$$

$$30x^2L^2 = 15x^4 + 7L^4 \quad [\div 15]$$

$$\Rightarrow 15x^4 + 7L^4 - 30x^2L^2 = 0$$

$$ax^2 \quad + \quad c \quad + bx = 0$$

$$\Rightarrow x^4 + \frac{7}{15}L^4 - (2)x^2(L^2) = 0 \qquad ax^2 + bx + c = 0$$

$$b \quad b$$

$$x^2 = + \frac{2L^2 \pm \sqrt{4L^4 - 4(1)\left(\frac{7}{15}\right)L^4}}{2(1)}$$

$$= \frac{2L^2 \pm \sqrt{4L^4 - 186L^4}}{2}$$

$$= \frac{2L^2 \pm 1.46L^2}{2}$$

$$x^2 = 1.73 \text{ or } 0.27 L^2$$

$$\boxed{x = 0.52L}$$

$$EIy = \frac{wLx^3}{36} - \frac{wx^5}{120L} + C_1 x + C_2$$

$$= \frac{wL(0.52L)^3}{36} - \frac{w(0.52L)^5}{120L} - \frac{7wL^3}{360}(0.52L)$$

$$EIy = \frac{wL}{36}(0.52L)^3 - \frac{w}{120L}(0.52L)^5 - \frac{7wL^3}{360}(0.52L)$$

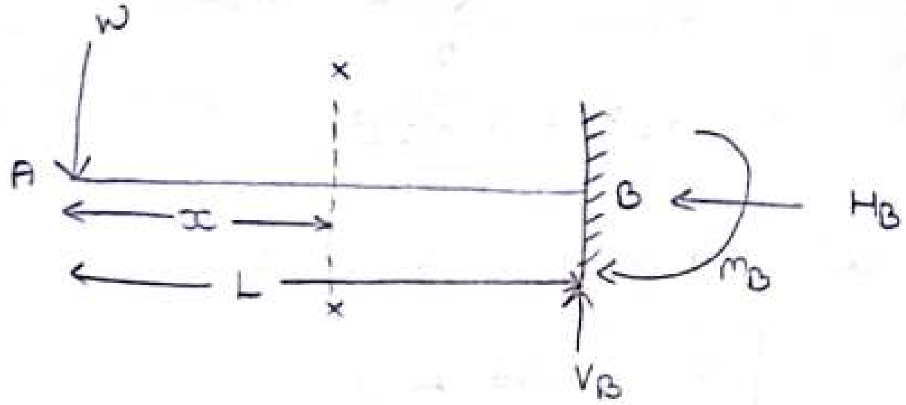
$$EIy = \frac{0.1406\,wL^4}{36} - \frac{\overset{0.0380}{\cancel{0.1406}}\,wL^{\cancel{5}\,4}}{120\,\cancel{L}} - \frac{3.64\,wL^4}{360}$$

$$EIy = wL^4\left[\frac{0.1406}{36} - \frac{\overset{0.0380}{\cancel{0.1406}}}{120} - \frac{3.64}{360}\right]$$

$$= wL^4\left[0.00390 - 0.00316 - 0.010\right.$$

$$EIy = -0.00652\,wL^4$$

$$\boxed{y = \frac{-0.00652\,wL^4}{EI}}$$



(1) Find the reactions

$\Sigma H = 0$

$H_B = 0$

$\Sigma V = 0$

$V_B - W = 0$

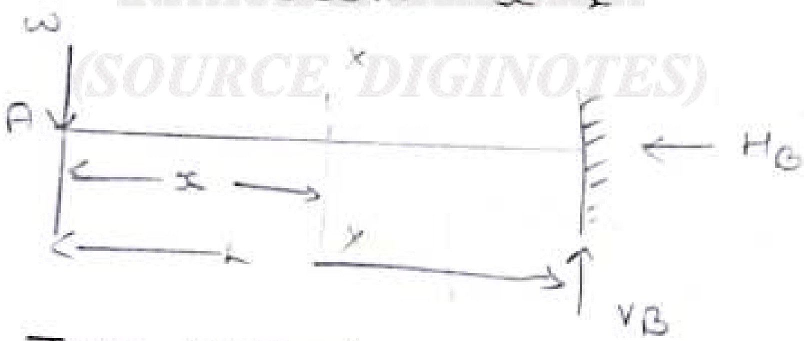
$\boxed{V_B = 0}$

$\Sigma M_B = 0$

$M_B - (W \times L) = 0$

$\Rightarrow \boxed{M_B = WL}$

(2) Consider Section $x - x$



(3) Take moment at section $x - x$

$$M_x = -Wx$$

(4) Consider the moment curvature relation

$$EI \frac{d^2y}{dx^2} = M_x = -Wx \quad \text{——} \quad (1)$$

step (5):- Integrate ① w.r.t 'x'

$$EI \frac{dy}{dx} = -\frac{wx^2}{2} + C_1 \longrightarrow ②$$

step (6):-

$$EIy = -\frac{wx^3}{6} + C_1x + C_2 \quad — ③$$

step (7):- Apply boundary conditions to find C_1 and C_2

(i)

at $x = L, \frac{dy}{dx} = 0$

$$② \Rightarrow 0 = -\frac{wL^2}{2} + C_1$$

$$\Rightarrow \boxed{C_1 = \frac{wL^2}{2}}$$

(ii) at $x = L \; y = 0$

$$eq^n ③ \Rightarrow 0 = -\frac{wL^3}{6} + \frac{wL^2}{2}L + C_2$$

$$\Rightarrow \frac{-wL^3}{6} + \frac{wL^3}{2} + C_2$$

$$\boxed{C_2 = \frac{-wL^3}{3}}$$

step (8):- Substitute the values of C_1 and C_2 in eq^n ② and ③

$$eq^n ② \Rightarrow EI\left(\frac{dy}{dx}\right) = -\frac{wx^2}{2} + \frac{wL^2}{2} \rightarrow ④$$

$$eq^n ③ \Rightarrow EIy = -\frac{wx^3}{6} + \frac{wL^2}{2}x - \frac{wL^3}{3} \rightarrow ⑤$$

a) maximum deflection and slope

(i) at $x = 0$

$$EI\left(\frac{dy}{dx}\right) = 0 + \frac{\omega L^2}{2}$$

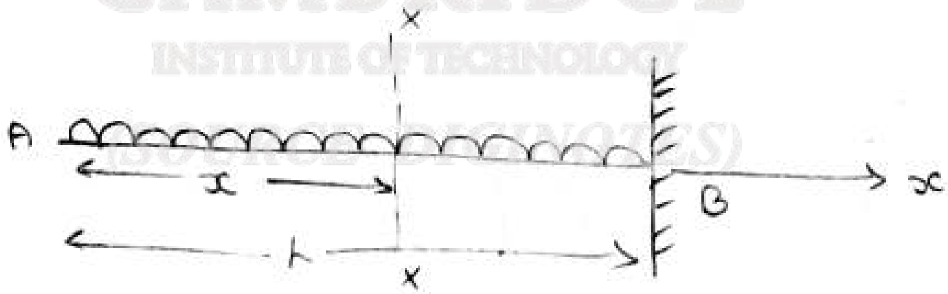
$$\boxed{\frac{dy}{dx} = \frac{\omega L^2}{2EI}}$$

(ii) at $x = 0$

$$EIy = -\frac{\omega L^3}{3}$$

$$\boxed{y = -\frac{\omega L^3}{3EI}}$$

Cantilever with a UDL :-



Step-1 :- Find the reactions :-

$\Sigma H = 0 \Rightarrow H_B = 0$

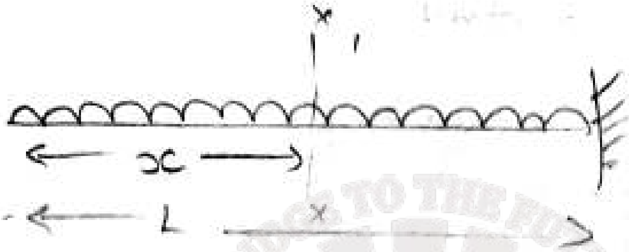
$\Sigma V = 0 \Rightarrow -(\omega \times L) + V_B = 0$

$\Rightarrow \boxed{V_B = \omega L}$

$\Sigma M_B = 0 \Rightarrow M_B - wL \cdot \frac{L}{2} = 0$

$$\Rightarrow \boxed{M_B = \frac{wL^2}{2}}$$

(ii) consider section x-x



$$M_x = \frac{wx^2}{2}$$

Step ③ :- Take a moment at section x-x

$$M_x = \frac{wx^2}{2}$$

Step ④ :- consider the moment curvature relation

$$EI\frac{d^2y}{dx^2} = -\frac{wx^2}{2} \longrightarrow ①$$

Step ⑤ :- Integrate ① w.r.t 'x'

$$EI\frac{dy}{dx} = -\frac{wx^3}{6} + C_1 \longrightarrow ②$$

Step ⑥ :-

$$EIy = \frac{-wx^4}{24} + C_1x + C_2 \longrightarrow ③$$

Step ⑦ :- Apply boundary conditions to find c_1 and c_2

(i) at $x = L$, $\frac{dy}{dx} = 0$; $y = 0$

② => $c_1 = +\omega L$

$$0 = -\frac{\omega L^3}{6} + c_1$$

$$\boxed{c_1 = \frac{\omega L^3}{6}}$$

(ii) at $x = L$, $y = 0$

$$0 = -\frac{\omega L^4}{24} + \frac{\omega L^3}{6} L + c_2$$

$$c_2 = \frac{\omega L^4}{24} - \frac{\omega L^4}{6}$$

$$\boxed{c_2 = -\frac{\omega L^4}{8}}$$

Step ⑧ : Substitute the value of c_1 and c_2 in eqⁿ ② and ③

$$EI\left(\frac{dy}{dx}\right) = \frac{-\omega x^3}{6} + \frac{\omega L^3}{6} \longrightarrow ④$$

$$EIy = -\frac{\omega x^4}{24} + \frac{\omega L^3}{6}x - \frac{\omega L^4}{8}$$

Step ⑨ :- Maximum deflection and slope

(i) at $x = 0$,

$$EI\frac{dy}{dx} = \frac{wL^3}{6}$$

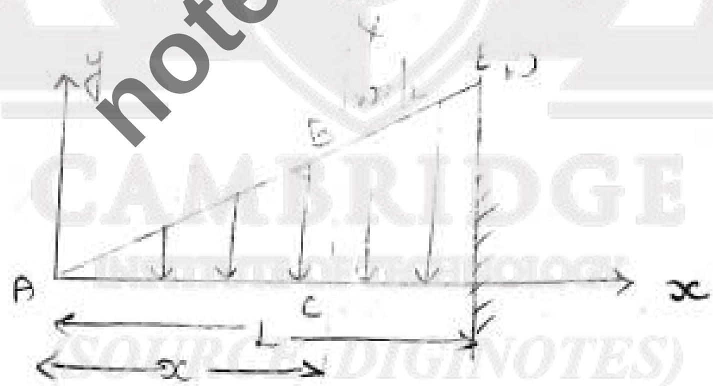
$$\Rightarrow \boxed{\frac{dy}{dx} = \frac{wL^3}{6EI}}$$

(ii) at $x=0$,

$$EIy = -\frac{wL^4}{8}$$

$$\Rightarrow \boxed{y = \frac{-wL^4}{8EI}}$$

Cantilever with UVL



Δ^{le} ABC & Δ^{le} ADE are similar

$$\frac{w}{L} = \frac{y}{x}$$

$$\Rightarrow \boxed{y = \frac{wx}{L}}$$

$$M_x = \left(-\frac{1}{2} \times x \cdot \frac{wx}{L}\right) \frac{x}{3}$$

$$= \frac{-wx^3}{6L}$$

$$EI \frac{d^2y}{dx^2} = M_x = \frac{-wx^3}{6L}$$

Integrate wrt x,

$$EI \frac{dy}{dx} = \frac{-w}{6L}\left(\frac{x^4}{4}\right) + C_1$$

$$= \frac{-wx^4}{24L} + C_1 \quad \text{—— (1)}$$

Integrate wrt x

$$EIy = \frac{-w}{24L}\left(\frac{x^5}{5}\right) + C_1 x + C_2$$

$$= \frac{-wx^5}{120L} + C_1 x + C_2 \quad \text{—— (2)}$$

To find C_1 & C_2

@ $x = L$, $\frac{dy}{dx} = 0$

(1) $\Rightarrow$ $0 = \frac{-wL^4}{24L} + C_1$

$$\Rightarrow \boxed{C_1 = \frac{wL^3}{24}}$$

@ $x = L$, $y = 0$

$$③ \Rightarrow 0 = \frac{-w.L^5}{120L} + \frac{wL^3}{24}.L + C_2$$

$$= \frac{-wL^4}{120} + \frac{wL^4}{24} + C_2$$

$$\boxed{C_2 = \frac{-wL^4}{30}}$$

Substitute C_1 and C_2 in eq^n ②

$$① \Rightarrow EI\frac{dy}{dx} = \frac{-wx^4}{24L} + \frac{wL^3}{24}$$

$$② \Rightarrow EIy = \frac{-wx^5}{120L} + \frac{wL^3}{24}x - \frac{wL^4}{30}$$

@ $x = 0$

$$① \rightarrow EI\frac{dy}{dx} = \frac{wL^3}{24}$$

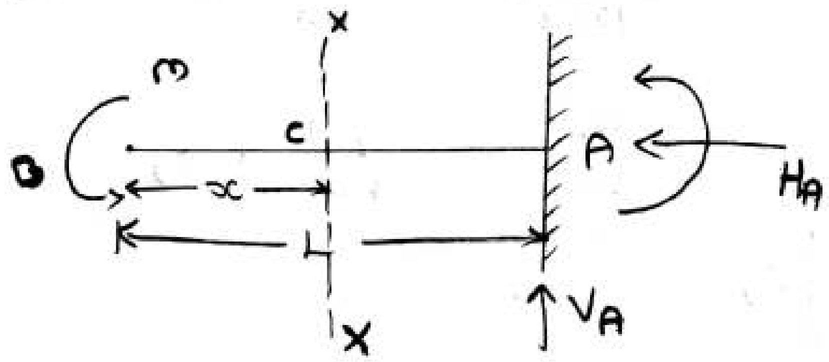
$$\rightarrow \boxed{\frac{dy}{dx} = \frac{wL^3}{24EI}}$$

at $x = 0$

$$② \Rightarrow EIy = \frac{-wL^4}{30}$$

$$\boxed{y = \frac{-wL^4}{30EI}}$$

Cantilever with a moment



$\Sigma H = 0 \Rightarrow H_A = 0$

$\Sigma V = 0 \Rightarrow V_A = 0$

$\Sigma m_B = 0 \Rightarrow m + m_A = 0$

$$\boxed{m_A = -m}$$

$m_x = -m$

$$EI \frac{d^2y}{dx^2} \Rightarrow m_x \Rightarrow -m$$

Integrate ① wrt 'x'

$$EI \frac{dy}{dx} \Rightarrow -mx + C_1 \longrightarrow ②$$

Integrate ② wrt x

$$EIy \Rightarrow -\frac{mx^2}{2} + C_1 x + C_2 \longrightarrow ③$$

To find C_1 & C_2

@ $x = L$, $\frac{dy}{dx} = 0$

$0 \Rightarrow -mL + C_1$

$$\boxed{C_1 = mL}$$

at $x = L$, $y = 0$

$0 \Rightarrow -\frac{mL^2}{2} + mL \times L + C_2$

$\frac{mL^2}{2} - mL^2 = C_2$

$$\boxed{C_2 = -\frac{mL^2}{2}}$$

Substitute values of C_1 and C_2 in (2) & (3)

$EI \frac{dy}{dx} = -mx + mL$

$EI y = -\frac{mx^2}{2} + mLx - \frac{mL^2}{2}$

maximum slope is at $x=0$

$EI \frac{dy}{dx} = mL$

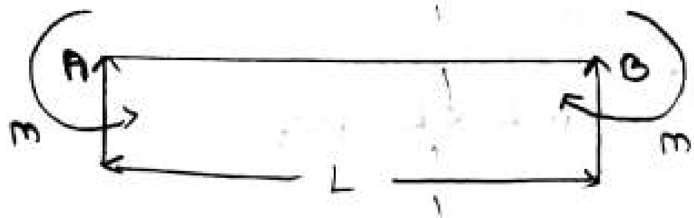
$$\boxed{\frac{dy}{dx} = \frac{mL}{EI}}$$

maximum deflection at $x=0$

$EIy = \frac{-mL^2}{2}$

$$\boxed{y = \frac{-mL^2}{2EI}}$$

Simply Supported beam Subjected to end couples



$\Sigma H = 0$, $\Sigma V = 0$, $R_A + R_B = 0$

$\Rightarrow \boxed{R_A = -R_B}$

$\Sigma M_A = 0 \Rightarrow (-R_B \times L) - \cancel{M} + \cancel{M} = 0$

$\Rightarrow \boxed{R_B = 0}$

$M_x = (R_A \times x) - M$

$\boxed{M_x = -M}$

Integrate ① w.r.t x

$EI \dfrac{d^2y}{dx^2} = -Mx + C_1 \longrightarrow ①$

Integrate ② w.r.t x

$EI \dfrac{dy}{dx} = -\dfrac{Mx^2}{2} + C_1 x + C_2$

$EI \dfrac{d^2y}{dx^2} = -M \longrightarrow ①$

Integrate eqⁿ ① wrt x

$EI \dfrac{dy}{dx} = -Mx + C_1 \longrightarrow ②$

Integrate again eq^n ② wrt x

$$EIy = -\frac{Mx^2}{2} + C_1x + C_2 \longrightarrow ③$$

To find C_1 and C_2

@ $x = \frac{L}{2}$, $\frac{dy}{dx} = 0$

$$① \Rightarrow 0 = -\frac{ML}{2} + C_1$$

$$\boxed{C_1 = \frac{ML}{2}}$$

@ $x = 0$

$$② \Rightarrow 0 = C_2$$

$$\boxed{C_2 = 0}$$

Substitute the values of C_1 and C_2 in ② and ③

② $\Rightarrow$

$$EI\frac{dy}{dx} = -Mx + \frac{ML}{2}$$

$$③ \Rightarrow EIy\frac{dy}{} = -\frac{Mx^2}{2} + \frac{ML}{2}x + 0$$

To find maximum slope and deflection

@ $x = \frac{L}{2}$,

$$② \Rightarrow EIy = -\frac{M\left(\frac{L}{2}\right)^2}{2} + \frac{ML\left(\frac{L}{2}\right)}{2}$$

$$= \frac{-mL^2}{8} + \frac{mL^2}{4}$$

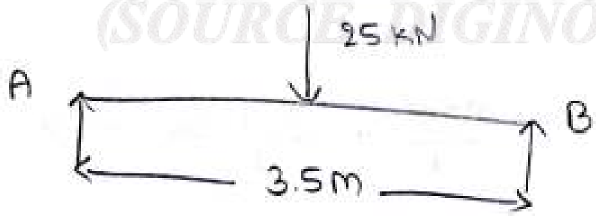
$$= \frac{mL^2}{8}$$

$$\Rightarrow \boxed{y = \frac{mL^2}{8EI}}$$

at $x = 0$

$$① \Rightarrow \boxed{\frac{dy}{dx} = \frac{mL}{2EI}}$$

① A 3.5 m long simply supported beam carries a point load of 25 kN at it's mid span. Determine the magnitude of UDL the beam can carry with the maximum deflection ~~is~~ to be the same as that in case of beam with a point load which is equal to 2.5 mm. Take E = 200 Gpa



<u>Given</u>:- $y_{max} = 2.5\ mm = 2.5 \times 10^{-3}\ m$

$E = 200\ Gpa = 2 \times 10^5\ N/mm^2$

Deflection

$$y = \frac{-WL^3}{24EI}$$

$25 \times 10^{-3} = 1$

$$\Rightarrow \quad 2.5 = -\frac{(25\times10^3)\times(3.5\times10^3)^3}{24 \times 2\times10^5 \times I}$$

$I = 8.932 \times 10^7 \ mm^4$

$$\boxed{I = 8.932 \times 10^{-5} \ m^4}$$

To find magnitude of UDL

$$y_{max} = \frac{-5WL^4}{384EI} \longrightarrow \omega_1 = ?$$

$$2.5 = \frac{-5\omega \times (3.5\times10^3)^4}{384 \times 2\times10^5 \times 8.932\times10^7}$$

$\Rightarrow \quad \omega_1 = -22.85 \ N/mm$

$$\boxed{\omega_1 = -22.85 \ KN/m}$$

MACAULEY'S METHOD

Macauley's method is an improved version of double integration method which can be used for finding the deflection of beams subjected to discontinous loads.

Basically this method involves equating the moment equation consisting of the terms related to all the loads on a beam to the term $EI \frac{d^2y}{dx^2}$, to get the expression involving the moment curvature relationship.

The deflection at any section on the beam can be found by choosing the terms related to the moment at that section.

In using this method we must remember these points.

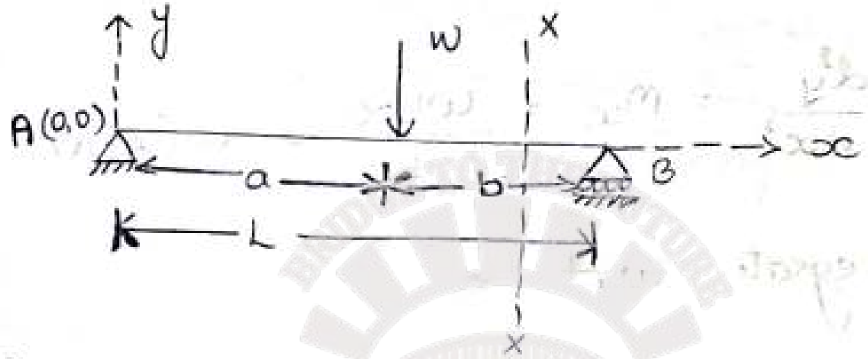
① Take the origin at the left most end of the beam.

② Express the bending moment equation for the right most segment of the beam.

③ The terms in brackets as in the term $w(x-a)$ should be integrated keeping them in brackets throughout. It is only in numerical calculations when the value of x has been specified that the brackets are opened for computation.

④ Use the terms selectively depending upon their validity. If the bracketed term is positive, it is included and if negative it becomes zero depending on the value of x

① A simply supported beam with eccentric loading.



<u>Step ①</u> :- Find the reactions at supports

$\Sigma V = 0$

$\Rightarrow R_A + R_B - w = 0$

$\rightarrow R_A + R_B = w$ —— ①

$\Sigma M_A = 0 \Rightarrow (-R_B \times L) + w(a) = 0$

$\Rightarrow \boxed{R_B = \frac{wa}{L}} \longrightarrow$ in ①

$R_A + \frac{wa}{L} = w$

$\Rightarrow R_A = w\left(1 - \frac{a}{L}\right)$

$= w\left(\frac{L-a}{L}\right)$

$\boxed{R_A = \frac{wb}{L}}$

Step ② :- Find moment at $x-x$

$$M_x = R_A \cdot x - w(x-a)$$

$$= \frac{wbx}{L} - w(x-a)$$

Step ③ :-

$$EI \frac{d^2y}{dx^2} = M_x = \frac{wbx}{L} - w(x-a)$$

Integrate wrt x

$$EI \frac{dy}{dx} = \frac{wbx^2}{2L} - \frac{w(x-a)^2}{2} + C_1 \rightarrow ②$$

Integrate wrt x

$$EIy = \frac{wbx^3}{6L} - \frac{w(x-a)^3}{6} + C_1x + C_2 \rightarrow ③$$

Step ④ :- To find C_1 and C_2

Apply boundary conditions

at $x=0$, $y=0 \Rightarrow ③$

$$\boxed{C_2 = 0}$$

at $x=L$, $y=0 \rightarrow ③$

$$0 = \frac{wbL^{\cancel{3}}}{6\cancel{L}} - \frac{w(L-a)^3}{6} + C_1L$$

$$0 = \frac{wbL^2}{6} - \frac{w(L-a)^3}{6} + C_1L$$

$$0 = \frac{wbL^2}{6} - \frac{wb^3}{6} + C_1 L$$

$$0 = \frac{wb}{6}(L^2 - b^2) + C_1 L$$

$$\Rightarrow C_1 = \frac{-wb}{6L}(L^2 - b^2)$$

$$= \frac{-wb}{6L}(L+b)(L-b)$$

$$\boxed{C_1 = \frac{-wab(L+b)}{6L}}$$

② $\Rightarrow$

$$EI\frac{dy}{dx} = \frac{wbx^2}{2L} - \frac{w(x-a)^2}{2} - \frac{wab}{6L}(L+b)$$

③ $\Rightarrow$

$$EIy = \frac{wbx^3}{6L} - \frac{w(x-a)^3}{6} - \frac{wabx}{6L}(L+b)$$

@ A, θ_A

@ $x=0$ in ②

$$EI\frac{dy}{dx} = \frac{-wab}{6L}(L+b)$$

$$\theta_A \Rightarrow \left.\frac{dy}{dx}\right|_A = \frac{-wab}{6EIL}(L+b)$$

@ $x = L$, in eqⁿ ②

$$EI\left.\frac{dy}{dx}\right|_B = \frac{wbL^2}{2L} - \frac{w(L-a)^2}{2} - \frac{wab}{6L}(L+b)$$

$$\Rightarrow EI\theta_B = \frac{WbL}{2} - \frac{Wb^2}{2} - \frac{Wab}{6L}(L+b)$$

$$= \frac{Wb}{2}\left[L - b - \frac{a}{3L}(L+b)\right]$$

$$= \frac{Wb}{6L}\left[3L^2 - 3bL - aL + ab\right]$$

$$= \frac{Wb}{2}\left[a - \frac{a}{3L}(L+b)\right]$$

$$= \frac{Wab}{2}\left[1 - \frac{(L+b)}{3L}\right]$$

$$= \frac{Wab}{6L}\left[3L - L - b\right]$$

$$= \frac{Wab}{6L}\left[2L - b\right]$$

$$EI\theta_B = \frac{Wab}{6L}\left[L + L - b\right]$$

$$= \frac{Wab}{6L}\left[L + a\right]$$

$$\Rightarrow \boxed{\theta_B = \frac{Wab}{6EIL}\left[L + a\right]}$$

Step ⑤ :- To find 'x' at which 'y' is max

$$\frac{dy}{dx} = 0 \Rightarrow y_{max}$$

② $\Rightarrow$ $$0 = \frac{Wbx^2}{2L} - \frac{W(x-a)^2}{2} - \frac{Wab}{6L}(L+b)$$

$$0 = \frac{Wbx^2}{2L} - W\left(\frac{x^2 + a^2 - 2xa}{2}\right)$$

$$- \frac{WabL}{6L} - \frac{Wab^2}{6L}$$

Assuming $a > \frac{L}{2}$. the maximum deflection occurs within the left span

$\therefore$ The value of x can be calculated using slope equation

$$0 = \frac{Wbx^2}{2L} - \cancel{\frac{W(x-a)^2}{2}}^{\to 0} - \frac{Wab}{6L}(L+b)$$

$$\Rightarrow 0 = \frac{\cancel{W}\cancel{b}x^2}{\cancel{2L}} - \frac{\cancel{W}a\cancel{b}}{\cancel{6L}_{3}}(L+b)$$

$$x^2 = \frac{a}{3}(L+b)$$

$$= \frac{(L-b)(L+b)}{3} = \frac{L^2-b^2}{3}$$

$$x = \sqrt{\frac{L^2-b^2}{3}}$$

$$EIy = \frac{Wbx^3}{6L} - \frac{W(x-a)^3}{6} - \frac{Wabx}{6L}(L+b)$$

To find maximum deflection

$$EIy_{max} = \frac{Wb\left(\sqrt{\frac{L^2-b^2}{3}}\right)^3}{6L} - \frac{W\left(\sqrt{\frac{L^2-b^2}{3}} - a\right)^3}{6}$$

$$= \frac{Wb}{2L}\left(\sqrt{\frac{L^2-b^2}{3}}\right)\frac{(L+b)(L-b)}{3} - \frac{Wab\left(\sqrt{\frac{L^2-b^2}{3}}\right)(L+b)}{6L}$$

$$= \frac{wb\left(\sqrt{\frac{L^2-b^2}{3}}\right)^{2}\left(\sqrt{\frac{L^2-b^2}{3}}\right)}{6L} - \frac{w\left(\sqrt{\frac{L^2-b^2}{3}}-a\right)^3}{6}$$

$$- wb\,\frac{\left(\sqrt{\frac{L^2-b^2}{3}}\right)\cdot\left(\frac{L^2-b^2}{3}\right)}{2L}$$

$$= \frac{wb}{2L}\left(\frac{L^2-b^2}{3}\right)\left(\sqrt{\frac{L^2-b^2}{3}}\right)\left[\frac{1}{3}-1\right] -$$

$$\frac{w\left(\sqrt{\frac{L^2-b^2}{3}}-a\right)^3}{6}$$

$$y_{max} = \frac{wb\,(L^2-b^2)^{3/2}}{9\sqrt{3}\,LEI}$$

where $a = \frac{L}{2}$, $x = \frac{L}{2}$

i.e For central load deflection occurs under the load and is maximum.

$$EIy = \frac{wbx^3}{6L} - \frac{w(x-a)^3}{6} - \frac{wabx}{6L}(L+b)$$

$$= \frac{wbx^3}{6L} - w$$

$$= \frac{wb}{6L}\left(\frac{L}{2}\right)^3 - \frac{w\left(\frac{L}{2} - \frac{L}{2}\right)^3}{6} - \frac{wab}{6L}$$

@ $x = \frac{L}{2}$, $a = b = \frac{L}{2}$

$$\Rightarrow EIy = \frac{w\left(\frac{L}{2}\right)\left(\frac{L}{2}\right)^3}{6L} - \frac{w\left(\frac{L}{2}\right)\left(\frac{L}{2}\right)\left(\frac{L}{2}\right)\left(L+\frac{L}{2}\right)}{6L}$$

$$= \frac{wL^3}{96} - \frac{w\left(\frac{L^3}{8}\right)\left(\frac{3L}{2}\right)}{6L}$$

$$= \frac{wL^3}{96} - \frac{3wL^3}{16 \times 6}$$

$$= \frac{-2wL^3}{96}$$

$$= \frac{-wL^3}{48}$$

$$\Rightarrow \boxed{y = \frac{-wL^3}{48EI}}$$

Case ③: when a = maximum deflection is assumed to be closer to the high right hand portion, $x = \frac{L}{\sqrt{3}}$ a

$$EIy = \frac{Wbx^3}{6L} - \frac{W(x-a)^3}{6} - \frac{Wabx}{6L}(L+b)$$

$$EIy = \frac{Wb\left(\frac{L}{\sqrt{3}}\right)^3}{6L} - \frac{W\left(\frac{L}{\sqrt{3}}-a\right)^3}{6} - \frac{Wab\left(\frac{L}{\sqrt{3}}\right)(L+b)}{6L}$$

$$= \frac{WbL^2}{(3\sqrt{3}\times 6)} - \frac{W(L-\sqrt{3}a)^3}{\sqrt{3}(6)\times 3} - \frac{Wab(L+b)}{6\sqrt{3}}$$

$$= \frac{W}{6\sqrt{3}}\left[\frac{bL^2}{3} - \frac{(L-\sqrt{3}a)^3}{3} - ab(L+b)\right]$$

$$= \frac{W}{18\sqrt{3}}\left[bL^2 - (L-\sqrt{3}a)^3 - 3ab(L+b)\right]$$

$$= \frac{W}{18\sqrt{3}}\left[bL^2 - \{L^3 - (\sqrt{3}a)^3 + 3L(\sqrt{3}a)^2(\sqrt{3}a)\} - 3abL - 3ab^2\right]$$

$$= \frac{W}{18\sqrt{3}}\left[bL^2 - L^3 + 3\sqrt{3}a^3 - (3\times 3La^2) - 3L^2\sqrt{3}a - 3abL - 3ab^2\right]$$

$$= \frac{W}{18\sqrt{3}}\left[(L-a)L^2 - L^3 + 3\sqrt{3}a^3 - 9La^2 - 3\sqrt{3}L^2a - 3a(L-a)L - 3a(L-a)^2\right]$$

$$= \frac{W}{18\sqrt{3}}\left[\cancel{L^3} - aL^2 - \cancel{L^3} + 3\sqrt{3}a^3 - 9La^2 - 3\sqrt{3}L^2a - 3aL^2 + 3a^2L - 3aL^2 - 3a^3 + 6a^2L\right]$$

$$= \frac{w}{18\sqrt{3}}\left[-aL^2 - 3\sqrt{3}aL^2 - 3aL^2 - 3aL^2 - 9La^2 + 3a^2L + 6a^2L - 3a^3 + 3\sqrt{3}a^3\right]$$

$$= \frac{w}{18\sqrt{3}}\left\{-aL^2\left[1+3\sqrt{3}+3+3\right] - a^3\left[3-3\sqrt{3}\right]\right\}$$

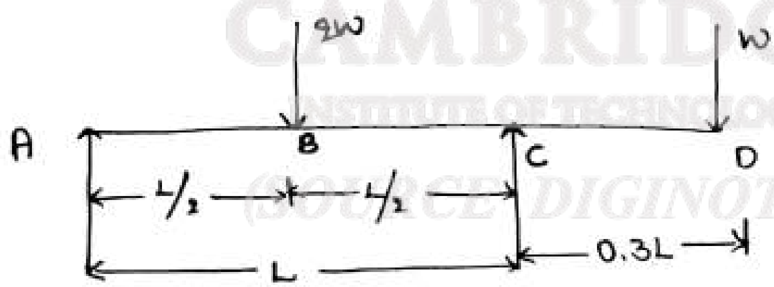
$$= \frac{w}{18\sqrt{3}}\left[-aL^2(7+3\sqrt{3}) - a^3\left[3-3\sqrt{3}\right]\right]$$

$$= \frac{-wa}{18\sqrt{3}}\left\{L^2(7+3\sqrt{3}) + a^2\left[3-3\sqrt{3}\right]\right\}$$

$$= \frac{-wa}{18\sqrt{3}}\left[12.2L^2 - 2.2a^2\right]$$

$$\boxed{y_{max} = \frac{-wa}{18\sqrt{3}EI}\left[12.2L^2 - 2.2a^2\right]}$$

② Find the slopes at A, C and D and deflection at B and D of the beam is given below



① Find reactions

$\Sigma V = 0 \Rightarrow R_A + R_C = 2W + W = 3W$ —— ①

$\Sigma M_A = 0 \Rightarrow 2W(L/2) - R_C(L) + W(1.3L) = 0$

$\Rightarrow WL - R_C L + W(1.3L) = 0$

$\Rightarrow R_C = \frac{2.3WL}{L}$

$$\boxed{R_C = 2.3\,w}$$

$$\boxed{R_A = 0.7\,w}$$

② $M_x = 0.7\,w(x) - 2w\left(x - L/2\right) + 2.3\,w\,(x-L)$

③

$$EI\,\frac{d^2y}{dx^2} = M_x = 0.7\,w_x - 2w\left(x - L/2\right) + 2.3\,w\,(x-L)$$

Step ④

$$EI\,\frac{dy}{dx} = 0.7\,\frac{wx^2}{2} - 2w\,\frac{\left(x - L/2\right)^2}{2} + 2.3\,w\,\frac{(x-L)^2}{2} + C_1 \quad \text{—— ②}$$

$$EIy = 0.7\,\frac{wx^3}{6} - 2w\,\frac{\left(x - L/2\right)^3}{6} + \frac{2.3\,w\,(x-L)^3}{6} + C_1x + C_2 \quad \text{—— ③}$$

Step ⑤ :- To find C_1 and C_2

@ $x=0$, $y=0$ in ③

$$\boxed{C_2 = 0}$$

@ $x=L$, $y=0$ in ③

$$0 = \frac{0.7\,wL^3}{6} - \frac{2w\left(L - L/2\right)^3}{6} + \cancel{\frac{2.3\,w\,(L-L)^3}{6}}^{\,0} + C_1L$$

$$\Rightarrow 0 = \frac{0.7}{6} wL^3 - 2w\left(\frac{L}{2}\right)^3 + C_1 L$$

$$0 = \frac{wL^3}{6}\left[0.7 - \frac{1}{4}\right] + C_1 L$$

$$0 = \frac{0.45\, wL^3}{6} + C_1 L$$

$$\boxed{C_1 = -0.075\, wL^2}$$

Step ⑥ :- Substitute the value of C_1 and C_2 in the equation ② and ③

$$EI\frac{dy}{dx} = 0.7\frac{wx^2}{2} - 2w\frac{(x-L/2)^2}{2} + 2.3w\frac{(x-L)^2}{2} + (-0.075\, wL^2)$$

$$EIy = 0.7\frac{wx^3}{6} - 2w\frac{(x-L/2)^3}{6} + \frac{2.3\, w(x-L)^3}{6} + (-0.075\, wL^2)$$

Step ⑦ :- at A, slope = ? $x=0$ in eqn ②

$$\Rightarrow EI\frac{dy}{dx} = \frac{0.7wx^2}{2} - 2\frac{w(x-0.5L)^2}{2} + \cancel{2.3\frac{w(x-L)^2}{2}} + C_1$$

$$\Rightarrow \boxed{\left.\frac{dy}{dx}\right|_A = \frac{-0.075\, wL^2}{EI}}$$

→ @ C , $x = L$; in eqⁿ ②

$$EI\frac{dy}{dx} = 0.7\frac{wL^2}{2} - 2w\frac{(L - L/2)^2}{2} + 2.3\,w\left(\frac{L-L}{2}\right)^2 + C_1$$

$$EI\frac{dy}{dx} = 0.7\frac{wL^2}{2} - 2w\frac{(L-0.5L)^2}{2} + \frac{2.3\,w(L-L)^2}{2} - 0.075\,wL^2$$

$$\boxed{\left.\frac{dy}{dx}\right|_C = \frac{wL^2}{40EI}}$$

⇒ @ D, $x = 1.3L$

$$EI\frac{dy}{dx} = \frac{0.7w(1.3L)^2}{2} - \frac{2w(1.3L-0.5L)^2}{2} + \frac{2.3\,w(1.3L-L)^2}{2} - 0.075\,wL^2$$

$$= \frac{1.183}{2}wL^2 - 0.64\,wL^2 + \frac{0.207\,wL^2}{2} - 0.075\,wL^2$$

$$EI\frac{dy}{dx} = \frac{-wL^2}{50}$$

$$\boxed{\left.\frac{dy}{dx}\right|_D = \frac{-wL^2}{50EI}}$$

Step (8) :-

Deflection at B

at $x = \frac{L}{2}$ in eqⁿ ③

$$EIy = \frac{0.7\, w\left(\frac{L}{2}\right)^3}{6} - 2w\cancel{\frac{\left(\frac{L}{2}-\frac{L}{2}\right)^3}{6}} +$$

$$\cancel{\frac{2.3\, w\left(\frac{L}{2}-L\right)^3}{6}} + \left(-0.075\, wL^2\right)$$

$$= \frac{0.7\, wL^3}{48} - \frac{0.075\, wL^3}{2}$$

$$EIy = \frac{-11wL^3}{480}$$

$$\boxed{y = \frac{-0.023\, wL^3}{EI}}$$

deflection at D

$x = 1.3L$ in eqⁿ ③

$$EIy = \frac{0.7\, w(1.3L)^3}{6} - \frac{2w\left(1.3L - L/2\right)^3}{6} +$$

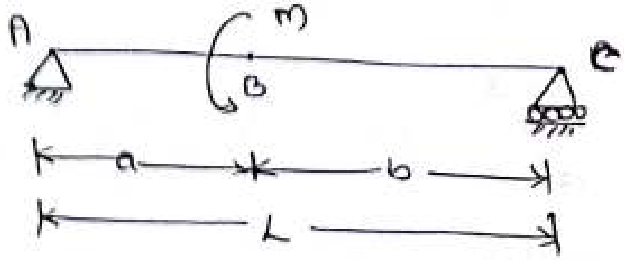
$$\frac{2.3\left(1.3L-L\right)^3}{6} - 0.075\, wL^2(1.3L)$$

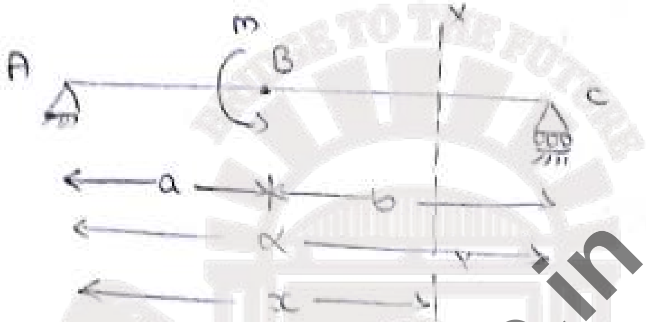
$$EIy = \frac{1.5379\, wL^3}{6} - \frac{1.024\, wL^3}{6} + \frac{0.0621\, wL^3}{6} - 0.0975\, wL^3$$

$$EIy = \frac{-3wL^3}{2000}$$

E.I.
$$\boxed{y_D = \frac{-0.0015\, wL^3}{EI}}$$

③ Compute the rotations of point 'A' and 'C' and the deflection at 'B' due to applied moment at B.





$\Sigma M_A = 0$

$\rightarrow \; -(R_c \times L) - M = 0$

$\Rightarrow \; \boxed{R_c = \frac{M}{L}} \qquad \boxed{R_A = \frac{M}{L}}$

$$M_x = \left[\frac{M}{L} \times x\right] - M(x-a)^0$$

<u>NOTE</u> :- In macaulay's method, other than the four rules mention initially, some special rules for particular problems have to be considered in order to reduce the difficulty in solving. In the above case, we have to consider $(x-a)$ term by solving the equation.

$$EI\frac{d^2y}{dx^2} = M_x = \frac{Mx}{L} - M(x-a)^0$$

$$EI\frac{dy}{dx} = \frac{Mx^2}{2L} - M(x-a) + C_1 \quad \text{—(2)}$$

$$EIy = \frac{Mx^3}{6L} - \frac{M(x-a)^2}{2} + C_1x + C_2 \quad \text{—(3)}$$

To find C_1 & C_2

Boundary conditions

@ $x=0$, $y=0$ in eq^n (3)

$$\boxed{C_2 = 0}$$

at $x=L$, $y=0$ in eq^n (3)

$$0 = \frac{M \cdot L^{\cancel{3}\,2}}{6\cancel{L}} - \frac{M(L-a)^2}{2} + C_1L$$

$$0 = \frac{ML^2}{6} - \frac{M(L-a)^2}{2} + C_1L$$

$$0 = \frac{ML^2}{6} - \frac{Mb^2}{2} + C_1L$$

$$0 = \frac{M}{6}\left[L^2 - 3b^2\right] + C_1L$$

$$\boxed{C_1 = \frac{-M}{6L}\left[L^2 - 3b^2\right]}$$

(2) $\Rightarrow$ $EI\frac{dy}{dx} = \frac{Mx^2}{2L} - M(x-a) - \frac{M}{6L}\left[L^2 - 3b^2\right]$

(3) $\Rightarrow$ $EIy = \frac{Mx^3}{6L} - \frac{M(x-a)^2}{2} - \frac{M}{6L}\left[L^2 - 3b^2\right]x$

Slope at A

$x = 0$ in (2)

$$EI\frac{dy}{dx} = \frac{-m}{6L}\left[L^2 - 3b^2\right]$$

$$\Rightarrow \boxed{\left.\frac{dy}{dx}\right|_A = \frac{-m}{6EIL}\left[L^2 - 3b^2\right]}$$

Slope at C

$x = L$

$$EI\frac{dy}{dx} = \frac{mL^2}{2L} - m(L-a) - \frac{m}{6L}\left[L^2 - 3b^2\right]$$

$$= \frac{mL}{2} - mb - \frac{m}{6L}\left[L^2 - 3b^2\right]$$

$$= \frac{m}{6L}\left[3L^2 - 6bL - L^2 + 3b^2\right]$$

$$= \frac{m}{2}\left[L - 2b - \frac{L^2}{3L} + \frac{3b^2}{3L}\right]$$

$$= \frac{m}{2}\left[a - b - \frac{L}{3} + \frac{b^2}{L}\right]$$

$$EI\frac{dy}{dx} = \frac{m}{2}\left[a - b - \frac{L}{3} + \frac{b^2}{L}\right]$$

$$= \frac{m}{2}\left[a - (L-a) - \frac{L}{3} + \frac{(L-a)^2}{L}\right]$$

$$= \frac{m}{2L}\left[\cancel{aL} - \cancel{L^2} + \cancel{aL} - \frac{L^2}{3} + \cancel{L^2} + a^2 - \cancel{2aL}\right]$$

$$= \frac{m}{2L}\left[\frac{-L^2}{3} + a^2\right]$$

$$= \frac{-m}{6L}\left[L^2 - 3a^2\right]$$

$$\boxed{\left.\frac{dy}{dx}\right|_c = \frac{-m}{6EIL}\left(L^2 - 3a^2\right)}$$

Deflection at B

$x = a$ in ③

$$EIy = \frac{ma^3}{6L} - \frac{m}{6L}\left[L^2 - 3b^2\right].a$$

$$= \frac{ma}{6L}\left[a^2 - L^2 + 3b^2\right]$$

$$= \frac{ma}{6L}\left[a^2 - (a+b)^2 + 3b^2\right]$$

$$= \frac{ma}{6L}\left[\cancel{a^2} - \cancel{a^2} - b^2 - 2ab + 3b^2\right]$$

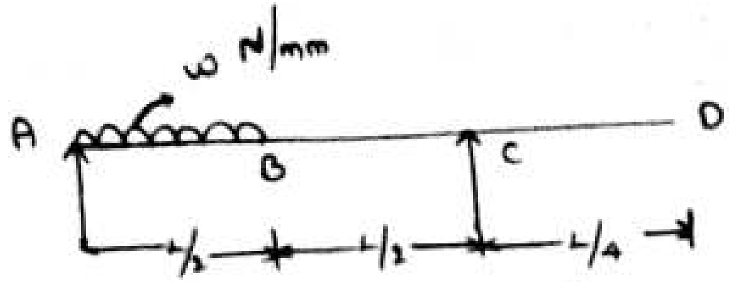
$$= \frac{ma}{6L}\left[2b^2 - 2ab\right]$$

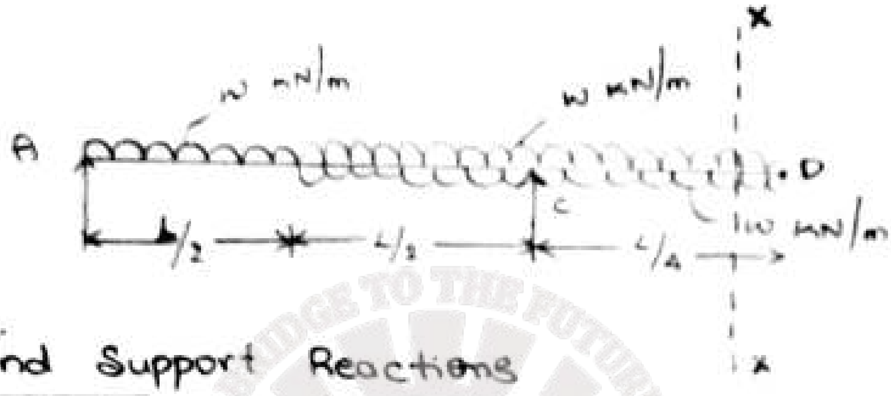
$$EIy = -\frac{mab}{6L}\left[2a - 2b\right]$$

$$= \frac{-mab}{3L}\left[a - b\right]$$

$$\Rightarrow \boxed{y_B = -\frac{mab}{3EIL}\left[a - b\right]}$$

④ Compute the slope at A and C and deflection at B and D





Find Support Reactions

$\Sigma V = 0 \Rightarrow R_A + R_C - w\left(\frac{5L}{4}\right) + w\left(\frac{3L}{4}\right) = 0$

$\Rightarrow R_A + R_C = \frac{5wL}{4} - \frac{3wL}{4}$

$\Rightarrow R_A + R_C = \frac{wL}{2}$ —— ①

$\Sigma M_A = 0$

$(-R_C \times L) + w\left(\frac{5L}{4}\right)\left(\frac{1}{2}\cdot\frac{5L}{4}\right) - w\left(\frac{3L}{4}\right)\left(\frac{1}{2}\cdot\frac{3L}{4} + \frac{L}{2}\right) = 0$

$-R_C L + \frac{25wL^2}{32} - \frac{21wL^2}{32} = 0$

$-R_C L + \frac{24wL^2}{32} = 0$

$\boxed{R_C = \frac{wL}{8}}$

R_A

$$R_A = \frac{wL}{2} - \frac{wL}{8}$$

$$= \frac{4wL - wL}{8}$$

$$\boxed{R_A = \frac{3wL}{8}}$$

$$M_x = \left[\frac{3wL}{8} \cdot x\right] - \left[w \cdot x \cdot \frac{x}{2}\right] + \frac{w}{2}\left(x-\frac{L}{2}\right)\left(x-\frac{L}{2}\right) + \frac{wL}{8}(x-L)$$

$$= \frac{3wLx}{8} - \frac{wx^2}{2} + \frac{w}{2}\left(x-L/2\right)^2 + \frac{wL}{8}(x-L)$$

$$EI\frac{d^2y}{dx^2} = M_x = \frac{3wLx}{8} - \frac{wx^2}{2} + \frac{w}{2}\left(x-L/2\right)^2 + \frac{wL}{8}(x-L)$$

Integrate wrt x

$$EI\frac{d^2y}{dx^2} = \frac{3wL}{8}\left(\frac{x^2}{2}\right) + \frac{w}{2}\left(\frac{x^3}{3}\right) + \frac{w}{2}\frac{\left(x - \frac{L}{2}\right)^3}{3} + \frac{wL}{8}\frac{\left(x-L\right)^2}{2} + c_1$$

$$EI\frac{d^2y}{dx^2} = \frac{3wLx^2}{16} - \frac{wx^3}{64} +$$

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To find C_1 and C_2 → Apply boundary conditions

Integrate wrt x

$$EI\frac{dy}{dx} = \frac{3WL}{8}\left(\frac{x^2}{2}\right) - \frac{W}{2}\left(\frac{x^3}{3}\right) + \frac{W}{2}\frac{(x-L/2)^3}{3} + \frac{WL}{8}\frac{(x-L)^2}{2} + C_1$$

$$= \frac{3WLx^2}{16} - \frac{Wx^3}{6} + \frac{W(x-L/2)^3}{6} + \frac{WL(x-L)^2}{16} + C_1 \quad \text{②}$$

$$EIy = \frac{3WL}{16}\left(\frac{x^3}{3}\right) - \frac{W}{6}\left(\frac{x^4}{4}\right) + \frac{W}{6}\frac{(x-L/2)^4}{4} + \frac{WL}{16}\frac{(x-L)^3}{3} + C_1x + C_2$$

$$EIy = \frac{WLx^3}{16} - \frac{Wx^4}{24} + \frac{W(x-L/2)^4}{24} + \frac{WL}{48}(x-L)^3 + C_1x + C_2 \quad \text{③}$$

To find C_1 and C_2

at $x=0$, $y=0$ in ③

$$\boxed{C_2 = 0}$$

@ $x=L$, $y=0$ in ③

$$0 = \frac{WL(L^3)}{16} - \frac{WL^4}{24} + \frac{W\left(\frac{L}{2}\right)^4}{24} + C_1L$$

$$\Rightarrow 0 = \frac{wL^4}{16} - \frac{wL^4}{24} + \frac{wL^4}{384} + C_1 L$$

$$0 = \frac{24wL^4 - 16wL^4\ wL^4}{384} + C_1 L$$

$$= \frac{\overset{3}{\cancel{9}}wL^4}{\underset{128}{\cancel{384}}} + C_1 L$$

$$\Rightarrow \boxed{C_1 = \frac{-3wL^3}{128}}$$

Equating the value of C_1 and C_2 in eqn (2) and (3)

(2) $\Rightarrow$

$$EI\frac{dy}{dx} = \frac{3wLx^2}{16} - \frac{wx^3}{6} + \frac{w(x-L/2)^3}{6} + \frac{wL(x-L)^2}{16} - \frac{3wL^3}{128}$$

(3) $\Rightarrow$

$$EIy = \frac{wLx^3}{16} - \frac{wx^4}{24} + \frac{w(x-L/2)^4}{24} + \frac{wL}{48}(x-L)^3 + \left(-\frac{3wL^3}{128}\right) + 0$$

slope at A

$x = 0$

$$EI\left(\frac{dy}{dx}\right) = 0 - 0 + 0 + 0 - \frac{3wL^3}{128}$$

$$EI\left(\frac{dy}{dx}\right) = -\frac{3wL^3}{128}$$

$$\boxed{\left.\frac{dy}{dx}\right)_A = -\frac{3wL^3}{EI\,128}}$$

Slope at C :-

$x = L$

$$EI\left(\frac{dy}{dx}\right)_C = \frac{3wL(L)^2}{16} - \frac{wL^3}{6} + w\frac{(L - L/2)^3}{6} + \cancel{\frac{wL(L-L)^2}{16}} - \frac{3wL^3}{128}$$

$$= \frac{3wL^3}{16} - \frac{wL^3}{6} + \frac{wL^3}{6\times 8} - \frac{3wL^3}{128}$$

$$= \frac{3wL^3}{16} - \frac{wL^3}{6} + \frac{wL^3}{48} - \frac{3wL^3}{128}$$

$$= \frac{7wL^3}{384}$$

$$\boxed{\left(\frac{dy}{dx}\right)_C = \frac{7wL^3}{384EI}}$$

deflection at B

$x = L/2$

$$EIy = \frac{3wL(L/2)^3}{48} - \frac{w\left(\frac{L}{2}\right)^4}{24} + \cancel{w\frac{\left(\frac{L}{2} - L/2\right)^4}{24}} + \cancel{\frac{wL\left(\frac{L}{2} - L\right)^3}{48}} - \frac{3wL^3\left(\frac{L}{2}\right)}{128}$$

$$EIy = \frac{3wL^4}{384} - \frac{3wL^4}{384} - \frac{3wL^4}{256}$$

$$y_B = \frac{-5wL^4}{768EI}$$

deflection at D

$$x = \frac{5L}{4}$$

$$EIy = \frac{3wL\left(\frac{5L}{4}\right)^3}{48} - \frac{w\left(\frac{5L}{4}\right)^4}{24} + \frac{w\left(\frac{5L}{4} - \frac{L}{2}\right)^4}{24}$$

$$+ \frac{wL\left(\frac{5L}{4} - L\right)^3}{48} - \frac{3wL^3\left(\frac{5L}{4}\right)}{128}$$

$$EIy = \frac{375wL^4}{3072} - \frac{625wL^4}{6144} + \frac{81wL^4}{6144}$$

$$\frac{wL^4}{3072} - \frac{15wL^4}{512}$$

$$EIy_D = \frac{7wL^4}{1536}$$

$$y_D = \frac{7wL^4}{1536EI}$$

A simply supported is subject to load varying from 'w' at the centre to '0' at the supports. Derive the general expressions for slope and Deflection and find the slope at supports and the maximum deflection EI is constant.

