Fourier Series :-

Periodic function :- A fn $f(x)$ is said to be periodic of period (T) if

$$f(x+T) = f(x), \forall x.$$

Ex : $\sin x$ & $\cos x$ are periodic functions with period 2π

Fourier series in $(c, c+2l)$

If a fn $f(x)$ is defined in $(c, c+2l)$, then the fourier series of this fn with period '$2l$' is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right) \rightarrow (1)$$

where

$$a_0 = \frac{1}{l}\int_c^{c+2l} f(x)\, dx.$$

$$a_n = \frac{1}{l}\int_c^{c+2l} f(x) \cos\left(\frac{n\pi x}{l}\right) dx$$

$$b_n = \frac{1}{l}\int_c^{c+2l} f(x) \sin\left(\frac{n\pi x}{l}\right) dx.$$

where a_0, a_n & b_n are called as fourier co-efficients.

Note :- period = upper lmt − lower lmt

case (i) :- Let $c = 0$ $l = \pi$

$\therefore (c, c+2l) = (0, 2\pi)$

$\therefore$ Fourier Series of $f(x)$ in $(0, 2\pi)$ is

(1) $\Rightarrow$ (0)

$$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{\pi}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{\pi}\right)$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx.$$

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x) \cdot dx.$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x) \cos nx \cdot dx.$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x) \sin nx \cdot dx.$$

Case ii

$c = -\pi, \quad l = \pi$

$\therefore (c, c+2l) = (-\pi, \pi)$

$\therefore$ Forier series of $f(x)$ in $(-\pi, \pi)$

put $l = \pi$

$\therefore$

$$f(x) = \frac{a_0}{2} + \sum a_n \cos nx + \sum b_n \sin nx.$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cdot dx$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos nx \cdot dx.$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cdot \sin nx \cdot dx.$$

1. obtain the fourier series of $f(x) = x$ in $(-\pi, \pi)$

period = U.L - L.L

$= -\pi - \pi = 2\pi$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx. \rightarrow (1)$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cdot dx.$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos nx \cdot dx.$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx \cdot dx.$$

Consider

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} x \cdot dx = \frac{1}{\pi}\left[\frac{x^2}{2}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{2\pi}\left[x^2\right]_{-\pi}^{\pi}$$

$$a_0 = \frac{1}{2\pi}\left[(\pi)^2 - (-\pi)^2\right]$$

$$= \frac{1}{2\pi}(0) = 0$$

$$\boxed{a_0 = 0}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x \cdot \cos nx \cdot dx.$$

$$= \frac{1}{\pi}\left[(x)\cdot\cancelto{0}{\left(\frac{\sin nx}{n}\right)} - (1)\left(\frac{-\cos nx}{n^2}\right)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi n^2}\left[\cos n\pi - \cos n\pi\right] = 0$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x \cdot \sin nx \cdot dx$$

$$= \frac{1}{\pi}\left[\frac{(x)(-\cos nx)}{n} + \cancel{\frac{(1)(\sin nx)}{n}}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{n\pi}\left[\pi \cos n\pi + \pi \cos n\pi\right]$$

$$= \frac{-1}{\pi n}\left[x \cos nx\right]_{-\pi}^{\pi}$$

$$= \frac{-1}{\pi n}\left[\pi \cos n\pi - (-\pi)\cos(-\pi x)\right]$$

$$= \frac{-1}{\pi n}\left[\pi(-1)^n + \pi(-1)^n\right]$$

$$\frac{-1}{\pi n} 2\pi(-1)^n$$

$$= \frac{-2}{n}(-1)^n$$

$$= \frac{2}{n}(-1)^{n+1}$$

Substituting the value of a_0, a_n, b_n in (1).

$$x = 0 + 0 + \sum_{n=1}^{\infty} \frac{2}{n}(-1)^{n+1} \sin nx.$$

$$\boxed{x = 2\sum_{n=1}^{\infty} \frac{1}{n}(-1)^{n+1} \sin nx}$$

2. Obtain the fourier series of $f(x) = x$ in $(0, 2\pi)$

period $= 2\pi$.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx.$$

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x) \cdot dx.$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cdot\cos nx \cdot dx$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx \cdot dx$$

consider

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} x \cdot dx$$

$$\frac{1}{\pi}\left.\frac{x^2}{2}\right]_0^{2\pi}$$

$$\frac{1}{2\pi}\left[x^2\right]_0^{2\pi}$$

$$\frac{1}{2\pi}\left[4\pi^2\right] = 2\pi$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} x \cdot \cos nx \cdot dx$$