Support reactions

$\Sigma V = 0 \Rightarrow$

$$V_A + V_B - \left(\tfrac{1}{2} \times L \times 2w\right) + \left(\tfrac{1}{2} \times \tfrac{L}{2} \times 2w\right) = 0$$

$$V_A + V_B = \frac{wL}{2} \quad \text{—} \ (1)$$

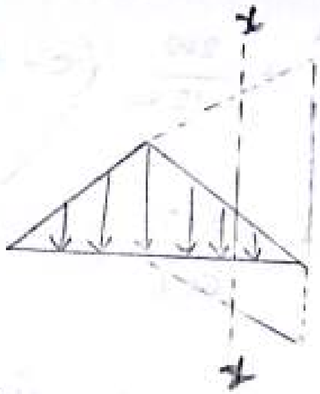
$\Sigma M_A = 0 \Rightarrow$

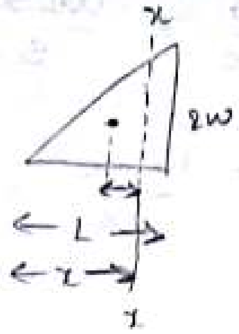
$$-(V_B \times L) + \left(\tfrac{1}{2} \times L \times 2w\right)\left(\tfrac{2L}{3}\right) - \left(\tfrac{1}{2} \times \tfrac{L}{2} \times 2w\right)\left[\tfrac{2}{3}\left(\tfrac{L}{2}\right) + \tfrac{L}{2}\right]$$

$$\Rightarrow \quad -V_B L + \frac{2wL^2}{3} - \frac{5wL^2}{12} = 0$$

$$-V_B = \frac{\left(\frac{5wL^2}{12} - \frac{2wL^2}{3}\right)}{L} = \frac{wL}{4}$$

$$V_A = \frac{wL}{4}$$



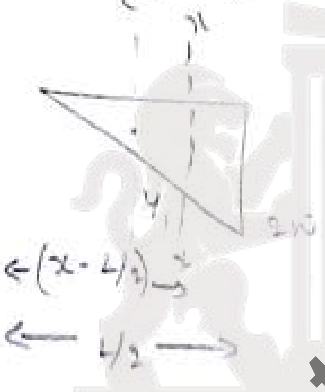


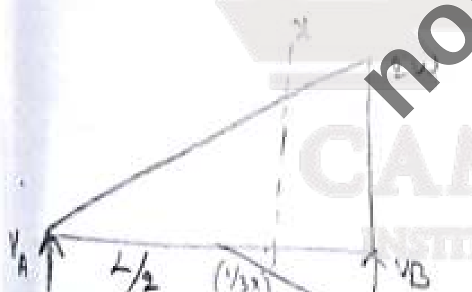
$$EI\frac{d^2y}{dx^2} = M_x$$

$$\frac{2w}{L/2} = \frac{y}{x}$$

$$y = \frac{2wx}{L/2}$$

$$M_x = \left(\frac{wL}{4}\times x\right) - \left(\frac{2wx}{L}\cdot\frac{1}{2}\cdot x\right)\left(\frac{x}{3}\right) + \frac{1}{2}\times\frac{4wx}{L}\times\left(x - L/2\right)\frac{\left(x - L/2\right)}{3}$$





$$M_x = +\left(\frac{wL}{4}\,x\right) - \left(\frac{2wx}{L}\cdot\frac{1}{2}\cdot x\right)\left(\frac{x}{3}\right) + \frac{1}{2}\times\frac{4w}{L}\left(x - L/2\right)\left(x - L/2\right)\left(\frac{x - L/2}{3}\right)$$

$$M_x = \frac{wLx}{4} - \frac{wx^3}{3L} + \frac{2w}{3L}\left(x - \frac{L}{2}\right)^3$$

$$EI\frac{dy}{dx} = \frac{wLx^2}{8} - \frac{wx^4}{12L} + \frac{2w}{12L}\left(x - L/2\right)^4 + C_1 \quad \text{—(2)}$$

$$EIy = \frac{wLx^3}{24} - \frac{wx^5}{60L} + \frac{2w}{60L}\left(x - L/2\right)^5 + C_1x + C_2 \quad \text{(3)}$$

Apply boundary conditions to find the value of C_1 and C_2

$$\left.\frac{dy}{dx}\right|_A = \frac{-5wL^3}{192EI}$$

$$\left.\frac{dy}{dx}\right|_B = \frac{5wL^3}{192EI}$$

$$y_{max} = \frac{-wL^4}{120EI}$$

$C_2 = 0$

$$0 = \frac{wL^4}{24} - \frac{wL^4}{60} + \frac{2w}{60L}(L/2)^5 + C_1L$$

$$= \frac{wL^4}{24} - \frac{wL^4}{60} + \frac{wL^4}{960} + C_1L$$

$$0 = \frac{40wL^4 - 16wL^4 + wL^4}{5} + C_1L$$

$$0 = \frac{\cancel{25}^{5}\,wL^4}{\cancel{960}_{192}} + C_1L$$

$$C_1 = \frac{-5wL^3}{192}$$

Equating the value of C_1 and C_2 in eq^n (2) and (3)

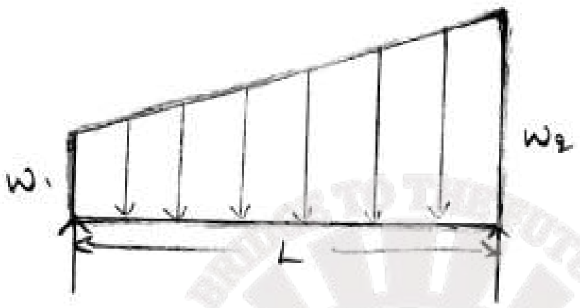
Eq^n (2) $\Rightarrow$

$$EI\frac{dy}{dx} = \frac{wLx^2}{8} - \frac{wx^4}{12L} + \frac{2w}{12L}\left(x - L/2\right)^4 + \left(\frac{-5wL^3}{192}\right)$$

③ $\Rightarrow$

$$EIy = \frac{wLx^3}{24} - \frac{wx^5}{60L} + \frac{2w}{60L}\left(x - \frac{L}{2}\right)^5 + \left(-\frac{5wL^3}{192}\right)$$

A simply supported beam carries a varying load varying from w_1 at the left end to w_2 at the right end. Find the slopes at the supports.



$$\Rightarrow \quad w_1 L + \frac{1}{2}(w_2 - w_1)L$$

$$\Rightarrow \quad w_1 L + \frac{w_2 L}{2} - \frac{w_1 L}{2}$$

Reactions at supports

$\Sigma V = 0$

$$V_A + V_B = w_1 L + \frac{w_2 L}{2} - \frac{w_1 L}{2}$$

$$V_A + V_B = \frac{w_1 L}{2} + \frac{w_2 L}{2} \rightarrow (1)$$

$$V_A = \frac{w_1 L}{2} + \frac{w_2 L}{2} - \frac{w_1 L}{6} - \frac{w_2 L}{3}$$

$$\boxed{V_A = \frac{w_1 L}{3} + \frac{w_2 L}{6}}$$

$\Sigma M_A = 0 \quad \circlearrowright +ve$

$$-V_B \times L + w_1 L \times \frac{L}{2} + \frac{(w_2 - w_1)}{2} L \left(\frac{2L}{3}\right) = 0$$

$$V_B L = \frac{w_1 L^2}{2} + \frac{w_2 L^2}{3} - \frac{w_1 L^2}{3}$$

$$V_B = \frac{w_1 L^2}{6L} + \frac{w_2 L^2}{3L}$$

$$\boxed{V_B = \frac{w_1 L}{6} + \frac{w_2 L}{3}}$$

$$M_x = \left(\frac{w_1 L}{3} + \frac{w_2 L}{6}\right)x - \frac{w_1 x^2}{2} - \left(\frac{(w_2 - w_1)}{L} x \times \frac{x}{2}\right)\left(\frac{x}{3}\right.$$

$$M_x = \frac{w_1 L x}{3} + \frac{w_2 L x}{6} - \frac{w_1 x^2}{2} - \frac{w_2 x^3}{6L} + \frac{w_1 x^3}{6L}$$

$$EI \frac{d^2y}{dx^2} = M_x = \frac{w_1 L x}{3} + \frac{w_2 L x}{6} - \frac{w_1 x^2}{2} - \frac{w_2 x^3}{6L} + \frac{w_1 x^3}{6L}$$

$$EI \frac{dy}{dx} = \frac{w_1 L x^2}{6} + \frac{w_2 L x^2}{12} - \frac{w_1 x^3}{6} - \frac{w_2 x^4}{24L} + \frac{w_1 x^4}{24L} + C_1 \quad (1)$$

$$EI . y = \frac{w_1 L x^3}{18} + \frac{w_2 L x^3}{36} - \frac{w_1 x^4}{24} - \frac{w_2 x^5}{120L} + \frac{w_1 x^5}{120L} +$$

$$C_1 x + C_2 \longrightarrow (3)$$

Finding C_1 and C_2

$x = 0$, $y = 0$ eq^n (3)

$$\boxed{C_2 = 0}$$

$x = L$, $y = 0$ eq^n (3)

$$0 = \frac{w_1 L^4}{18} + \frac{w_2 L^4}{36} - \frac{w_1 L^4}{24} - \frac{w_2 L^4}{120} + \frac{w_1 L^4}{120} + C_1 L$$

$$C_1 = \frac{-11 w_1 L^4}{120 L} - \frac{7 w_2 L^4}{360 L}$$

$$\boxed{C_1 = \frac{-w_1 L^3}{45} - \frac{7 w_2 L^3}{360}}$$

Slope at $x=0$ eqⁿ (2)

$$EI\frac{dy}{dx} = -\frac{w_1L^3}{45} - \frac{7w_2L^3}{360}$$

$$\frac{dy}{dx} = \frac{-w_1L^3}{45EI} - \frac{7w_2L^3}{360EI}$$

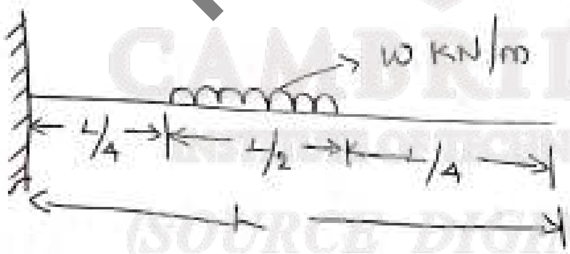
Slope at $x=L$ eqⁿ (2)

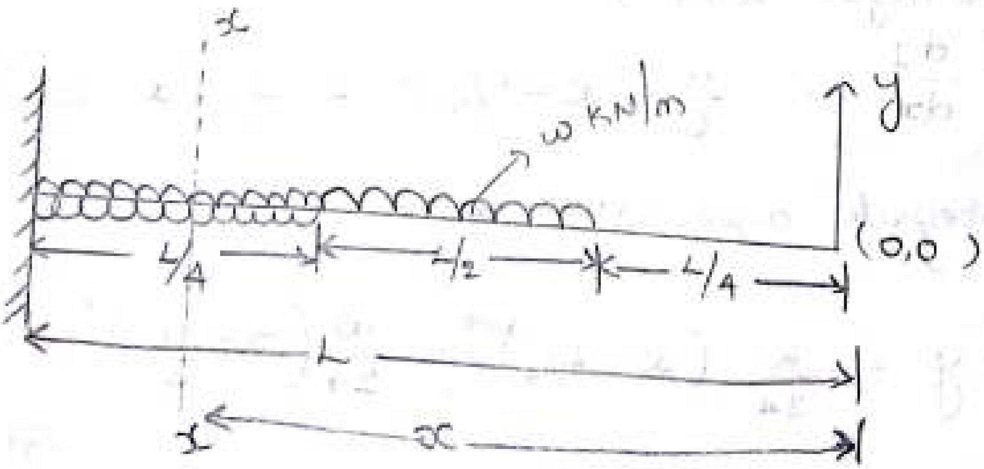
$$EI\frac{dy}{dx} = \frac{w_1L^3}{6} + \frac{w_2L^3}{12} - \frac{w_1L^3}{6} - \frac{w_2L^3}{24} + \frac{w_1L^3}{24} - \frac{w_1L^3}{45} - \frac{7w_2L^3}{360}$$

$$EI\frac{dy}{dx} = \frac{7w_1L^3}{360} + \frac{w_2L^3}{45}$$

$$\frac{dy}{dx} = \frac{7w_1L^3}{360EI} + \frac{w_2L^3}{45EI}$$

7. Find maximum deflection and slope at free end.





Since the load 'w' is discontinous, in order to Solve for the deflection and slope using the machuley's method, the UDL must be extended to a length of $L/4$ at the left hand corner and should be balanced by an equal load applied at the bottom as shown in figure

$H_A = 0$

$V_A - w \times \frac{L}{2} = 0$ $\qquad$ $-M + \frac{wL}{2} \times \frac{L}{2} = 0$

$\boxed{V_A = \frac{wL}{2}}$ $\qquad$ $\boxed{M = \frac{wL^2}{4}}$

$$M_x = w \times (x - L/4) \times \frac{(x - L/4)}{2} - w \times \left(x - \frac{3L}{4}\right) \times \frac{\left(x - \frac{3L}{4}\right)}{2}$$

$$M_x = \frac{w}{2}(x - L/4)^2 - \frac{w}{2}(x - 3L/4)^2 = EI\frac{d^2y}{dx^2}$$

$$EI\frac{d^2y}{dx^2} = M_x = \frac{w}{2}(x - L/4)^2 - \frac{w}{2}(x - 3L/4)^2$$

Integrate w.r.t x

$$\Rightarrow EI\frac{dy}{dx} = \frac{w}{6}(x - L/4)^3 - \frac{w}{6}\left(x - \frac{3L}{4}\right)^3 + C_1 \rightarrow (1)$$

Integrate again wrt x

$$EIy = \frac{w}{24}(x - L/4)^4 - \frac{w}{24}\left(x - \frac{3L}{4}\right)^4 + C_1x + C_2 \longrightarrow (2)$$

To find c_1 and c_2

Boundary conditions at $x = L$ in eq^n ① ;

$$\frac{dy}{dx} = 0$$

$$\Rightarrow 0 = \frac{w}{6}\left(3L/4\right)^3 - \frac{w}{6}\left(L/4\right)^3 + c_1$$

$$\Rightarrow 0 = \frac{27wL^3}{384} - \frac{wL^3}{384} + c_1$$

$$\Rightarrow \frac{-26wL^3}{384} + c_1 = 0$$

$$\boxed{c_1 = \frac{-13wL^3}{192}}$$

Boundary conditions at $x = L$ in eq^n ② $y = 0$

$$0 = \frac{w}{24}\left(3L/4\right)^4 - \frac{w}{24}\left(L/4\right) - \frac{13wL^4}{192} + c_2$$

$$0 = -\frac{81wL^4}{6144} + \frac{wL^4}{6144} + \frac{13wL^4}{192} + c_2$$

$$\boxed{c_2 = \frac{7wL^4}{128}}$$

Equating the value of c_1 and c_2 in eq^n ① and eq^n ②

① ⇒

$$EI\frac{dy}{dx} = \frac{w}{6}\left(x - L/4\right)^3 - \frac{w}{6}\left(x - 3L/4\right)^3 + \left(-\frac{13wL^3}{192}\right)$$

eq^n (2) $\Rightarrow$

$$EIy = \frac{w}{24}(x - L/4)^4 - \frac{w}{24}\left(x - \frac{3L}{4}\right)^4 - \frac{13wL^3}{192} + \frac{7wL^4}{128}$$

Deflection at the centre

at $x = 0$

$$EI\frac{dy}{dx} = \frac{w}{6}(x - L/4)^3 - \frac{w}{6}(x - 3L/4)^3 + \frac{-13wL^3}{192}$$

$$EI\frac{dy}{dx} = \frac{-13wL^3}{192}$$

$$\boxed{\frac{dy}{dx} = \frac{-13wL^3}{192EI}}$$

Slope at the centre at $x = L/2$

$$EI\frac{dy}{dx} = \frac{w}{6}(x - L/4)^3 - \frac{w}{6}(x - 3L/4)^3 + \left(\frac{-13wL^3}{192}\right)$$

$$= \frac{w}{6}\left(\frac{L}{2} - \frac{L}{4}\right)^3 - \frac{w}{6}\left(\frac{L}{2} - \frac{3L}{4}\right)^3 - \frac{13wL^3}{192}$$

$$= \frac{w}{6}\left(\frac{2L - L}{4}\right)^3 - \frac{w}{6}\left(\frac{2L - 3L}{4}\right)^3 - \frac{13wL^3}{192}$$

$$= \frac{w}{6}\left(\frac{L}{4}\right)^3 + \frac{w}{6}\cancel{\left(\frac{L}{4}\right)^3}^{0} - \frac{13wL^3}{192}$$

$$= \frac{wL^3}{384} - \frac{13wL^3}{192}$$

$$\boxed{\frac{dy}{dx} = \frac{-25wL^3}{384EI}}$$

Deflection at the free end

at $x = 0$

$$EIy = \frac{w}{24}(\cancel{x} - L/4)^4 - \frac{w}{24}\left(0 - \frac{3L}{4}\right)^4 + C_1x + C_2$$

$$-\frac{13wL^3}{192}x + \frac{7wL^4}{128}$$

$$EIy = \frac{7wL^4}{128}$$

$$\boxed{y = \frac{7wL^4}{128EI}}$$

Slope at the free end

at $x = L/2$

$$EIy = \frac{w}{24}\left(\frac{L}{2} - \frac{L}{4}\right)^4 - \frac{w}{24}\left(\frac{L}{2} - \frac{3L}{4}\right)^4 -$$

$$\frac{13wL^3}{192} + \frac{7wL^4}{128}$$

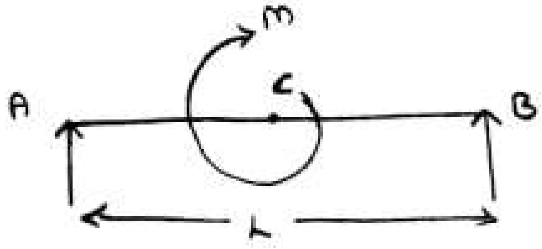
$$= \frac{w}{24}\left[\frac{2L-L}{4}\right)^4 - \frac{w}{6}\left(\cancel{\frac{2L-3L}{4}}\right)^4 - \frac{13wL^4}{384}$$

$$+ \frac{7wL^4}{128}$$

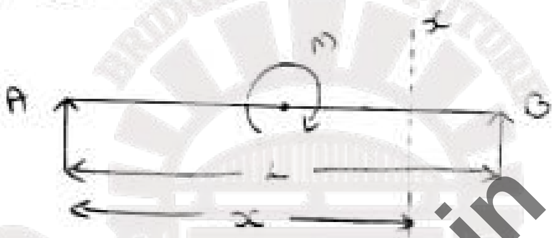
$$EIy = \frac{w}{24} \quad 2L-L \quad \frac{w}{24}\left(\frac{L}{4}\right)^4 - \frac{13wL^4}{384} + \frac{7wL^4}{128}$$

$$EIy = \frac{wL^4}{6144} - \frac{13wL^4}{384} + \frac{7wL^4}{128}$$

$$\boxed{y = \frac{43wL^4}{2048EI}}$$



A simply supported beam of length 'L' is Subjected to a clockwise 'm' at it's mid Span. Determine a slope at its Supports maximum deflection and its location



<u>To find support reactions</u>

$\Sigma V = 0 \Rightarrow R_A + R_B = 0$ —— ①

$\Sigma M_A = 0 \Rightarrow -(R_B \times L) + m = 0$

$\Rightarrow \boxed{R_B = \frac{m}{L}}$ $\boxed{R_A = \frac{-m}{L}}$

$M_x = R_A \cdot x + m\left(x - L/2\right)^0 = EI \frac{d^2y}{dx^2}$

$EI \frac{dy}{dx} = -\frac{m}{L} \cdot \frac{x^2}{2} + m\left(x - L/2\right) + C_1$ —— ②

$EI\, y = \frac{-mx^3}{6L} + \frac{m}{2}\left(x - L/2\right)^2 + C_1 x + C_2$ —— ③

at $x = 0$ $y = 0$

$0 = 0 + 0 + 0 + C_2$

$\boxed{C_2 = 0}$

at $x = L$, $y = 0$

$$0 = \frac{-mL^3}{6L} + \frac{m}{2}\left(L - L/2\right)^2 + C_1 L + C_2$$

$$0 = -\frac{mL^2}{6} + \frac{mL^2}{8} + C_1 L + 0$$

$$C_1 L = \frac{mL^2}{6} + \frac{mL^2}{8}$$

$$C_1 \cancel{L} = + \frac{mL^{\cancel{2}}}{24}$$

$$\boxed{C_1 = \frac{mL}{24}}$$

$$EI\frac{dy}{dx} = -\frac{m}{L}\cdot\frac{x^2}{2} + m\left(x - L/2\right) + \frac{mL}{24}$$

$$EI\,y = \frac{-mx^3}{6L} + \frac{m}{2}\left(x - L/2\right)^2 + \frac{mL}{24}x$$

Slope at A

Slope at $x = 0$

$$EI\frac{dy}{dx} = \frac{mL}{24}$$

$$\boxed{\left.\frac{dy}{dx}\right)_A = \frac{mL}{24EI}}$$

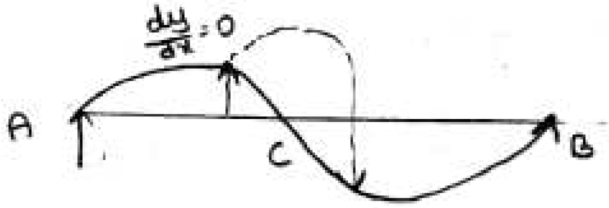
Slope at B

$\rightarrow$ at $x = L$

$$EI\frac{dy}{dx} = \frac{-m}{\cancel{L}}\cdot\frac{L^{\cancel{2}}}{\cancel{2}} + m\left(L - L/2\right) + \frac{mL}{24}$$

$$= \frac{-m\cancel{L}}{\cancel{2}} + \frac{m\cancel{L}}{\cancel{2}} + \frac{mL}{24}$$

$$\boxed{\left.\frac{dy}{dx}\right)_B = \frac{mL}{24EI}}$$



Since $\frac{dy}{dx}=0$ at maximum deflection and considering the portion AC

To find location of maximum deflection

apply $\frac{dy}{dx}=0$ in eqⁿ ②

$$0 = \frac{-mx^2}{2L} + \frac{mL}{24}$$

$$\Rightarrow \frac{x^2}{2L} = \frac{L}{24}$$

$$\Rightarrow x^2 = \frac{L^2}{12}$$

$$\boxed{x = 0.289\,L}$$

Substituting $x = 0.289\,L$ in eqⁿ ③

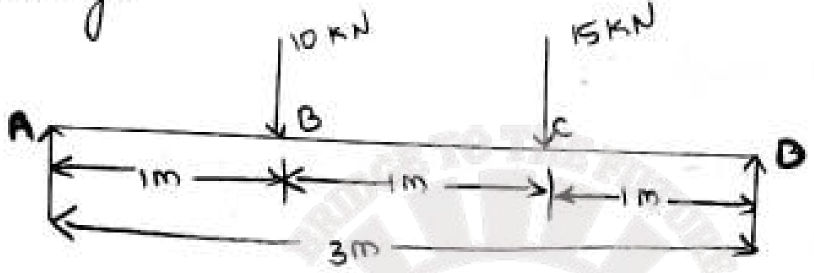
$$EI\,y_{max} = \frac{-m}{6L}(0.289L)^3 + \frac{mL}{24}(0.289\,L)$$

$$EIy = \frac{-m(0.0241)L^{\cancel{3}}}{6\cancel{L}} + \frac{0.289\,mL^2}{24}$$

$$= \frac{-0.0241\,mL^2}{6} + \frac{0.289\,mL^2}{24}$$

$$\boxed{y = \frac{0.008\,mL^2}{EI}}$$

A simply supported beam is loaded with two concentrated loads as shown in the figure. Find the slopes at the supports and deflection at B and C if $\omega_1 = 10$ KN, $\omega_2 = 15$ KN at a distance of 1m and 2m from the support A respectively.



Find the support reactions

$\Sigma V = 0$

$$R_A + R_D - 10 - 15$$

$$R_A + R_D - 25 = 0$$

$$\boxed{R_A + R_D = 25 \text{ KN}}$$

$\Sigma M_A = 0$

$$10 \times 1 + 15 \times 2 - R_D \times 3 = 0$$

$$R_D \times 3 = 40$$

$$\boxed{R_D = 13.33 \text{ KN}}$$

$$R_A + 13.33 = 25$$

$$\boxed{R_A = 11.67 \text{ KN}}$$

$$M_x = R_A \times x - 10 \times (x-1) - 15 \times (x-2)$$

$$M_x = 11.66x - 10(x-1) - 15(x-2)$$

$$EI \frac{d^2y}{dx^2} = M_x = 11.66x - 10(x-1) - 15(x-2)$$

Integrate w.r.t x

$$EI\frac{dy}{dx} = \frac{11.66x^2}{2} - \frac{10(x-1)^2}{2} - \frac{15(x-2)^2}{2} + C_1$$

$$EIy = \frac{11.66x^3}{6} - \frac{10(x-1)^3}{6} - \frac{15(x-2)^3}{6} + C_1x + C_2$$

To find C_1 and C_2

Boundary conditions at $x=0$ $y=0$

$$0 = C_2 \quad \boxed{C_2 = 0}$$

at $x=3$, $y=0$

$$EIy = \frac{11.66(3)^3}{6} - \frac{10(3-1)^3}{6} - \frac{15(3-2)^3}{6}$$

$$= 52.49 - 13.33 - 2.5$$

$$= 52.515 - 13.33 - 2.5 + C_1(3) + 0$$

$$3C_1 = 36.685$$

$$\boxed{C_1 = -12.22}$$

Substituting the value of C_1 and C_2 in eq^n (1) and (2)

$$EI\frac{dy}{dx} = \frac{11.66x^2}{2} - \frac{10(x-1)^2}{2} - \frac{15(x-2)^2}{2} - 12.22$$

$$EIy = \frac{11.66x^3}{6} - \frac{10(x-1)^3}{6} - \frac{15(x-2)^3}{6} - 12.22x + 0$$

Slope at Supports

i.e at $x = 0$

$$EI\frac{dy}{dx} = -12.22$$

$$\boxed{\frac{dy}{dx} = \frac{-12.22}{EI}}$$

Support D

at $x = 3$

$$EI\frac{dy}{dx} = \frac{11.66(3)^2}{2} - \frac{10(3-1)^2}{2} - \frac{15(3-2)^2}{2} - 12.22$$

$$\frac{dy}{dx} = \frac{11.66\times 9}{2} - \frac{10\times 4}{2} - \frac{15\times 1}{2} - 12.22$$

$$\boxed{\left.\frac{dy}{dx}\right|_D = \frac{12.75}{EI}}$$

Deflection at B

at $x = 1$

$$EIy_B = \frac{11.66\times 1^3}{6} - \frac{10(1-1)^3}{6} - \frac{15(1-2)^3}{6} - 12.22\times 1$$

$$\boxed{y_B = \frac{-10.276}{EI}}$$

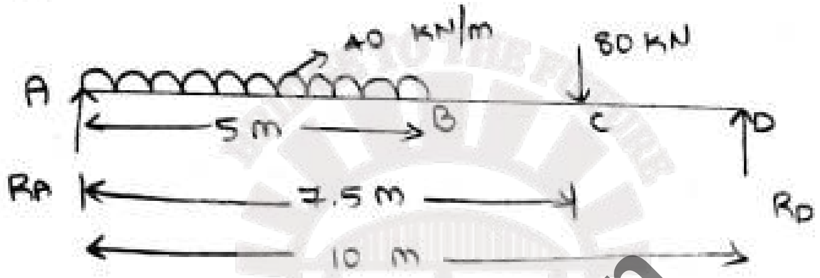
Deflection at D

at $x = 2$

$$EIy_C = \frac{11.66\times 2^3}{6} - \frac{10(2-1)^3}{6} - \frac{15(2-2)^3}{6} - 12.22\times 2$$

$$\boxed{y_C = \frac{-10.56}{EI}}$$

A simply supported beam of span 10 m carries at UDL of load 40 KN/m over a length of 5m of its left half and a concentrated load of 80 KN at 7.5 m from the left end. Determine the slopes at the ends and the deflections under the concentrated load and at mid span take 'E' to be equal to 200×10^6 KN/m² and $I = 56{,}000$ cm⁴



$\Sigma V = 0$

$R_A + R_D = 40 \times 5 + 80$

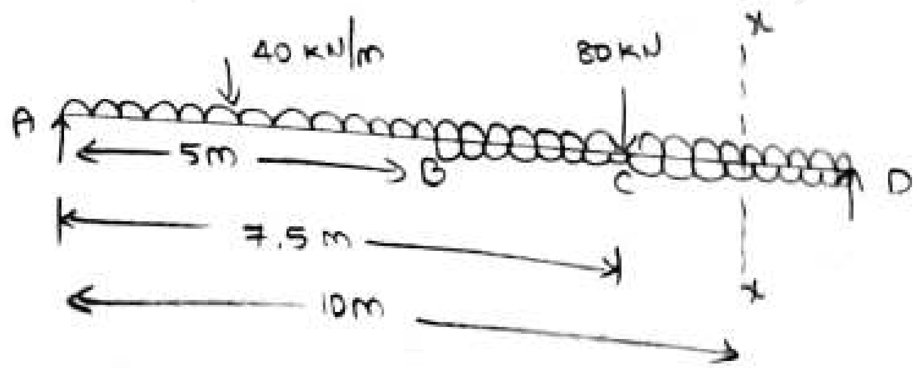
$R_A + R_D = 280$

$\Sigma M = 0$

$(40 \times 5) + (80 \times 7.5) + (R_D \times 10) = 0$

$\boxed{R_D = 110 \text{ KN}}$

$\boxed{R_A = 170 \text{ KN}}$



$M_x = 170 \times x - 40 \times x \times x/2 - 80 \times (x - 7.5)$

$+ 40 \times (x-5) \times \frac{(x-5)}{2}$

$$EI\frac{d^2y}{dx^2} = 170x - 20x^2 - 80(x-7.5) + 20(x-5)^2$$

$$EI\frac{dy}{dx} = \frac{170x^2}{2} - \frac{20x^3}{3} - \frac{80(x-7.5)^2}{2} + \frac{20(x-5)^3}{3} + C_1 \rightarrow (1)$$

$$EI\,y = \frac{170x^3}{6} - \frac{20x^4}{12} - \frac{80(x-7.5)^3}{6} + \frac{20(x-5)^4}{12} + C_1x + C_2 \rightarrow (2)$$

Finding C_1 and C_2

at $x=0$ $y=0$

$$\boxed{C_2 = 0}$$

at $x=10$ $y=0$

$$0 = \frac{170(10^3)}{6} - \frac{20(10)^4}{12} - \frac{80(10-7.5)^3}{6} + \frac{20(10-5)^4}{12} + C_1(10)$$

$$\boxed{C_1 = -1250}$$

Substituting the value of C_1 and C_2 in (1) and (2)

$$EI\frac{dy}{dx} = \frac{170x^2}{2} - \frac{20x^3}{3} - \frac{80(x-7.5)^2}{2} + \frac{20(x-5)^3}{3} - 1250$$

$$EI\,y = \frac{170x^3}{6} - \frac{20x^4}{12} - \frac{80(x-7.5)^3}{6} + \frac{20(x-5)^4}{12} + (-1250)x$$

MOMENT - AREA METHOD

The moment - Area method involves the use of a simple techinque of finding slopes and deflections due to bending from the bending moment diagram.

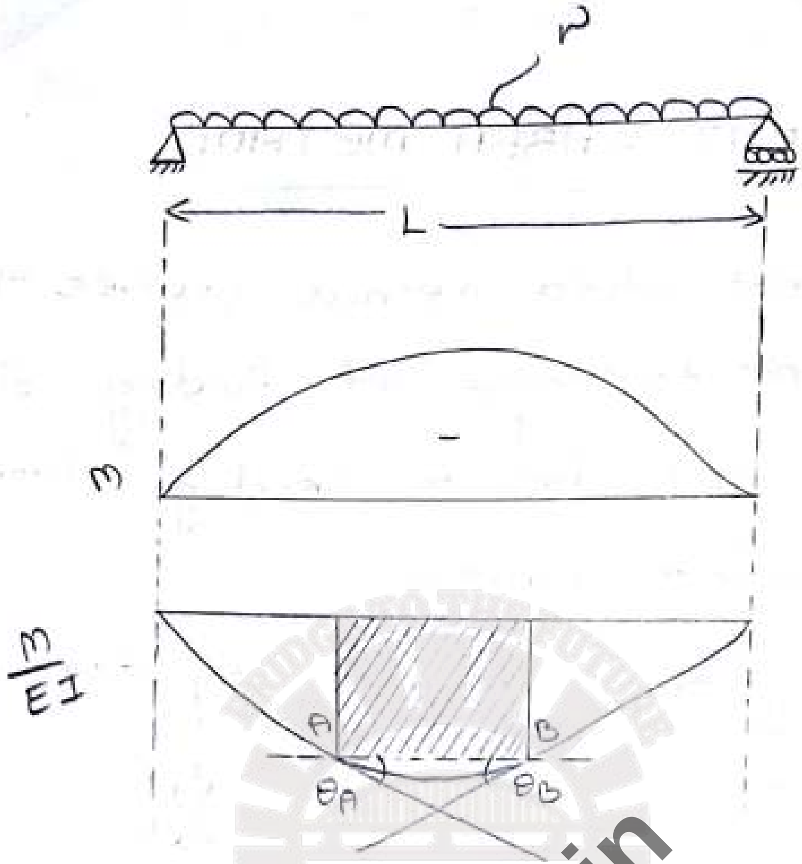
Starting from the basic equation $\frac{M}{EI}$ is equal to $\frac{M}{EI} = \frac{1}{R} = \frac{d^2y}{dx^2}$

It relates the area and moment of area of the bending moment diagram to the deformations in beams.

This method is particularly useful in finding slopes and deflections at specified points only. It is also useful when the cross sectional dimensions of the beam vary.

The First Moment Area diagram

It states that the angle between the tangents to the elastic curve between any two points is equal to the area of the M/EI daigram between the two points.



Consider a beam having a load 'w' as shown in the figure. The bending moment diagram is as shown in the figure and the curvature diagram also known as the $\frac{M}{EI}$ diagram is as shown in the figure.

From the moment curvature equation

WKT $\quad \frac{1}{R} = \frac{d^2y}{dx^2} = \frac{M}{EI}$

where $\frac{1}{R}$ = curvature of the elastic curve

The above equation can also be written as

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{M}{EI}$$

Here w.r.t. $\frac{dy}{dx} = \theta$

$$\therefore \frac{d\theta}{dx} = \frac{M}{EI} \quad \text{or} \quad d\theta = \frac{M}{EI} dx$$

considering the two points A and B on the elastic curve in which the slopes have to be found ~~that~~ out

we consider the integrating above equation from A to B

$$\int_{\theta_A}^{\theta_B} d\theta = \int_A^B \frac{M}{EI} dx$$

Here, $\theta_B - \theta_A$ = is ~~the~~ angle between the tangents drawn at points A and B of the elastic curve and integration of $\int_A^B \frac{M}{EI} dx$ is the area of the $\frac{M}{EI}$ diagram between these points

$$\boxed{\theta_B - \theta_A = \int_A^B \frac{M}{EI} dx}$$

This equation states what is called the first moment area theorem or the Mohr's theorem.

The Second moment Area Theorem

" It states that the moment of the area of $\frac{M}{EI}$ diagram between two points of a beam about one of these points is equal to the vertical intercept named by the tangent drawn at one point on a vertical line through the second point about which the moment is taken.

Consider an elastic curve as shown in the figure if 'R' is the Radius of the curvature for a small curve [part of an elastic curve] 'dx' making an angle 'dθ', to find the vertical intercept between points A and B, draw a tangents through A and

B as shown in the figure. Also draw tangents to the points forming 'dx' and extended to the vertical intercept as shown in the figure. Here 'x' is equal to the distance between point B and the small curve 'dx'. From the figure we know that

$$R\,d\theta = dx \quad \text{and similarly} \quad x\,d\theta = dy$$

Integrating the above equation we get

$$\int x\,d\theta = \int dy$$

we also know that $d\theta = \frac{M}{EI}\,dx$

$$\therefore \int x \cdot \frac{M}{EI}\,dx = \int dy$$

Integrating from points A to B we get

$$\boxed{\int_{y_A}^{y_B} dy = \bar{x}\int_A^B \frac{M}{EI}\,dx}$$

where $\bar{x}$ is the centroidal distance from the point in consideration.

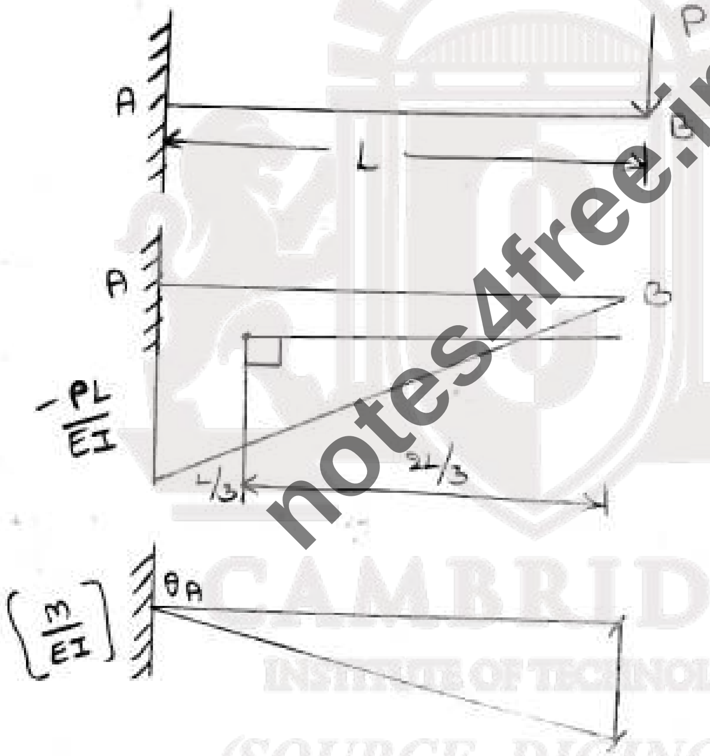
This proves the second moment area theorem.

<u>Note</u> :- The intercept will be made at the

point about which the moment is taken.

problem

① A cantilever of span 'L' carries a load 'p' at the free end. calculate the slope and deflection at the free end in terms of EI which is constant.



$$\Sigma M_A = 0 \Rightarrow (P \times L) - M_A$$

$$\Rightarrow M_A = PL$$

$$\theta_B - \cancel{\theta_A}^{0} = \int_A^B \frac{M}{EI} ds$$

$$\int_A^B \frac{M}{EI} dx = \frac{1}{2} \cdot L \cdot \frac{PL}{EI}$$

$$= \frac{PL^2}{2EI}$$

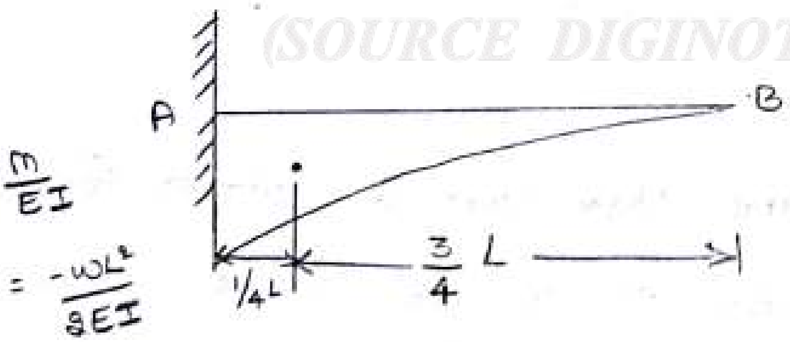
$$\boxed{\theta_B = \frac{-PL^2}{2EI}}$$

$$y_B - \cancel{y_A}^{\to 0} = \bar{x} \int_A^B \frac{M}{EI}\, dx$$

$$y_B = \frac{2L}{3}\left(\frac{-PL^2}{2EI}\right)$$

$$\boxed{y_B = \frac{-PL^3}{3EI}}$$

(2) A cantilever of uniform cross section carries a uniformly distrubed load 'ω' throughout it's length. calculate the slope and deflection at the free end.



Area of the parabola $= \frac{1}{3} b.h$

$\Sigma M_A = 0 \quad wL \times \frac{L}{2} - M_A$

$$M_A = \frac{wL^2}{2}$$

$$\theta_B - \theta_A = \int_A^B \frac{M}{EI} dx$$

$$\int_A^B \frac{M}{EI} dx = \frac{1}{3} \times L \times -\frac{wL^2}{2EI}$$

$$= \frac{wL^3}{6EI}$$

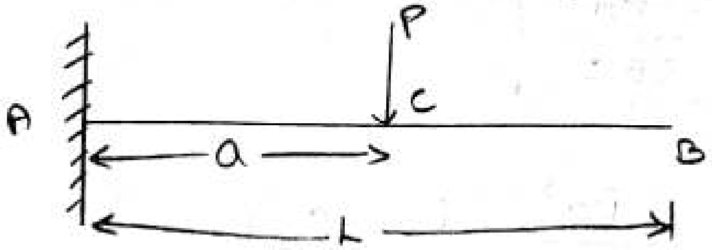
$$\boxed{\theta_B = \frac{wL^3}{6EI}}$$

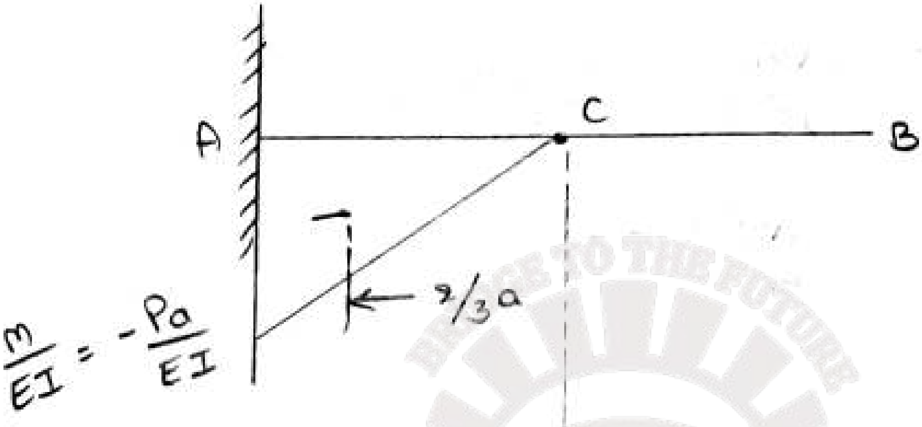
$$y_B - y_A = \bar{x} \int_A^B \frac{M}{EI} dx$$

$$y_B = \frac{3L}{4} - \frac{wL^3}{\cancel{6}_2 EI}$$

$$\boxed{y_B = \frac{-wL^4}{8EI}}$$

③ Calculate the slope and deflection for a cantilever as shown in the figure under the load and at the free end. If the load acts at a distance 'a' from 'A' where $a < L$.





Portion AC to find θ_c

1st moment Area theorem.

$$\theta_c - \cancel{\theta_A}^{0} = \int_A^C \frac{M}{EI}\, dx$$

$$\theta_c = \frac{1}{2} \cdot a \cdot -\frac{Pa}{EI}$$

$$\boxed{\theta_c = \frac{-Pa^2}{2EI}}$$

AC

$$y_c - \cancel{y_A}^{0} = \bar{x} \int_A^C \frac{M}{EI}\, dx$$

$$y_c = \frac{2}{3} a \left(\frac{-Pa^2}{2EI} \right)$$

$$\boxed{y_C = \frac{-Pa^3}{3EI}}$$

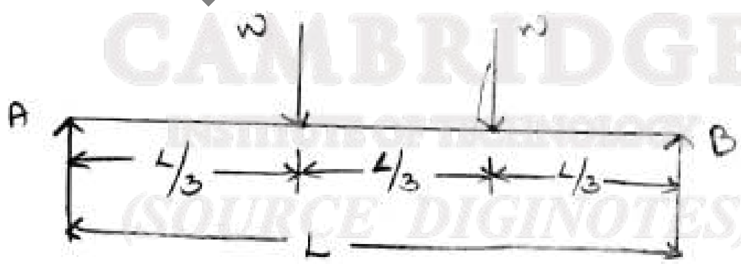
$$y_B = \frac{-Pa^2}{2EI}\left(l - \frac{a}{3}\right)$$

$$= \frac{-Pa^2}{2EI}\left(\frac{3l - a}{3}\right)$$

$$= \frac{-3Pa^2 l + Pa^3}{6EI}$$

$$\boxed{y_B = \frac{-Pa^2}{6EI}(3l - a)}$$

④ A simply supported beam carries two concentrated loads as shown in the figure. calculate the slope at the ends and the deflection under the loads



$\Sigma V = 0 \Rightarrow$ $R_A + R_B = 2w$ —— ①

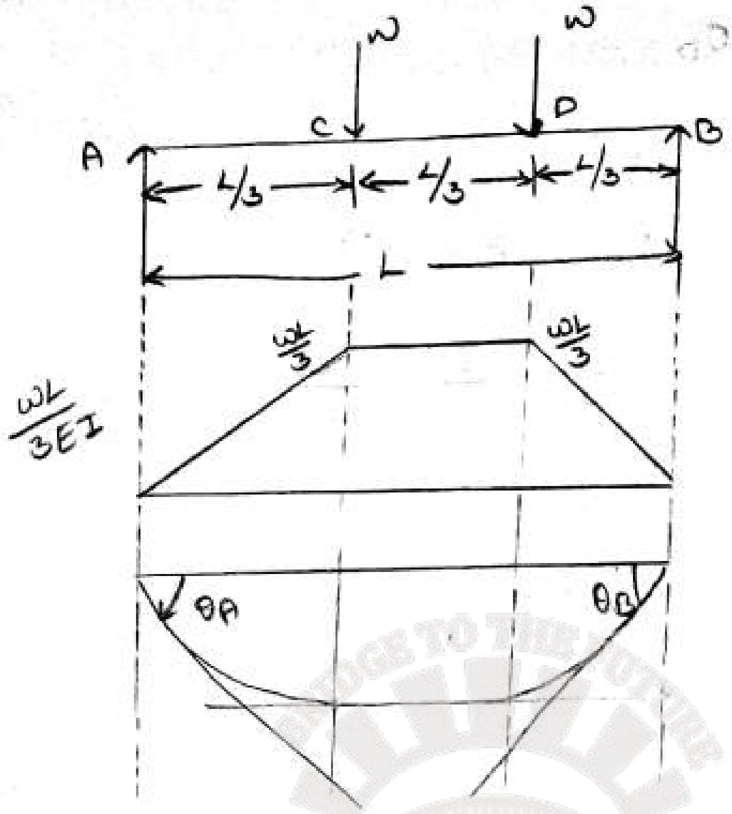
$\Sigma M_A = 0 \Rightarrow$ $(-R_B \times L) + w\left(\frac{L}{3}\right) + w\left(\frac{2L}{3}\right) = 0$

$$\boxed{R_B = w}$$

$R_A + R_B = 2w$

$R_A = 2w - w$

$\boxed{R_A = w \text{ kN}}$



θ_A and θ_B [AC]

$$\theta_A - \cancel{\theta_C}^{0} = \int_A^C \frac{M}{EI}\,dx$$

$$\theta_A = \frac{1}{2} \cdot \frac{L}{3} \cdot \frac{wL}{3EI}$$

$$\boxed{\theta_A = \frac{wL^2}{18EI}}$$

Deflection @ C [AC]

$$-y_C + \cancel{y_A}^{0} = \bar{x} \int_A^C \frac{M}{EI}\,dx$$

$$-y_C = \frac{wL^2}{18EI} \cdot \frac{1}{3} \cdot \frac{L}{3}$$

$$\boxed{y_C = \frac{-wL^3}{162EI}}$$

θ_B and θ_D

$$\theta_B - \theta_D = \int_B^D \frac{M}{EI}\,dx$$

$$= \frac{1}{2} \quad \frac{L}{3} \quad \frac{wL}{3EI}$$

$$\boxed{\theta_B = \frac{wL^2}{18EI}}$$

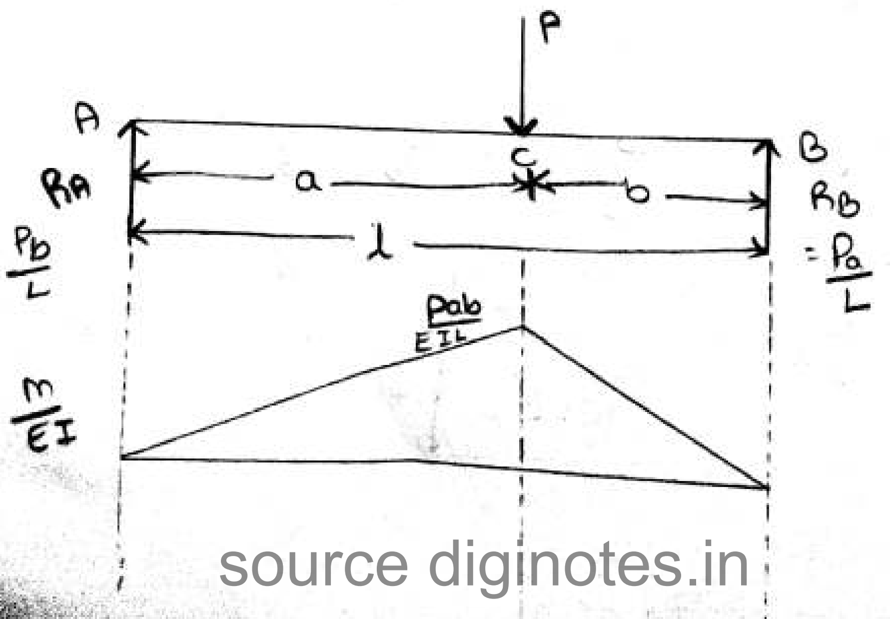
Deflection at D

$$\cancel{y_B}^{0} - y_D = \bar{x} \int_B^D \frac{M}{EI}\,dx$$

$$-y_D = \frac{wL^2}{18EI} \cdot \frac{1}{3} \cdot \frac{L}{3}$$

$$\boxed{y_D = \frac{-wL^3}{162EI}}$$

⑤ A simply supported beam carries a excentric load 'P' at distance 'a' from the left end. compute the slopes at the ends and deflection under the load.



$\Sigma Y = 0 \quad R_A + R_B = P$

$\Sigma M_A = 0$

$\Rightarrow$

$(-R_B \times l) + P(a) = 0$

$+R_B = +\frac{Pa}{l}$

$$\boxed{R_B = \frac{Pa}{L}}$$

$R_A + R_B = P$

$R_A = P - \frac{Pa}{L}$

$= \frac{PL - Pa}{L}$

$$\boxed{R_A = \frac{Pb}{L}}$$

[AC]

$\theta_A - \cancel{\theta_C} = \int_A^C \frac{m}{EI} \, dx$

$\theta_A = \frac{1}{2} \cdot a \cdot \frac{Pab}{EIL}$

$$\boxed{\theta_A = \frac{Pa^2 b}{2EIL}}$$

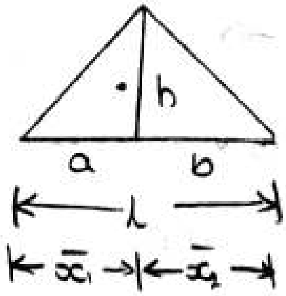
$\theta_B - \cancel{\theta_D}^{0} = \int_B^D \frac{m}{EI} \, dx$

[BC]

$\theta_B = \frac{1}{2} \cdot b \cdot \frac{Pab}{EIL}$

$$\boxed{\theta_B = \frac{Pab^2}{2EIL}}$$

Note :- For a triangle which is not a right angled triangle as shown in the figure.



The area $= \frac{bh}{2}$

centroid $= \bar{x}_1 = \frac{l+a}{3}$

$\bar{x}_2 = \frac{l+b}{3}$

Deflection at AC

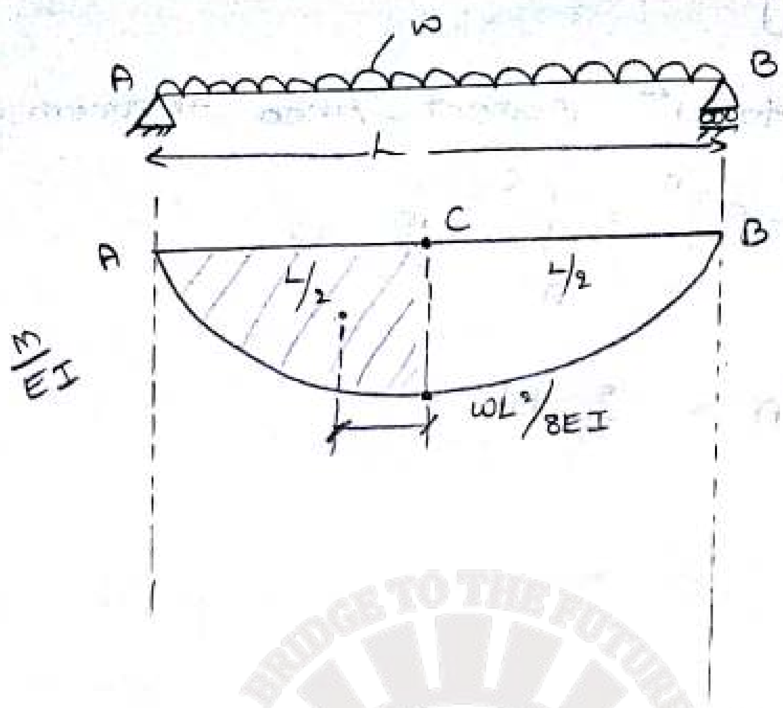
$$y_A - y_C = \frac{l+a}{3} \cdot \frac{Pa^2 b}{2EIL}$$

$$-y_C = \frac{Pa^2 Lb}{6EIL} + \frac{Pa^3 b}{6EIL}$$

$$-y_C = \frac{Pa^2 b}{6EI}\left[1 + \frac{a}{L}\right]$$

$$\boxed{y_C = \frac{-Pa^2 b}{6EIL}[L+a]}$$

⑥ A simply supported beam as shown in the figure carries a UDL of intensity 'w' on the total span of the beam. calculate the slopes at the ends and deflection at mid span.



Support Reactions

$\Sigma V = 0 \Rightarrow V_A + V_B = w \times L$ —— (1)

$\Sigma M = 0 \Rightarrow (-V_B \times L) + \left(w \cdot L \cdot \frac{L}{2}\right) = 0$

$\Rightarrow \boxed{V_B = \frac{wL}{2}}$

$V_A + V_B = w \times L$

$V_A + \frac{wL}{2} = wL$ $\qquad \frac{2wL - wL}{2} = \frac{wL}{2}$

$V_A = wL - \frac{wL}{2}$

$\boxed{V_A = \frac{wL}{2}}$

$BM_C = R_A \times \frac{L}{2} - \left(w \cdot \frac{L}{2} \cdot \frac{L}{4}\right)$

$= \frac{wL^2}{4} - \frac{wL^2}{8}$

$= \frac{+wL^2}{8}$

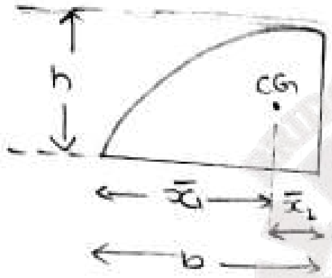
[AC]

According to 1st moment - Area theorem

$$\theta_A - \cancel{\theta_C}^{\,0} = \int_A^C \frac{m}{EI}\,dx$$

$$\theta_A = \frac{2}{3}\,bh$$

Note :-



$$\bar{x}_1 = \frac{5b}{8} \;;\; \bar{x}_2 = \frac{3b}{8}$$

$$\theta_A = \frac{2}{3}\cdot\frac{L}{2}\cdot\frac{\omega L^2}{8EI}$$

$$\boxed{\theta_A = \frac{\omega L^3}{24EI}}$$

$$\theta_B - \theta_D = \int_B^D \frac{m}{EI}\,dx$$

$$\theta_B = \frac{2}{3}\cdot\frac{L}{2}\cdot\frac{\omega L^2}{8EI}$$

$$\boxed{\theta_B = \frac{\omega L^3}{24EI}}$$

Deflection

According to 2nd moment Area theorem

[AC]

$$\cancel{y_A}^{0} - y_c = \bar{x}\int_A^C \frac{M}{EI}\,dx$$

$$-y_c = \frac{5}{8}\cdot\frac{L}{2}\left(\frac{wL^3}{24EI}\right)$$

$$\boxed{y_c = \frac{-5wL^4}{384EI}}$$

(7) A simply supported beam carrying concentrated load at the centre is as shown in the figure. compute the slope at the edges and deflection under the load.

Support reaction

$$R_A + R_B = w$$

$$\Sigma M_A = 0 \quad (-R_B \times L) + \left(w \times \frac{L}{2}\right)$$

$$R_B = \frac{w}{2}$$

$$R_A + \frac{w}{2} = w$$

$$R_A = w - \frac{w}{2} \qquad \boxed{R_A = \frac{w}{2}}$$

$$\theta_A - \cancel{\theta_C}^{\,0} = \int_A^C \frac{M}{EI}\,dx$$

$$= \frac{1}{2} \times \frac{L}{2} \cdot \frac{wL}{4EI}$$

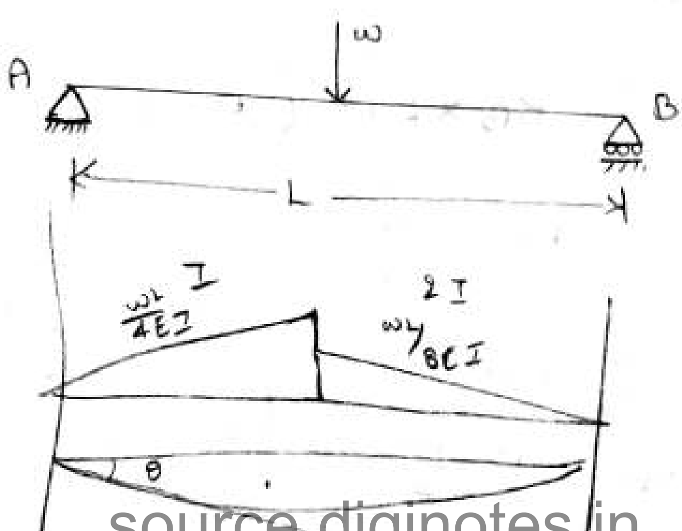
$$\boxed{\theta_A = \frac{wL^2}{16EI}}$$

$$\cancel{y_A}^{\,0} - y_C = \bar{x} \int_A^C \frac{M}{EI}\,dx$$

$$-y_C = \frac{\cancel{2}}{3} \cdot \frac{L}{\cancel{2}} \cdot \frac{wL^2}{16EI}$$

$$\boxed{y_C = \frac{-wL^3}{48EI}}$$

⑧ A simply supported beam of span 'L' is subjected to a concentrated load 'w' at its centre. The flexural rigidity of the cross section of the beam for the right half is twice that of left half span. Find the slope at supports and deflection at the mid point.



$\frac{M}{EI}$ $\frac{wL}{2}$ $\frac{1}{2EI}$

Note:- For cantilever beams in notice that the fixed end has no slope this means that the tangent at 'A' to the elastic curve is the same as the unbent beam axis.

In the case of a cantilever beam if you take a point at a any distance 'x' from the fixed end the angle between the tangent gives directly the slope and the vertical intercept directly gives a deflection.

The area of the shaded $\frac{M}{EI}$ diagram between A and C is the slope at point 'C'. Similarly the moment of the $\frac{M}{EI}$ diagram between A and C about 'C' directly gives the deflection at 'C'. The moment area method thus apt for solving cantilever beams.

⇒ For a simply supported beams as shown in the figure to find the slope at 'A', we proceed as follows.

* Find the moment of area of the $\frac{M}{EI}$ diagram between A and B about 'B'. This gives the vertical intercepts Y_{AB} [vertical distance at B from the tangent A]

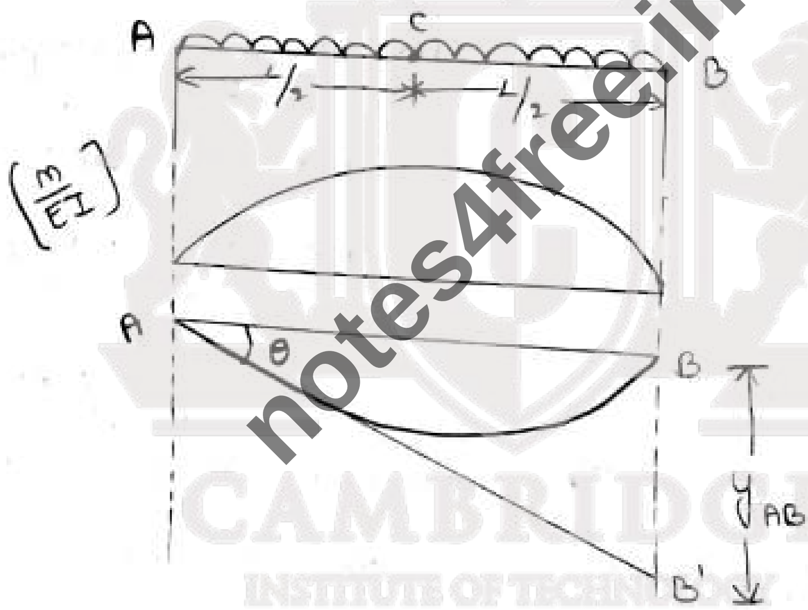
* From the triangle $AB'B$, $\tan\theta = \frac{B'B}{AB}$

$\tan\theta = \frac{B'B}{AB} = \frac{y_{AB}}{\alpha}$ as θ is very small

$\tan\theta$ is almost equal to θ in radians ($\tan\theta = \theta$)

this gives the slope at A.

* Slope at B can be similarly foundout by taking the moment about point A.

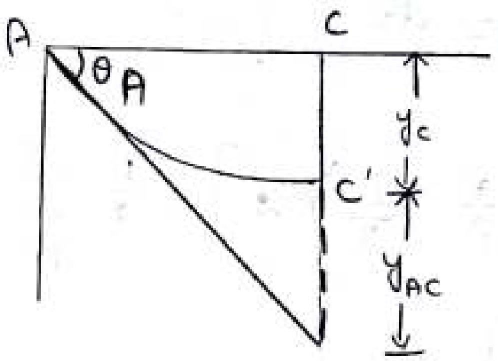


* To find deflection

(1) If you find the moment of area of $\frac{M}{EI}$ diagram between A and C about C. This gives the length $C'C''$.

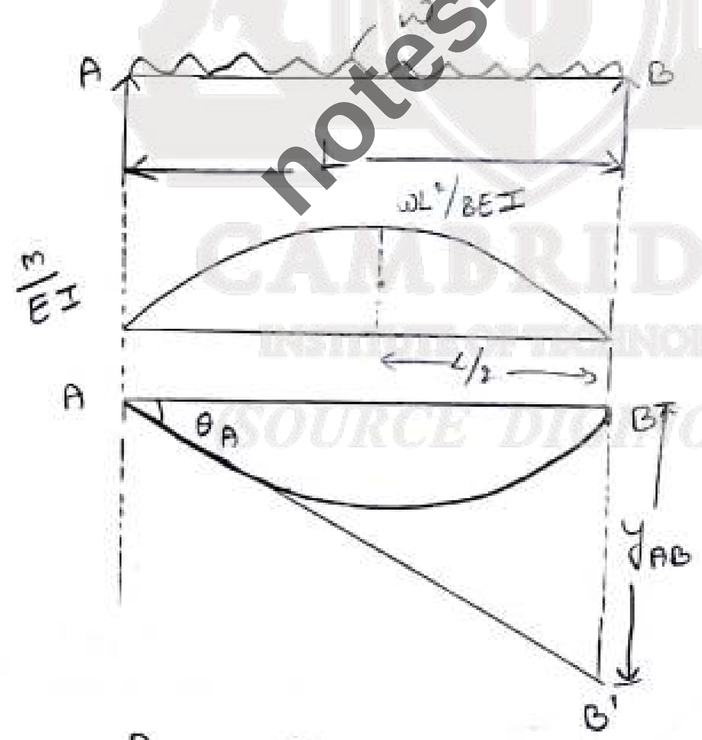
* From the triangle $AC''C$

$\frac{C''C}{AC} = \tan\theta$



Knowing the slope at A $C''C$ can be calculated. Deflection at C = $CC' = C'C - C'C''$

(1) Calculate the slope at the ends and the deflection at the centre of the given simply supported beam of span L



$$R_A = R_B = \frac{wL}{2}$$

<u>Step - 1</u> To find slope

* @ A

$$y_{AB} = \bar{x}\int_{A}^{B} \frac{m}{EI}\,dx$$

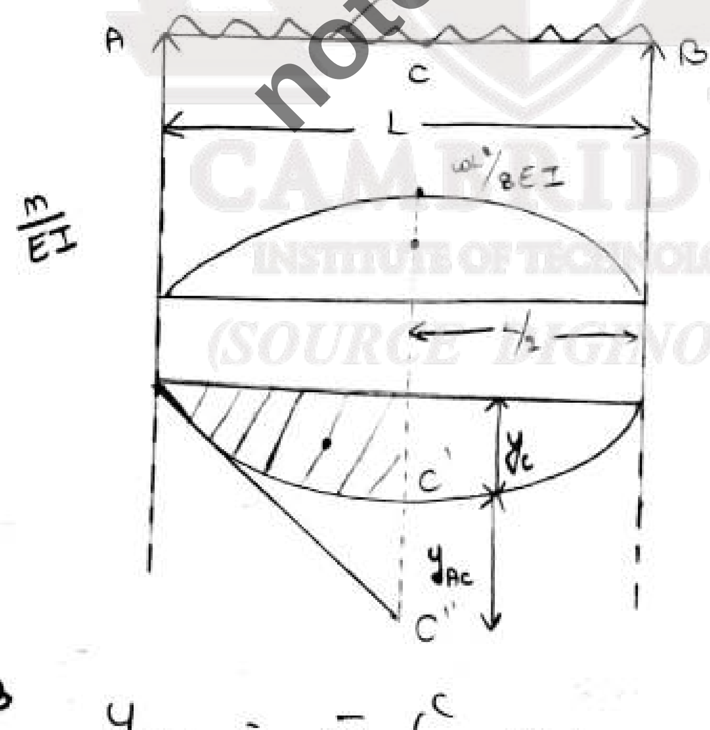
$$= \left[\left(\frac{wL^2}{\cancel{8}_{4}EI} \times \frac{\cancel{2}}{3}\cdot\frac{L}{\cancel{2}}\right)\cancel{2}\right]\frac{L}{2}$$

$$\boxed{y_{AB} = \frac{wL^4}{24EI}}$$

$$\theta_A = \frac{y_{AB}}{L} = \frac{wL^{\cancel{4}\,3}}{24EI\cancel{L}}$$

$$\boxed{\theta_A = \frac{wL^3}{24EI}}$$

<u>Step 2</u> :- <u>To find Deflection</u>



$$y_{AC} = \bar{x}\int_{A}^{C} \frac{m}{EI}\,dx$$

$$= \left(\frac{\cancel{5}}{8}\times\frac{L}{\cancel{2}}\right)\times\left(\frac{\cancel{2}}{3}\cdot\frac{L}{2}\cdot\frac{wL^2}{8EI}\right)$$

$$\boxed{y_{AC} = \frac{\omega L^4}{128EI}}$$

$$\theta_A = \frac{CC''}{L/2} \Rightarrow \frac{\omega L^3}{24EI} \times \frac{L}{2} = CC''$$

$$CC'' = \frac{\omega L^4}{48EI}$$

$$y_C = CC' + C'C''$$

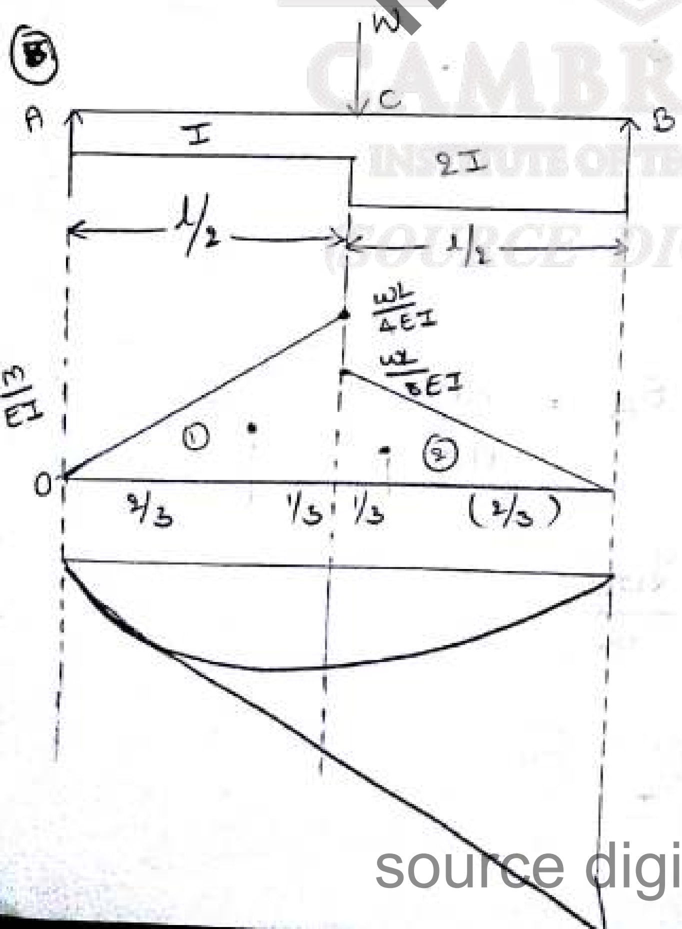
$$= y_C + y_{AC}$$

$$y_C = CC'' - y_{AC}$$

$$= \frac{\omega L^4}{48EI} - \frac{\omega L^4}{128EI}$$

$$\boxed{y_C = \frac{5\omega L^4}{384EI}}$$

(5)



$R_A = R_B = \frac{\omega}{2}$

$BM_C = \frac{\omega}{2} \times \frac{L}{2} = \frac{\omega L}{4}$

$\frac{\omega L}{4} \quad \frac{1}{2EI} = \frac{\omega L}{8EI}$

$$\theta_A = \frac{y_{AB}}{L}$$

$$y_{AB} = \bar{x}\int_A^B \frac{m}{EI}\,dx$$

$$= \bar{x}\left\{\left[\frac{1}{2}\times\frac{L}{2}\times\frac{wL}{4EI}\right]\times\left(\frac{L}{2}+\frac{L}{6}\right) + \left[\frac{1}{2}\times\frac{L}{2}\times\frac{wL}{8EI}\right]\times\frac{2}{3}\,\frac{L}{2}\right\}$$

$$y_{AB} = \frac{wL^2}{16EI}\left(\frac{4L}{6}\right) + \frac{wL^2}{32EI}\left(\frac{L}{3}\right)$$

$$= \frac{wL^3}{24EI} + \frac{wL^3}{96EI}$$

$$\boxed{y_{AB} = \frac{5wL^3}{96EI}}$$

$$\theta_A = \frac{y_{AB}}{L} = \frac{5wL^3}{96EI}\Big/L$$

$$\boxed{\theta_A = \frac{5wL^2}{96EI}}$$

Δ^{le} ABA' $\quad \tan\theta_B = \dfrac{AA'}{AB}$

$$\tan\theta_B = \theta_B = \frac{y_{BA}}{\alpha}$$

$$y_{B.A} = \bar{x}\int_B^A \frac{m}{EI}\,dx$$

$$y_{BA} = \left\{\left(\frac{1}{2} \times \frac{L}{2} \times \frac{\omega L}{4EI}\right) \times \left(\frac{2}{3} \cdot \frac{L}{2}\right)\right\} + \left\{\left(\frac{1}{2} \times \frac{L}{2} \times \frac{\omega L}{8EI}\right)\left(\frac{L}{2} + \frac{L}{6}\right)\right\}$$

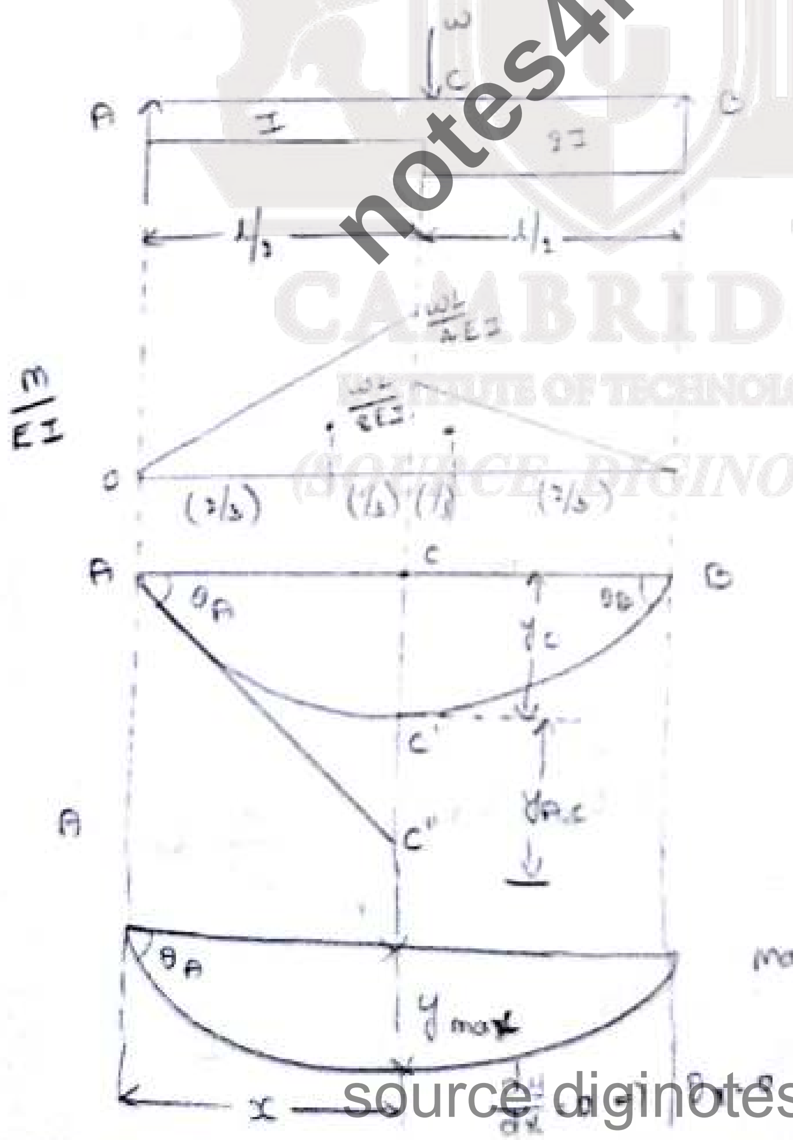
$$= \frac{\omega L^3}{48EI} + \frac{\omega L^3}{48EI}$$

$$= \frac{\omega L^3}{24EI}$$

$$\theta_B = \frac{y_{BA}}{L} = \frac{\omega L^3}{24EI} \Big/ L$$

$$\boxed{\theta_B = \frac{\omega L^2}{24EI}}$$

$x = 0.4565L$



maximum deflection

$$y_{A,C} = \bar{x}\int_A^C \frac{M}{EI}\,dx \quad \text{wrt 'c'}$$

$$= \left\{\frac{1}{2}\times\frac{L}{2}\times\frac{wL}{4EI}\right\}\times\left(\frac{1}{3}\cdot\frac{L}{2}\right)$$

$$y_{A,C} = \frac{wL^3}{96EI}$$

$$\theta_A = \frac{CC''}{AC} \Rightarrow \theta_A \times AC = CC''$$

$$\Rightarrow CC'' = CC' + C'C''$$

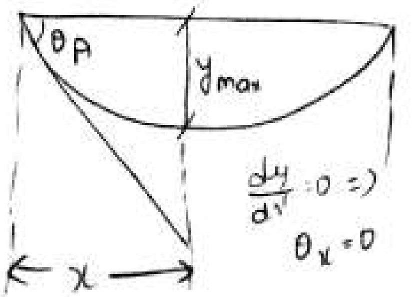
$$= y_c + y_{A,C}$$

$$\theta_A \times A_C = y_c + y_{A,C}$$

$$\frac{5wL^2}{96EI}\times\frac{L}{2} = y_c + \frac{wL^3}{96EI}$$

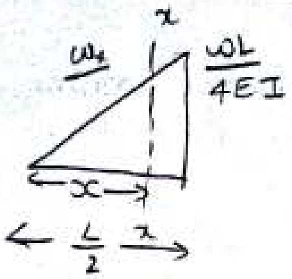
$$\frac{5wL^3}{192EI} - \frac{wL^3}{96EI} = y_c$$

$$\boxed{y_c = \frac{wL^3}{64EI}}$$ downward



$$\theta_A - \cancel{\theta_x}^{\,0} = \int_A^x \frac{M}{EI}\,dx$$

$$\theta_A = \frac{5wL^2}{96EI} = \frac{1}{2}\times x$$



$$\frac{\frac{wL}{4EI}}{L/2} = \frac{y}{x}$$

$$\Rightarrow \frac{wx}{2EI} = y$$

$$\theta_A = \frac{5wL^2}{96EI} = \frac{1}{2} \times x \cdot \frac{wx}{2EI}$$

$$\Rightarrow \frac{5L^2}{96} = \frac{x^2}{4}$$

$$y_{A,x} = \bar{x} \int_A^x \frac{m}{EI}\, dx$$

$$= \left[\frac{1}{3}(x)\right] \left[\frac{1}{2} \times x \times \frac{wx}{2EI}\right]$$

$$y_{A,x} = 0.079 \frac{wL^3}{EI}$$

$$A \times x'' \Rightarrow \theta_A = \frac{x x''}{A x}$$

$$\frac{5wL^2}{96EI} = \frac{x x' + x' x''}{x}$$

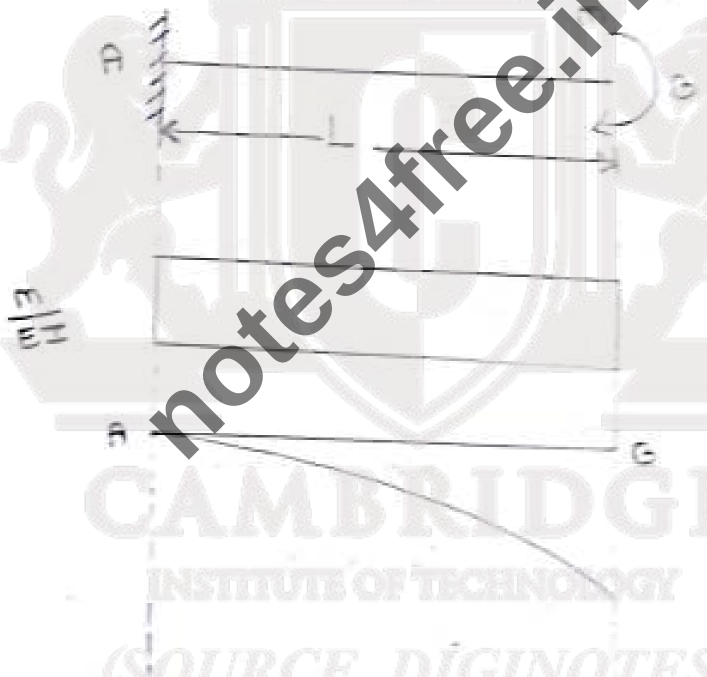
$$\frac{5wL^2}{96EI} \cdot x = y_{max} + y_{A,x}$$

$$\Rightarrow y_{max} = \frac{5wL^2 x}{96EI} - y_{A,x}$$

$$\boxed{y_{max} = \frac{wL^3}{63.09EI}}$$

problems on cantilever beam using moment area method

① A cantilever beam AB of length 'l' is subjected to a bending moment or a couple 'm' as shown in the figure. Find the slope and deflection at the free end. Take EI as constant.



$\Sigma V = 0$, $\Sigma H = 0$

$\Sigma M_B = 0 \Rightarrow -m_A + m = 0$

$$\boxed{m_A = m}$$

$$\theta_B - \cancel{\theta_A}^{\,0} = \int \frac{M}{EI}\,dx = \frac{M}{EI}\cdot L$$

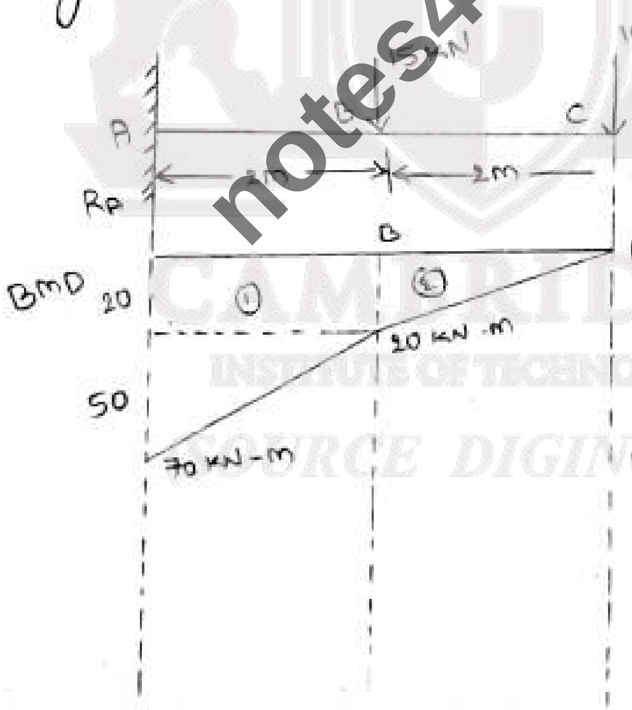
$$\boxed{\theta_B = \frac{ML}{EI}}$$

$$y_B - \cancel{y_A}^{\,0} = \bar{x}\int \frac{M}{EI}\,dx$$

$$y_B = \frac{L}{2}\cdot\frac{ML}{EI}$$

$$\boxed{y_B = \frac{ML^2}{2EI}}$$

② Calculate the slope and deflection at B and C of the cantilever beam as shown in the figure.



$BM_C = 0$

$BM_B = 10 \times 2$
$= 20$ kN-m

$BM_A = 70$ kN-m

$$R_A - 15 - 10 = 0$$

$$R_A - 25 = 0$$

$$\boxed{R_A = 25 \text{ kN}}$$

$\Sigma M_A = 0 \Rightarrow$

$$M_A = (10 \times 4) + (15 \times 2)$$

$$\boxed{M_A = 70 \text{ KN-m}}$$

$$\theta_B - \cancel{\theta_A}^{\,0} = \int_B^A \frac{M}{EI}\,dx$$

$$\theta_B = \text{Area } [① + ②] \text{ of } \frac{M}{EI} \text{ diagram}$$

$$= \frac{(2 \times 20) + (\frac{1}{2} \times 2 \times 50)}{EI}$$

$$\boxed{\theta_B = \frac{90}{EI}}$$

$$\theta_C - \cancel{\theta_A}^{\,0} = \int_C^A \frac{M}{EI}\,dx$$

$$= \frac{(2 \times 20) + (\frac{1}{2} \times 2 \times 50) + (\frac{1}{2} \times 2 \times 20)}{EI}$$

$$\boxed{\theta_C = \frac{110}{EI}}$$

Take EI to be equal to $5.5 \times 10^{+3}$ KN-m^2

$$\theta_B = \frac{90}{EI} = \frac{90}{5.5 \times 10^3} = 0.01636$$

$$\theta_C = \frac{110}{EI} = \frac{110}{5.5 \times 10^3} = 0.02$$

Deflection

$$y_B - \cancel{y_A}^{\,0} = \bar{x}\int_B^A \frac{m}{EI}\,dx$$

$$y_B = \frac{\left(2\times 20\times \frac{x}{x}\right)+\left(\frac{1}{2}\times 50\times \frac{2}{\cancel{..}}\right)\left(\frac{2}{3}\times 2\right)}{EI}$$

$$y_B = \frac{106.67}{EI}$$

$$y_B = \frac{106.67}{5.5\times10^3} = 0.0193 \text{ m}$$

y_B = 19 mm acting downwards

$$y_C - \cancel{y_A}^{\,0} = \bar{x}\int_C^A \frac{m}{EI}\,dx$$

$$y_C = \frac{\left[20\times 2\times\left(2+\frac{2}{2}\right)\right]+\left[\frac{1}{2}\times 50\times \cancel{2}\times\left(2+\frac{2}{3}\times 2\right)\right]+\left[\frac{1}{\cancel{2}}\times\cancel{2}\times 20\times\frac{2}{3}\times 2\right]}{EI}$$

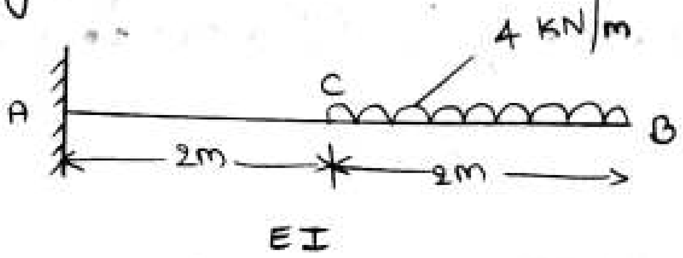
$$y_C = \frac{120+\frac{500}{3}+\frac{80}{3}}{EI}$$

$$y_C = \frac{313.33}{EI} = \frac{313.33}{5.5\times10^3} = 0.05696 \text{ m}$$

$y_C = 0.05696$ m ↓

$\simeq$ 56.96 mm

③ Determine the slope and deflection at the free end for the cantilever beam as shown in the figure.



$\Sigma V = 0$

$V_A - 8 = 0$

$\boxed{V_A = 8\ KN}$

$\Sigma m_A = 0$

$-m_A + 8 \times 3 = 0$

$\boxed{m_A = 24\ KN\text{-}m}$

$m_A = -4 \times 2 \times 3$

$\boxed{m_A = -24\ KN\text{-}m}$

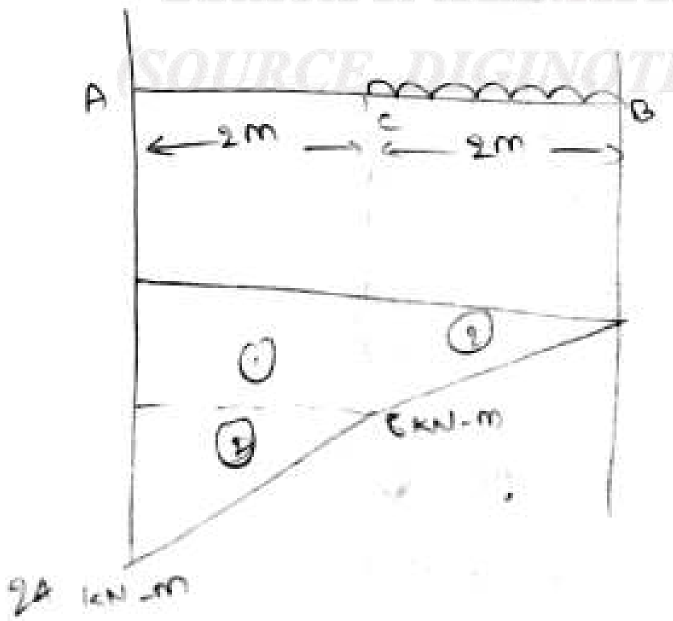
$m_C = -4 \times 2 \times 1$

$\boxed{m_C = -8\ KN\text{-}m}$

$m_B = 8 \times 4 - 8 \times 1 - 24$

$m_B = 32 - 32$

$\boxed{m_B = 0}$



Slope at B :-

portion AB :-

$$\theta_B - \cancel{\theta_A}^{\,0} = \int_B^A \frac{M}{EI}\,dx$$

$$\theta_B = \text{Area of } \left[1 + 3 + 2 \text{ of } \frac{M}{EI} \text{ dia} \right]$$

$$\theta_B = \frac{[2\times 8] + \left[\frac{1}{2}\times 2\times 16\right] + \left[\frac{1}{3}\times 2\times 8\right]}{EI}$$

$$\theta_B = \frac{16 + 16 + \frac{16}{3}}{EI}$$

$$\theta_B = \frac{112}{3EI} \quad \text{or} \quad \frac{37.33}{EI}$$

Deflection at B :-

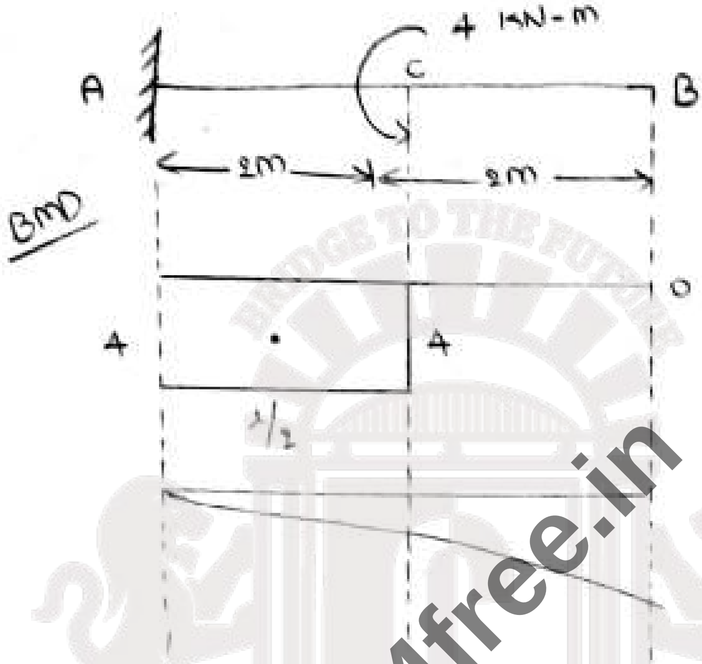
$$y_B - \cancel{y_A}^{\,0} = \int_B^A \frac{M}{EI}\,x\,dx$$

$$y_B = \frac{\left[2\times 8 \times\left(2+\frac{\cancel{2}}{\cancel{2}}\right)\right] + \left[\frac{1}{2}\times 16\times 2\times\left(2\times\frac{2}{3}\times 2\right)\right] + \left[\frac{1}{\cancel{3}}\times 2\times 8\times\left(\frac{\cancel{3}}{4}\times 2\right)\right]}{EI}$$

$$y_B = \frac{48 + \frac{160}{3} + 8}{EI}$$

$$y_B = \frac{328}{3EI} \quad \text{or} \quad \frac{109.33}{EI}$$

④ A cantilever beam as shown in the figure is subjected to a moment of 4 kN-m at its mid span. calculate the slope and deflection at it's free end.



$\Sigma V = 0$

$\Sigma M_A = 0$

$\boxed{V_A = 0}$ $\boxed{H_A = 0}$

$-M_A = 4$

$\boxed{M_A = -4 \text{ kN-m}}$

Slope at B :-

$$\theta_B - \cancel{\theta_A}^{0} = \int_B^A \frac{M}{EI}\, dx$$

$$\theta_B = \frac{4 \times 2}{EI}$$

$$\boxed{\theta_B = \frac{8}{EI}}$$

Deflection at B :

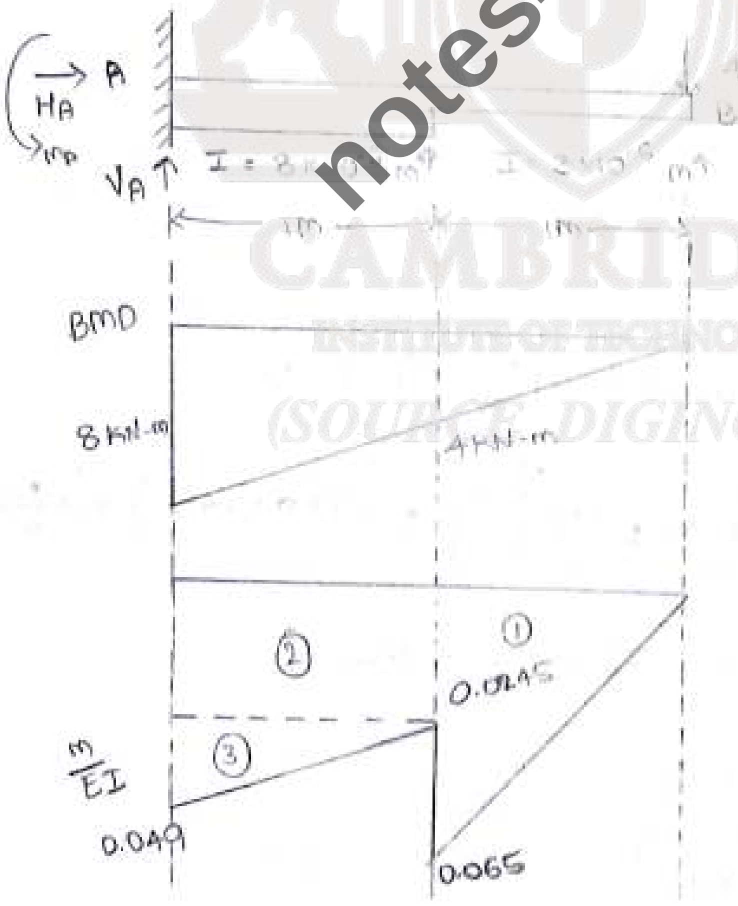
$$y_B - \cancel{y_A}^{0} = \bar{x} \int_B^A \frac{m}{EI} dx$$

$$y_B = \frac{4 \times 2 \times \left(2 + \frac{2}{2}\right)}{EI}$$

$$y_B = \frac{8 \times (3)}{EI}$$

$$\boxed{y_B = \frac{24}{EI}}$$

(5) Find the deflection and slope of the cantilever beam AB shown in the figure. Take $E = 204 \times 10^5 \text{ kN/m}^2$



Support reactions

$\Sigma H = 0 \Rightarrow H_A = 0$

$\Sigma V = 0 \Rightarrow V_A - 4 = 0$

$$\boxed{V_A = 4\,KN}$$

$\Sigma M = 0 \Rightarrow -M_A + (4 \times 2) = 0$

$$\Rightarrow \boxed{M_A = 8\,KN\text{-}m}$$

$$\theta_B - \cancel{\theta_A}^{0} = \int_A^B \frac{M}{EI}\,dx$$

$$\theta_B = \left(\frac{1}{2} \times 1 \times 0.065\right) + (0.0245 \times 1) + \left(\frac{1}{2} \times 1 \times 0.0245\right)$$

$$\boxed{\theta_B = 0.069 \text{ radians}}$$

$$y_B - \cancel{y_A}^{0} = \int_A^B \frac{M}{EI}\,x\,dx$$

$$y_B = \left[\left(\frac{1}{2} \times 1 \times 0.065\right) \times \left(\frac{2}{3} \times 1\right)\right] +$$

$$\left[(0.0245 \times 1) \times \left(\frac{1}{2} + 1\right)\right] + \left[\left(\frac{1}{2} \times 1 \times 0.0245\right) \times \left(1 + \frac{2}{3}(1)\right)\right]$$

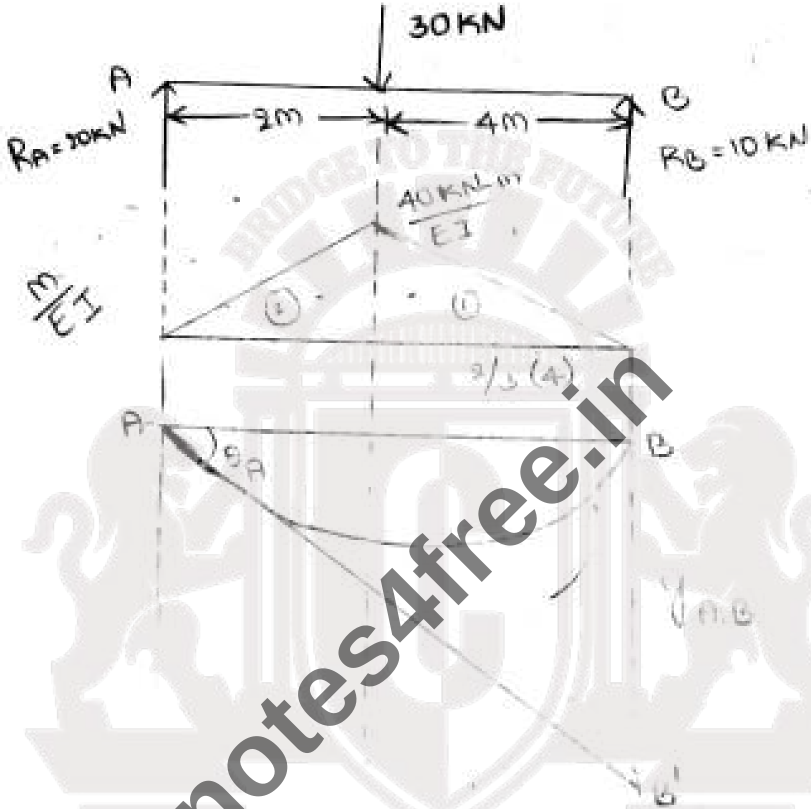
$$= 0.0216 + 0.03675 + 0.02041$$

$$= -0.0788 \text{ m}$$

$$\boxed{y_B = -78.8 \text{ mm}}$$

problems on simply supported beams

① Find the maximum slope and deflection for the beam as shown in figure using the moment - area method take $EI = 10.2 \times 10^3$ kN/m^2



$\Sigma V = 0$

$R_A + R_B = 30$ KN

$\Sigma M_A = 0$ $(-R_B \times 6) + (30 \times 2) = 0$

$(-R_B \times 6) + 60 = 0$

$-R_B = -10$

$\boxed{R_B = 10 \text{ KN}}$

$R_A + R_B = 30$

$R_A + 10 = 30$

$R_A = 30 - 10$

$\boxed{R_A = 20 \text{ KN}}$

B.m @ A and B = 0

B.m @ C = 20 × 2 = 40 KN-m

$$\theta_A = \frac{y_{AB}}{L}$$

$$y_{AB} = \bar{x} \int_A^B \frac{M}{EI}\, dx$$

$$= \left[\left(\frac{1}{2} \times 4 \times \frac{40}{EI}\right)\left(\frac{2}{3} \times 4\right)\right] + \left[\left(\frac{1}{2} \times 2 \times \frac{40}{EI}\right)\left(4 + \left(\frac{1}{3} \times 2\right)\right)\right]$$

$y_{AB} =$

$y_{AB} = 0.039$ m

$$\theta_A = \frac{y_{A,B}}{L}$$

$$= \frac{0.039}{6}$$

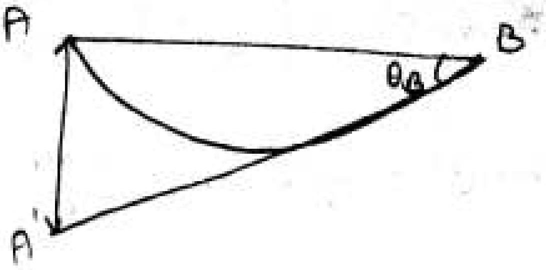
$$\boxed{\theta_A = 6.53 \times 10^{-3} \text{ radians}}$$

$= 0.00653$

$$\theta_B = \frac{y_{B,A}}{L}$$

$$y_{B,A} = \left[\left(\frac{1}{2} \times 4 \times \frac{10}{EI}\right)\left(2 + \frac{1}{3}(4)\right)\right] + \left[\left(\frac{1}{2} \times 2 \times \frac{40}{EI}\right)\left(\frac{2}{3} \times 2\right)\right]$$

$y_{B.A} = 0.0312 \text{ m}$

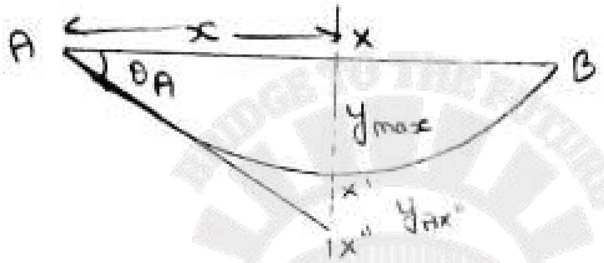


$\theta_B = \frac{y_{B.A}}{L}$

$= 0.0312$

$$\boxed{\theta_B = 5.2 \times 10^{-3} \text{ radians}}$$

$= 0.0052$



$\theta_A = \frac{XX''}{x}$

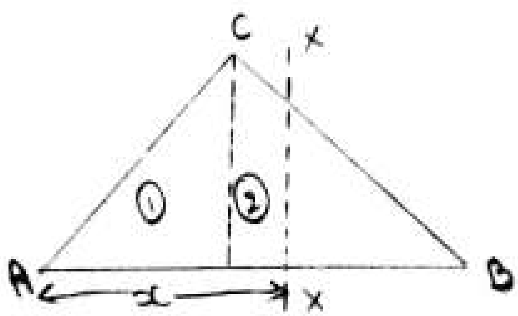
$\theta_A \cdot x = XX' + X'X''$ — (1)

To find 'x'

Portion AX

$\theta_A - \cancel{\theta_X}^{\,0} = \int_A^X \frac{M}{EI}\, dx$

$\theta_A =$



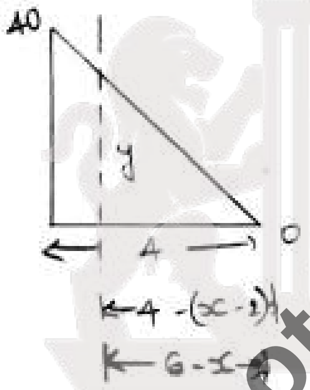
$$\text{Area } ① = \left(\frac{1}{2} \times 2 \times \frac{40}{EI}\right)$$

$$\text{Area } ② = \frac{1}{2} \times (x-2)\left[\frac{40}{EI} + \frac{10(6-x)}{EI}\right]$$

$$= \frac{1}{2}(x-2)\left[\frac{40+60-10x}{EI}\right]$$

$$= \frac{1}{2}(x-2)\left[\frac{100-10x}{EI}\right]$$

$$= \frac{1}{2EI}\left[100x - 200 - 100x^2 + 20x\right]$$



$$\frac{40}{4} = \frac{y}{(6-x)}$$

$$\Rightarrow \frac{40(6-x)}{4} = y$$

$$\frac{10(6-x)}{EI} = y$$

$$\text{Area } ② = \frac{\left[-10x^2 + 120x - 200\right]}{2EI}$$

$$= \frac{-5x^2}{EI} + \frac{60x}{EI} - \frac{100}{EI}$$

$$\theta_A = \text{Area } ① + \text{Area } ②$$

$$= \frac{40}{EI} - \frac{5x^2}{EI} + \frac{60x}{EI} - \frac{100}{EI}$$

$$= \frac{-60}{EI} - \frac{5x^2}{EI} + \frac{60x}{EI}$$

$$\theta_A \times EI = -5x^2 + 60x - 60$$

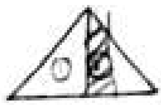
$$66.504 = -5x^2 + 60x - 60$$

$$\Rightarrow \quad -5x^2 + 60x - 126.504 = 0$$

$$\boxed{x = 2.729\ m}$$

$$\theta_A \cdot x = xx' + x'x''$$

$$y_{Ax''} = x'x'' = \bar{x}\int_A^x \frac{m}{EI}\,dx$$



$$= \left(\frac{1}{2} \times 2 \times \frac{40}{EI}\right) \times \left[x - \frac{2}{3}(2)\right]$$

$$+ \left[\frac{1}{2} \times (x-2)\left(\frac{40}{EI} + \frac{10(6-x)}{EI}\right)\frac{(x-2)}{2}\right]$$

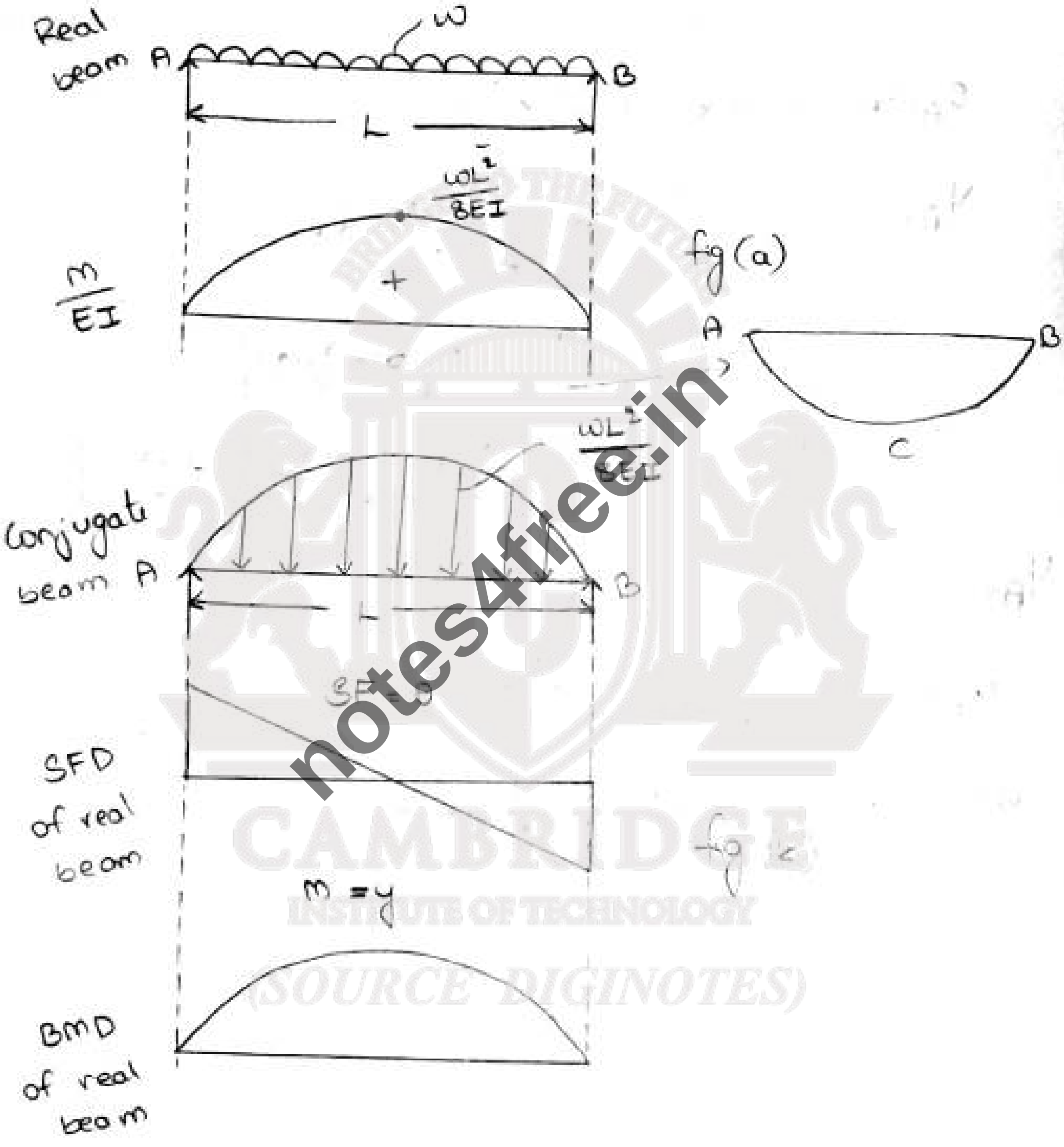
$$y_{Ax''} = x'x''$$

$$\theta_A \times x = y_{max} + x'x''$$

$$6.53 \times 10^{-3} \times 2.729 = y_{max} +$$

CONJUGATE BEAM METHOD

The conjugate beam is a beam carrying the the $\frac{M}{EI}$ diagram of a real beam as load



Consider a simply supported beam carrying a UDL of intensity as shown in fig (a) to be the real beam. The $\frac{M}{EI}$ diagram of the real beam is also as shown in the fig (a).

By defination, the $\frac{m}{EI}$ diagram of the real beam becomes the loading in the conjugate beam as shown in fig (6).

In the double integration method we derived the fundamental relationships relating bending moment curvature.

$$\frac{1}{R} = \frac{d^2y}{dx^2} = \frac{m}{EI}$$

we also derive the relationships

$$\frac{dm}{dx} = SF \quad \text{and} \quad \frac{d(SF)}{dx} = w$$

$$\therefore \frac{d^4y}{dx^4} = w$$

$$\frac{d^3y}{dx^3} = SF$$

and $\frac{d^2y}{dx^2}$ = moment (m)

These relationships show a similarity that should exist between w, SF and m and $\frac{m}{}$ m $\frac{dy}{dx}$ and y. They also show that the deflection in certain cases can be found from the load intensity fuction as in the simply supported beam carrying a UDL as shown in the fig (6).

∴ From the above relationships, we can co-relate the shear force diagram of the ream beam to the slope. we can also co-relate the bending moment diagram of the real beam with deflection.

Conjugate Beam theorems

Theorem - 1

The slope at any point of a beam is equal to the shear force at that point of the conjugate beam loaded with the $\frac{M}{EI}$ diagram of the real beam.

Considering the beam as shown in the fig (a) from the moment area method.

W.K.t $\quad \theta_A = \frac{y_{A.B}}{L}$

If we consider a beam, a load on which is the $\frac{M}{EI}$ diagram as shown in fig (b). we note that θ_A = reaction at the end 'A' of Such a beam that kind of beam is known as conjugate beam

∴ From the first moment area diagram theorem

$$\theta_A - \theta_C = \int_A^C \frac{M}{EI}\, dx$$

$$\theta_C = \theta_A - \text{Area of } \frac{M}{EI} \text{ diagram}$$

Slope at C = Reaction at A − $\frac{M}{EI}$ loading between a A and C of conjugate beam.

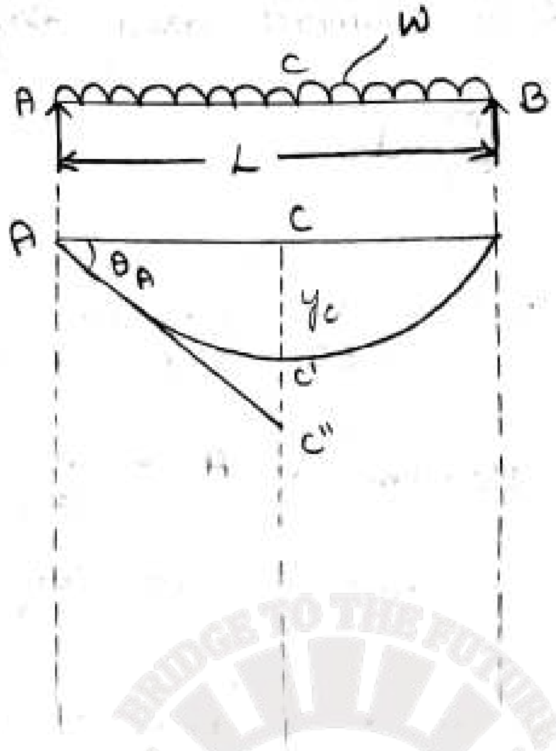
∴ θ_C = slope at C = shear force at the point 'C' of the conjugate beam.

Theorem -2

The deflection at any point of a beam is equal to the bending moment at the corresponding point of the conjugate beam loaded with the $\frac{M}{EI}$ diagram of a load real beam.

proof :- considering the deflection curve as shown in figure 'c'

The deflection at 'C' is given by



The deflection at 'C' is given by

$$y_c = CC'' - C'C''$$

$$= \theta_A \cdot \frac{L}{2} - y_{AC}$$

$\therefore$ y_c = Reaction at 'A' $\times \frac{L}{2}$ – moment of area of $\frac{M}{EI}$ diagram between 'A' and 'C' at 'C'

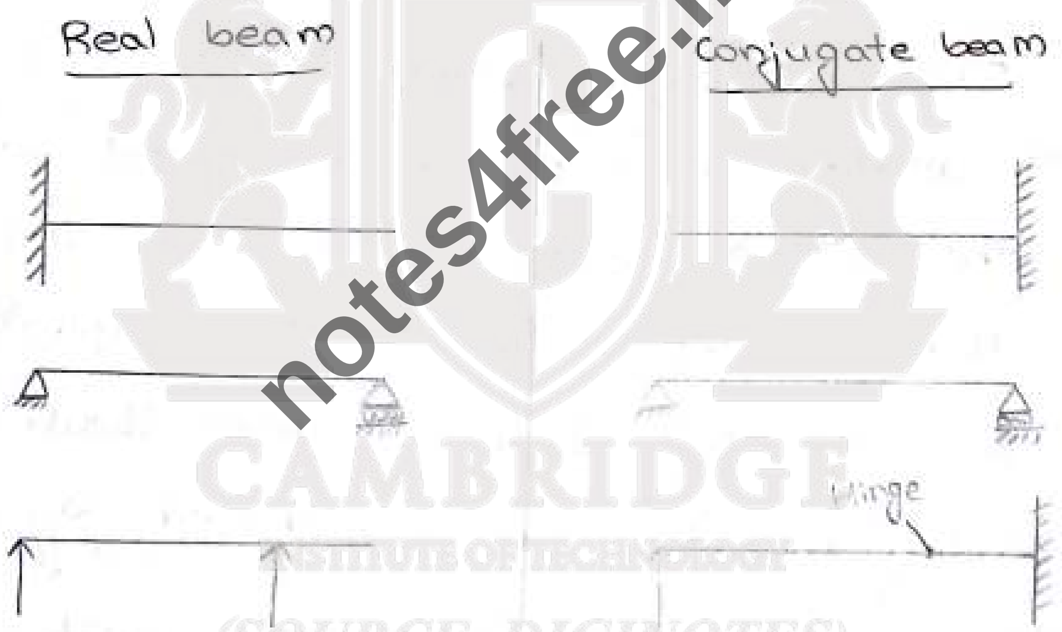
$\therefore$ y_c = Bending moment at 'C' of the conjugate beam

Approach to conjugate beam method

The conjugate and real beam have the same length or span because for every point of the real beam at which the slope or deflection is required to be calculated there should be a corresponding point on the conjugate beam.

* The conjugate beam should be supported in such a way so as to satisfy the slope and deflection conditions of the real beam.

In the following figures the support condition of the real beam and the corresponding conjugate beam are shown in different cases.



(i) In a cantilever, the supports get interchanged in the real beam the deformation conditions have no slope and deflection at the fixed end but slope and deflection exist at the free end. Correspondingly the conjugate beam should have shear force and as well as bending moment at that point corresponding to the free end. But no shear force and bending moment at the point corresponding to the fixed end of real beam.

(ii) A simply supported beam continous to remain so. The condition that the deflection is zero at both support points is satisfied, because there is no bending moment at the ends of the simply supported beam.

(iii) The over hanging beam shown is a more complex example of a conjugate beam. At the interior support there is no deflection but there is slope. So at the corresponding point on the conjugate beam there should be no bending moment. This is achieved by providing a hinge at that point, which ensures that there is no bending moment in the conjugate beam due to the $\frac{M}{EI}$ load.

In summary a free end becomes the fixed end a fixed end becomes a free end then a simple support remains a simple support and the simple support in the interior becomes a hinge.

(iii) The sign conventions

① A positive bending moment in the real beam is a downward load on a conjugate beam. A negative bending moment becomes an upward load

② A positive shear force in the conjugate beam corresponds to positive slope. A negative shear force corresponds to negative slope.

③ A positive bending moment in the conjugate beam means downward deflection. A negative bending moment consequently means the upward deflection.

Problems :-

① Find the slope and deflection for the cantilever beam at its free end as shown in the figure.

Step 1 :- To calculate the support reactions and to draw the $\frac{M}{EI}$ diagram.

$\Sigma H = 0, \Rightarrow H_A = 0$

$\Sigma V = 0, \Rightarrow V_A - P = 0$

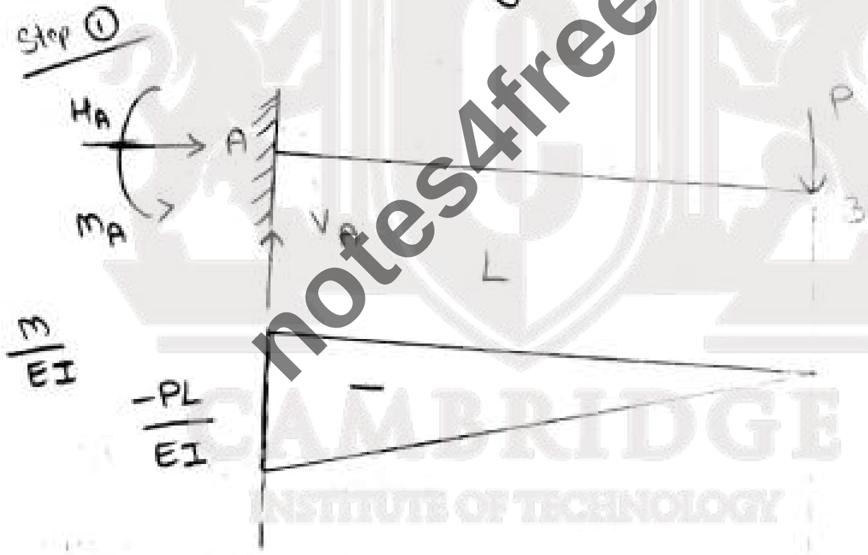
$$\boxed{V_A = P}$$

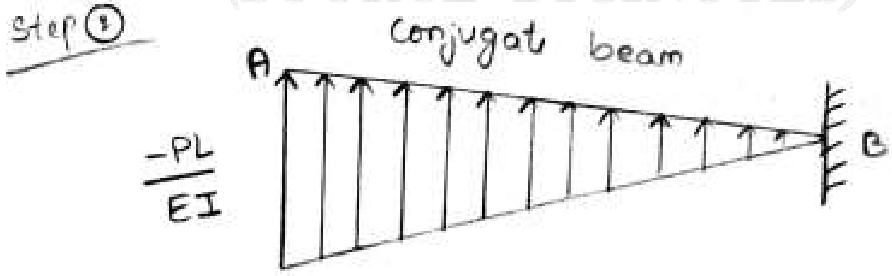
$\Sigma M = 0,$

$\Rightarrow -M_A + (P \times L) = 0$

$$\boxed{M_A = PL}$$

Step ② :- To draw the conjugate beam with the $\frac{MI}{EI}$ diagram on the loading.





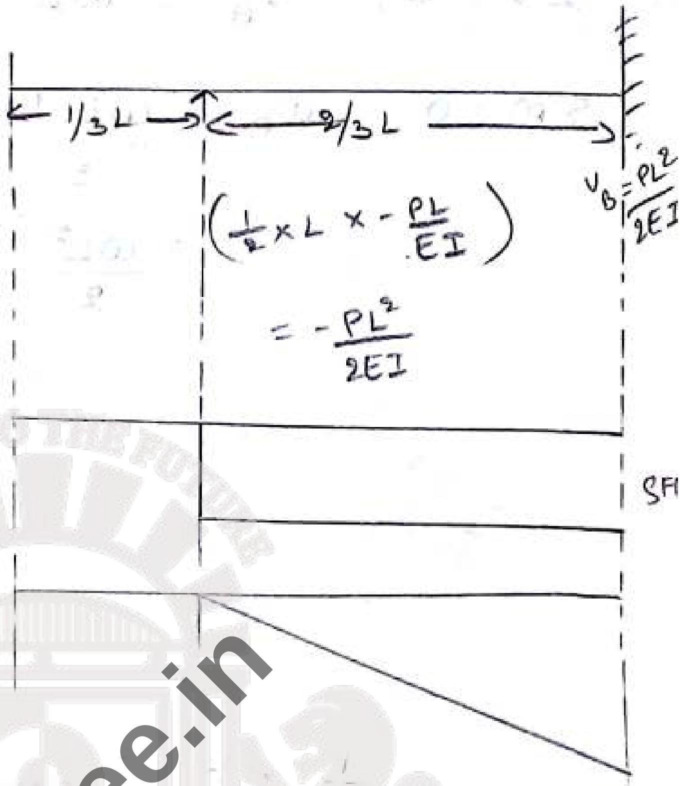
Step ③ :- To draw the SFD and BMD of the conjugat beam

θ_B = Reaction at B

SF at B

$= \frac{1}{2} \times L \times \frac{PL}{EI}$

$= \frac{-PL^2}{2EI}$

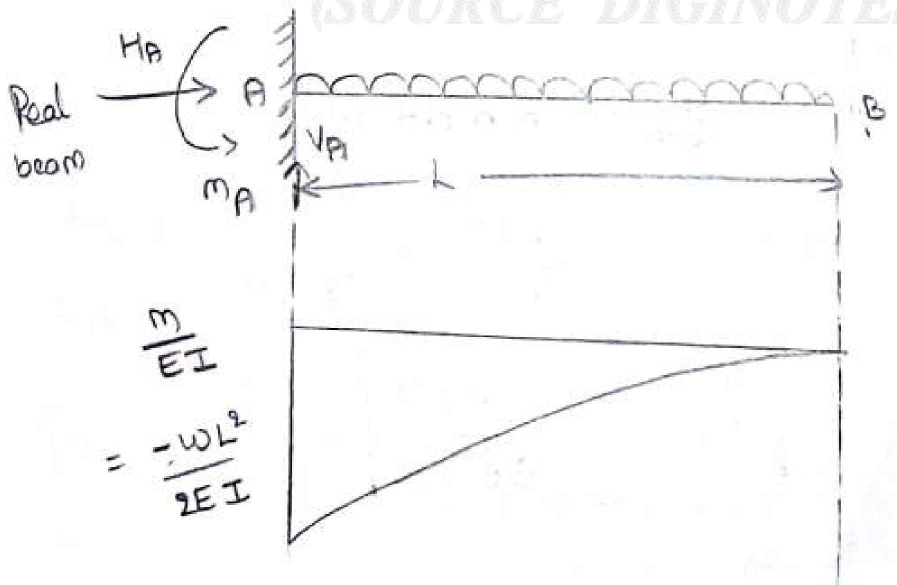


y_B = moment at B

$= \frac{-PL^2}{\cancel{2}EI} \times \frac{\cancel{2}}{3} L$

$= \frac{-PL^3}{3EI}$

(2) A cantilever beam is loaded with a UDL as shown in the figure. calculate the slope and deflection at it's free end. and take EI to be constant

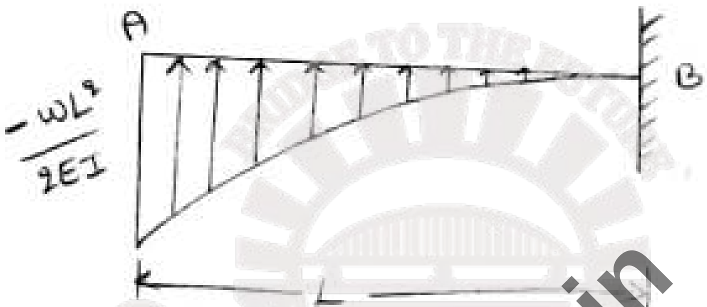


$\Sigma H = 0, \quad H_A = 0$

$\Sigma V = 0, \quad V_A = \omega L$

$\Sigma M = 0, \quad M_A = \frac{\omega L \cdot L}{2}$

$= \frac{\omega L^2}{2}$



$\frac{1}{3} \cdot b.h$ = Area of the parabola

$\frac{1}{3} \times L \times \frac{-\omega L^2}{2EI} = \frac{-\omega L^3}{6EI}$

θ_B = Slope at B

$\theta_B = \frac{1}{3} \cdot L \times \frac{-\omega L^2}{2EI}$

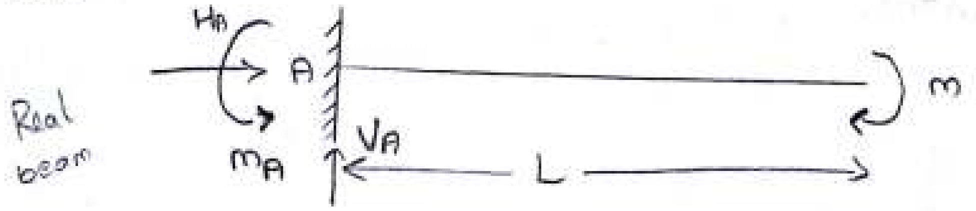
$$\boxed{\theta_B = \frac{-\omega L^3}{6EI}}$$

y_B = moment at B

$= \frac{-\omega L^3}{6EI} \times \frac{3}{4} \cdot L$

$$\boxed{y_B = \frac{-\omega L^4}{8EI}}$$

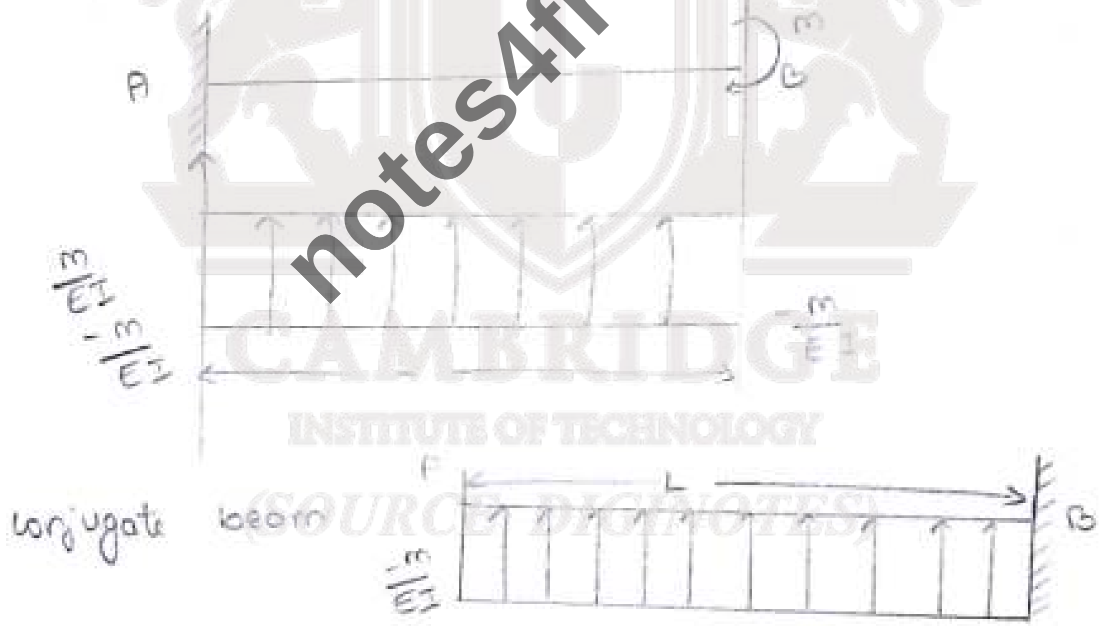
③ Determine the slope and deflection of cantilever beam with a couple 'm' acting at the free end as shown in the figure at it's free end.



$\Sigma H_A = 0$ $\boxed{H_A = 0}$

$\Sigma V = 0$ $\boxed{V_A = 0}$

$\Sigma m = 0$ $\boxed{m_A = m}$



θ_B = S.F at B

$\theta_B = \frac{-m}{EI} \times L$

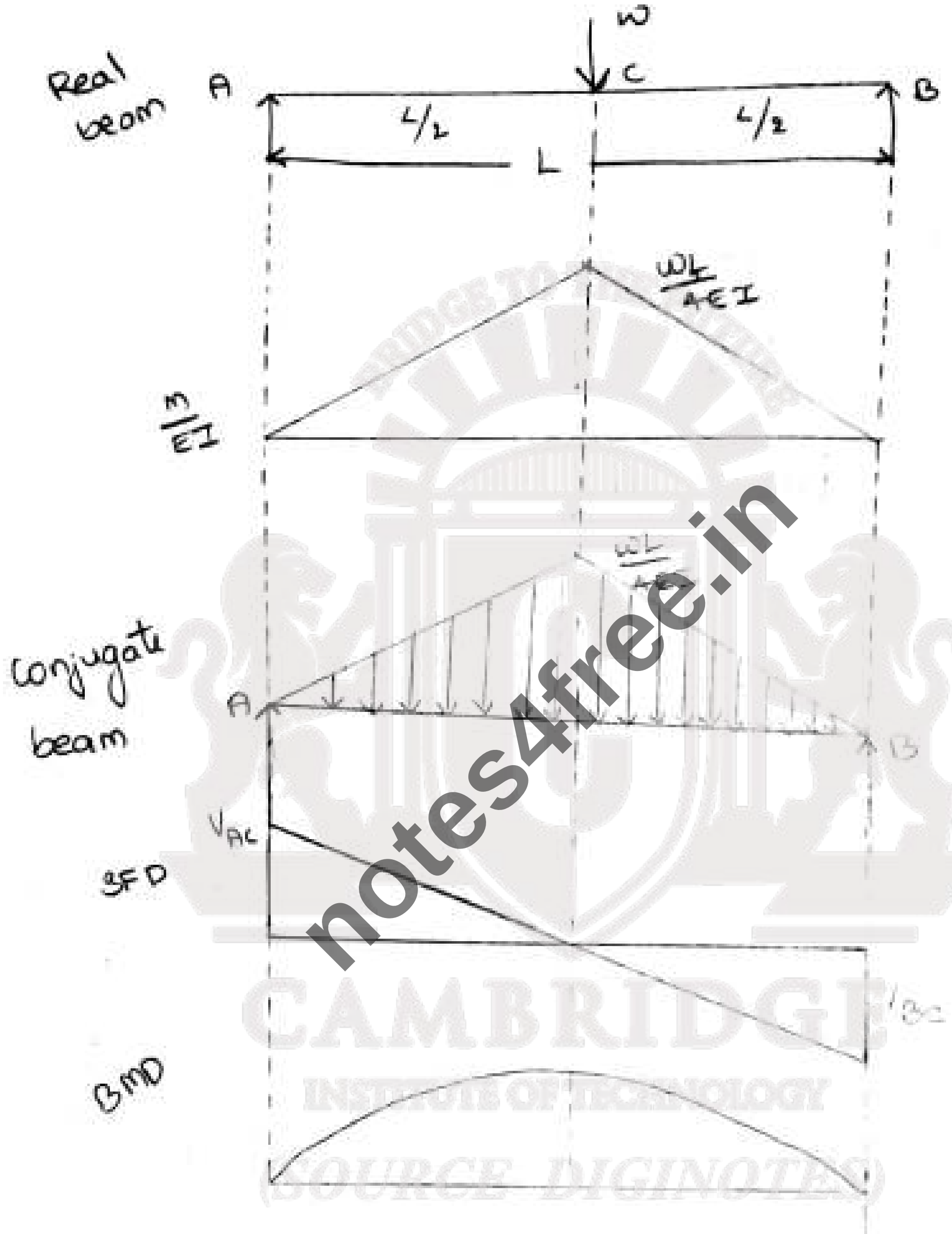
$$\boxed{\theta_B = \frac{-mL}{EI}}$$

y_B = moment at B

$y_B = \frac{-mL}{EI} \times \frac{L}{2}$

$$\boxed{y_B = \frac{-mL^2}{2EI}}$$

④ calculate the slope and deflection of a simply supported beam carrying a point load at the centre.



① Support reactions

$$V_A = V_B = \frac{W}{2}$$

② $\frac{M}{EI}$ - diagram

$$BM_C = R_A \cdot \frac{L}{2} = \frac{WL}{4}$$

③ Draw conjugate beam

(i) Supports

(ii) $\frac{M}{EI}$ diagram $\rightarrow$ Loading

④ SFD and BMD of conjugate beam

$$\Sigma V = 0 \Rightarrow V_A + V_B = \frac{1}{2} \times L \times \frac{wL}{4EI}$$

$$V_A + V_B = \frac{wL^2}{8EI}$$

$$\Sigma M_A = 0 \Rightarrow (-V_{B_c} \times L) + \left(\frac{1}{2} \times \frac{L}{2} \times \frac{wL}{4EI}\right)\left(\frac{2}{3} \cdot \frac{L}{2}\right) + \left(\frac{1}{2} \times \frac{L}{2} \times \frac{wL}{4EI}\right)\left(\frac{L}{2} + \frac{1}{3} \cdot \frac{L}{2}\right) = 0$$

$$V_{B_c} L = \frac{wL^2}{16EI} \times \frac{L}{3} + \frac{wL^2}{16EI} \times \frac{2L}{3}$$

$$= \frac{wL^3}{48EI} + \frac{2wL^3}{48EI} = \frac{3wL^3}{48EI}$$

$$V_{B_c} L = \frac{wL^3}{16EI}$$

$$\Rightarrow \boxed{V_{B_c} = \frac{wL^2}{16EI}}$$

$$\boxed{V_{A_c} = \frac{wL^2}{16EI}}$$

⑤ Slope

θ_A = SF_A of the conjugate beam

$$\boxed{\theta_A = \frac{wL^2}{16EI}}$$

$\theta_B = SF_B$ of conjugate beam

$$\boxed{\theta_B = \frac{wL^2}{16EI}}$$

Deflection:- Acc to

$y_c = BM_c$ of the conjugate beam

$$= V_{A_c} \cdot \frac{L}{2} - \left(\frac{1}{2} \times \frac{L}{2} \cdot \frac{wL}{4EI}\right)\left(\frac{1}{3} \cdot \frac{L}{2}\right)$$

$$= \frac{wL^2}{16EI} \cdot \frac{L}{2} - \frac{wL^2}{16EI} \frac{L}{6}$$

$$= \frac{wL^3}{32EI} - \frac{wL^3}{96EI}$$

$$\boxed{y_c = \frac{wL^3}{48EI}}$$

(5) calculate the slope and deflection of a simply supported beam carrying a UDL.

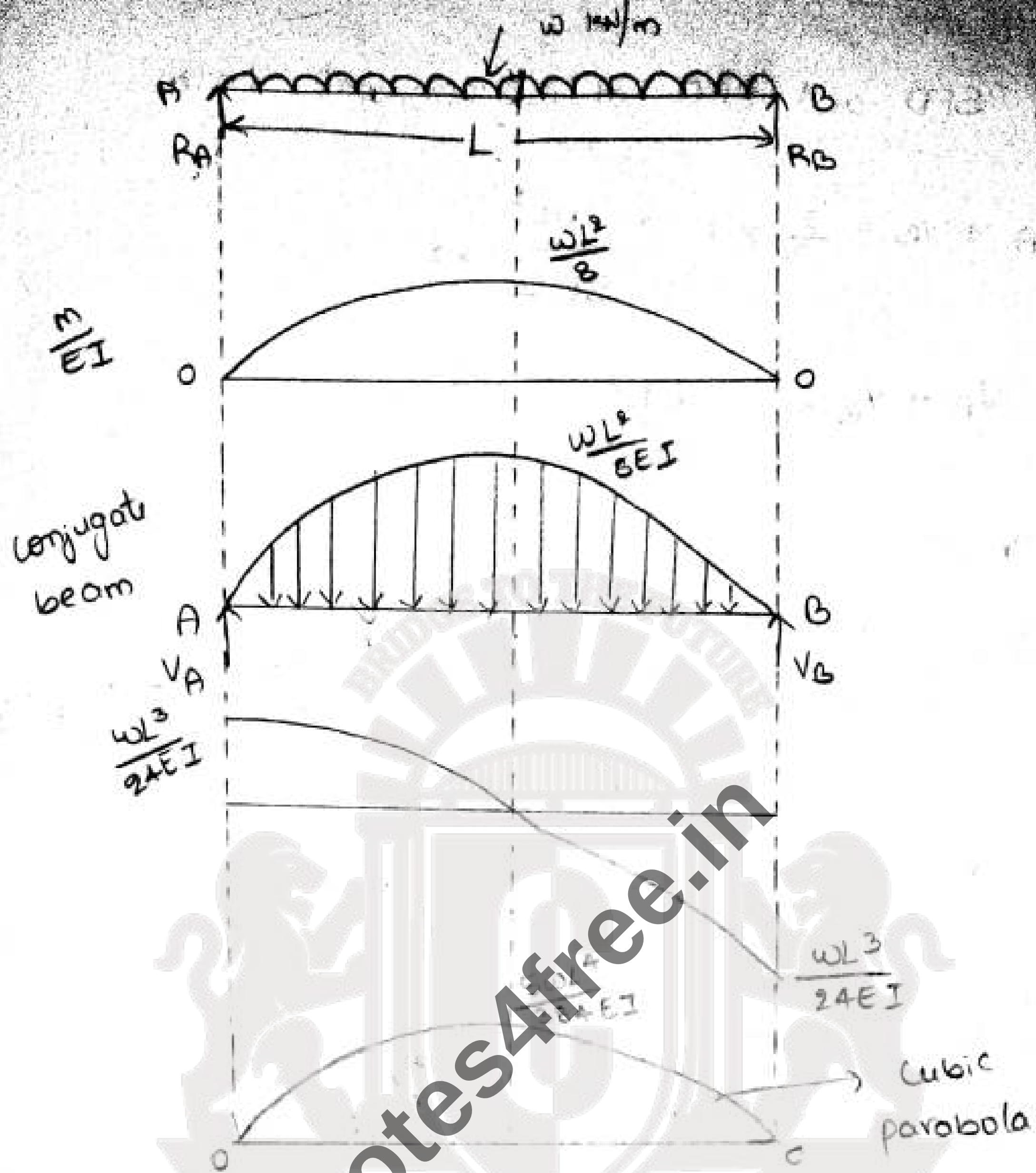
(1) Support Reactions:-

$$R_A + R_B = wL$$

$\Sigma M_A = 0$ $\quad wL \times \frac{L}{2} - R_B \times L = 0$

$$\frac{wL^2}{2} - R_B \times L = 0$$

$$\boxed{R_B = \frac{wL}{2}} \qquad \boxed{R_A = \frac{wL}{2}}$$



② $\frac{M}{EI}$ diagram

$$BM_C = \frac{wL}{2} \times \frac{L}{2} - w \times \frac{L}{2} \times \frac{L}{4}$$

$$= \frac{wL^2}{4} - \frac{wL^2}{8}$$

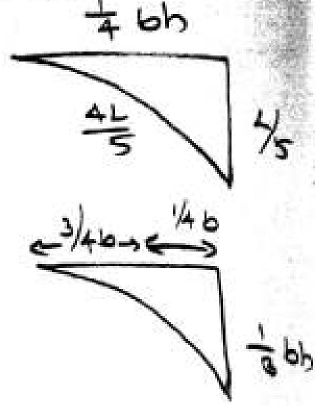
$$= \frac{wL^2}{8}$$

③ Draw conjugate beams

(i) Supports

(ii) $\frac{M}{EI}$ diagram - UDL

④ SFD and BMD of a conjugate beam



$$V_A + V_B = \frac{2}{3} \times L \times \frac{wL^2}{8EI} \times 2$$

$$V_A + V_B = \frac{wL^3}{6EI}$$

$\Sigma M_A = 0$

$$+\frac{2}{3} \times \frac{wL^2}{8EI} \times \frac{L}{2} \times \frac{5}{8} \times \frac{L}{2} + \frac{2}{3} \times \frac{wL^2}{8EI} \times \frac{L}{2} \times \left(\frac{L}{2} + \frac{3}{8} \times \frac{L}{2}\right)$$

$$- V_B L = 0$$

$$V_B \times L = \frac{5wL^4}{384EI} + \frac{11wL^4}{384EI}$$

$$\boxed{V_B = \frac{wL^3}{24EI}} \qquad \boxed{V_A = \frac{wL^3}{24EI}}$$

SF at $A_R = \frac{wL^3}{8EI}$

SF at $C = \frac{wL^3}{24EI} - \frac{2}{3} \times \frac{L}{2} \times \frac{wL^2}{8EI}$

$= \frac{wL^3}{12EI}$ 0

SF at $B_L = \frac{wL^3}{8EI} - \frac{wL^3}{12EI} - \frac{wL^3}{12EI}$

$= \frac{-wL^3}{24EI}$

Slope :- θ_A

Slope :- $\theta_A = SF_A$ at A

$$\boxed{\theta_A = \frac{wL^3}{24EI}}$$

$\theta_B = SF_B$ at B

$$\boxed{\theta_B = \frac{-wL^3}{24EI}}$$

Deflection :-

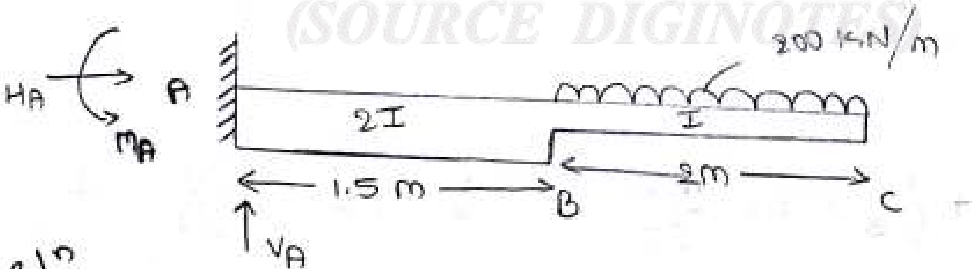
Deflection at C $\Rightarrow$ $y_c = \frac{wL^3}{24EI} \times \frac{L}{2} - \frac{\cancel{2}}{3} \times \frac{L}{\cancel{2}} \times$

$$\frac{wL^2}{8EI} \times \frac{3}{8} \times \frac{L}{2}$$

$$\boxed{y_c = \frac{5wL^4}{384EI}}$$

(6) Calculate the slope and deflection for B and C for the beam as shown in the figure.

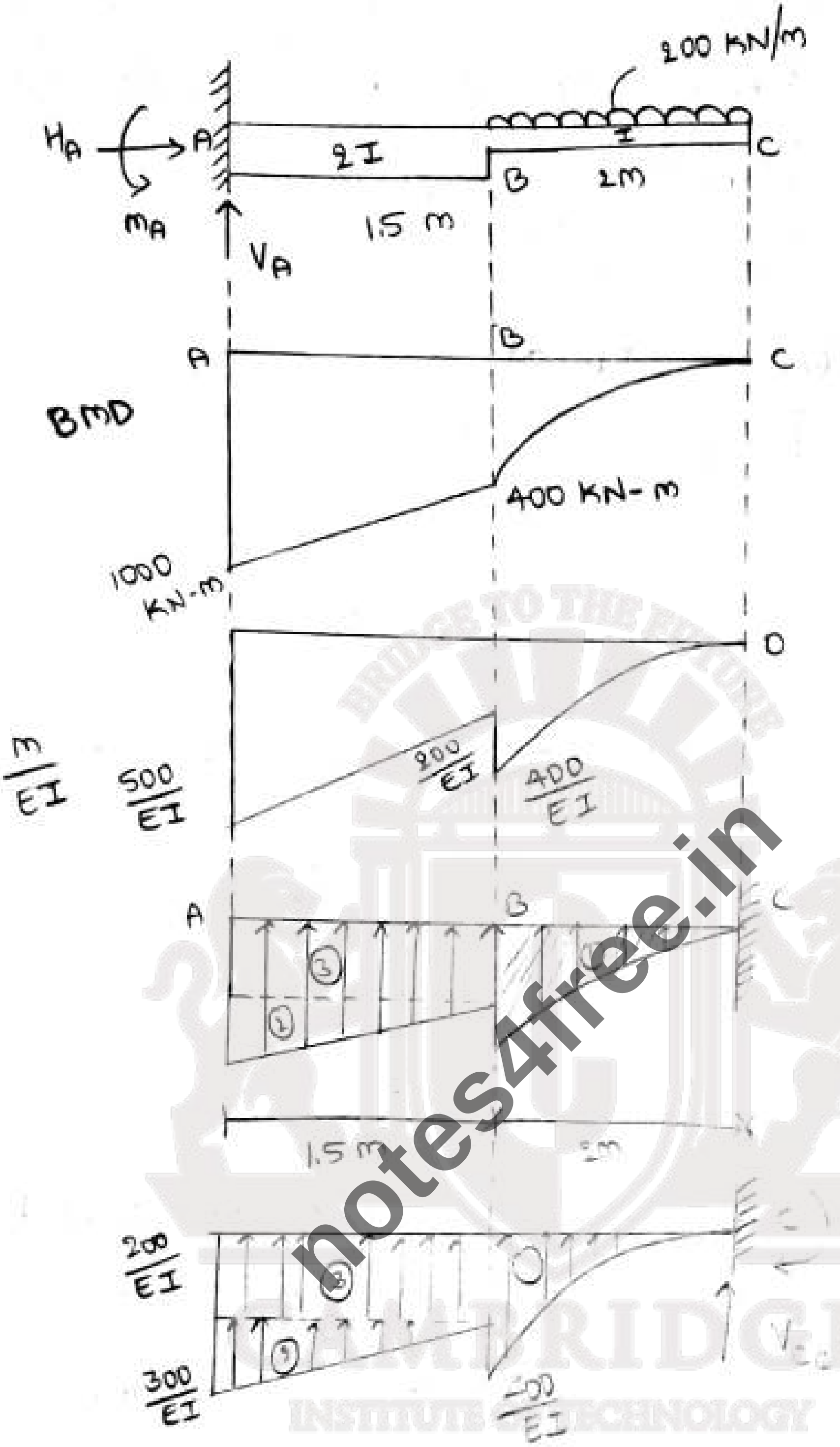
Take $E = 210 \times 10^9 \ N/m^2$, $I = 120 \times 10^6 \ mm^4$



Sol^n :-

$\Sigma H = 0$, $\boxed{H_A = 0}$

$\Sigma V = 0$ $\boxed{V_A = 400 \ kN}$



$\Sigma V = 0$

$$\Rightarrow \quad V_{c_c} + \overbrace{\left(\frac{1}{3} \times 2 \times \frac{400}{EI}\right)}^{①} + \overbrace{\left(\frac{1}{2} \times 1.5 \times \frac{300}{EI}\right)}^{②} + \left(\frac{200}{EI} \times 1.5\right) = 0$$

$$\Rightarrow \boxed{V_{c_c} = -\frac{791.67}{EI}}$$

$\Sigma M_{A_C} = 0$

$$\Rightarrow \quad M_{C_C} - \left(\frac{1}{3} \times 2 \times \frac{400}{EI}\right)\left(\frac{1}{4} \times 2\right) + 1.5)$$

$$\left(-\frac{1}{2} \times 1.5 \times \frac{300}{EI}\right)\left(\frac{1}{3} \times 1.5\right) - \left(\frac{200}{EI} \times 1.5\right)\left(\frac{1.5}{2}\right) = 0$$

$$\boxed{M_{CC} = \frac{879.833}{EI}}$$

Slope at B

θ_B = SF @ B of conjugate beam

$$\theta_B = \frac{525}{EI}$$

$$\boxed{\theta_B = 0.02083 \text{ rads}}$$

θ_C = SF @ C of conjugate beam

$$\theta_C = \frac{791.67}{EI}$$

$$\boxed{\theta_C = 0.0314 \text{ radians}}$$

y_B = BM @ B of conjugate beam.

$$= \left(\frac{1}{2} \times 1.5 \times \frac{300}{EI}\right)\left(\frac{2}{3} \times 1.5\right) + \left(\frac{200}{EI} \times 1.5\right)\left(\frac{1.5}{2}\right)$$

$$y_B = \frac{450}{EI} = \frac{450}{210 \times 10^9 \times 120 \times 10^{-6}}$$

$$= 1.7857 \times 10^{-5} \text{ m}$$

$$\boxed{y_B = 0.0178 \text{ mm}}$$

$$y_C = \left(\frac{1}{2} \times 1.5 \times \frac{300}{EI} \times \left(2 + \frac{2}{3} \times 1.5\right)\right) +$$

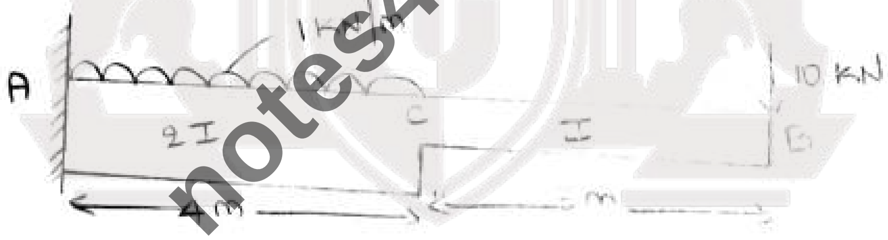
$$\left(\frac{200}{EI} \times 1.5 \times \left(2 + \frac{1.5}{2}\right)\right) + \left(\frac{1}{3} \times 2 \times \frac{400}{EI} \times \frac{3}{4} \times 2\right)$$

$$y_c = \frac{675}{EI} + \frac{825}{EI} + \frac{400}{EI}$$

$$y_c = \frac{1900}{210 \times 10^9 \times 120 \times 10^{-6}}$$

$$\boxed{y_c = 0.07539 \text{ mm}}$$

(7) Determine the slope and deflection of the loaded cantilever beam as shown in the figure at the free end by conjugate beam method.



$\Sigma V = 0$

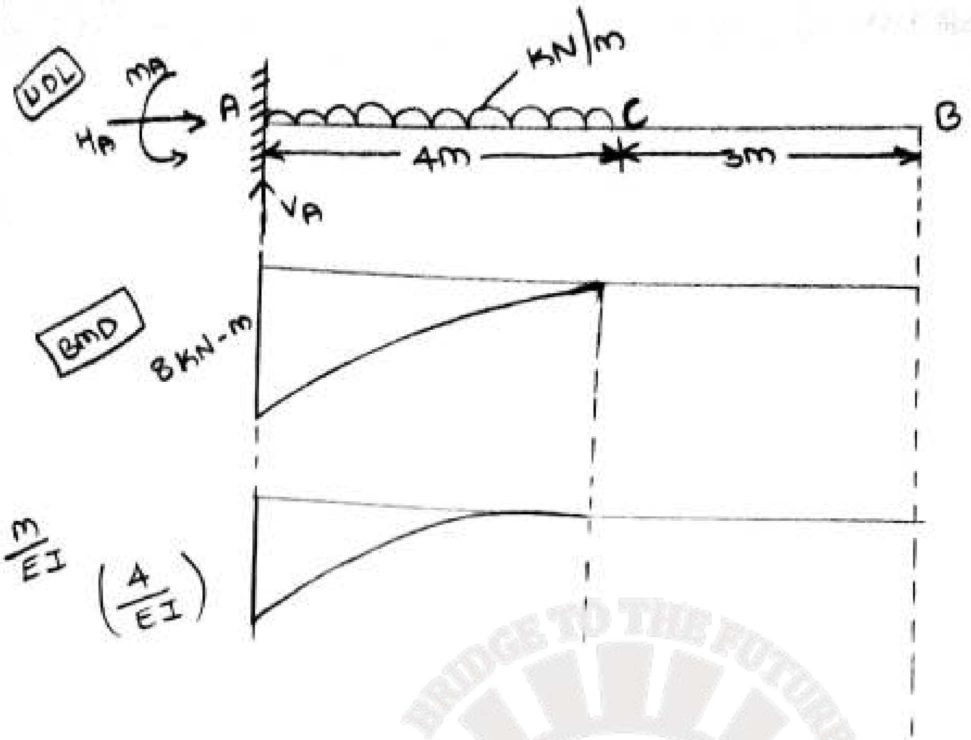
$$V_A = (1 \times 4) + 10$$

$$\boxed{V_A = 14 \text{ kN}}$$

$\Sigma M = 0$

$$-M_A + (1 \times 4 \times 2) + (10 \times 7)$$

$$\boxed{M_A = 78 \text{ kN/m}}$$



For only UDL

$\Sigma H = 0$, $\boxed{H_A = 0}$

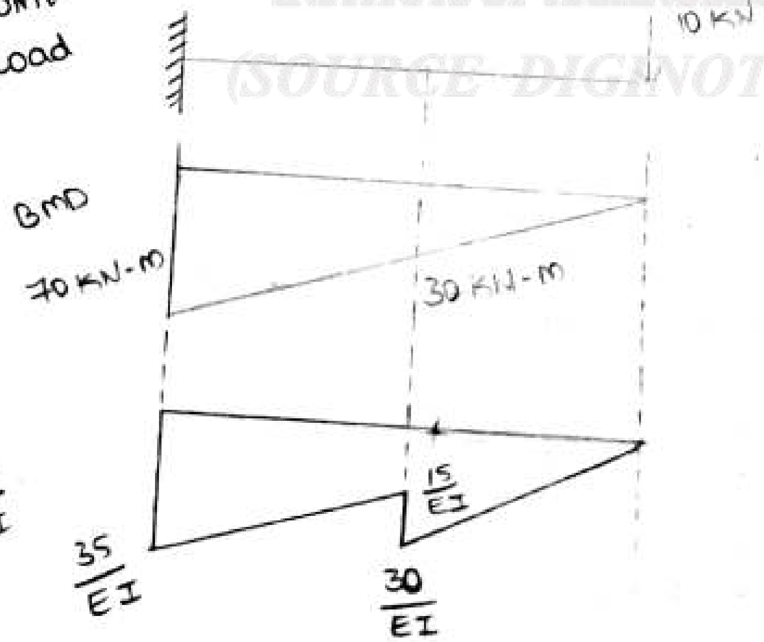
$\Sigma V = 0$, $V_A = 1 \times 4$

$\boxed{V_A = 4\ kN}$

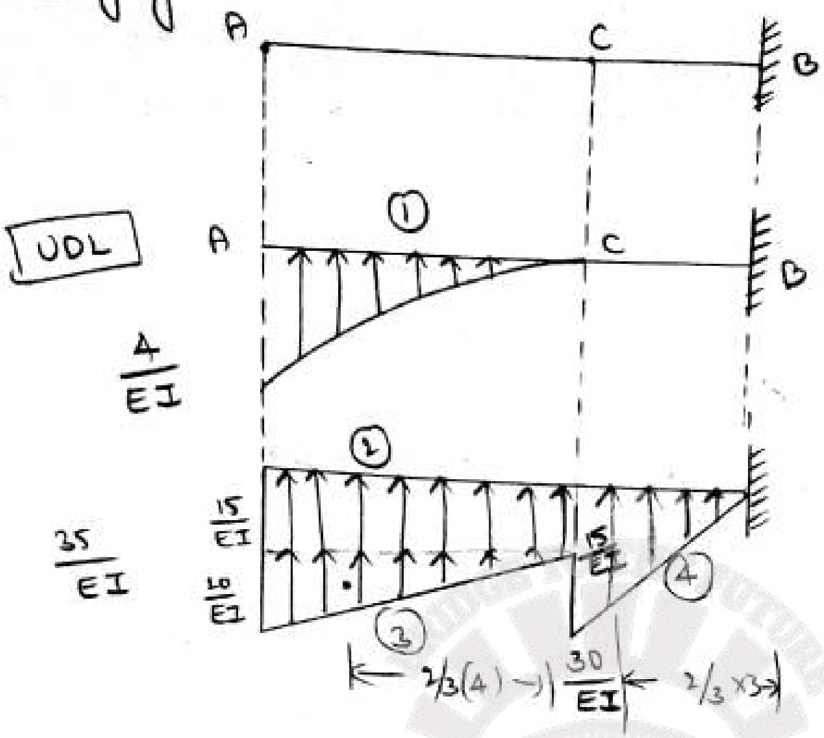
$\Sigma M = 0$, $-M_A + (1 \times 4 \times 2)$

$\boxed{M_A = 8\ kN\text{-}m}$

For point load



conjugate beam



θ_B = SF @ B of CB

y_B = BM @ B of CB

$$\theta_B = \left(\frac{1}{3} \times 4 \times \frac{4}{EI}\right) + \left(\frac{15}{EI} \times 4\right) + \left(\frac{1}{2} \times 4 \times \frac{20}{EI}\right) + \left(\frac{1}{2} \times 3 \times \frac{30}{EI}\right)$$

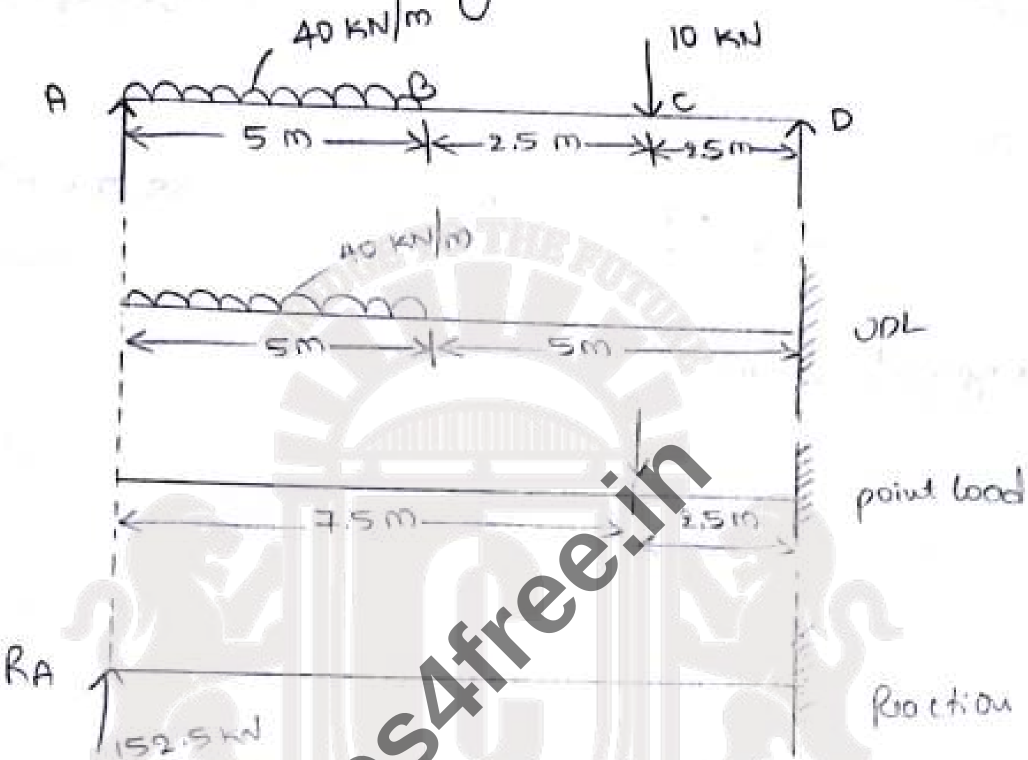
$$\theta_B = \frac{5.33}{EI} + \frac{60}{EI} + \frac{40}{EI} + \frac{45}{EI}$$

$$\boxed{\theta_B = \frac{150.33}{EI}}$$

$$y_B = \left(\frac{1}{3} \times 4 \times \frac{4}{EI}\right)\left(\frac{3}{4} \times 4 + 3\right) + \left(4 \times \frac{15}{EI}\right)(5) + \left(\frac{1}{2} \times 4 \times \frac{20}{EI}\right)\left(\frac{2}{3} \times 4 + 3\right) + \left(\frac{1}{2} \times 3 \times \frac{30}{EI}\right)(2)$$

$$\boxed{y_B = \frac{648.67}{EI}}$$

⑧ Calculate the slope at the supports and the deflection at mid span and under the 10 KN load for the simply supported beam as shown in the figure.



$\Sigma V = 0$

$R_A + R_B = (40 \times 5) + 10$

$= 210$ — ①

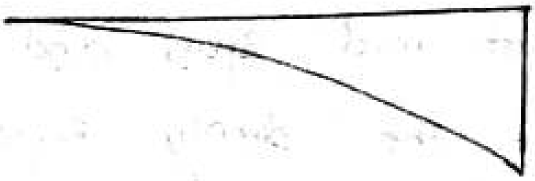
$\Sigma m_A = 0$

$\Rightarrow (-R_B \times 10) + (40 \times 5 \times 2.5) + (10 \times 7.5) = 0$

$\Rightarrow \boxed{R_B = 57.5 \text{ KN}}$

$\boxed{R_A = 152.5 \text{ KN}}$

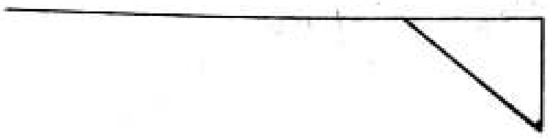
UDL



$(40 \times 5 \times 7.5)$

$\left(\frac{M}{EI}\right)$

point load



(10×2.5)

(R_A)

(152.5×10)

Conjugate beam

$\Sigma V = 0$

$$R_{A_c} + R_{B_c} + \left(\frac{1}{\cancel{3}} \times 10 \times \frac{\overset{500}{\cancel{1500}}}{EI}\right) + \left(\frac{1}{2} \times 2.5 \times \frac{25}{EI}\right)$$

$$- \left(\frac{1}{\cancel{2}} \times \overset{5}{\cancel{10}} \times \frac{1525}{EI}\right) = 0$$

$$\Rightarrow R_{A_c} + R_{B_c} + \frac{5000}{EI} + \frac{31.25}{EI} - \frac{7625}{EI} = 0$$

$$\Rightarrow R_{A_c} + R_{B_c} = \frac{2593.75}{EI} \quad \text{— (2)}$$

$\Sigma M_{Ac} = 0$

$$\Rightarrow (-R_{Bc} \times 10) - \left(\frac{1}{3} \times 10 \times 1500\right)\left(\frac{3}{4} \times 10\right) - \left(\frac{1}{2} \times 2.5 \times \frac{25}{EI}\right)\left(7.5 + \left(\frac{2}{3} \times 2.5\right)\right) + \left(\frac{1}{2} \times 10 \times \frac{1525}{EI}\right)\left(\frac{2}{3} \times 10\right) = 0$$

$$\boxed{R_{Bc} = \frac{1304.68}{EI} \text{ kN-m}}$$

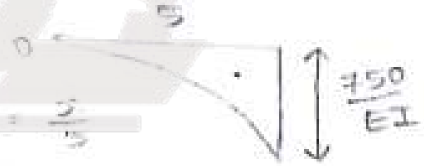
From (2) $\Rightarrow$ $\boxed{R_{Ac} = \frac{1289.07}{EI}}$

$\theta_A = \frac{1289.07}{EI}$ radians

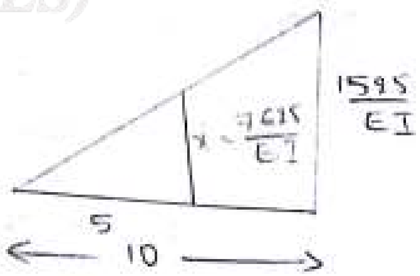
$\theta_B = \frac{-1304.68}{EI}$ radians

<u>Deflection</u>

@ mid span



BM_c of conjugate Beam $= y_c = (R_{Ac} \times 5) + \left(\frac{1}{3} \times 5 \times \frac{750}{EI}\right)\left(\frac{3}{4} \times 5\right) - \left(\frac{1}{2} \times 5 \times \frac{762.5}{EI}\right)\left(\frac{1}{3} \times 5\right)$



$= 6445.3 + 1562.5 - 3177.083$

$$\boxed{y_c = \frac{4830.716}{EI}} \downarrow$$

Downwards

1525×5

$\frac{1525}{EI \times 10} = \frac{x}{5}$

$x = \frac{762.5}{EI}$

BMD of conjugate beam $(y_D) = (R_{AC} \times 7.5) + \left(\frac{1}{3} \times 7.5 \times 1125 \times \left(\frac{1}{4} \times 7.5\right)\right) - \left(\frac{1}{2} \times 7.5 \times 1143.75 \times \left(\frac{1}{3} \times 7.5\right)\right)$

$y_D = 9667.95 + 5273.4375 - 10722.656$

$\boxed{y_D = \frac{4218.731}{EI}}$ [downwards]

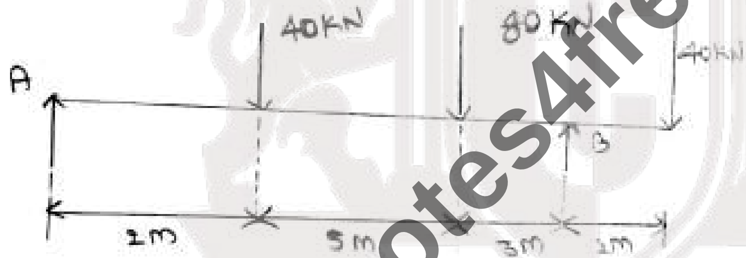
① For $\frac{1500}{10} = \frac{x_1}{7.5}$

$x_1 = 1125$

② For $\frac{1525}{10} = \frac{x_2}{7.5}$

$x_2 = 1143.75$

Draw BMD.



$R_A + R_B = 40 + 80 + 40$

$R_A + R_B = 160 \text{ KN}$

$\Sigma M_A = 0$

$40 \times 2 + 80 \times 7 - R_B \times 10 + 40 \times 12 = 0$

$80 + 560 - R_B \times 10 + 480 = 0$

$\boxed{R_B = 112 \text{ KN}}$

$R_A + R_B = 160$

$R_A + 112 = 160$

$\boxed{R_A = 48 \text{ KN}}$

SF Values:-

SF at A_R = 48 KN

SF at C_L = 48 KN

SF at C_R = 8 KN = 48-40

SF at D_L = 8 KN = 48-40

SF at D_R = -72 KN = 48-40-80

SF at B_L = -72 KN = 48-40-80

SF at B_R = 40 KN = 48-40-80+110

SF at E_L = 40 KN

Bm Values :-

Bm at A is Zero

Bm at C = 48 × 2 = 96 KN-m

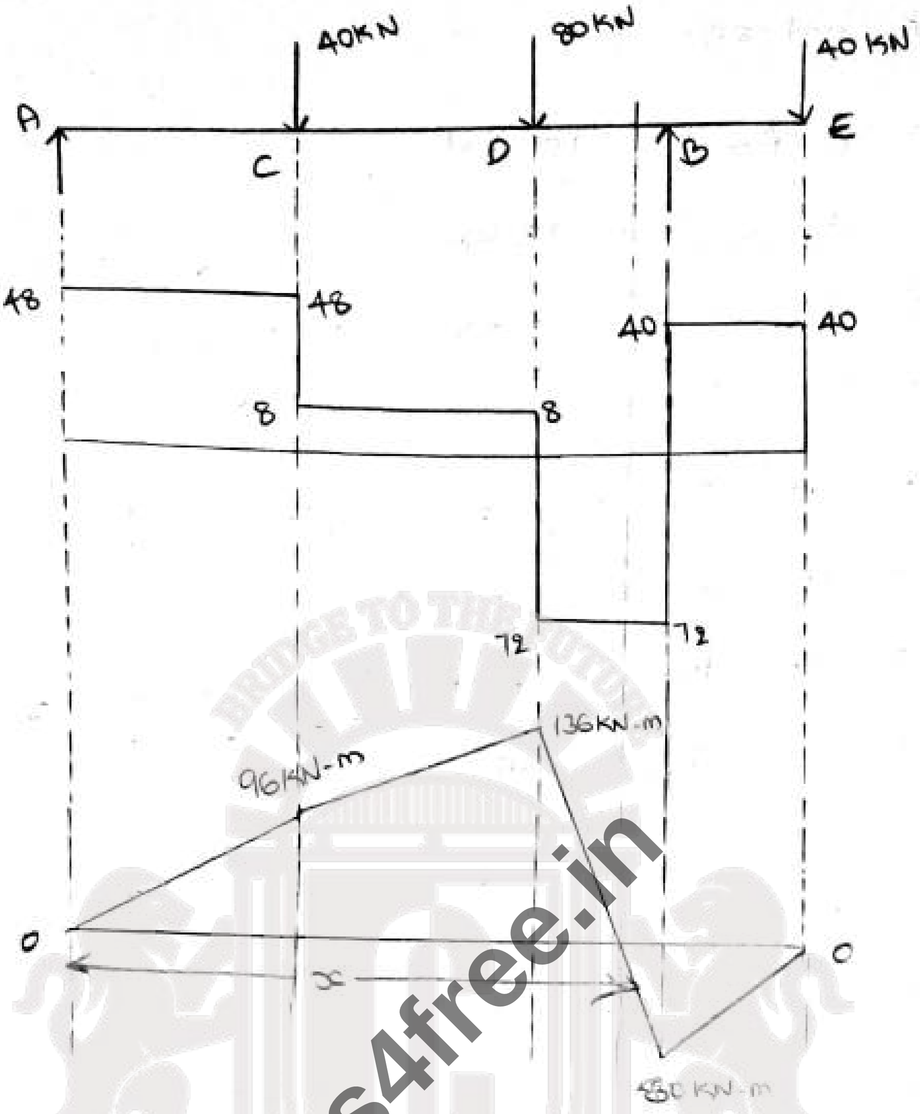
Bm at D = 48×7 - 40×5 = 136 KN-m

Bm at B = 48×10 - 40×8 - 80×3

= -80 KN-m

Bm at E = 48×12 - 40×10 - 80×5 + 112×2

= 0 KN-m



$\Sigma M_{x} = 0$

$48 \times x - 40 \times (x-2) - 80(x-7) = 0$

$48x - 40x + 80 - 80x + 560 = 0$

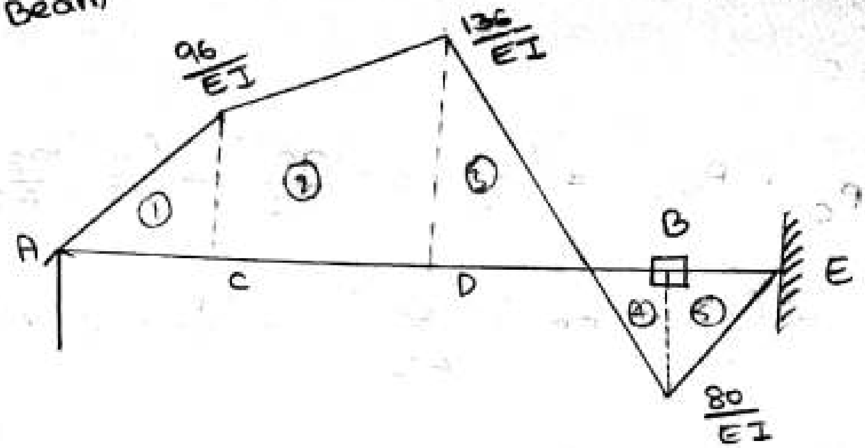
$48x - 40x - 80x = -560 - 80$

$-72x = 480 - 640$

$x = \frac{-640}{-72}$

$\boxed{x = 8.88m}$

Conjugate Beam



Bm @ Hinge = 0

$\Sigma m_B = 0$

$\Rightarrow (+R_{A_c} \times 10) - \left(\frac{1}{2} \times 2 \times \frac{96}{EI}\right)\left(8 + \frac{1}{3} \times 2\right) - \frac{1}{2}\left(\frac{96}{EI} + \frac{136}{EI}\right)(5)\left(\frac{5}{2}+3\right) - \left(\frac{1}{2} \times 1.89 \times \frac{136}{EI}\right)\left(\left(\frac{2}{3} \times 1.89\right) + 1.11\right) + \left(\frac{1}{2} \times 1.11 \times \frac{80}{EI}\right)\left(\frac{1}{3} \times 1.11\right) = 0$

$(R_{Ac} \times 10) - \frac{832}{EI} - \frac{3190}{EI} - \frac{304.59}{EI} + \frac{16.428}{EI} = 0$

$R_{AC} \times 10 = 4310.162$

$\boxed{R_{AC} = 431.0162}$

$\boxed{R_{AC} = \frac{431.02}{EI}}$

θ_A = SF @ A of CB

$\boxed{\theta_A = \frac{431.02}{EI} \text{ radians}}$

$$\Sigma V = 0$$

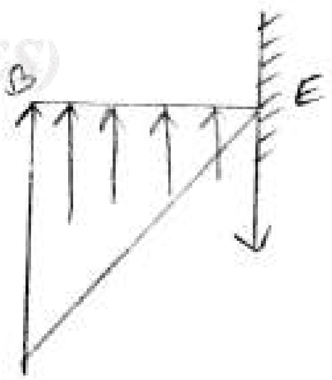
$$\Rightarrow R_{AC} + R_{BC} = \left(\frac{1}{2} \times 2 \times \frac{96}{EI}\right) + \left(\frac{1}{2}\left(\frac{96}{EI} + \frac{136}{EI}\right)5\right)$$

$$+ \left(\frac{1}{2} \times 1.89 \times \frac{136}{EI}\right) - \left(\frac{1}{2} \times 1.11 \times \frac{80}{EI}\right)$$

$$\frac{431.02}{EI} + R_{BC} = \frac{96}{EI} + \frac{580}{EI} + \frac{128.52}{EI} - \frac{44.4}{EI}$$

$$\frac{431.02}{EI} + R_{BC} = \frac{760.12}{EI}$$

$$\boxed{R_{BC} = \frac{329.1}{EI}} \qquad \theta_B =$$



module - 4

ARCHES AND CABLES

Beams transfer the applied load to the end Supports by bending and shear action. In this process either one or two points at a particular Section is Subjected to maximum Stress. The material is most of the portion is under stress and hence under utilized. For larger Spans beams are very uneconomical and many a times the Self weight of the beam contributes to the stress in Such large portions that it is difficult to design beams for longer Spans. Hence for larger spans [bridges] arches are provided instead of beams. Arches are nothing but curved beams that transfer load in their plane.

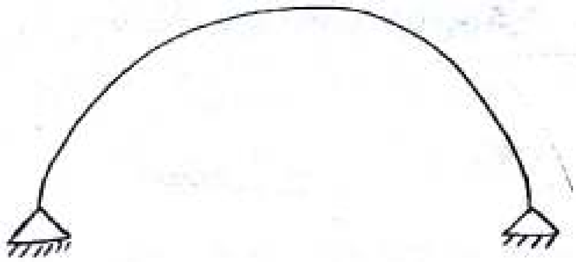
→ Arches transfer loads to abutments.

→ Hinges may be provided at three points. The topmost point of an arch is called a crown which Sometimes has a hinge.

→ The height of the Crown above the Support level is called rise.

→ There are three types of arches depending upon the number of hinges provided.

(i) Two hinged arches :-



(ii) Hingeless arch / fixed arch :-

(iii) Three hinged Arch :-

THREE HINGED ARCHES :-

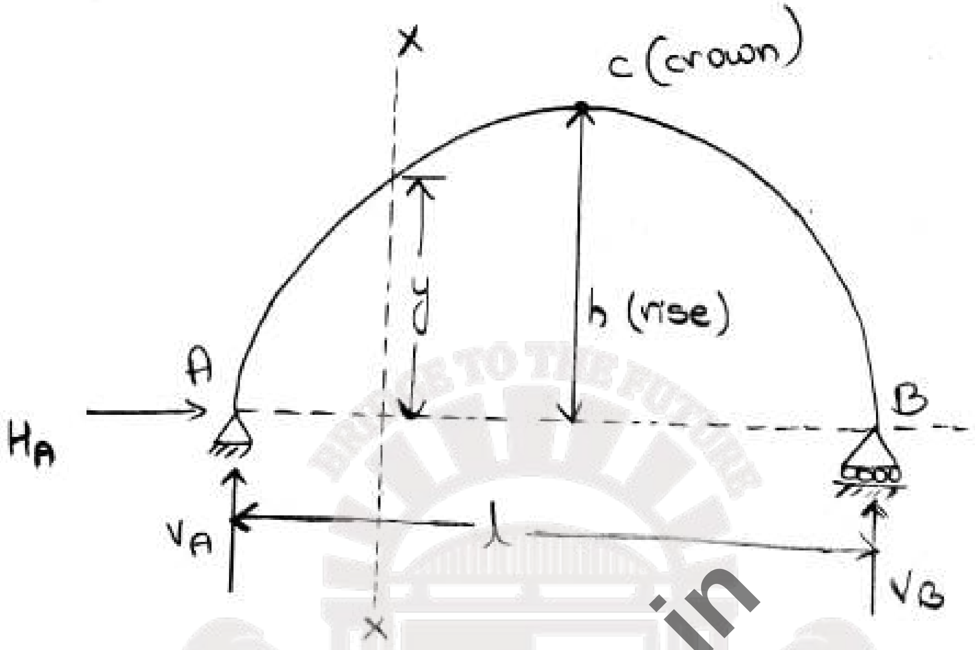
Three hinged Arch is a determinant structures where as two hinged and fixed arches are indeterminant structures.

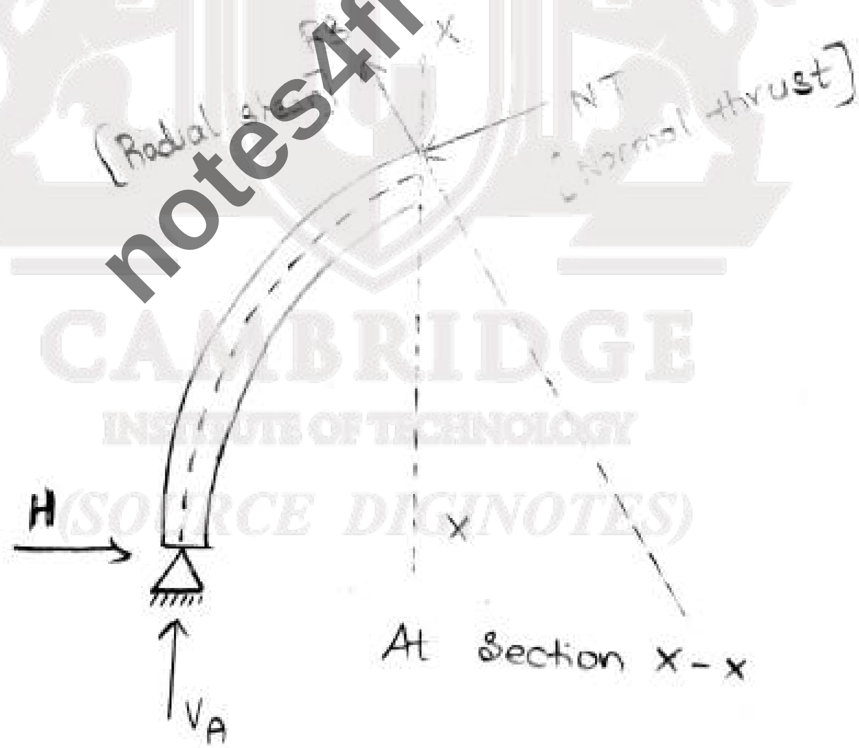
There are two types of three hinged arches depending upon their shape

① circular arch

② parabolic arch

Important parameters for a three hinged parabolic Arch





At Section x - x

* Equation of a parabola $\boxed{y = \frac{4hx}{l^2}(l-x)}$

This equation is also known as the rise equation of a parabolic arch.

→ slope equation

$$\boxed{\frac{dy}{dx} = \tan\theta = \frac{4h}{l^2}(l-2x)}$$

→ Normal thrust (NT) and radial shear :-

Normal thrust and radial shear are two forces that excist due to the inherint property of the arches.

Normal thrust is the force which acts along the longitudinal axis of the arch (axial force).

Radial shear is the shearing force created due to the radius of the arch acting radially.

$$\boxed{\begin{aligned} N.T &= H\cos\theta + V\sin\theta \\ R.S &= H\sin\theta - V\cos\theta \end{aligned}}$$

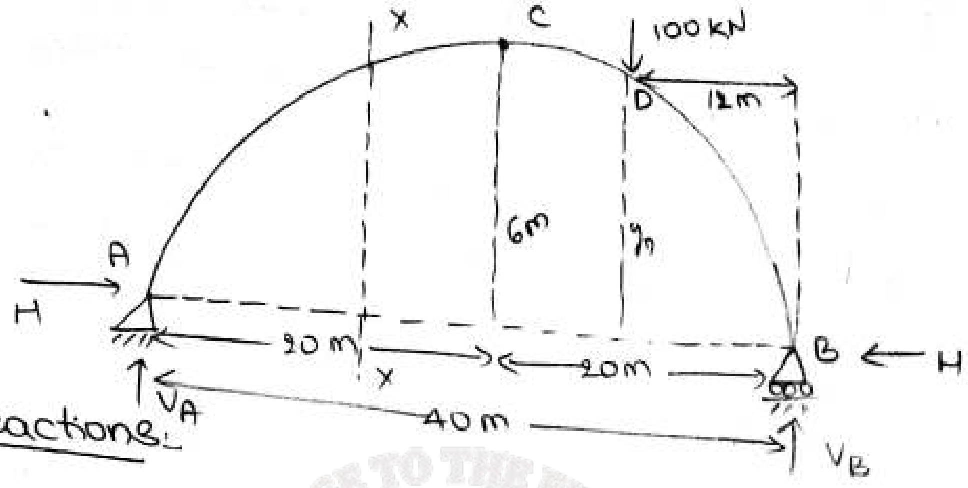
where 'V' is the shear force at the section

(1) Three hinged parabolic arch of span 40m and central rise 6m carries a point load of 100 KN at distance of 12 m from the right hinge.

Calculate (i) Reactions

(ii) Bending moment diagram

(iii) Normal thrust and radial shear at a section 10 m from the left.



(i) Reactions:-

$\Sigma H = 0 \Rightarrow H - H = 0$

$H = H$

$\Sigma V = 0 \Rightarrow V_A + V_B - 100 = 0$

$V_A + V_B = 100 \longrightarrow (1)$

$\Sigma m_A = 0$

$\Sigma m_B = 0$

$\Sigma m_C = 0$

$\Sigma m_A = 0$

$-V_B \times 40 + 100 \times 28 = 0$

$\boxed{V_B = 70 \text{ kN}}$

$\therefore$ from eqn (1)

$\boxed{V_A = 30 \text{ kN}}$

Consider $\Sigma m_C = 0$

$V_A \times 20 - H \times 6 = 0$

$$30 \times 20 - H \times 6 = 0$$

$$\boxed{H = 100 \text{ KN}}$$

$\therefore$ The reactions are $V_A = 30$ KN,

$V_B = 70$ KN and $H = 100$ KN

(ii) BMD

→ Rise and slope equation

$$y = \frac{4hx}{l^2}(l-x)$$

$$y = \frac{4 \times 6 \times x}{40^2}(40-x)$$

$$y = \frac{24x(40-x)}{1600}$$

$$y = 0.6x - 0.015x^2$$

$$\frac{dy}{dx} = \tan\theta = \frac{4h}{l^2}(l-2x)$$

$$\frac{dy}{dx} = \frac{4 \times 6}{40^2}(40-2x)$$

$$\frac{dy}{dx} = 0.015(40-2x)$$

$$\frac{dy}{dx} = 0.6 - 0.03x$$

BM calculation :-

$$M_A = M_B = M_C = 0$$

$$M_D = (V_A \times 28) - (H \times y_D)$$

From rise equation

Substitute $x = 12$ m

$$y_D = 0.6(12) - 0.015(12)^2$$

$$y_D = 5.04$$

$$M_D = (30 \times 28) - (100 \times 5.04)$$

$$\boxed{M_D = 336 \text{ KN-m}}$$

Portion AC :-

At section $x - x$

$$M_x = V_A \times x - (H \times y_x)$$

$$M_x = 30x - 100(0.6x - 0.015x^2)$$

$$M_x = 30x - 60x + 1.5x^2$$

$$M_x = -30x + 1.5x^2$$

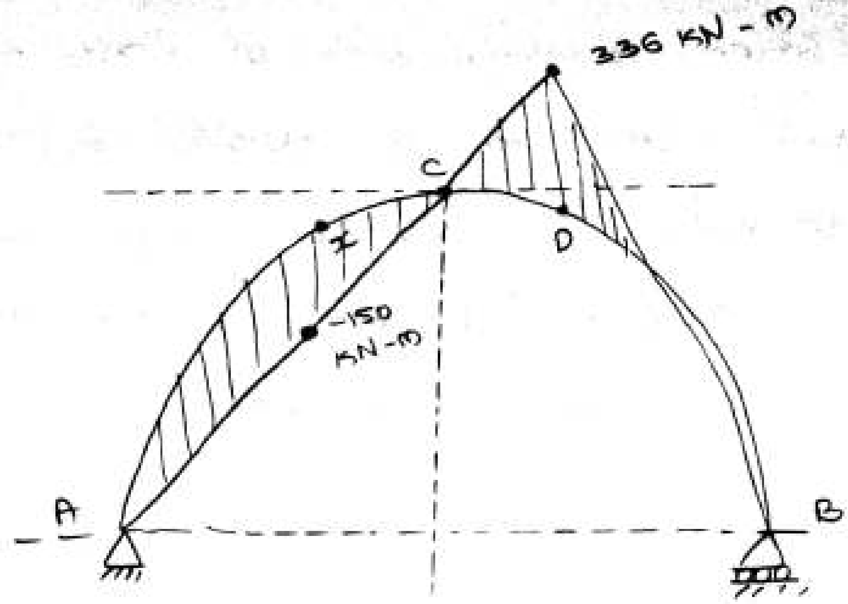
$$\frac{\partial M_x}{\partial x} = 0$$

$$\frac{\partial M_x}{\partial x} = -30 + 3x = 0$$

$$\boxed{x = 10 \text{ m}}$$

$$M_x = -30 \times 10 + 1.5 \times 10^2$$

$$\boxed{M_x = -150 \text{ KN-m}}$$



(iii) Normal thrust and radial shear at 10m from the left

at $x = 10$ m

$$\tan\theta = \frac{4h}{l^2}(l-2x) \quad \text{From (slope equation)}$$

$$\tan\theta = \frac{4\times 6}{40^2}(40 - 2\times 10)$$

$$\tan\theta = 0.3$$

$$\boxed{\theta = 16.69}$$

$H = 100$ KN

$V = 30$ KN

$$N.T = H\cos\theta + V\sin\theta$$

$$= 100\times\cos(16.69) + 30\times\sin(16.69)$$

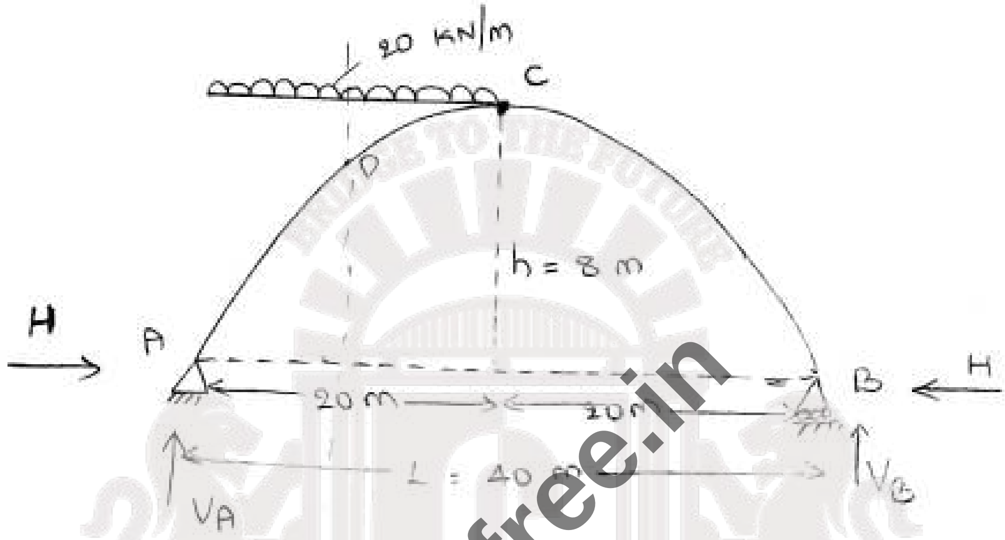
$$\boxed{N.T = 104.403 \text{ KN}}$$

$$R.S = H\sin\theta - V\cos\theta$$

$$R.S = 100\times\sin(16.69) - 30\times\cos(16.69)$$

$$\boxed{R.S = -0.0168 \text{ KN}}$$

② Three hinged parabolic arch of span 40 m and central rise 8 m is subjected to a UDL of 20 kN/m over the left hand. Calculate (i) reactions (ii) BMD (iii) NT and RS at section 10m from the left.



(i) Reactions:-

$\Sigma H = 0$

$\Sigma V = 0$

$$V_A + V_B (-20 \times 20) = 0$$

$$\Rightarrow \quad V_A + V_B = 400 \quad \text{—— ①}$$

$$\Sigma M_A = 0 \Rightarrow -V_B \times 40 + 20 \times 20 \times 10 = 0$$

$$- \quad V_B \times 40 = 4000$$

$$\boxed{V_B = 100 \text{ KN}}$$

$$V_A + V_B = 400$$

$$V_A + 100 = 400$$

$$\boxed{V_A = 300 \text{ KN}}$$

$\Sigma M_C = 0$

$$\Rightarrow (V_A \times 20) - (H \times 8) - \left(20 \times 20 \times \frac{20}{2}\right) = 0$$

$$300 \times 20 - H \times 8$$

$$\boxed{H = 250 \text{ KN}}$$

(ii) BMD

* Rise and slope equation

$$y = \frac{4hx}{L^2}(L-x)$$

$$= 4 \times 250$$

$h = 8\text{ m}$; $L = 40\text{ m}$

$$y = \frac{4 \times 8 \times x}{(40)^2}(40-x) = 0.02x(40-x)$$

$$\boxed{y = 0.8x - 0.02x^2}$$

Rise

$$\frac{dy}{dx} = \tan\theta = \frac{4h}{l^2}(l-2x)$$

$$\frac{dy}{dx} = \frac{4 \times 8}{(40)^2}(40-2x)$$

$$= 0.02(40-2x)$$

$$\boxed{\frac{dy}{dx} = 0.8 - 0.04x}$$

* BM calculation.

$M_A = M_B = M_C = 0$ [∵ Hinged]

portion AC

M_D @ $x = 10\,m$

$$M_D = (V_A \times 10) - H \times y_D - \left(20 \times 10 \times \frac{10}{2}\right)$$

$$y_D = 0.8\,(10) - 0.02\,(10^2)$$

$$= 8 - 2$$

$$\boxed{y_D = 6\,m}$$

$H = 250$

$V_A = 300$

$$M_D = (300 \times 10) - 250 \times 6 - \left(20 \times 10 \times \frac{10}{2}\right)$$

$$\boxed{M_D = 500\ KN\text{-}m}$$

For portion BC

$$M_x = V_B \times x - H y_x$$

$$= 100x - 250 - 250\,(0.8x - 0.02x^2)$$

$$M_x = -100x + 5x^2$$

For maximum Bending moment

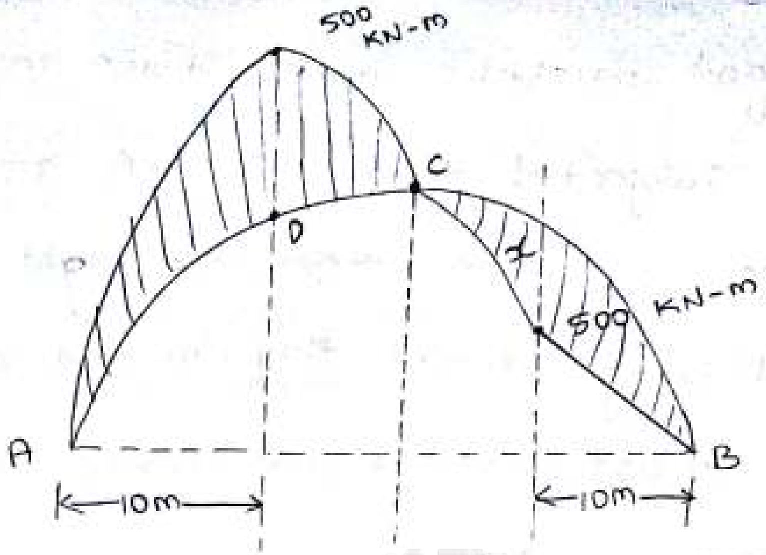
$$\frac{\partial M_x}{\partial x} = 0 \Rightarrow -100 + 10x = 0$$

$$\Rightarrow \boxed{x = 10\,m}$$

$$M_x = -100x + 5x^2$$

$$= -100\,(10) + 5\,(10)^2$$

$$\boxed{M_x = -500\ KN\text{-}m}$$



(iii) NT and RS @ 10m from left

$NT = H\cos\theta + V\sin\theta$

$RS = H\sin\theta - V\sin\theta$

$\tan\theta$

@ $x = 10$ m $= 0.8 - 0.04(10)$

$= 0.4$

$V = V_A - (20\times10)$

$\Rightarrow 100$

$\boxed{\theta = 21.8'}$

$x=10 \quad \tan\theta = \frac{4h}{l^2}(l-2x) = \frac{4(8)}{(40)^2}(40-2(10))$

$\tan\theta = 0.4 \Rightarrow \theta = \tan^{-1}(0.4)$

$\boxed{\theta = 21.80}$

$NT = 250\cos(21.80) + 100\sin(21.8)$

$= 232.121 + 37.1367$

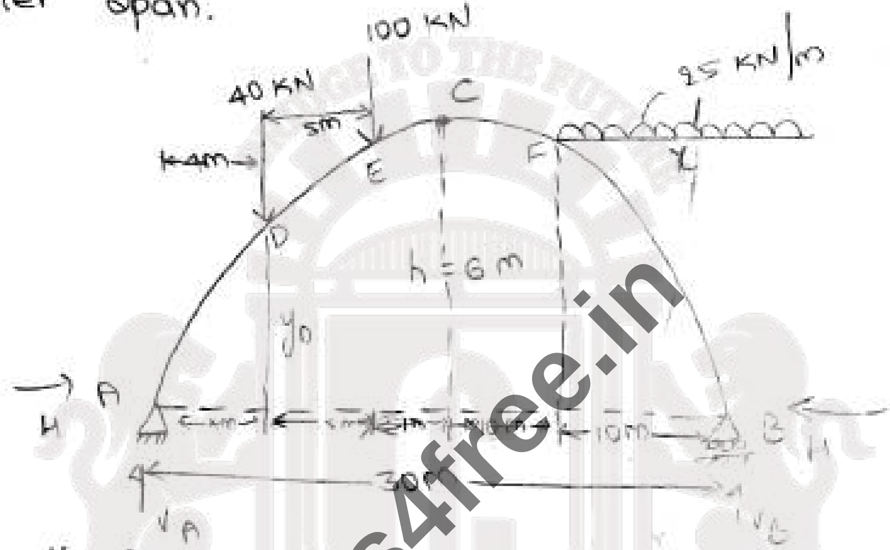
$\boxed{NT = 269.25}$

$RS = 250\sin(21.8) - 100\cos(21.8)$

$= 92.842 - 92.8485$

$RS = -0.0065$

③ A 3 hinged parabolic arch shown in the figure is subjected to a UDL of 25 kN/m for a distance of 10 m from the right hinge. Draw the bending moment diagram and determine the normal thrust and radial shear at quarter span.



(i) Reactions:

$\Sigma H = 0$

$H - H = 0$

$H = H$

$\Sigma V = 0 \quad V_A + V_B = 40 + 100 + (25 \times 10)$

$V_A + V_B = 390$

$\Sigma M_A = 0$

$-V_B \times 30 + (25 \times 10 \times 25) + 100 \times 9 + 40 \times 4 = 0$

$\cancel{+}V_B \times 30 = \cancel{+} 310$

$\boxed{V_B = 243.67 \text{ kN}}$

$V_A + V_B = 390$

$\boxed{V_A = 146.33 \text{ kN}}$

consider $\Sigma M_C = 0$

$$146.33 \times 15 - 40 \times 11 - 100 \times 6 - H \times 6 = 0$$

$$H \times 6 = 1154.95$$

$$\boxed{H = 192.49 \text{ KN}}$$

$\therefore$ The reactions are $V_A = 146.33$ KN,

$V_B = 243.67$ KN

$H = 192.49$ KN

(iii) BMD

→ rise and slope equation

$$y = \frac{4hx}{l^2}(l - x)$$

$$y = \frac{4 \times 6 \times x}{30^2}(30 - x)$$

$$y = 0.0266\,x(30 - x)$$

$$y = 0.8x - 0.0266x^2$$

$$\frac{dy}{dx} = \tan\theta = \frac{4h}{l^2}(l - 2x)$$

$$\frac{dy}{dx} = \frac{4 \times 6}{30^2}(30 - 2x)$$

$$\frac{dy}{dx} = 0.8 - 0.0533x$$

BM calculation

$$M_A = M_B = M_C = 0$$

Portion AD :-

$$M_D = 146.33 \times 4 - (192.49 \times y_D)$$

From rise equation

Substitute $x = 4$ m

$$y_D = 0.8(4) - 0.0266(4)^2$$

$$y_D = 2.774$$

$M_D = 146.33 \times 4 - (192.49 \times 2.774)$

$\boxed{M_D = 51.35 \text{ KN-m}}$

portion AE :-

$M_E = 146.33 \times 9 - (40 \times 5) - (H \times y_E)$

From rise eqn

$x = 9 \text{ m}$

$y_E = 0.8 \times 9 - 0.0266 (9)^2$

$y_E = 5.045$

$M_E = 146.33 \times 9 - (40 \times 5) - (192.49 \times 5.045)$

$\boxed{M_E = 145.85 \text{ KN-m}}$

portion FB :-

$M_F = 243.67 \times 10 - (25 \times 10 \times 5) - (192.49 \times y_F)$

From rise equation , $x = 10$

$y_F = 0.8 \times 10 - 0.0266 \times 10^2$

$y_F = 5.34$

$M_F = 243.67 \times 10 - (25 \times 10 \times 5) - (192.49 \times 5.34)$

$\boxed{M_F = 158.80 \text{ KN-m}}$

portion xB :-

$M_x = 243.67 \times 5 - (192.49 \times y_x) - (25 \times 5 \times 2.5)$

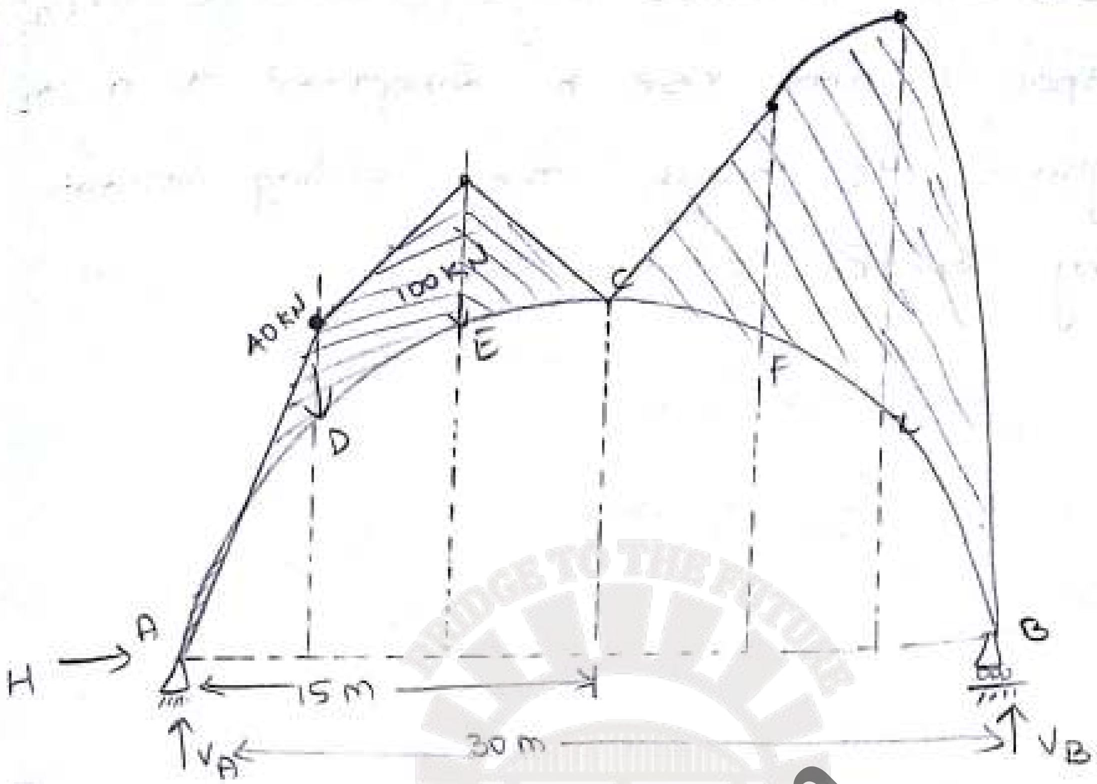
From rise equation , $x = 5$

$y_x = 0.8 \times 5 - 0.0266 \times 5^2$

$y_x = 3.335$

$M_x = 243.67 \times 5 - (192.49 \times 3.335) - (25 \times 5 \times 2.5)$

$\boxed{M_x = 263.89 \text{ KN-m}}$



(iii) N.T and R.S at quarter of the span

$= \frac{30}{4} = 7.5\,m$

at $x = 7.5\,m$

$\tan\theta = \frac{4h}{l^2}(l - 2x)$

$\tan\theta = 0.8 - 0.0533x$

$\tan\theta = 0.8 - 0.0533 \times 7.5$

$\tan\theta = 0.400$

$\boxed{\theta = 21.80}$

$H = 192.49$ KN

$V = 146.33$ KN

$NT = H\cos\theta + V\sin\theta$

$NT = 192.49\cos(21.80) + 146.33\sin(21.80)$

$\boxed{N.T = 233.066\ KN}$

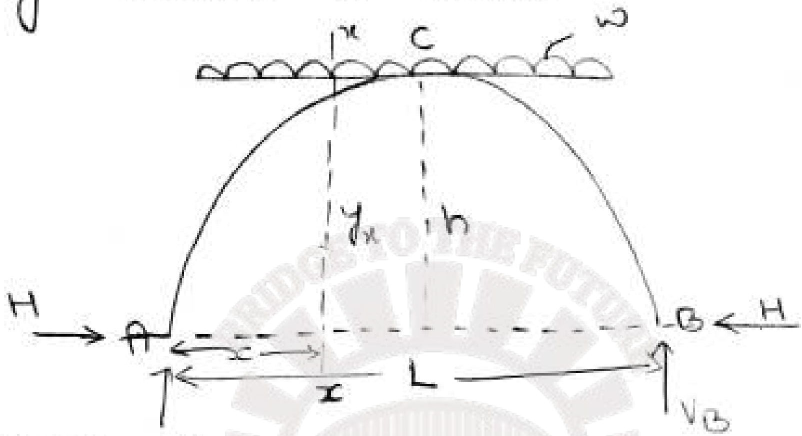
$R.S = H\sin\theta - V\cos\theta$

$R.S = 192.49\sin(21.80) - 146.33\cos(21.80)$

$\boxed{R.S = -64.38\ KN}$

☆☆

(A) prove that a three hinge parabolic arch of span 'L' and rise 'h' Subjected to a UDL throughout its length, The bending moment at any Section is zero.



$\Sigma H = 0$

$\Sigma V = 0 \Rightarrow V_A + V_B = wL$

$\Sigma M_A = 0$

$-V_B \times L + w \times L \times \frac{L}{2} = 0$

$\boxed{V_B = \frac{wL}{2}}$ $\boxed{V_A = \frac{wL}{2}}$

$\Sigma M_c = 0$

$\Rightarrow V_A \times \frac{l}{2} - H \times h - w \frac{l}{2} \frac{l}{4} = 0$

$\frac{wL^2}{4} - Hh - \frac{wL^2}{8} = 0$

$\boxed{H = \frac{wL^2}{8h}}$

BMD

☆ Rise and slope

Rise $= y = \frac{4hx}{L^2}(L-x)$

$$\frac{dy}{dx} = \tan\theta = \frac{4h}{L^2}(L-2x)$$

★ <u>BM calculations</u> :-

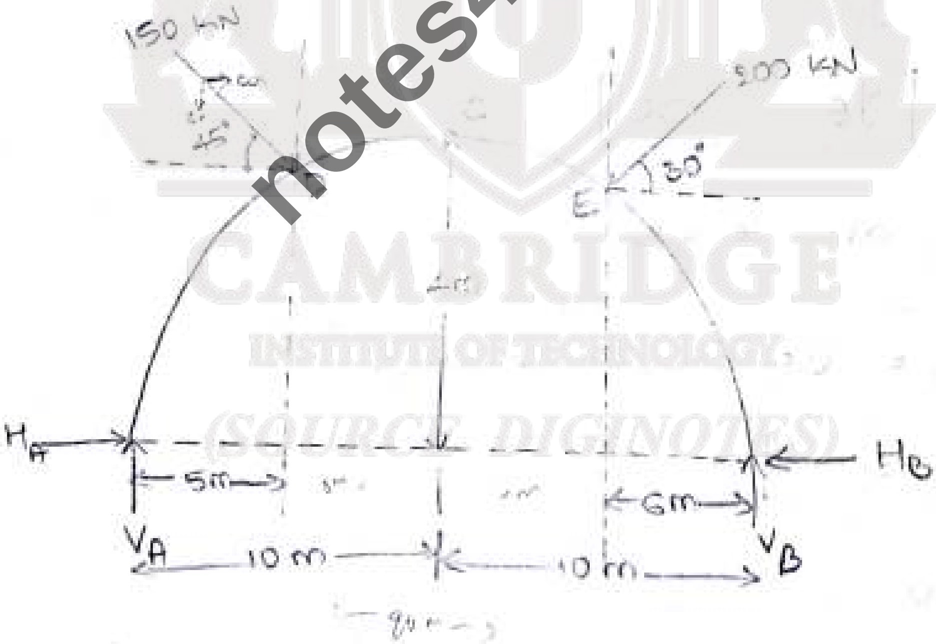
$$M_x = V_A \cdot x - H y_x - w \cdot x \cdot \frac{x}{2}$$

$$= \frac{wLx}{2} - \frac{wL^2}{8h}\left[\frac{4hx}{L^2}(L-x)\right] - \frac{wx^2}{2}$$

$$= \frac{wLx}{2} - \frac{wLx}{2} + \frac{wx^2}{2} - \frac{wx^2}{2}$$

$$\boxed{M_x = 0}$$

⑤ For the three hinged parabolic arch shown. Determine the reactions at the supports also determine the maximum BM in the arch



$\Sigma H = 0$

$$H_A - H_B = -150\cos 45^\circ + 200\cos 30^\circ$$

$$H_A - H_B = 67.139$$

$\Sigma V = 0$ $\quad V_A + V_B = 150 \sin 45^\circ + 200 \sin 30^\circ$

$$V_A + V_B = 206.06$$

$\Sigma M_A = 0 \qquad \Sigma M_B = 0 \qquad \Sigma M_C = 0$

$\Sigma M_A = 0$

$-V_B \times 20 + 200 \sin 30^\circ \times 14$

Rise and slope equation

$$y = \frac{4hx}{l^2}(l-x)$$

$$y = \frac{4 \times 4 \times x(20-x)}{20^2}$$

$$y = 0.8x - 0.04x^2$$

y_E at $x = 14$

$$y_E = 0.8 \times 14 - 0.04(14)^2$$

$$\boxed{y_E = 3.36}$$

y_D at $x = 5$

$$y_D = 0.8 \times (5) - 0.04(5)^2$$

$$\boxed{y_D = 3}$$

$\Sigma M_A = 0$

$$-V_B \times 20 + 200 \sin 30^\circ \times 14 - 200 \cos 30^\circ \times 3.36$$
$$+ 150 \sin 45^\circ \times 5 + 15 \cos 45^\circ \times 3 = 0$$

$$\boxed{V_B = 83.327 \text{ kN}}$$

$$V_A + V_B = 206.06$$

$$V_A + 83.327 = 206.06$$

$$\boxed{V_A = 122.73 \text{ KN}}$$

$\Sigma M_C = 0$

$$122.73 \times 10 - (H_A \times 4) - (150 \sin 45^\circ \times 5) - (150 \cos 45^\circ \times 1) = 0$$

$$\boxed{H_A = 147.72 \text{ KN}}$$

$$H_A - H_B = 67.139$$

$$147.72 - H_B = 67.139$$

$$\boxed{H_B = 80.581 \text{ KN}}$$

<u>Note</u>:- If resultants of the support reaction is asked in the problem, then we have to find the resultant and its angle by using the following equations.

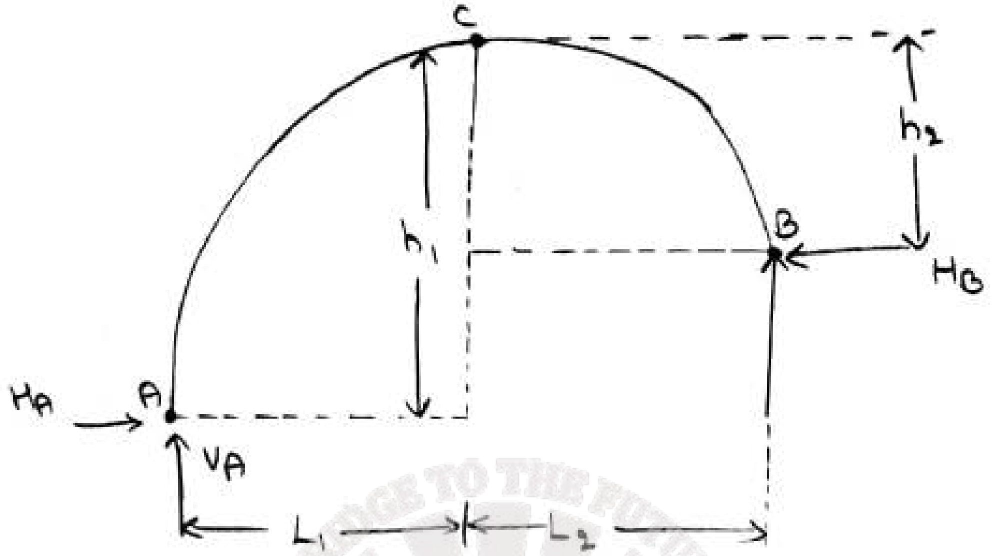
$$R_A = \sqrt{H_A^2 + V_A^2}$$

$$R_B = \sqrt{H_B^2 + V_B^2}$$

$$\theta_A = \tan^{-1}\left(\frac{V_A}{H_A}\right)$$

$$\theta_B = \tan^{-1}\left(\frac{V_B}{H_B}\right)$$

ABUTMENTS AT DIFFERENT LEVELS



* Horizontal Distances

$$\frac{L_1}{\sqrt{h_1}} = \frac{L_2}{\sqrt{h_2}} \quad \& \quad L_1 + L_2 = L$$

or
$$L_1 = \frac{L\sqrt{h_1}}{\sqrt{h_1}+\sqrt{h_2}} \quad \& \quad L_2 = \frac{L\sqrt{h_2}}{\sqrt{h_1}+\sqrt{h_2}}$$

* Rise equations

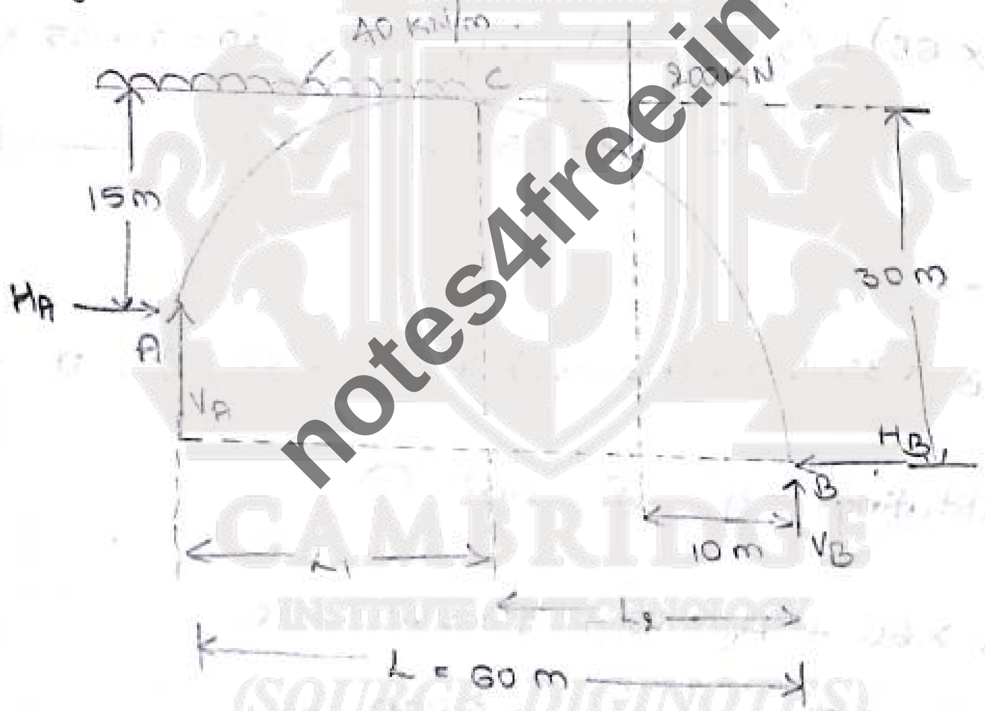
$$y = \frac{4hx}{L^2}(L-x) \rightarrow \text{For portion AC}$$

$$\Rightarrow y = \frac{4h_1x}{(2L_1)^2}(2L_1-x)$$

For portion BC

$$\Rightarrow y = \frac{4h_2x}{(2L_2)^2}(2L_2-x)$$

① Three hinged parabolic arch ACB as a span of 60 m its abutments A and B are at a depths 15 m and 30 m from the crown, the arch carries a UDL of 40 kN/m over a portion AC and a concentrated load of 200 kN. at a point 10 m from B. Find the reaction. Calculate N.T, R.S and bending moment at 15 m from A and also find out maximum bending moment at A.



* H^L Distances

$$L_1 = \frac{L\sqrt{h_1}}{\sqrt{h_1}+\sqrt{h_2}}$$

$h_1 = 15m$; $L = 60m$

$h_2 = 30m$

$L_1 = 24.85 \ m$

$L = L_1 + L_2 \Rightarrow$

$$L = L_1 + L_2$$

$$60 = 24.85 + L_2$$

$$60 - 24.85 = L_2$$

$$\boxed{L_2 = 35.15\text{ m}}$$

$$V_A + V_B = 200 + (40 \times 24.85)$$

$$V_A + V_B = 1194\text{ KN} \quad \text{—— (1)}$$

$$\Sigma m_A = 0$$

$$(-V_B \times 60) + (200 \times 50) + (H_B \times 15) + \cancel{(40 \times 24.85 \times 12.425)}$$

$$\left(40 \times 24.85 \times \frac{24.85}{2}\right) = 0 \quad \text{—— (2)}$$

$$\Sigma m_B = 0$$

$$(-V_B \times 35.15) + (H_B \times 30) + (200 \times 25.15) = 0 \quad \text{—— (3)}$$

Substituting eq^n (2) and (3)

$$-V_B \times 60 + H_B \times 15 + 22350 = 0$$

$$-V_B \times 35.15 + H_B \times 30 + 5030 = 0$$

$$\boxed{V_B = 467.53\text{ kN}}$$

$$V_A + V_B = 1194$$

$$V_A + 467.53 = 1194$$

$$\boxed{V_A = 726.47\text{ kN}}$$

$$H_A = H_B = H = 380.12\text{ kN}$$

* BM calculation

(i) Rise and slope equation

For portion AC

$$y = \frac{4h_1 x}{(2L_1)^2}(2L_1 - x)$$

$$= \frac{4 \times 15 \times x}{(2 \times 24.85)^2} \times (2 \times 24.85 - x)$$

$$y = 1.2072x - 0.0242x^2$$

$$\frac{dy}{dx} = 1.2072 - 0.0484x$$

For portion CB

$$y = \frac{4h_2 x}{(2L_2)^2}(2L_2 - x)$$

$$y = \frac{4 \times 30 \times x}{(2 \times 35.15)^2}(2 \times 35.15 - x)$$

$$y = 1.7069x - 0.0242x^2$$

$$\frac{dy}{dx} = 1.7069 - 0.0484x$$

For portion AC

N.T and R.S and B.M at 15 m from AC

BM at 15 m

$$y = 12072 \times 15 - 0.0242 \times 15^2$$

$$y = 12.595 \text{ m}$$

BM @ $x = 15$ m

$$M_{x=15m} = (V_A \times 15) - (H_A \times y_{AC(x=15)}) - 40 \times 15 \times \frac{5}{2}$$

$$M_{x_{15}} = 726.46 \times 15 - 379.99 \times 12.595 - \left(40 \times 15 \times \frac{15}{2}\right)$$

$$M_{x=15} = 1610.925 \text{ KN-m}$$

N.T = $H\cos\theta + V\sin\theta$

$H_A = 379.99$

$V_A = 726.46 + 600$

$N.T = 379.99 \cos(25.69) + (726.46 - 600)\sin(25.69)$

$\tan\theta = 1.2079 - 0.0484\,x$

$$\boxed{NT = 397.24 \text{ KN}}$$

$\tan\theta = 1.2079 - 0.0484 \times 15$

$RS = H\sin\theta - V\cos\theta$

$\tan\theta = 0.4812$

$R.S = 379.99 \sin(25.69) - (726.46 - 600)\cos(25.69)$

$\theta = 25.69$

$$\boxed{RS = 50.3374 \text{ KN}}$$

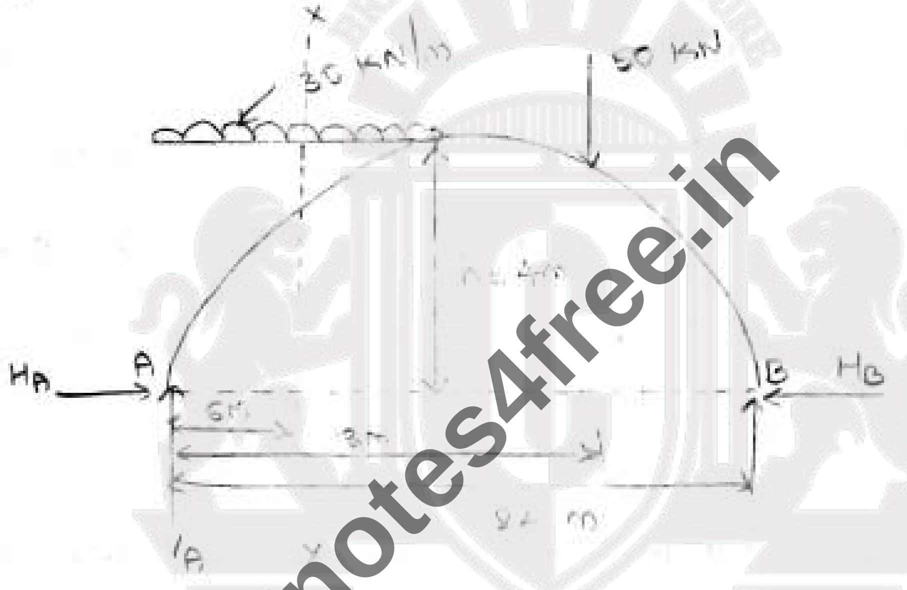
Maximum bending moment at x

$$M_x = 726.46 \times x - 467.54\,(1.2072 - 0.0484x) - \left(40 \times x \times \frac{x}{2}\right)$$

$$M_x = 726.46\,x - 564.41 + 22.62\,x$$

$$\frac{\partial M_x}{\partial x} = 726.46 + 22.62 - 20x^2 - 40x = 0$$

① A three hinged parabolic arch hinged at supports and at crown span of 24 m and the central rise of 4m. It carries concentrated load of 50 KN at 18 m from left support and UDL of 130 KN-m over the left half portion. Determine the bending moment, NT and RS at a section 6m from the left support. Sketch BMD.



② A three hinged arch of span 20 m, hinges are provided at supports and crown of the arch. The rise at crown is 5m and arch is subjected to point load of 200 KN at 6m from the left support. Find the reactions at supports and calculate normal thrust, and radial thrust shear at 6m from left support. Draw the BM diagram and also indicates the portion of maximum positive and negative BM.

③ A three hinged parabolic arch Symmetrical Span and 5 m rise, carries UDL of 14 KN/m on the entire Span and a point load of 200 KN and 5m from right end. Determine the reactions and also determine the BM, NT, and RS at 5m from left end.

④ A three hinged parabolic arch has a Span of 20m and rise of 5m. It carries a UDL of 25 KN/m over the left half of the Span and point load of 130 KN at 5m from right end. Find the BM, NT, RS at a Section 4m from the left end.

⑤ A three hinged parabolic arch has a Span of 20m and central rise of 4m. It is loaded with UDL of intensity 2 KN/m. on the left 8m length. Find (i) direction and magnitude of reactions at the hinge A and B (ii) maximum positive and negative Bm and position (iii) Draw BMD.

CABLES

cables are used to support loads over long spans such as suspension bridges, transmission lines, aerial tramway, guywires for high tower etc.

The only force in the cable is direct tension. Since the cables are flexible they carry zero bending moment. Analysis of cables involves the determination of reactions at the supports and tension over different parts of the cable. To determine the reactions at supports and tension in any part of the cable, equilibrium conditions are used. In addition to that the condition that the bending moment at any point of the cable is zero is used.

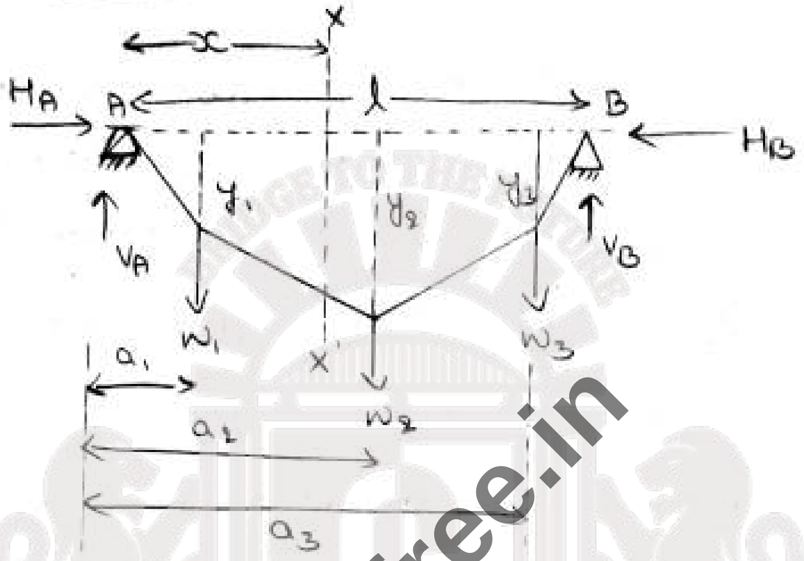
① Cable subjected to concentrated load

Assumptions :-

* The horizontal span of the vertical concentrated loads are known.
* weight of the cable is negligable.
* cable is flexible that is resistance to bending is small or bending moment is zero.

* portions of the cable between successive loads may be treated as two force members

The approach to solve cables with concentrated load



Consider a cable as shown in the figure with concentrated loads W_1, W_2 and W_3 at distances a_1, a_2 and a_3 from A respectively.

* To find the vertical support reactions
  (i) consider a cable to be a beam and analyse for V_A and V_B by using the equilibrium equations.

* To find the horizontal thrust
  (i) consider the section at a distance 'x' and deflection 'y_x' ∴ Bending moment at the section 'x' is given by

$M_x = V_A \cdot x - W_1(x-a_1) + H \cdot y_x$

Since the cable is flexible M_x = zero

$-(V_A x - W_1(x-a_1)) = H y_x$

But we know that the term on the left hand side of the above equation is nothing but the moment of the beam at section 'x'.

$\therefore \quad M_{Beam} = H \cdot y_x$

$\therefore \quad \boxed{H = \frac{M_{beam}}{y_x}}$ or $\boxed{y_x = \frac{M_{beam}}{H}}$

Hence the deflected shape is similar to the bending moment diagram of the beam. If M_1, M_2 and M_3 are the beam moments at load points 1, 2 and 3 then the deflections y_1, y_2 and y_3 are given by

$y_1 = \frac{M_1}{H} \quad y_2 = \frac{M_2}{H} \quad y_3 = \frac{M_3}{H}$

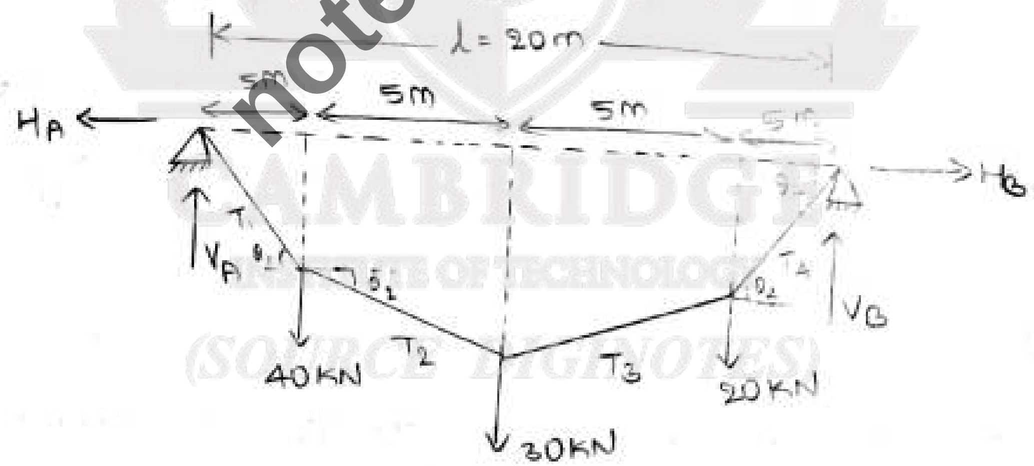
Hence if the horizontal thrust is known, the position of cable at any one point is known, the deflections at all points can be calculated and the shape of the cable is found. After finding the deflections, slope of various segments can be found.

* To find the tension in the table

(i) consider the portion between successive load to be a two force member and analyse the joints to find tension

problem :-

① A light cable is supported at two points 20 cm apart which are at the same level. The cable supports three concentrated loads as shown in the figure. The deflection at the first point is found to be 0.8 m. Determine the tension in the different segments and total length of the cable.



Step-1 :- To find vertical support reactions

Consider the beam

$\Sigma V = 0$

$$V_A + V_B = 40 + 30 + 20 = 90 \quad \text{— ①}$$

$\Sigma M_A = 0$

$\Rightarrow (-V_B \times 20) + (40 \times 5) + (20 \times 15) + (30 \times 10)$

$-V_B \times 20 + 300 + 300 + 200 = 0$

$-V_B \times 20 = -800$

$\boxed{V_B = 40 \text{ KN}}$

$\boxed{V_A = 50 \text{ KN}}$

Step-2 : To find the horizontal thrust

Beam moments at at ① ② & ③

$M_1 = 50 \times 5 = 250 \text{ KN-m}$

$M_2 = (50 \times 10) - (40 \times 5) = 300 \text{ KN-m}$

$M_3 = (50 \times 15) - (40 \times 10) - (30 \times 5) = 200 \text{ KN-m}$

WKT $y_1 = 0.8 m$

$y_1 = \frac{M_1}{H} = \frac{250}{\quad}$

$0.8 = \frac{250}{H}$

$H = \frac{250}{0.8}$ $\boxed{H = 312.5 \text{ KN}}$

$y_2 = \frac{M_2}{H} = \frac{300}{312.5} = 0.096$

$y_3 = \frac{M_3}{H} = \frac{200}{312.5} = 0.64$

Step-4 To find the slope

$$\theta_1 = \tan^{-1}\left(\frac{y_1}{5}\right)$$

$$= \tan^{-1}\left(\frac{0.8}{5}\right)$$

$$\boxed{\theta_1 = 9.09^\circ}$$

$$\theta_2 = \tan^{-1}\left(\frac{y_2 - y_1}{5}\right)$$

$$= \tan^{-1}\left(\frac{0.96 - 0.8}{5}\right)$$

$$\theta_2 = 1.83^\circ$$

$$\theta_3 = \tan^{-1}\left(\frac{y_2 - y_3}{5}\right)$$

$$= \tan^{-1}\left(\frac{0.96 - 0.64}{5}\right)$$

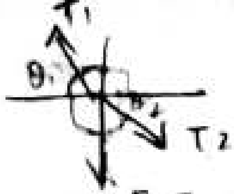
$$\theta_3 = 3.66^\circ$$

$$\theta_4 = \tan^{-1}\left(\frac{y_3}{5}\right) = \tan^{-1}\left(\frac{0.64}{5}\right)$$

$$\theta_4 = 7.29^\circ$$

Step-5: To find tension

Let tensions in the segments be T_1, T_2, T_3 and T_4. Then consider the point 1.



$90-\theta_1+90+\theta_2$

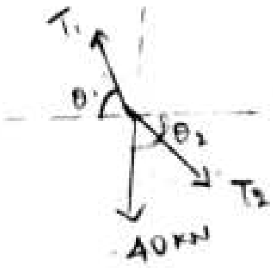
$180+\theta_2-\theta_1$

40θ T_1 316.5

T_2 316.79

point ①

F.B.D



Consider Lami's theorem

$$\frac{T_1}{\sin(90-\theta_2)} = \frac{T_2}{\sin(90+\theta_1)} = \frac{40}{\sin(180-\theta_1+\theta_2)}$$

$$\frac{T_1}{\sin(90-\theta_2)} = \frac{40}{\sin(180-\theta_1+\theta_2)}$$

$$\frac{T_1}{\sin(90-1.83)} = \frac{40}{\sin(180-9.09+1.83)}$$

$$T_1(\sin(180-9.09+1.83)) = 40(\sin(90-1.83))$$

$$T_1 = \frac{39.9795}{0.12637}$$

$$T_1 = 318.36$$

$$\boxed{T_1 = 316.54}$$

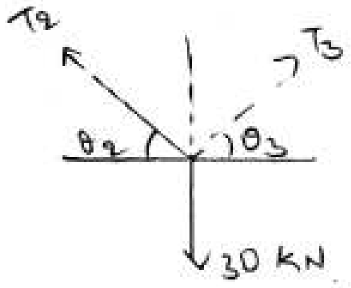
$$\frac{T_2}{\sin(90+9.09)} = \frac{40}{\sin(180-9.09+1.83)}$$

$$T_2 \times 0.1263 = 40 \times 0.9874$$

$$T_2 = \frac{40 \times 0.9874}{0.1263}$$

$$\boxed{T_2 = 312.71}$$

point ②



$$\frac{T_2}{\sin(90+\theta_3)} = \frac{T_3}{\sin(90+\theta_2)} = \frac{30}{\sin(180-\theta_2+\theta_3)}$$

$$\frac{T_3}{\sin(90+1.83)} = \frac{30}{\sin(180-1.83+3.66)}$$

$T_3 \times$

$$L = \frac{5}{\cos\theta_1} + \frac{5}{\cos\theta_2} + \frac{5}{\cos\theta_3} + \frac{5}{\cos\theta_4}$$

$$= \frac{5}{\cos(9.09)} + \frac{5}{\cos(1.83)} + \frac{5}{\cos(3.66)} + \frac{5}{\cos(7.29)}$$

$$= 5.0635 + 5.0025 + 5.0102 + 5.0407$$

$$L = 20.1169 \approx 20.117 \text{ m}$$

(2) A chord supported at its ends 40 m apart carries loads of 20 KN, 10 KN and 12 KN at distances 10 m, 20 m, 30 m respectively from the left end. If the point on the chord where the 10 KN load is supported at 13 m below the level of the end supports, Determine (i) Reaction at Supports (ii) The tension in different parts of the chord (iii) The total length of the chord.

Reaction

$$V_A + V_B = 20 + 10 + 12$$

$$V_A + V_B = 42$$

$M_A = 0$, $-V_B \times 40 + 12 \times 30 + 10 \times 20 + 20 \times 10 = 0$

$$-40V_B + 360 + 200 + 200 = 0$$

$$-40V_B = -760$$

$$\boxed{V_B = 19 \text{ KN}}$$

$$V_A + V_B = 42$$

$$V_A + 19 = 42$$

$$V_A = 42 - 19$$

$$\boxed{V_A = 23 \text{ KN}}$$

Step-2

$$M_1 = 23 \times 10 = 230 \quad KN-m$$

$$M_2 = (V_A \times 20) - (20 \times 10) = (23 \times 20) - (20 \times 10)$$

$$M_2 = 260 \quad KN-m$$

$$M_3 = (V_A \times 30) - (20 \times 20) - (10 \times 10) = 190 \quad KN-m$$

WKT

$$y_2 = 13$$

$$y_2 = \frac{M_2}{H} \Rightarrow 13 = \frac{260}{H}$$

$$H \times 13 = 260$$

$$\boxed{H = 20 \text{ m}}$$

$$y_1 = \frac{M_1}{H} = \frac{230}{20} = 11.5 \text{ m}$$

$$y_3 = \frac{M_3}{H} = \frac{190}{20} = 9.5 \text{ m}$$

To find slope

$$\theta_1 = \tan^{-1}\left(\frac{y_1}{10}\right) = \tan^{-1}\left(\frac{11.5}{10}\right)$$

$$= 48.99$$

$$\theta_2 = \tan^{-1}\left(\frac{y_2 - y_1}{10}\right) = \tan^{-1}\left(\frac{13 - 11.5}{10}\right)$$

$$= 8.53^\circ$$

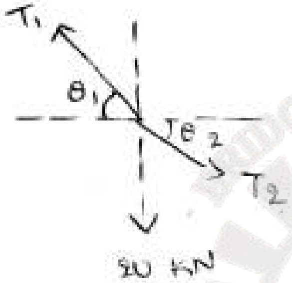
$$\theta_3 = \tan^{-1}\left(\frac{y_2 - y_3}{10}\right) = \tan^{-1}\left(\frac{13 - 9.5}{10}\right)$$

$$= 19.29$$

$$\theta_4 = \tan^{-1}\left(\frac{y_3}{10}\right) = \tan^{-1}\left(\frac{9.5}{10}\right)$$

$$= 43.53$$

<u>step-5</u> <u>To find tension:</u>



consider Lami's theorem

$$\frac{T_1}{\sin(90-\theta_2)} = \frac{T_2}{\sin(90+\theta_1)} = \frac{20}{\sin(180-\theta_1+\theta_2)}$$

$$\frac{T_1}{\sin(90-\theta_2)} = \frac{20}{\sin(180-\theta_1+\theta_2)}$$

$$T_1\left(\sin(180-48.99+8.53)\right) = 20\left(\sin(90-8.53)\right)$$

$$T_1(0.6480) = 19.7787$$

$$T_1 = 30.48 \approx$$

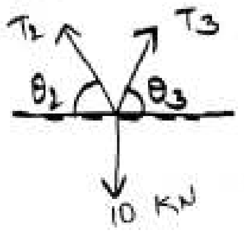
$$\boxed{T_1 = 30.5}$$

$$\frac{T_2}{\sin(90+\theta_1} = \frac{20}{\sin(180-\theta_1+\theta_2)}$$

$$T_2\left(\sin(180-48.99+8.53)\right) = 20\left(\sin(90+48.99)\right)$$

$$T_2 = \frac{13.1238}{0.6489} \qquad \boxed{T_2 = 20.22}$$

point - 2



$$\frac{T_2}{\sin(90+\theta_3)} = \frac{T_3}{\sin(90+\theta_2)} = \frac{10}{\sin(180-\theta_2-\theta_3)}$$

$$\frac{T_3}{\sin(90+\overset{8.53}{\cancel{19.29}})} = \frac{10}{\sin(180-8.53-19.29)}$$

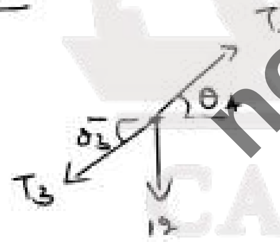
$T_2 \;\epsilon\; 10$

$$T_3\left(\sin(180-8.53-19.29)\right) = 10 \times \left(\sin(90+8.53)\right)$$

$$T_3 = \frac{9.8893}{0.4666}$$

$$\boxed{T_3 = 21.19}$$

point - 3



$$\frac{T_3}{\sin(90+\theta_4)} = \frac{T_4}{\sin(90-\theta_3} = \frac{12}{\cancel{\sin(180-\theta_4+\theta_3)}\ \sin(180-\theta_4+\theta_3)}$$

$$\frac{T_4}{\sin(90-19.29)} = \frac{12}{\sin(180-43.53+19.29)}$$

$$T_4\left(\sin(180-43.53+19.29)\right) = 12\left(\sin(90-19.29)\right)$$

$$\boxed{T_4 = 28.019}$$

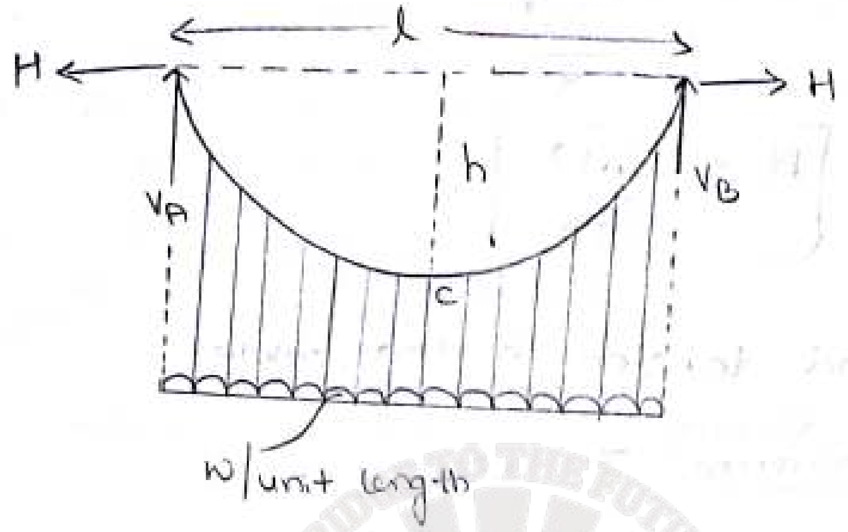
$$L = \frac{10}{\cos\theta_1} + \frac{10}{\cos\theta_2} + \frac{10}{\cos\theta_3} + \frac{10}{\cos\theta_4}$$

$$= \frac{10}{\cos(49)} + \frac{10}{\cos(8.53)} + \frac{10}{\cos(19.29)} + \frac{10}{\cos(43.53)}$$

$$= 15.242 + 10.11 + 10.59 + 13.79$$

$$\boxed{L = 49.73\ m}$$

cable Subjected to UDL



Let a cable of length 'L' be supported at points A and B which are at a horizontal distance 'l' and are at the same level. The cable is subjected to a UDL of w/unit length as shown in the figure.

(1) To find vertical reactions

Consider the beam action and find the vertical reactions V_A and V_B by considering equilibrium equations

(2) To find the horizontal thrust

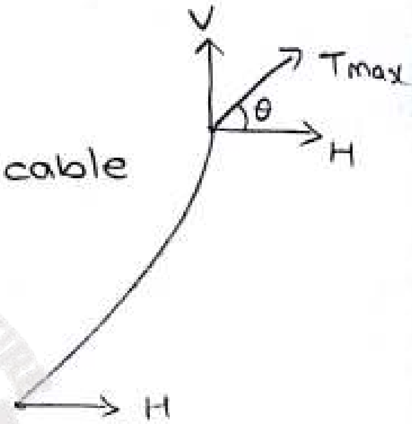
We know that bending moment at any point on the cable is equal to zero. Consider the point 'c' therefore $M_c = 0$

$$M_c = 0 \Rightarrow V_A \times \frac{l}{2} - w.\frac{l}{2}.\frac{l}{4} - Hy = 0$$

$$\Rightarrow \quad H.h = \frac{wl^2}{4} - \frac{wl^2}{8}$$

$$= \frac{wl^2}{8}$$

$$\boxed{H = \frac{wl^2}{8h}}$$



(3) To find tension in the cable

Maximum Tension

we know that maximum tension occurs at the supports

$\therefore$ T_{max} can be given by

$$\boxed{T_{max} = \sqrt{H^2+V_2}}$$

The slope is given by $\theta = \tan^{-1}\left(\frac{V}{H}\right)$

Minimum tension

minimum tension occurs at the central span of the cable given by

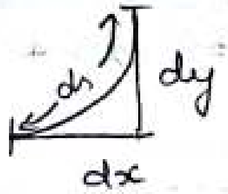
$$T_{min} = \sqrt{H^2+0^2}$$

Since there are no vertical forces

$$\boxed{T_{min} = H}$$

(4) To find the length of the cable

consider a small part of the length of the cable 'ds' as shown in the figure



WKT $\frac{ds}{dx} = \sqrt{1+\left(\frac{dy}{dx}\right)^2}$

$$= 1 + \frac{1}{2}\left(\frac{dy}{dx}\right)^2$$

But we know that for a parabola $\frac{dy}{dx} =$

$$\frac{dy}{dx} = \frac{4h}{l^2}(l-2x)$$

$$\therefore \frac{ds}{dx} = 1 + \frac{1}{2}\left[\frac{4h}{l^2}(l-2x)\right]^2$$

The total length of the cable is given by

$$\therefore L = \int_0^l 1 + \frac{1}{2}\left[\frac{4h}{l^2}(l-2x)\right]^2 dx$$

$$L = \int_0^l 1 + \frac{1}{2}\left[\frac{16h^2}{l^4}(l^2+4x^2-4lx)\right] dx$$

$$= \int_0^l 1 + \frac{1}{2}\left\{\frac{16h^2}{l^2} + \frac{64h^2x^2}{l^4} - \frac{64h^2x}{l^3}\right\} dx$$

$$= \int_0^l \left(1 + \frac{8h^2}{l^2} + \frac{32h^2x^2}{l^4} - \frac{32h^2x}{l^3}\right) dx$$

$$= \left[x + \frac{8h^2}{l^2}.x + \frac{32h^2.x^3}{3l^4} - \frac{32h^2.x^2}{2l^3}\right]_0^l$$

$$= l + \frac{8h^2.l}{l^2} + \frac{32h^2.l^3}{3l^4} - \frac{32h^2.l^2}{2l^3}$$

$$= l + \frac{8h^2}{l} + \frac{32}{3}\frac{h^2}{l} - \frac{16h^2}{l}$$

$$= l + \frac{8h^2}{l}\left[1 + \frac{4}{3} - 2\right]$$

$$\boxed{L = l + \frac{8h^2}{3l}}$$

problem

① A light flexible cable 18 m long is supported at 2 ends which are at the same level. The supports are 16 m apart. The cable is subjected to UDL of 1 KN/m. of horizontal length over its entire span. Determine the reactions developed at the support.

Step - 1

To find vertical reactions

$$V_A + V_B = 16 \text{ KN}$$

$$M_A = 0 \quad - V_B \times 16 + (1 \times 16)(16/2)$$

$$= -16 V_B + 128 = 0$$

$$\boxed{V_B = 8 \text{ KN}}$$

$$V_A = 16 - 8$$

$$\boxed{V_A = 8 \text{ KN}}$$

step-2

To find horizontal thrust

$$H = \frac{wl^2}{8h}$$

$$= \frac{1\times(16)^2}{8\times h}$$

$$H = \frac{1\times(16)^2}{8\times 3.47}$$

$$\boxed{H = 9.24\ KN}$$

$$L = l + \frac{8h^2}{3l}$$

$$18 = 16 + \frac{8h^2}{3\times 16}$$

$$18 = 16 + 0.1666 \times h^2$$

$$18 = 16.1666 \times h^2$$

$$h^2 =$$

$$\boxed{h = 3.47\ m}$$

(iii) To find maximum Tension

$$T_{max} = \sqrt{H^2+V^2}$$

$$= \sqrt{(9.24)^2+(8)^2}$$

$$\boxed{T_{max} = 12.22\ KN}$$

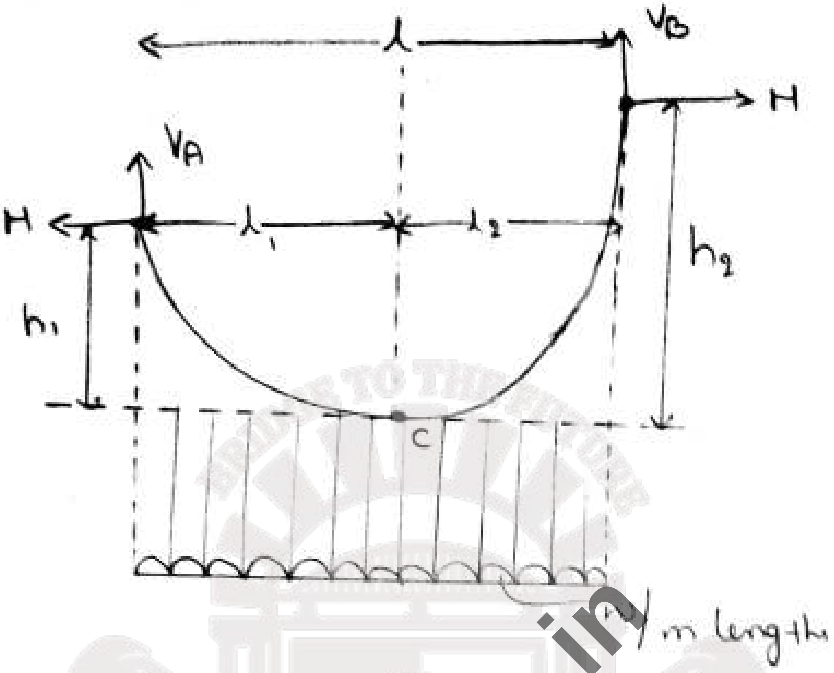
$$T_{min} = H$$

$$\boxed{T_{min} = 9.24\ KN}$$

$$\theta = \tan^{-1}\left(\frac{V}{H}\right) = \left(\frac{8}{9.24}\right)$$

$$\boxed{\theta = 40.88^\circ}$$

CABLES AT ENDS WITH DIFFERENT LEVELS



The equation of a parabola in general

$y = kx^2$ or $\frac{x^2}{y}$ is constant as well as $\frac{y}{x^2}$ is constant. Applying these equations to points A and B we get

$$H_1 = \frac{\omega l_1^2}{2H}$$

$$h_2 = \frac{\omega l_2^2}{2H}$$

Similarly $l_1 = \frac{l\sqrt{h_1}}{\sqrt{h_1} + \sqrt{h_2}}$

$$l_2 = \frac{l\sqrt{h_2}}{\sqrt{h_1} + \sqrt{h_2}}$$

To find 'H' calculate the moment at 'C'

① considering LHS

$$m_c = LHS = V_A \cdot l_1 - Hh_1 - wl_1 \cdot \frac{l_1}{2} = 0$$

$$\Rightarrow V_A \cdot l_1 - Hh_1 - \frac{wh^2}{2} = 0$$

$$V_A = \frac{Hh_1}{l_1} + \frac{wl_1}{2} \quad \text{—— ①}$$

② considering RHS

$$(-V_B \times l_2) + H \times h_2 + \frac{wl_2^2}{2} = 0$$

$$\boxed{V_B = \frac{Hh_2}{l_2} + \frac{wl_2}{2}} \quad \text{—— ②}$$

Adding equations ① and ② we get

$$V_A + V_B = \frac{Hh_1}{l_1} + \frac{wl_1}{2} + \frac{Hh_2}{l_2} + \frac{wl_2}{2}$$

$$V_A + V_B = \frac{H.h_1}{\frac{l\sqrt{h_1}}{\sqrt{h_1}+\sqrt{h_2}}} + \frac{w}{2}\left(\frac{l\sqrt{h_1}}{\sqrt{h_1}+\sqrt{h_2}}\right) + \frac{Hh_2}{\frac{l(\sqrt{h_2})}{\sqrt{h_1}+\sqrt{h_2}}}$$

$$+ \frac{w}{2}\left(\frac{l\sqrt{h_2}}{\sqrt{h_1}+\sqrt{h_2}}\right)$$

$$V_A + V_B = H\left[\frac{h_1(\sqrt{h_1}+\sqrt{h_2})}{l(\sqrt{h_1})}\right] + \frac{wl}{2}\left[\frac{\sqrt{h_1}+\sqrt{h_2}}{\sqrt{h_1}+\sqrt{h_2}}\right]$$

$$+ H\left[\frac{h_2(\sqrt{h_1}+\sqrt{h_2})}{l\sqrt{h_2}}\right]$$

$$\Sigma V = 0 \Rightarrow V_A + V_B - wl = 0$$

$$V_A + V_B = wl \quad \text{—— (3)}$$

$$\Rightarrow wl = \frac{H}{l}\left[\frac{h_1(\sqrt{h_1}+\sqrt{h_2})}{\sqrt{h_1}} + \frac{h_2(\sqrt{h_1}+\sqrt{h_2})}{\sqrt{h_2}}\right] + \frac{wl}{2}$$

$$\frac{wl^2}{2} = H(\sqrt{h_1}+\sqrt{h_2})\left[\frac{h_1}{\sqrt{h_1}} + \frac{h_2}{\sqrt{h_2}}\right]$$

$$H = \frac{wl^2}{2(\sqrt{h_1}+\sqrt{h_2})\left[\frac{h_1}{\sqrt{h_1}} + \frac{h_2}{\sqrt{h_2}}\right]}$$

$$= \frac{wl^2}{2(\sqrt{h_1}+\sqrt{h_2})\left(\frac{(\sqrt{h_1})^2}{\sqrt{h_1}} + \frac{(\sqrt{h_2})^2}{\sqrt{h_2}}\right)}$$

$$\boxed{H = \frac{wl^2}{2(\sqrt{h_1}+\sqrt{h_2})^2}}$$

Length of the cable ACB is equal to half of sum of length of ACB' + sum of length of A'CB

Length of the cable ACB $= \frac{1}{2} \times$ [Sum of length of ACB' + sum of length of A'CB]

$$\left(\because L = l + \frac{8h^2}{3l}\right)$$

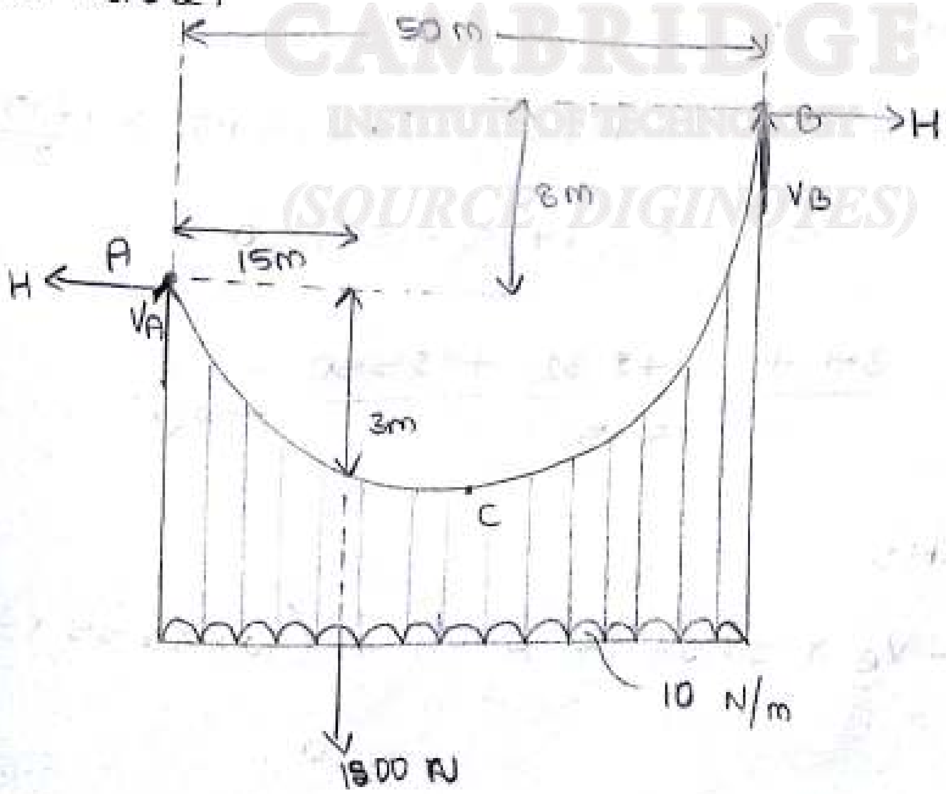
$$L = \frac{1}{2}\left[\text{Sum of } ACB' + A'CB\right]$$

$$= \frac{1}{2}\left[2l_1 + \frac{8h_1^2}{6l_1} + 2l_2 + \frac{8h_2^2}{6l_2}\right]$$

$$= l + \frac{4}{6}\left(\frac{h_1^2}{l_1} + \frac{h_2^2}{l_2}\right)$$

$$\boxed{L = l + \frac{2}{3}\left(\frac{h_1^2}{l_1} + \frac{h_2^2}{l_2}\right)}$$

① A flexible cable weighing 10 N/m hangs between two supports 50 m apart. The left support is 8 m below the right support. The cable also supports a point load of 1200 N. at a point 15 m from the left support and 3m below the left support. Assuming that the weight of cable is spread uniformly on a horizontal span. Find the maximum tension for cable.



$$h_2 = h_1 + 8 = 11\,m$$

$$h_1 = 3\,m$$

$$l = 50\ m$$

$$l_1 = \frac{l\,(\sqrt{h_1})}{\sqrt{h_1}+\sqrt{h_2}} \qquad l_2 = \frac{l\,(\sqrt{h_2})}{\sqrt{h_1}+\sqrt{h_2}}$$

$$= \frac{50\times(\sqrt{3})}{(\sqrt{3})+(\sqrt{11})} \qquad = \frac{50\times(\sqrt{11})}{\sqrt{3}+\sqrt{11}}$$

$$l_1 = 17.15\ m \qquad l_2 = 32.85\ m$$

② step   Reactions

$$\Sigma V = 0\ ; \quad V_A + V_B - (10\times 50) - 1200 = 0$$

$$V_A + V_B = 1700\ N \quad \text{—— ①}$$

$$M_c = LHS$$

$$= (V_A \times 17.15) - H(3) - 10\times 17.15 \times \frac{17.15}{2} - 1200\times(17.15-15) = 0$$

$$V_A = \frac{3H + 1472.32 + 2580}{17.15} \longrightarrow ②$$

$$M_c = RHS$$

$$M_c = -V_B \times 32.85 + H\times 11 + \left(10\times 32.85\times\frac{32.85}{2}\right) = 0$$

$$V_B = \frac{11H + 5395.61}{32.85} \rightarrow (3)$$

$$V_A + V_B = \frac{3H + 1472.33 + 2580}{17.15} + \frac{11H + 5395.61}{32.85}$$

$$1700 = \frac{98.55H + 48309.21 + 84753 + 188.65H + 92534.7115}{563.3775}$$

$$957741.75 = 287.2H + 225596.9215$$

$$H = \frac{732144.8285}{287.2}$$

$$H = 2549.25$$

$$H = 2550\,m$$

$$V_A = \frac{3 \times 2550 + 1472.32 + 2580}{17.15}$$

$$V_A = 673.604 \qquad \boxed{V_A = 682.25\ N}$$

$$V_B = \frac{11(2580) + 5395.61}{32.85}$$

$$\boxed{V_B = 1018.13\ N}$$

At point B, maximum tension occurs

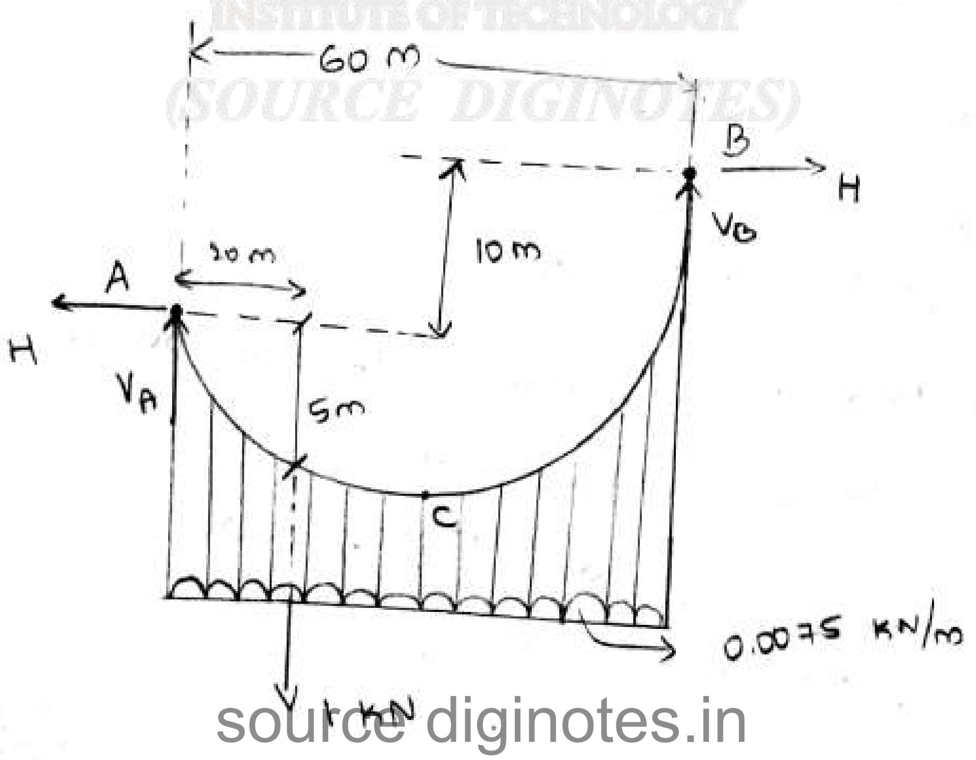
$$T_{max} = \sqrt{V_B^2 + H^2}$$

$$= \sqrt{(1018.13)^2 + (2550)^2}$$

$$\boxed{T_{max} = 2745.74}$$

Note: In order to avoid confusion consider the point at which the point load is applied to be the point 'c'.

① The flexible suspension cable of weight 0.0075 KN-m hangs between two vertical walls 60 m apart. The left end being attached to the wall at a point 10 m below the right end. A concentrated load of 1 KN is attached to the cable in such a manner that the point of attachment of the load is 20 m from the left hand side and 5 m below the left support. Show that the maximum tension occurs at the right hand support and find its value and also find the total length of the cable.



$h_1 = 5\,m$

$h_2 = h_1 + 10 = 15\,m$

$$l_1 = \frac{l\sqrt{h_1}}{\sqrt{h_1}+\sqrt{h_2}} \qquad l_2 = \frac{l\sqrt{h_2}}{\sqrt{h_1}+\sqrt{h_2}}$$

$$= \frac{60 \times \sqrt{5}}{\sqrt{5}+\sqrt{15}} \qquad = \frac{60 \times \sqrt{15}}{\sqrt{5}+\sqrt{15}}$$

$l_1 = 21.96\ m \qquad l_2 = 38.03\ m$

$l_1 = 20\,m \qquad l_2 = 40\,m$

Reactions :-

$V_A + V_B - 1 - 0.45 = 0$

$V_A + V_B = 1.45\ kN$

$M_C = LHS$

$V_A \times 20 - (H \times 5) - \left(0.0075 \times 20 \times \frac{20}{2}\right) = 0$

$V_A = \frac{5H + 1.5}{20} \longrightarrow (1)$

$M_C = RHS$

$-V_B \times 40 + (H \times 11) + \left(0.0075 \times 40 \times \frac{40}{2}\right) = 0$

$V_B = \frac{11H + 6}{40} \longrightarrow (2)$

$V_A + V_B = \frac{5H + 1.5}{20} + \frac{11H + 6}{40}$

$1.45 = 10$

MODULE - 5

Influence Line diagram and moving Loads

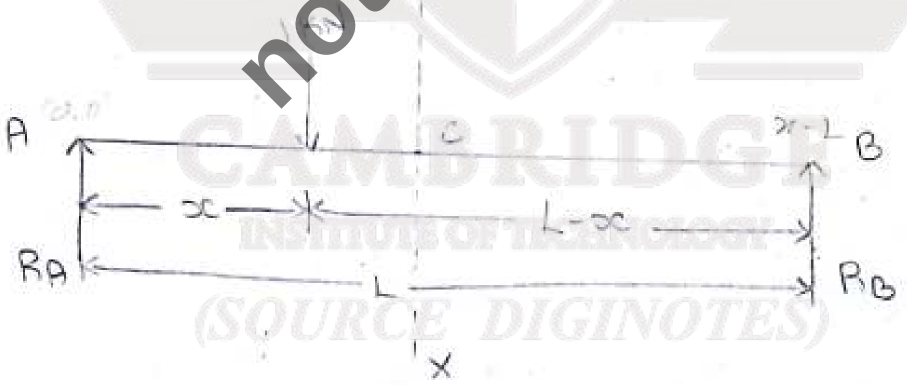
The loads due to Self weight [dead loads and live loads acting on a structure are known as fixed loads. These fixed loads create fixed reactions, shear force and bending moment. But there are certain loads which change with respect to time such as moving loads or rolling wheel rollers loads. which account for the vehicles passing over a bridge, flyover, etc., These loads may act at the most critical point producing the maximum effects.

Due The Varaitation in the structural response is expressed in terms of reactions shear force and bending moment which occur due to moving loads. To identify the positions of loads for maximum shear force and bending moment at specified sections Influence Line diagram are used.

A curve or a graph that represents a function like reaction at the Support, Shear force at a Section, bending moment at a Section of a Structure for a various position of a unit load on the Span of the Structure is called influence line diagram [ILD].

Influence line diagrams for Simply Supported beams

Case (i) ILD for a Simply Supported beam Subjected to a Single point load



ILD for reactions

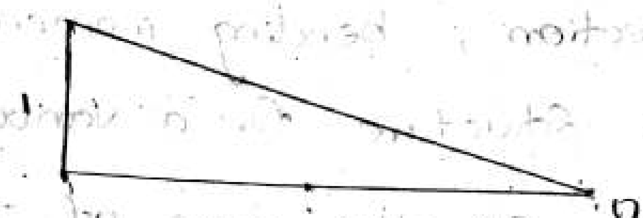
ILD for R_A

$\Sigma M_B = 0 \implies +(R_A \times L) - 1(L-x) = 0$

$$\boxed{R_A = \frac{L-x}{L}}$$

At $x = 0$, $R_A = 1KN$

At $x = L$, $R_A = 0$



ILD for R_A

ILD for R_B

$\Sigma M_A = 0 \Rightarrow (-R_B \times L) + 1x = 0$

$$\boxed{R_B = \frac{x}{L}}$$

At $x = L$, $R_B = 1 KN$

At $x = 0$, $R_A = 0 KN$

ILD for R_B

ILD for Shearforce

Load is in portion AC

$SF @ C = -R_B = \frac{-x}{L}$

@ $x = 0$, $SF @ A = 0$

@ $x = a$, SF @ C = $\frac{-a}{L}$

@ $x = L$, SF @ B = nothing

Load to be in portion (BC)

SF @ C = $+R_A$

$= \frac{L-x}{L}$

SF @ A $_{x=0}$ = 1

SF @ C $_{x=a}$ = $\frac{L-a}{L} = 1 - \frac{a}{L}$

($x = L$) SF @ B = 0

Note :- If the load is between points A and C . the shear force at 'C' would be $-R_B$ as it would be in the case of a fixed load [RHS].

* when the load is in between C and B, shear force at C would be $+R_A$ as it would be in the fixed beam [LHS]

ILD for BM

Load is in portion AC

$M_C = R_B (L-a)$

$= \frac{x}{L} (L-a)$

@ $x=0 \Rightarrow M@A = 0$

@ $x=a \Rightarrow M@C = \frac{a(L-a)}{L} = \frac{ab}{L}$

@ $x=L \Rightarrow M@B = L-a$

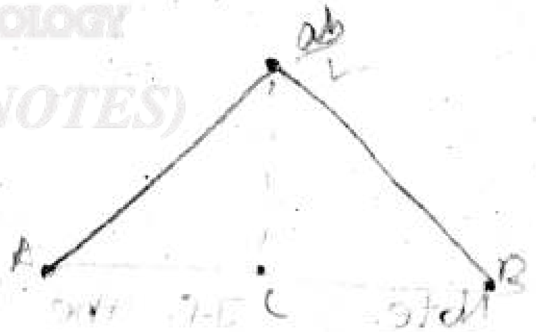
Load is in portion BC

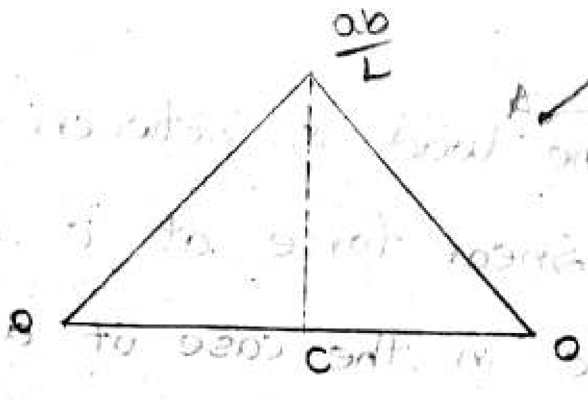
$M_C = R_A \cdot a = \left(\frac{L-x}{L}\right)a$

@ $x=0$, $M_A = a$

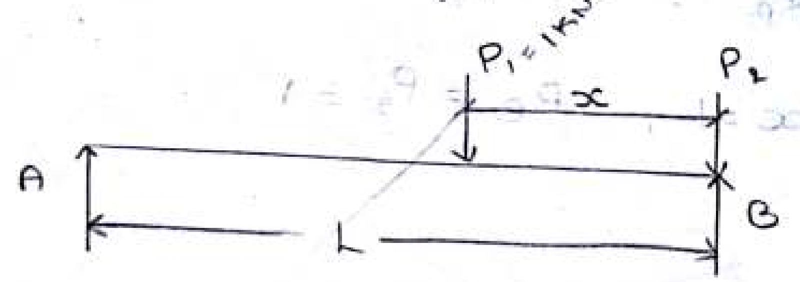
@ $x=a$, $M_C = \frac{(L-a)a}{L} = \frac{ab}{L}$

@ $x=L$, $M_B = 0$





Influence line diagram for a Simply Supported beam Subjected to two point loads



ILD for reactions

ILD for R_A

$\Sigma M_B = 0 \Rightarrow (R_A \times L) - (P_1 x) = 0$

$\Rightarrow \boxed{R_A = \frac{P_1 x}{L}}$

@ $x = 0$, $R_A = 0$

@ $x = L$, $R_A = P_1 = 1$

ILD for R_A

ILD for R_B

$\Sigma M_A = 0$

$(-R_B \times L) + P_1(L - x) + P_2(L) = 0$

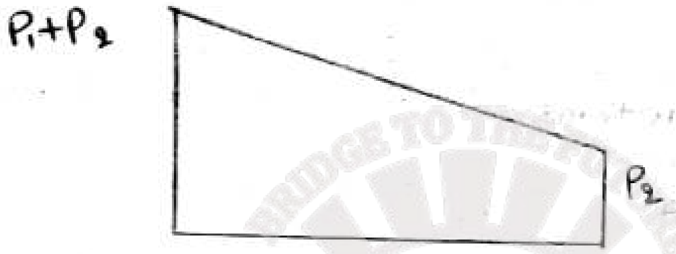
$R_B \times L = P_1(L-x) + (P_2 \times L)$

$$\boxed{R_B = \frac{P_1(L-x)}{L} + P_2}$$

@ $x = 0$

$$R_B = P_1 + P_2$$

@ $x = L$, $R_B = P_2 = 1$



ILD for R_B

ILD for shear force

Load is in portion AC

SF @ C $= -R_B$

$$= -\left(\frac{P_1(L-x)}{L} + P_2\right)$$

SF @ A $= 0$, @ $x = 0$

@ $x = 0$, SF @ A $= -(P_1 + P_2)$

@ $x = a$, SF @ C $= -\left(\frac{P_1(L-a)}{L} + P_2\right)$

SF @ C $= -\left(\frac{P_1 b}{L} + P_2\right)$

SF @ $x = L$ S.F @ B $= -P_2$

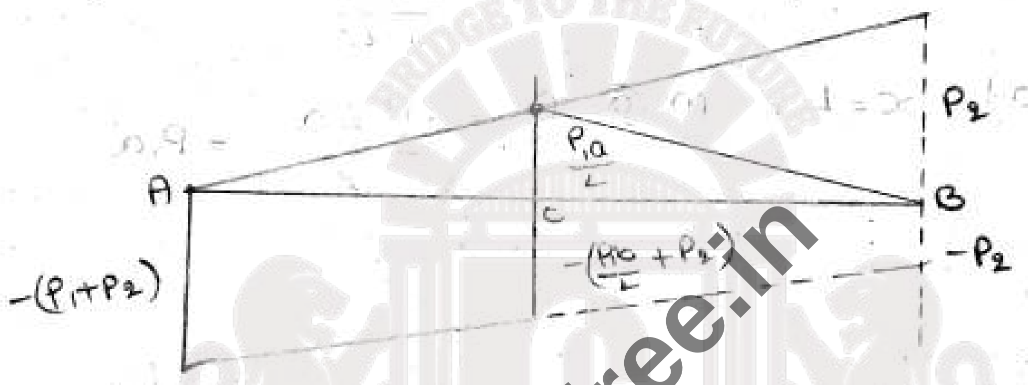
Load is in portion BC

$$SF @ \cancel{C} = +R_A = \frac{P_1 x}{L}$$

at $x=0$ $SF @ A = 0$

at $x=a$ $SF @ C = \frac{P_1 a}{L}$

at $x=L$, $SF @ B = \frac{P_1 \cancel{L}}{\cancel{L}} = P_1$



ILD for S.F

ILD for BM

If load is in portion AC

$$m_C = R_B \times (L-a)$$

$$m_C = \left(\frac{P_1(L-x)}{L} + P_2\right)(L-a)$$

at $x=0$, $m @ A = \left(\frac{P_1 \cancel{L}}{\cancel{L}} + P_2\right)(L-a)$

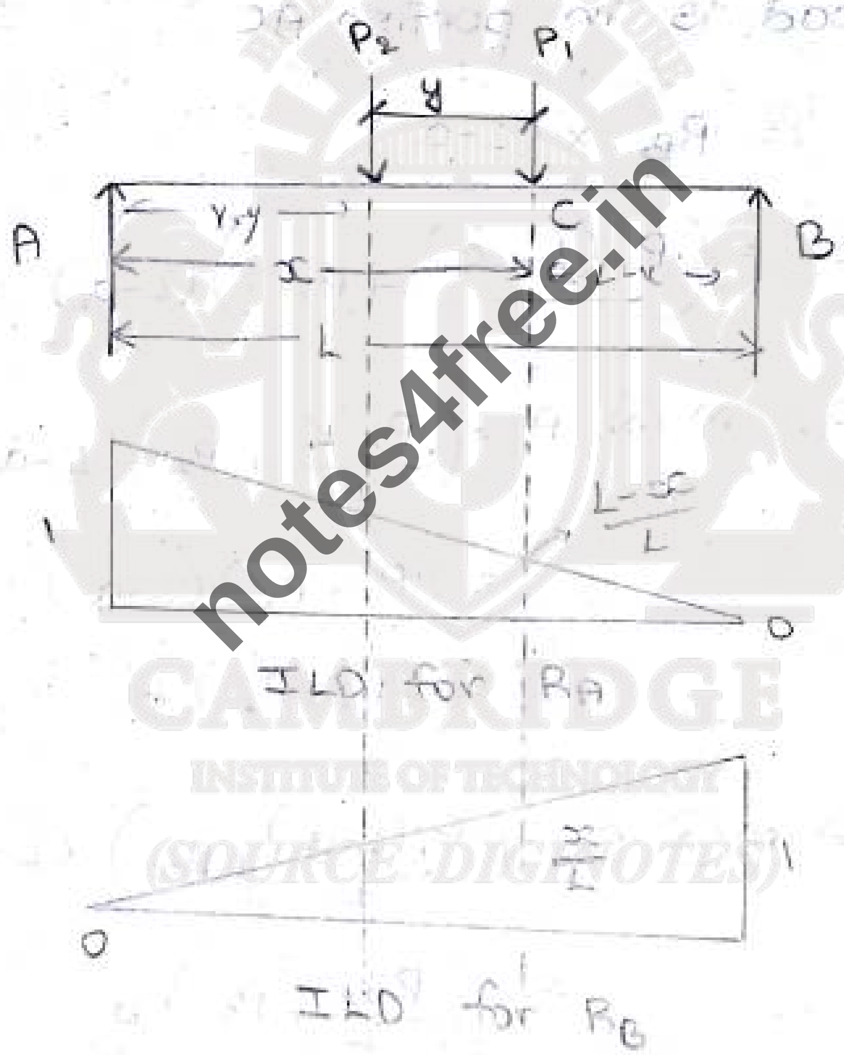
$$= (P_1+P_2)(L-a)$$

$$= (P_1+P_2)\,b$$

at $x=a \Rightarrow m @ C = \left(\frac{P_1(L-a)}{L} + P_2\right)(L-a)$

$$= \left(\frac{P_1 b}{L} + P_2\right) b$$

Influence line diagrams for a Simply Supported beam Subjected to two point loads



To find the reactions R_A and R_B for the influence line diagram of the reactions used found out in the problem involving a single point load is referred.

To find reaction R_A

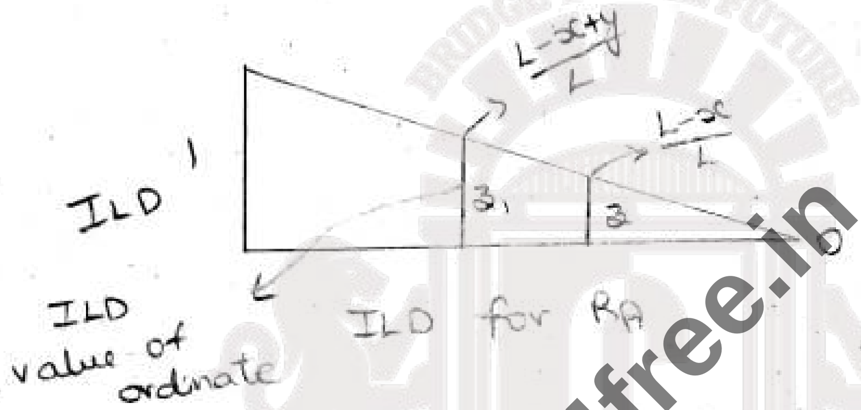
ILD for R_A

$$\frac{1}{L_1} = \frac{z}{L-x}$$

$$z = \frac{L-x}{L}$$

$$\frac{1}{L} = \frac{z}{L-x+y}$$

$$z_1 = \frac{L-x+y}{L}$$



ILD for R_A

Reaction at A

R_A = Load × ILD value of ordinate

$$R_A = P_2(z_1) + P_1(z)$$

$$R_A = P_2\left(\frac{L-x+y}{L}\right) + P_1\left(\frac{L-x}{L}\right)$$

To find reaction R_B

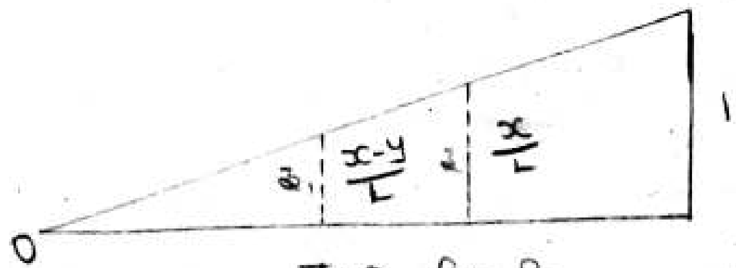
ILD for R_B

$$\frac{1}{L} = \frac{z}{x}$$

$$z = \frac{x}{L}$$

$$\frac{1}{L} = \frac{z_1}{x-y}$$

$$z_1 = \frac{x-y}{L}$$



ILD for R_B

Reaction at B

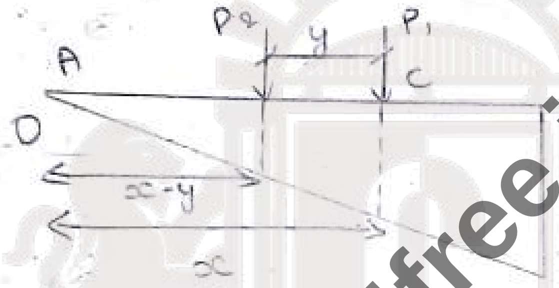
8

R_B = Load x ILD value of ordinate

$$R_B = P_2\left(\frac{x-y}{L}\right) + P_1\left(\frac{x}{L}\right)$$

ILD for Shear force

For portion AC = $-R_B$



ILD for SF (max (-ve))

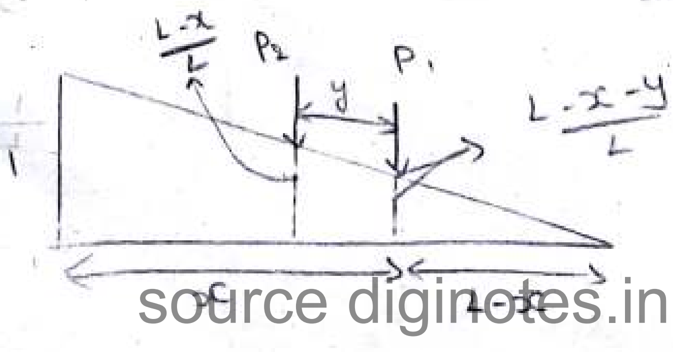
maximum shear force @ C (-ve)

$$= -\left[P_1\left(\frac{x}{L}\right) + P_2\left(\frac{x-y}{L}\right)\right]$$

$$= -P_1\left(\frac{x}{L}\right) - \frac{P_2(x-y)}{L}$$

Absolute maximum -B @ (x=L) $= -P_1 - \frac{P_2(L-y)}{L}$

For portion CB = R_A



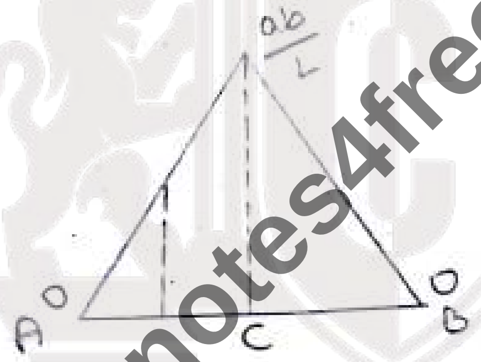
Maximum shear force (+ve) 9

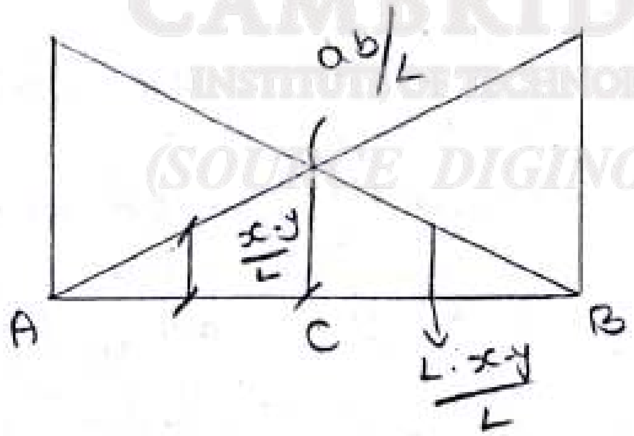
$$= P_1\left(\frac{L-x-y}{L}\right) + P_2\left(\frac{L-x}{L}\right)$$

Absolute maximum at 'A'

at $(x=0)$ =

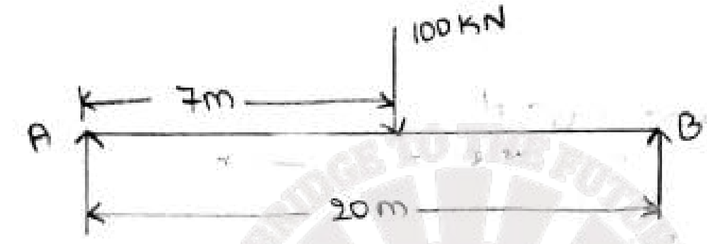
ILD for BM





ILD problems

① Determine the reactions R_A and R_B, shear force at 'c', maximum bending moment at 'c' by using influence line diagram for the beam loaded as shown in the figure.



ILD for Reaction R_A

Lets consider a unit load at a point c

$\Sigma M_B = 0$

$R_A \times 20 - 1(20 - x) = 0$

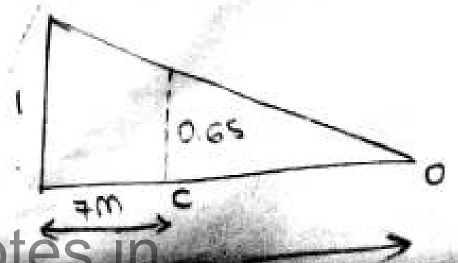
$\Rightarrow \boxed{R_A = \frac{20 - x}{20}}$

@ $x = 0$, $R_A = 1$

@ $x = 7$, $R_A = 0.65$

@ $x = 20$, $R_A = 0$

$\frac{1}{20} = \frac{x}{13}$



R_A = Load × ILD value of ordinate

$= 100 \times 0.65 = 65$ KN

$\therefore \boxed{R_A = 65 \text{ KN}}$

<u>ILD for</u> <u>Reaction R_B</u>

$\Sigma M_A = 0$

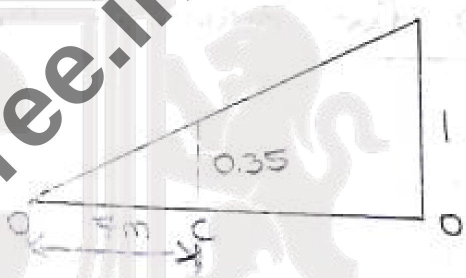
$\Rightarrow -R_B \times 20 + 1 \times x = 0$

$\Rightarrow \boxed{R_B = \frac{x}{20}}$

@ $x = 0$, $R_B = 0$

@ $x = 7$, $R_B = 0.35$

@ $x = 20$, $R_B = 1$



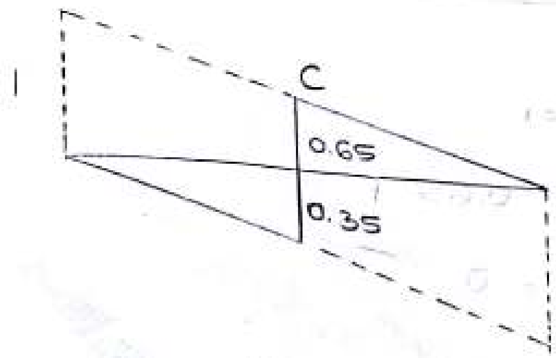
ILD for R_B

R_B = Load × ILD value of ordinate

$= 100 \times 0.35 = 35$ KN

$\therefore \boxed{R_B = 35 \text{ KN}}$

<u>ILD for</u> <u>Shear force</u> <u>@ C</u>



ILD for SF

$SF @ C$ = Load × ILD value of ordinate

$= (100 \times 0.65) + [100 \times (-0.35)]$

$\boxed{SF @ C = 30\ KN}$

maximum +ve SF

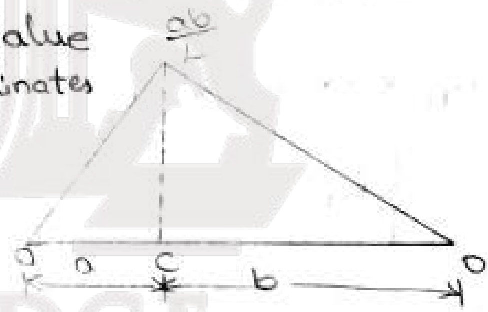
$SF @ C$ (+ve) $= 100 \times 0.65 = 65\ KN$

maximum -ve SF

$SF @ C$ (-ve) $= 100 \times -0.35 = -35\ KN$

<u>ILD for BM</u>

$BM @ C = M_C = \frac{ab}{L}$ = ILD value of ordinates



$= \frac{7 \times 13}{20} = 4.55\ KN.m$

$BM @ C = Load \times \frac{ab}{L}$

$= 100 \times 4.55$

$= 455\ KN\text{-}m$

Absolute maximum Bending moment

when the load is at the centre, BM → maximum

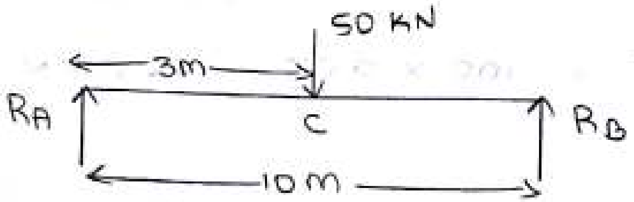
$\therefore a = 10,\ b = 10,\ L = 20$

i.e $M_C = \frac{10 \times 10}{20} = 5\ KN\text{-}m$

$\therefore$ Absolute maximum BM $= 100 \times 5$

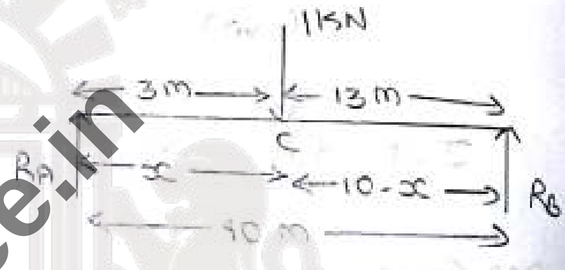
$= 500\ KN\text{-}m$

② Determine the reactions R_A and R_B, shear force at C, maximum bending moment at 'C' by using influence line diagram for the beam loaded as shown in the figure.



ILD for Reaction R_A

Lets consider a unit load at a point C



$\Sigma M_B = 0$

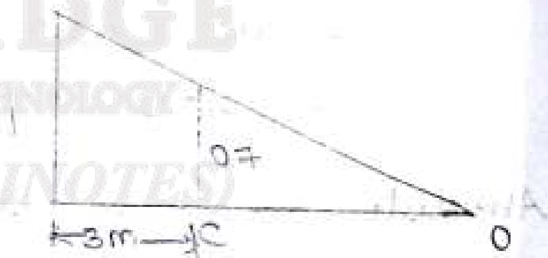
$R_A \times 10 - 1(10-x) = 0$

$\boxed{R_A = \frac{10-x}{10} \text{ KN}}$

@ $x=0$, $R_A = 1$

@ $x=3$, $R_A = 0.7$

@ $x=10$, $R_A = 0$



R_A = Load × ILD value of ordinate

$= 50 \times 0.7$

$\boxed{R_A = 35 \text{ KN}}$

ILD for Reaction R_B

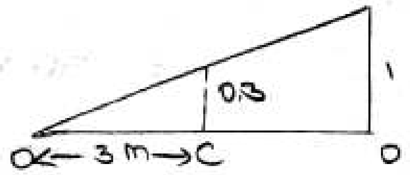
$\Sigma M_A = 0$, $\Rightarrow -R_B \times 10 + 1 \times x = 0$

$\boxed{R_B = \frac{x}{10}}$

@ $x = 0$, $R_B = 0$

@ $x = 3$, $R_B = 0.3$

@ $x = 10$, $R_B = 1$

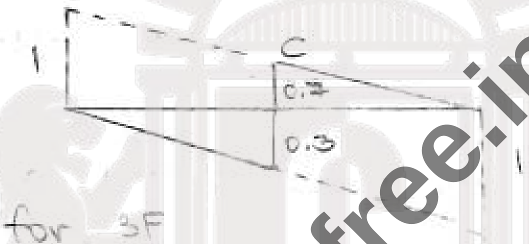


R_B = Load × ILD value of ordinate

$= 100 \times 0.3 = 30$ KN

$\boxed{R_B = 30 \text{ KN}}$

ILD for Shearforce @ C



ILD for SF

SF @ C = Load × ILD value of ordinate

$= (\overset{30}{100} \times 0.7) + (\overset{50}{100} \times -(0.3))$

SF @ C = 20 KN

maximum +ve SF

SF @ C (+ve) $= 50 \times 0.7 = 35$ KN

maximum -ve SF

SF @ C (-ve) $= 50 \times (-0.3) = -15$ KN

ILD for BM

BM @ C $= M_C = \frac{ab}{L}$ = ILD value of coordinate

$= \frac{3 \times 7}{10} = 2.1$ kN-m

$BM @ C = Load \times \frac{ab}{L}$

$= 50 \times 2.1$

$= 105 \ KN-m$

Absolute maximum bending moment

when the load is at centre, BM → maximum

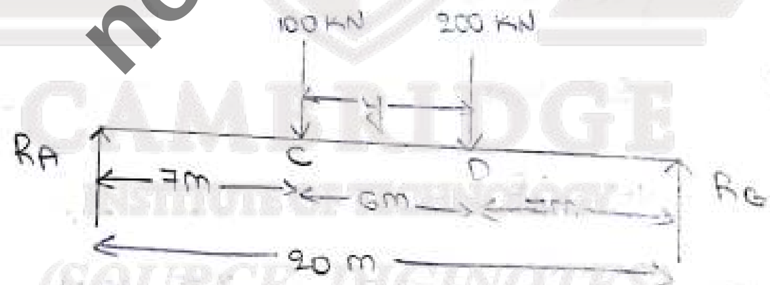
$\therefore \ a = 5, \quad b = 5, \quad L = 10$

i.e $M_c = \frac{5 \times 5}{10} = 2.5 \ KN-m$

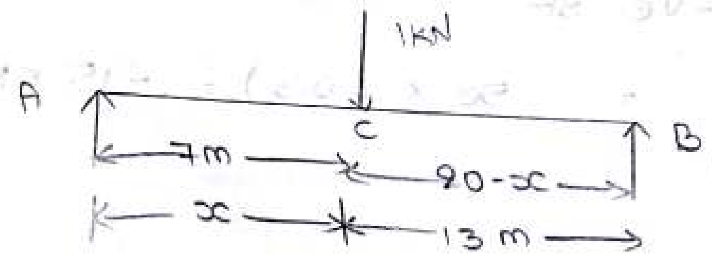
$\therefore$ Absolute maximum $BM = 50 \times 2.5$

$= 125 \ KN-m$

③ Determine the reactions R_A and R_B, SF and BM at the point C when it is subjected to loads as shown in the figure.



ILD for Reaction R_A



$(R_A \times 20) - 1(20-x)$

$R_1 = \frac{20-x}{20}$

$(-R_B \times 20)(1 \times x)$

Consider a unit load at a point C

$\Sigma M_B = 0$

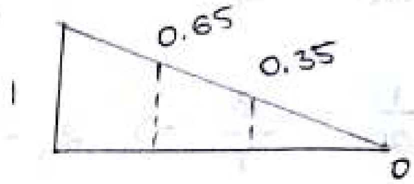
$\boxed{R_A = \frac{20-x}{20}}$

@ $x = 0$, $R_A = 1$

@ $x = 7m$, $R_A = 0.65$

@ $x = 13m$, $R_A = \frac{20-13}{20} = 0.35$

@ $x = 20$ – $R_A = 0$



ILD for R_A

R_A = Load x ILD value of ordinates

$= (100 \times 0.65) + (200 \times 0.35)$

$\boxed{R_A = 135 \text{ KN}}$

ILD for reaction R_B

$\Sigma m_A = 0$,

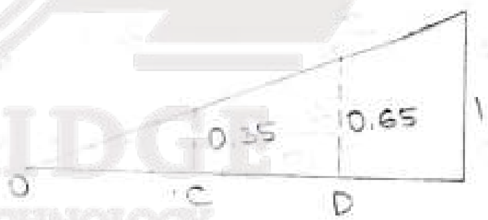
$\Rightarrow \boxed{R_B = \frac{x}{20}}$

@ $x = 0$, $R_B = 0$

@ $x = 7m$, $R_B = 0.35$

@ $x = 13m$, $R_B = 0.65$

@ $x = 20m$, $R_B = 1$



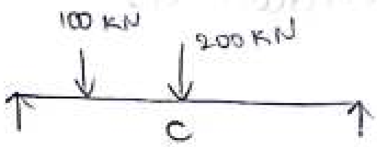
ILD for R_B

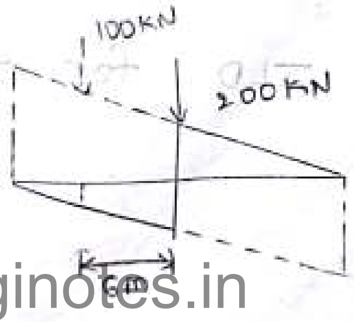
R_B = Load x ILD value of ordinates

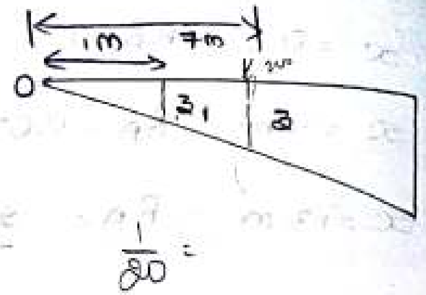
$= (100 \times 0.35) + (200 \times 0.65)$

$\boxed{R_B = 165 \text{ KN}}$

maximum -ve SF







$\frac{1}{20} = \frac{z}{7} \Rightarrow z = \frac{7}{20} = 0.35$

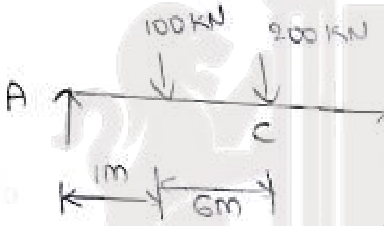
$\frac{1}{20} = \frac{z_1}{1} \Rightarrow z_1 = \frac{1}{20} = 0.05$

Maximum -ve SF

$SF @ C_{(-ve)} = (100 \times (-0.05)) + (200 \times (-0.35))$

$= -75$ KN

Maximum +ve SF



$\frac{1}{20} = \frac{z}{13} \Rightarrow z = \frac{13}{20} = 0.65$

$\frac{1}{20} = \frac{z_1}{19} \Rightarrow z_1 = 0.095$

$\therefore$ maximum +ve SF =

Load $\times$ ILD value of ordinates

$(100 \times 0.095) + (200 \times 0.65)$

$SF @ C_{(+ve)} = 225$ KN-m

But the ILD for SF is drawn as

ILD for SF

max +ve SF

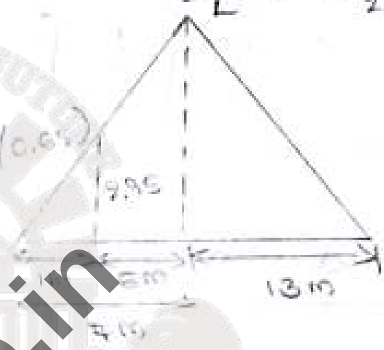
occurs when 100 KN is at C

$\therefore$ max +ve SF $= (100 \times 0.65) + (200 \times 0.35)$

$= 135$ KN

Maximum BM

when 200 KN is acting at point 'C', BM. → max



$\therefore$ maximum BM =

Load × ILD value of ordinate

$= (100 \times 2.95) + (200 \times 4.55)$

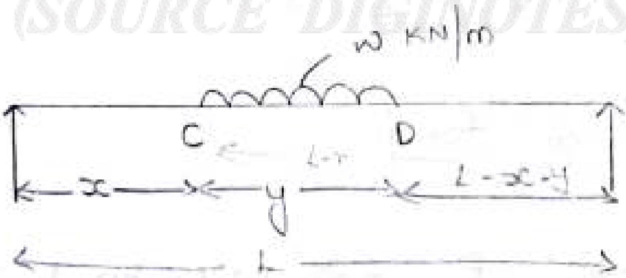
$= 1205$ KN-m

$\frac{4.55}{7} = \frac{z}{1}$

$\Rightarrow z = 0.65$

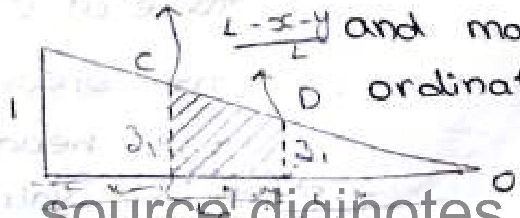
10

ILD for simply supported beam having a UDL shorter than the span



To find

Reaction R_A :- Draw the ILD for reaction R_A and mark the values of ordinates corresponding to sections C and D



$\frac{1}{L} = \frac{z}{L-x}$

$z = \frac{L-x}{L}$

∴ Reaction at R_A

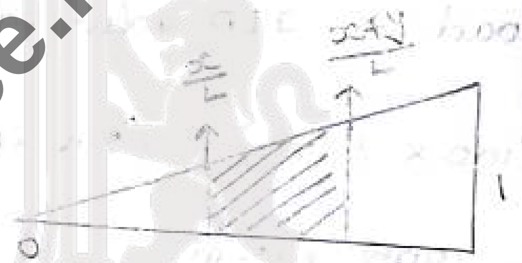
R_A = Load × Area of ILD between C and D

$$= w \times \left[\frac{1}{2} \cdot y \left(\frac{L-x}{L} + \frac{L-x-y}{L}\right)\right]$$

$$= w\left[\frac{y}{2}\left[\frac{L-x+L-x-y}{L}\right]\right]$$

$$\boxed{R_A = w\left[\frac{y}{2}\left[\frac{2L-2x-y}{L}\right]\right]}$$

To find the reaction at R_B



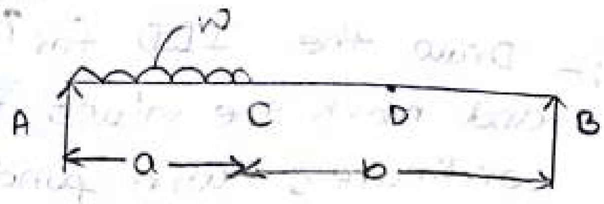
R_B = Load × Area of ILD between c and D

$$= w\left[\frac{1}{2} \cdot y\left[\frac{x}{L} + \frac{x+y}{L}\right]\right]$$

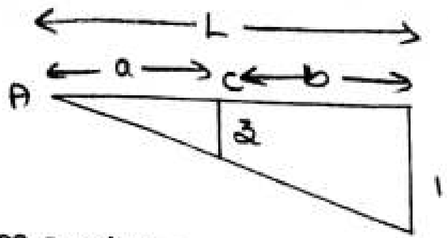
$$= w\left[\frac{y}{2}\left[\frac{2x+y}{L}\right]\right]$$

To find shear force

Maximum -ve SF → UDL head is at c



To find maximum shear force at c. Draw the ILD for shear force when the UDL head is at 'c' and mark the values of the ordinate corresponding to the section.



$$\frac{1}{L} = \frac{z}{a}$$

$$z = \frac{a}{L}$$

$\therefore$ maximum $-$ve SF,

SF @ C = Load $\times$ Area between A and C

$$= w\left(\frac{1}{2} \times a \times \frac{a}{L}\right) -$$

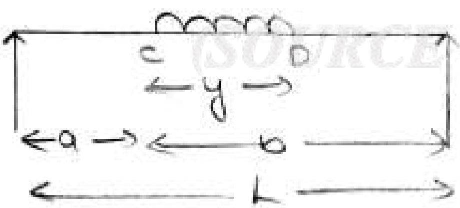
$$= \frac{wa^2}{2L}$$

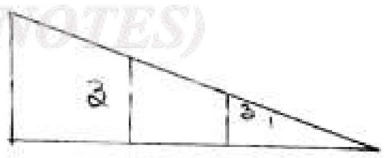
Then to find maximum +ve shear force

~~The find~~ The find maximum positive shear force will occur when the head of the UDL moves ahead of 'c' and tail is at 'c'

Draw the ILD for SF at c and mark the values of ordinates corresponding to the section

$\therefore$ maximum +ve shear force $\rightarrow$ UDL, tail is at 'c'





$$\frac{1}{L} = \frac{z}{b}$$

$$z = \frac{b}{L}$$

$$\frac{1}{L} = \frac{z_1}{b-y}$$

$$z_1 = \frac{b-y}{L}$$

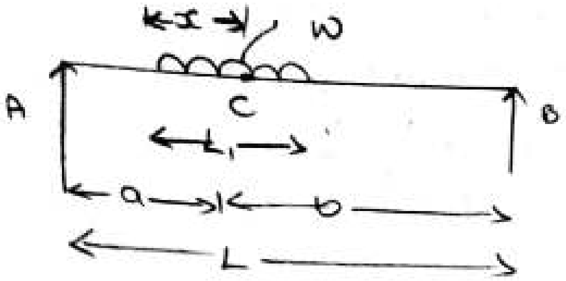
Maximum +ve SF

= w $\times$ Area between C and D

$$= w\left[\frac{1}{2} \times y\left(\frac{b}{L} + \frac{b-y}{L}\right)\right]$$

To find BM

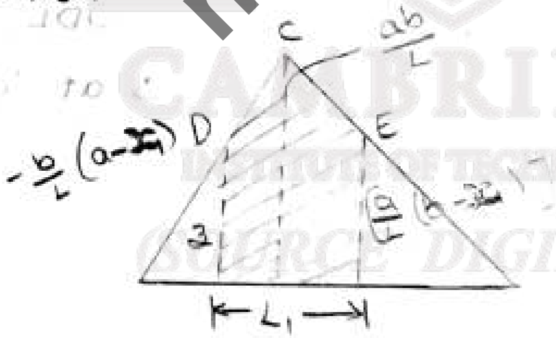
To find maximum BM



To get the max BM at 'C' the point 'C' should divide the UDL in the same proportion as it divides the span.

i.e $\frac{x}{L_1} = \frac{a}{L}$

Draw the ILD for the BM and mark the values of the ordinates corresponding to the section

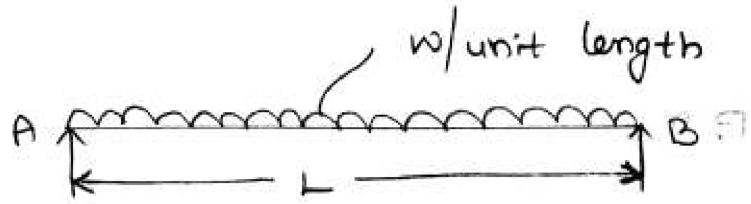


ILD for BM

∴ Maximum BM at 'C' =

Intensity of load x Area of the ILD between D and E

Influence line diagram for a simply supported beam subjected to a UDL longer than the span

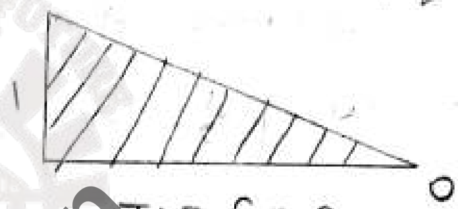


Reaction R_A

R_A = Load × Area b/w A and B

$= w \times \frac{1}{2} \times L \times 1$

$$\boxed{R_A = \frac{wL}{2}}$$



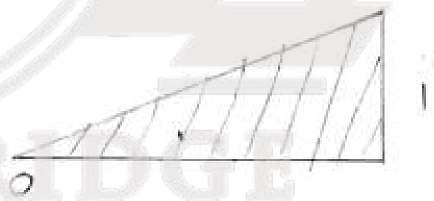
ILD for R_A

Reaction R_B

R_B = Load × Area b/w B and A

$= w \times \frac{1}{2} \times L \times 1$

$$\boxed{R_B = \frac{wL}{2}}$$



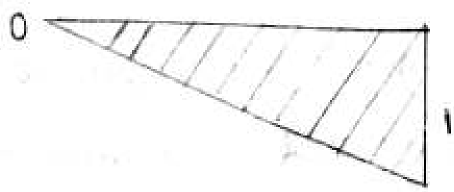
ILD for R_B

Maximum -ve SF

Maximum -ve SF $= -R_B$

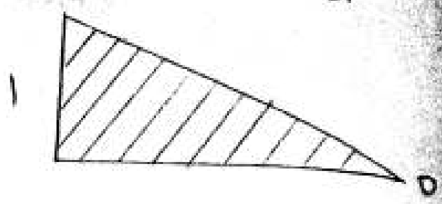
$= \frac{-wL}{2}$

max -ve SF



Maximum +ve SF

Max +ve SF $= \frac{wL}{2}$

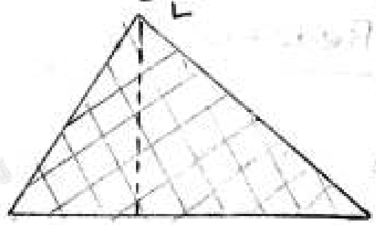


To find max B.M

M_c = Load × Area of ILD between A and B

$= w\left[\frac{1}{2} \times L \times \frac{ab}{L}\right]$

$= \frac{wab}{2}$



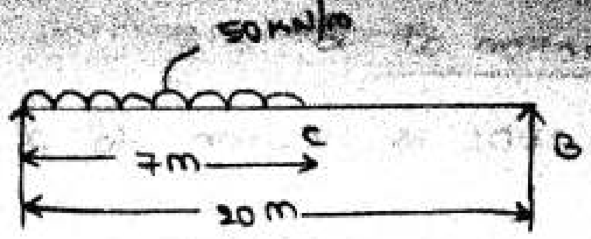
Absolute Max BM

$M = \frac{w\left(\frac{L}{2}\right)\left(\frac{L}{2}\right)}{2}$

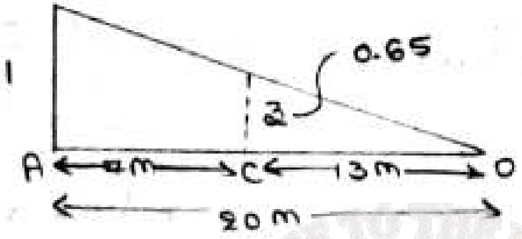
$\boxed{M = \frac{wL^2}{8}}$

Problems

① A UDL of intensity 50 kN/m and of length 7m is rolling over a simply supported gurder of span 20 m. Determine (i) Reaction at A and reaction at B when the head of the UDL is at 7m at from left end (ii) maximum Reaction R_A and Maximum Reaction R_B (iii) maximum positive and negative force at section 'c' maximum Bending moment and Absolute maximum BM of the sect



(i) Reaction at A



$\frac{1}{20} = \frac{2}{13}$

$\Rightarrow 2 = 0.65$

R_A = Load x Area of ILD between A and C

$= 50 \times \left[\frac{1}{2} \, 7 \, (1 + 0.65) \right]$

$\boxed{R_A = 288.75 \text{ kN}}$

Reaction at B :-

$\frac{1}{20} = \frac{2}{7}$

$\Rightarrow 2 = 0.35$

ILD for R_B

R_B = Load x Area of ILD between A and C

$= 50 \times \left[\frac{1}{2} \times 7 \times 0.35 \right]$

$\boxed{R_B = 61.25 \text{ kN}}$

(ii) Maximum Reaction at A

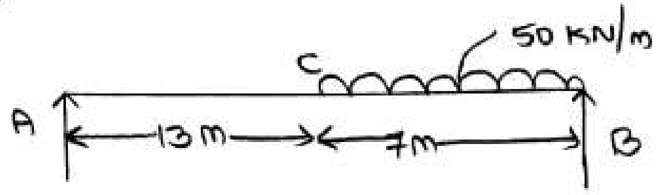
Occure when the UDL is in the portion AC or nearer to A

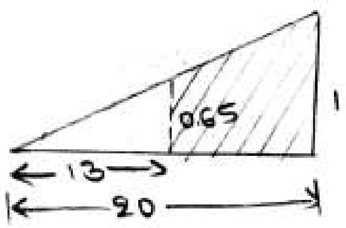
$\boxed{R_{A\,max} = 288.75 \text{ kN}}$

Maximum Reaction at B

occurs when UDL is nearer to B

i.e,



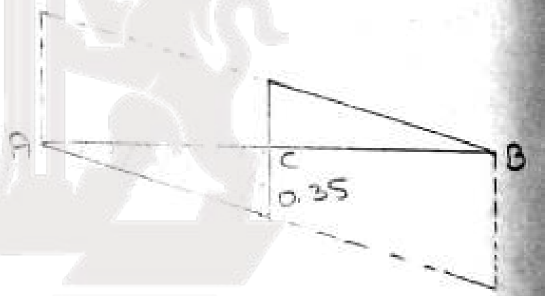


R_B = Load × Area of ILD between C and B

$= 50 \times \frac{1}{2} \times 7 \left[1 + 0.65\right]$

$$\boxed{R_{B\,max} = 288.75 \text{ kN}}$$

(iii) maximum positive Shear force



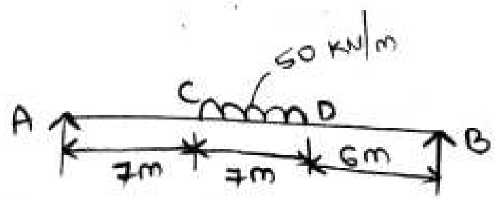
max (-ve) SF =

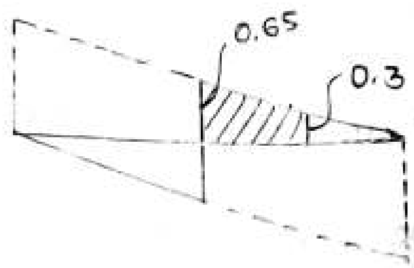
Load × Area of ILD between A and C

$\therefore \quad = 50 \times \left\{ -\left(\frac{1}{2} \times 7 \times 0.35\right)\right\}$

max (-ve) SF = −61.25 kN

maximum positive Shear force



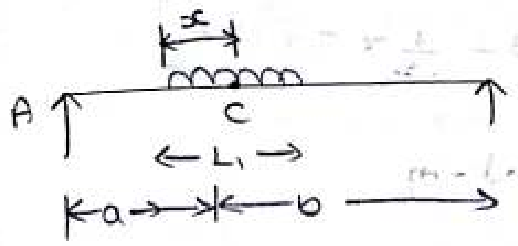


max (+ve) SF = Load × Area of ILD between C and D

$= 50 \times \frac{1}{2} \times 7 \left(0.65 + 0.3\right)$

$= 166.25$ kN

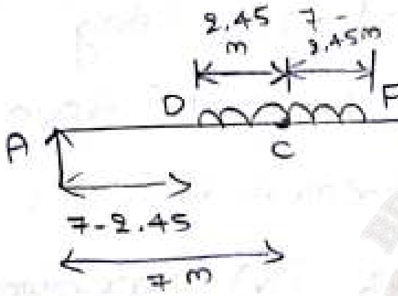
(iv) maximum bending moment

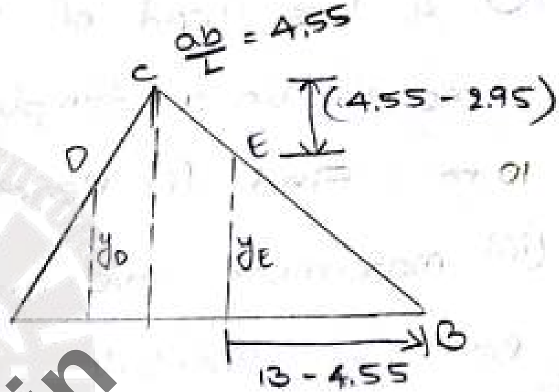


$$\frac{x}{L_1} = \frac{a}{L}$$

$$\frac{x}{7} = \frac{7}{20}$$

$$x = \frac{49}{20} = 2.45$$





$$\frac{y_D}{4.55} = \frac{\frac{7\times13}{20}}{7} \qquad \frac{y_E}{(13-4.55)} = \frac{\frac{ab}{L}}{13}$$

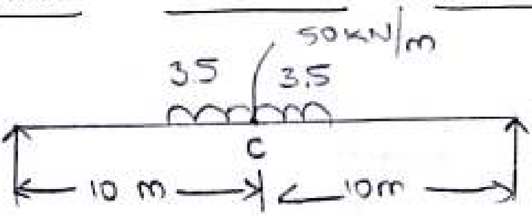
$$y_D = 2.95 \qquad y_E = 2.95$$

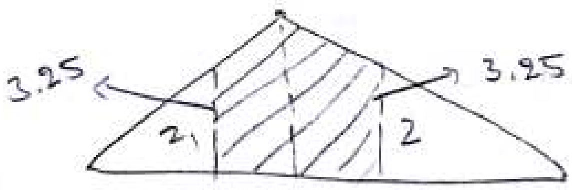
max BM = Load x Area of ILD b/w D and E

$$= 50 \times \left[\left(\tfrac{1}{2}\times 7\times 1.6\right) + (7\times 2.95)\right]$$

max BM = 1312.5 KN-m

(v) Absolute maximum bending moment





$$\frac{5}{10} = \frac{2}{6.5}$$

$$Z = 3.25$$

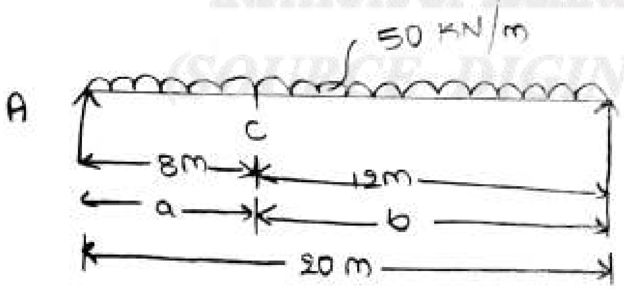
Absolute max BM

$$= 50\left[7 \times 3.25 + \tfrac{1}{2} \times 7 \times 1.75\right]$$

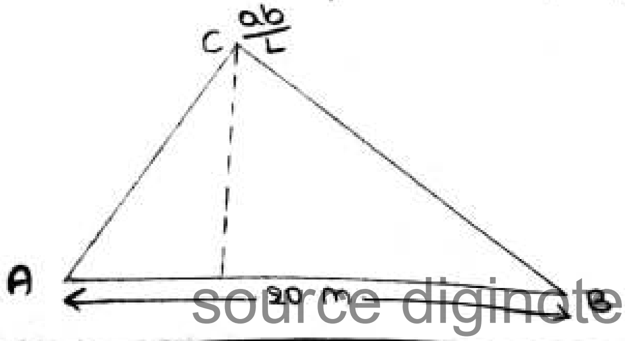
$$= 1443.75 \text{ KN-m}$$

② A live load of 5 KN/m and 8m long moves on a simply supported beam of span 10 m. Find (i) Reactions (ii) maximum reactions (iii) maximum +ve and -ve forces (iv) maximum BM (v) Absolute maximum BM at 4m from the left hand.

③ A UDL of 50 KN/m rolls over a simply supported beam of 20 m span. Determine maximum BM which can occur at a section 8m from the left end. Also find the maximum bending moment, shear force and reactions.



(i) max BM @ a point 8m from 'A'



max BM @ C.

M_C = Load × Area of ILD between A and B

$= 50 \times \left[\frac{1}{2} \times 20 \times \frac{8\times12}{20}\right]$

$\boxed{M_C = 2400 \text{ KN-m}}$

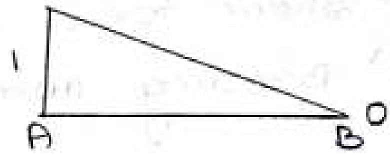
(ii) Maximum Shear force [Absolute max]

$SF_{(+ve)}$ = Load × Area of ILD b/w A and B

Absolute max SF_{+ve} =

$50 \times \left[\frac{1}{2} \times 20 \times 1\right]$

= 500 KN



ILD for +ve SF

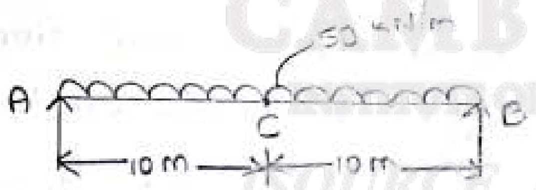
Absolute max $SF_{(-ve)}$ = Load × Area of ILD b/w B and A

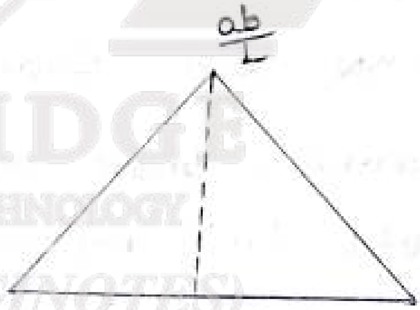
$= 50 \times \left[\frac{1}{2} \times 20 \times 1\right]$

= −500 KN

ILD for -ve SF

(iii) Absolute max BM





Absolute max BM = Load × Area of ILD b/w A & B

$= 50 \times \left[\frac{1}{2} \times 20 \times \frac{100}{20}\right]$

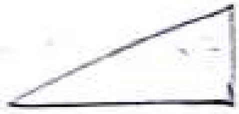
Absolute max BM = 2500 KN-m

(iv) Reactions :-

R_A = Load × Area of ILD b/w A and B

$\boxed{R_A = 500 \text{ KN}}$

R_B = Load x Area of ILD b/w A and B



$\boxed{R_B = 500 \text{ KN}}$

④ A UDL of intensity 30 KN/m longer than the span prosses a guarder of span 20 m from left to right. calculate (i) maximum Reactiene (ii) Shear force at a section 8 m from A (iii) Bending moment at a section 8 m from A

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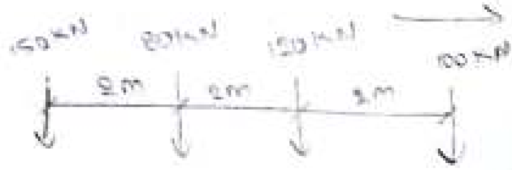
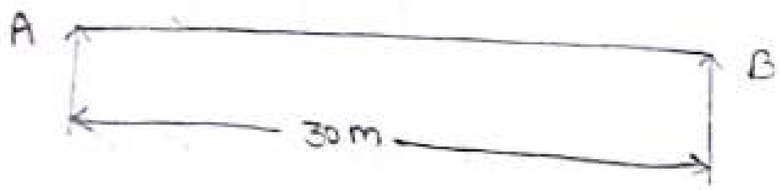
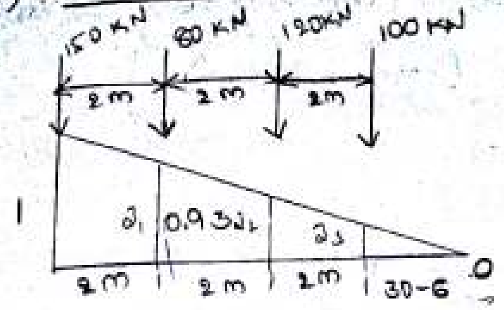
Problems on simply supported beams subjected to multiple concentrated loads

① The multiple point loads 100 KN, 120 KN, 80 KN and 150 KN with a spacing of 2m crosses a garder of span 30 m from left to right with a 100 KN load leading. calculate (i) Reactions (ii) maximum shear force at a section 10 m from the left (iii) maximum BM at a section 10 m from the left (iv) Absolute max SF (v) Absolute maximum bending moment.





(i) Reactions



ILD for RA

$\frac{1}{30} = \frac{z_1}{28}$ $\frac{1}{30} = \frac{z_2}{26}$

$z_1 = 0.933$ $z_2 = 0.866$

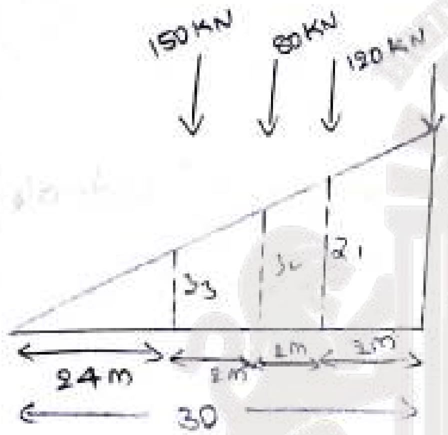
$\frac{1}{30} = \frac{z_3}{24}$

$z_3 = 0.8$

R_A = Load × ILD value of ordinates

$= (150 \times 1) + (80 \times 0.933) + (120 \times 0.866) + (100 \times 0.8)$

$= 407.6$



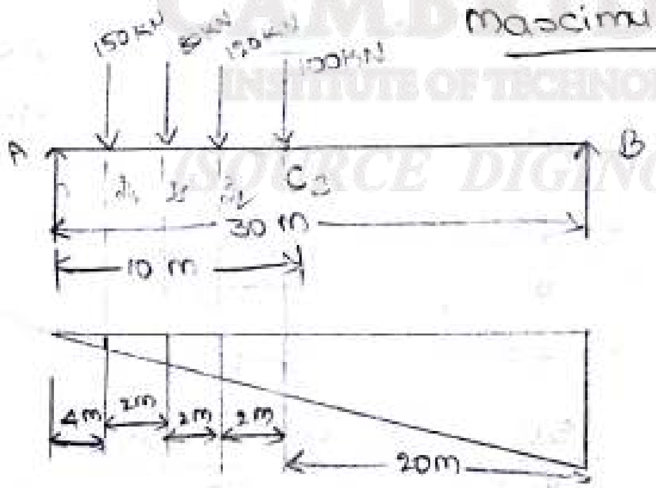
$\frac{1}{30} = \frac{120}{28}$

R_B = Load × ILD value of ordinates

$= (100 \times 1) + (120 \times 0.93) + (80 \times 0.86) + (150 \times 0.8)$

R_B = 400.4 KN

Maximum (-ve) Shear force



ILD for -ve SF

$\frac{1}{30} = \frac{z_1}{10}$

$z_1 = 0.33$

$\frac{1}{30} = \frac{z_2}{8}$

$z_2 = 0.26$

$\frac{1}{30} = \frac{z_3}{6}$

$z_3 = 0.2$

$\frac{1}{30} = \frac{z_4}{4}$

$z_4 = 0.13$

$$\text{max -ve SF} = -\left[(100\times0.33)+(120\times0.26)+(80\times0.2)+(150\times0.13)\right]$$

$$= -99.7 \text{ KN}$$

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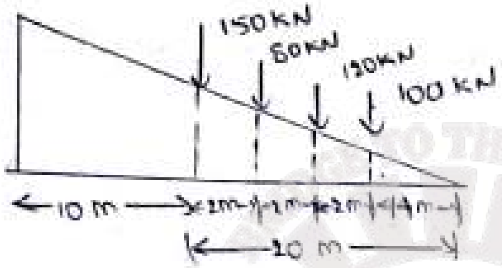
max positive shearforce

$\frac{1}{30} = \frac{z_1}{20}$ $\quad z_1 = 0.66$

$\frac{1}{30} = \frac{z_2}{18}$ $\quad z_2 = 0.6$

$\frac{1}{30} = \frac{z_3}{16}$ $\quad z_3 = 0.53$

$\frac{1}{30} = \frac{z_4}{14}$ $\quad z_4 = 0.46$



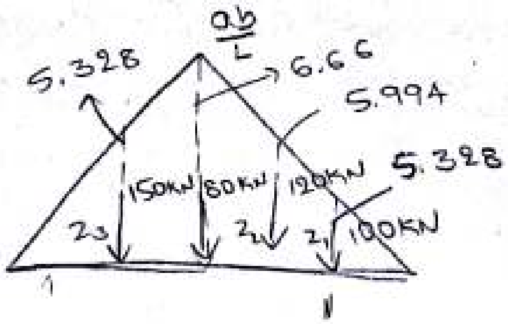
max +ve shearforce = Load × ILD value of ordinate

$$= (150\times0.66) + (80\times0.6) + (120\times0.53) + (100\times0.46)$$

$$= 256.6 \text{ KN}$$

SL No	Load crossing the Section	Average load in portion AC	Average load in portion CB	Remarks
1	100 KN	$\frac{150+80+120}{10} = 35$	$\frac{100}{20} = 5$	AC is greater than CB
2	120 KN	$\frac{80+150}{10} = 23$	$\frac{120+100}{20} = 11$	
3	80 KN	$\frac{150}{10} = 15$	$\frac{80+120+100}{20} = 15$	
4	150 KN			

max BM at section 10 m from left 5 m,



$\frac{1}{10} \times$

19

$\frac{1}{20} = \frac{3}{18}$

$\frac{80 \times 10}{30}$

$$\text{max BM} = (150 \times 5.325) + (80 \times 6.66) + (120 \times 5.995) + (100 \times 5.325)$$

$$= \underline{2584.08 \text{ KN-m}}$$

To find absolute max. BM

<u>Step 1</u>: To find the Resultant and its position

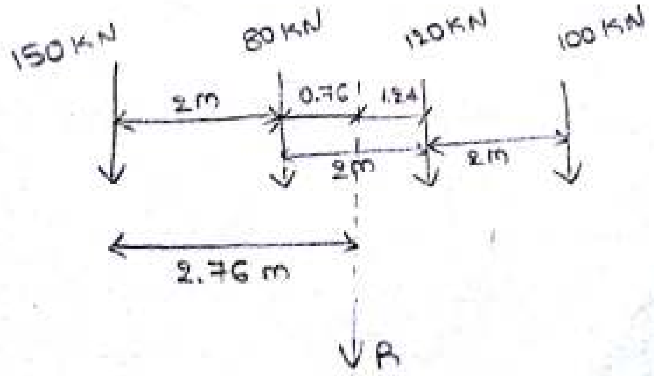
Resultant $R = 150 + 80 + 120 + 100$

$= 450$ KN

position of the Resultant w.r.t 1st load

$$\bar{x} = \frac{(150\times0)+(80\times2)+(120\times4)+(100\times6)}{150+80+120+100}$$

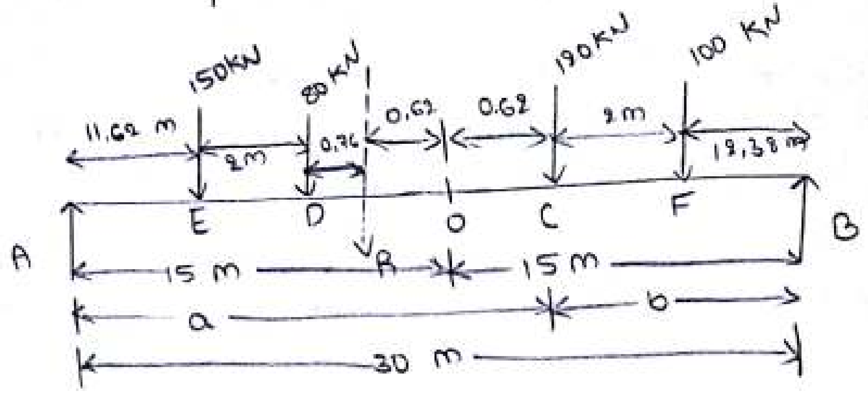
$$\bar{x} = 2.76 \text{ m}$$



<u>Step 2</u> :- Select the heaviest load nearer to R

Heavier Load = 120 KN

Step ③ :- The position of loads on the beam.

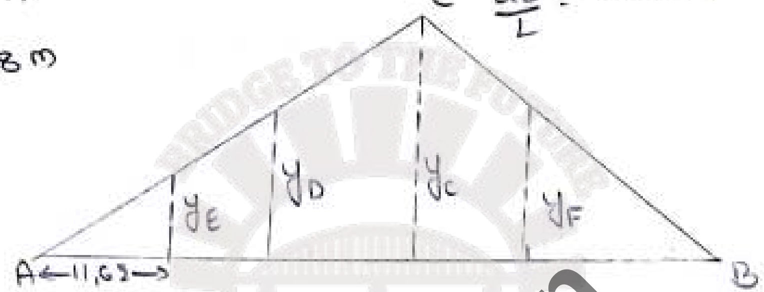


⑩

$a = 15.62$ m

$b = 14.38$ m

$\frac{ab}{L} = 7.48$ m



ILD for Absolute Max BM

Step ④ :- calculation of BM

M_C = Load × ILD value of ordinate

$$\frac{7.48}{15.62} = \frac{y_D}{13.62} \qquad \frac{7.48}{14.38} = \frac{y_F}{12.38}$$

$y_D = 6.52$ $\qquad y_F = 6.432$ m

$$\frac{7.48}{15.62} = \frac{y_E}{11.62}$$

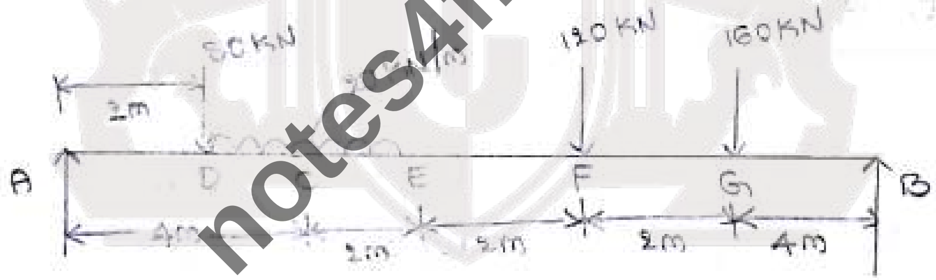
$y_E = 5.564$ m

$$M_C = (190 \times 7.48) + (80 \times 6.52) + (150 \times 5.56) + (100 \times 6.43)$$

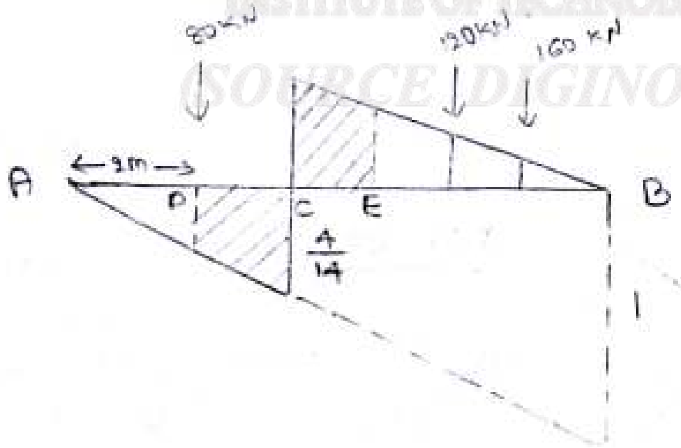
$M_C = 2896.2 \approx 2897.1$

(5) For a simply supported beam of span 25 m, compute by Influence line principle (i) maximum BM at 8m from left support (ii) Absolute maximum BM (iii) maximum Reaction. A series of concentrated loads to be taken as a rolling load system is as shown in the figure. (21)

(6) Using ILD determine shear force and BM at section 'c' in the simply supported beam as shown in the figure



ILD for SF

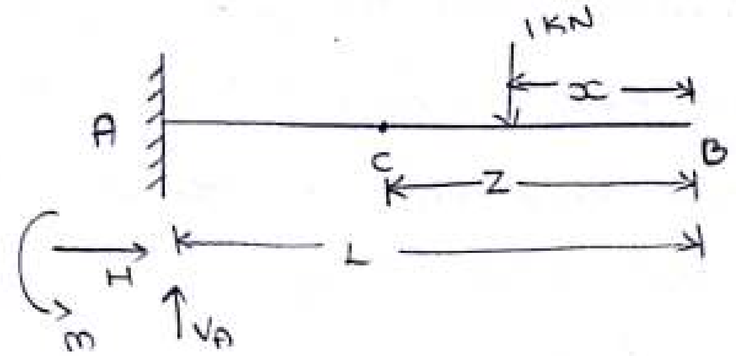


$$\frac{1}{L} = \frac{z}{4}$$

$$\Rightarrow z = \frac{4}{L}$$

ILD for cantilever beam

(5)



$\Sigma_V = 0$

$\Rightarrow \quad V_A - 1 = 0$

$\boxed{V_A = 1\,KN}$

$\Sigma M_B = 0 \quad \Rightarrow (-V_A \times L) + (1 \times x) - M_A = 0$

$\boxed{M_A = x - L}$

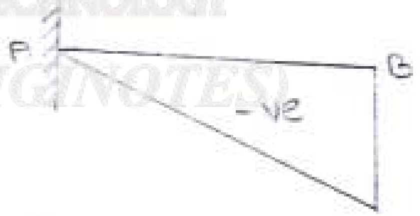
ILD for V_A

ILD for M_A

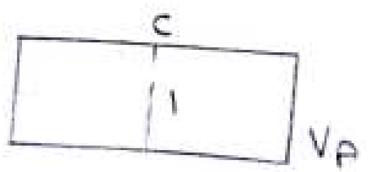
@ $x = 0$, $M_A = -L$

@ $x = L$, $M_A = 0$

@ $x = z$, $M_A = z - L$

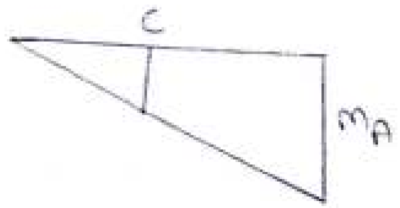


For SF



$V_A = SF$

For BM



ILD for over hanging beam

For an over hanging beam we have to find out the reaction at A and B, Shear force at the Section B. Bending moment at Section D. Shear force at section E and BM at Section E.

(6)

ILD for R_A

$\Sigma M_B = 0$

$(R_A \times L) - 1 \times (L - x) = 0$

$$\boxed{R_A = \frac{L-x}{L}}$$

@ $x = 0$, $R_A = 1$

@ $x = z$, $R_A = \frac{L-z}{L}$

@ $x = L$, $R_A = 0$

@ $x = L+a$,

$R_A = \frac{L-(L+a)}{L}$

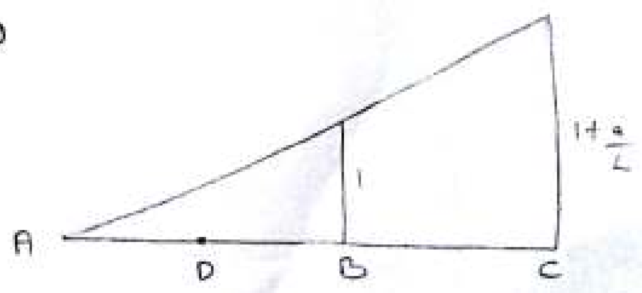
$= \frac{-a}{L}$

ILD for R_A

ILD for R_B

$-R_B \times L + 1 \times x = 0$

$$\boxed{R_B = \frac{x}{L}}$$



@ $x = 0$, $R_B = 0$

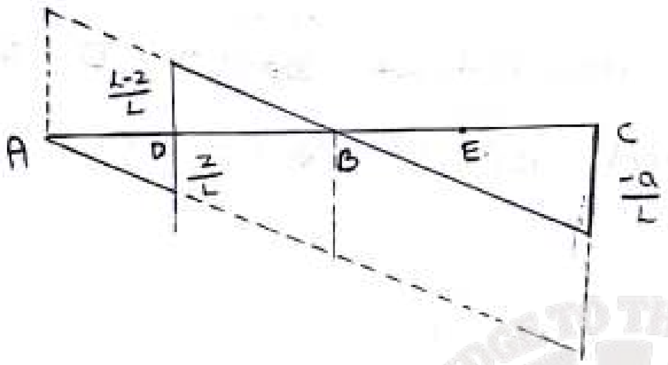
@ $x = z$, $R_B = \frac{z}{L}$

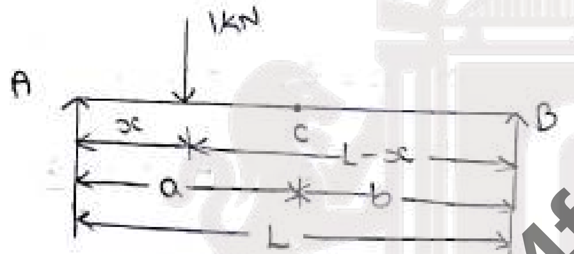
@ $x = L$, $R_B = 1$

@ $x = L+a$, $R_B = \frac{L+a}{L} = 1 + \frac{a}{L}$

ILD for Shear force

7





Load

In AC, $m_C = R_B(b)$

In CB, $m_C = R_A(a)$

Load in AD

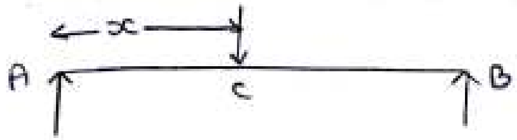
@ $x = 0$, $m_D = 0$

@ $x = z$, $m_D = 1$

@ $x = L$, $m_D = L-z$

Simply supported beam

case-1 → Single load



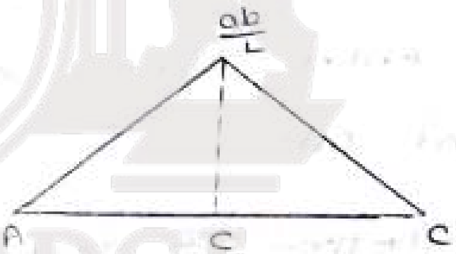
Reactions

⇒ load × ILD of ordinate

⇒ load × ILD of ordinate

Shear force

Bending moment



Portion AC = $-R_B$

portion BC = R_A

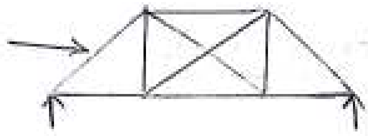
load × ILD value of ordinate

Case - 2

ILD for Trusses

Trusses are usually used in bridges that carry moving loads. There are two types of truss bridges

(1) Through Deck Bridge



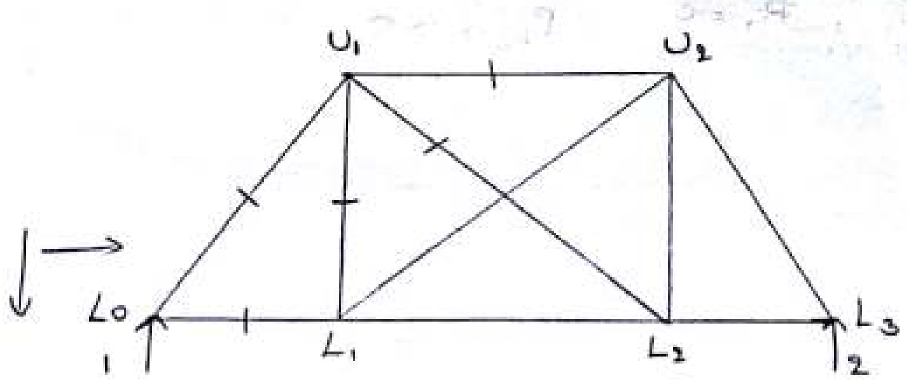
(2) Deck Bridge

To draw the influence line diagram for the trusses, we should first consider the type of bridge and analyised. Truss bridge consist of reactions at two of its supports from which the entire forces in the axial members are found out.

method of finding ILD'S for thease reactions is same as that of beams and method of joints and method of sections are used to find ILD's for forces in the truss members.

problem:-

① Draw the influence line diagram for forces in the members $L_0 U_1$, $U_1 U_2$, $L_0 L_1$, $L_1 L_2$, $U_1 L_2$ and $U_1 L_1$



Here load is moving over the bottom chord

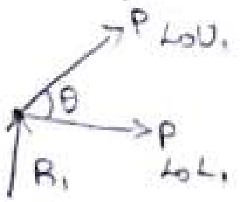
∴ ILD for R_1 can be drawn as follows

ILD for R_1

ILD for R_2

① ILD for L_0U_1

Joint L_0



$\Sigma V = 0 \Rightarrow R_1 + P_{L_0U_1} \sin\theta = 0$

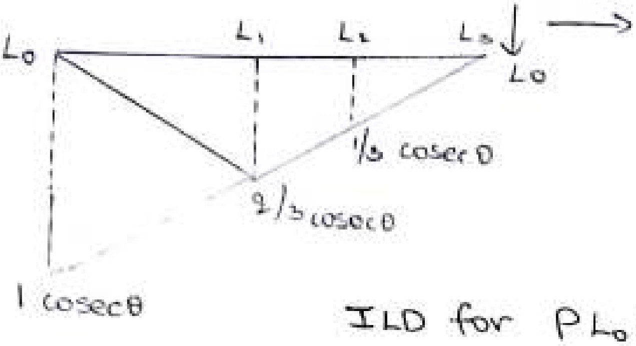
$\Rightarrow P_{L_0U_1} = \dfrac{-R_1}{\sin\theta} = -\operatorname{cosec}\theta \, R_1$

@ L_0, $R_1 = 1$, $P_{L_0U_1} = -\operatorname{cosec}\theta$

@ L_1, $R_1 = 2/3$, $P_{L_0U_1} = \frac{-2}{3}\operatorname{cosec}\theta$

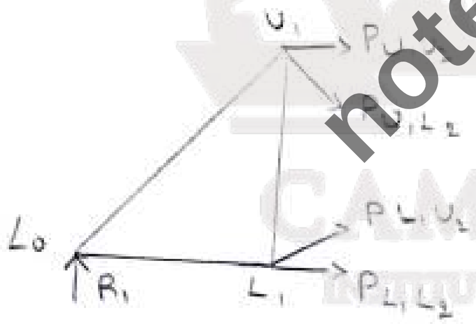
@ L_2, $R_1 = \frac{1}{3}$, $P_{L_0U_1} = \frac{-1}{3}\operatorname{cosec}\theta$

@ L_3, $R_1 = 0$, $P_{L_0U_1} = 0$



ILD for $P_{L_0U_1}$

ILD for U_1U_2

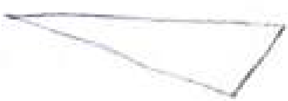


$\Sigma M_{L_2} = 0$

$P_{U_1U_2} \times h + \left(R_1 \times \frac{2}{3}\right) + \left[\cancel{P_{L_1U_2} \sin\theta \left(\frac{1}{3}\right)}^{\,0}\right] = 0$

$P_{U_1U_2} \times h = -\frac{R_1 2}{3}$

$$\boxed{P_{U_1U_2} = \frac{-2R_1}{3h}}$$



$\Rightarrow P_{L_0L_1} + P_{L_0U_1} \cos\theta = 0$

$\Rightarrow P_{L_0L_1} = -\left(\frac{-R_1}{\sin\theta} \cos\theta\right) = R_1 \cot\theta$