$$= \frac{1}{\pi}\left[x \cdot \cancel{\frac{\sin nx}{n}}^{0} - (1)(-)\frac{\cos nx}{n^2}\right]$$

$$= \frac{1}{\pi n^2}\left[\cos nx\right]_0^{2\pi}$$

$$\frac{1}{\pi n^2}\left[\cos n2\pi - 1\right] = \frac{1}{\pi n^2}\left[1-1\right] = 0 //$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx \cdot dx$$

$$\frac{1}{\pi}\int_0^{2\pi} x \sin nx \, dx$$

$$\frac{1}{\pi}\left[\cancel{x\left(\frac{\sin nx}{n}\right)}^{0} - (1)\left(-\frac{\cos nx}{n^2}\right)\right]_0^{2\pi}$$

$$\frac{1}{\pi n^2}$$

make 16 + 17

$$\frac{1}{\pi}\left[\frac{x(\cos nx)}{n} + (1)\frac{\cancel{(\sin nx)}}{\cancel{n}}\right]_0^{2\pi}$$

$$-\frac{1}{n\pi}\left[x\cos nx\right]_0^{2\pi}$$

$$\frac{-1}{n\pi}\left[2\pi\cos n 2\pi - 0\cos 0\right]$$

$$\frac{-1}{n\pi}\left[2\pi - 0\right]$$

$$b_n = \frac{-2}{n}$$

$$f(x) = \frac{2\pi}{2} + 0 + \sum_{n=1}^{\infty}\frac{-2}{n}\sin nx$$

$$f(x) = \pi - 2\sum_{n=1}^{\infty}\frac{1}{n}\sin nx$$

3. obtain the FS of $f(x) = x^2$ in $(-\pi, +\pi)$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos nx + \sum_{n=1}^{\infty} b_n\sin nx$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cdot\cos nx\,dx$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cdot\sin nx\,dx$$

consider

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\,dx = \frac{1}{\pi}\frac{x^3}{3}$$

$$= \frac{1}{3\pi}\left[x^3\right]_{-\pi}^{\pi} = \pi^3 + \pi^3$$

$$= \frac{2\pi^3}{3\pi}$$

$$= \frac{2}{3}\pi^2$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \cdot \cos nx \cdot dx$$

$$= \left[(x^2)\cancel{\frac{\sin nx}{n}} - 2x\frac{(-\cos nx)}{n^2} + \cancel{\frac{2(-\sin nx)}{n^3}}\right]$$

$$= \frac{1}{\pi n^2}\left[2x \cos nx\right]_{-\pi}^{\pi}$$

$$\frac{2}{\pi n^2}\left[\pi \cos n\pi + \pi \cos n\pi\right]$$

$$\frac{4\cancel{\pi}\cos n\pi}{\cancel{\pi} n^2}$$

$$a_n = \frac{4(-1)^n}{n^2}$$ //

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \cdot \sin nx \cdot dx$$

$$= \frac{1}{\pi}\left[(x^2)\left(\frac{-\cos nx}{n}\right) - (2x)\cancel{\left(\frac{-\sin nx}{n^2}\right)} + 2\left(\frac{\cos nx}{n^3}\right)\right]_{-\pi}^{\pi}$$

$$\frac{1}{\pi}\left[\frac{2}{n^3}\left[\cos nx\right]_{-\pi}^{\pi} - \frac{1}{n}\left[x^2 \cos nx\right]_{-\pi}^{\pi}\right]$$

$$\frac{1}{\pi}\left[\frac{2}{n^3}\left[\cancel{\cos n\pi - \cos n\pi}\right]^{0} - \frac{1}{n}\left[\cancel{\pi^2 \cos n\pi - (-\pi)^2 \cos(-n)\pi}\right]^{0}\right]$$

$$b_n = 0$$ //

$$f(x) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2}\cos nx + 0$$ //

4. obtain the FS for $f(x) = x^2$ in $(0, 2\pi)$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x) \cdot dx$$

Date ____
Page ____

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cdot \cos nx \cdot dx$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\ \sin nx \cdot dx$$

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} x^2 \cdot dx = \frac{1}{3\pi}\left[x^3\right]_0^{2\pi}$$

$$\frac{1}{3\pi}\, 8\pi^3$$

$$\boxed{a_0 = \frac{8}{3}\pi^2}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} x^2 \cdot \cos nx \cdot dx$$

$$= \frac{1}{\pi}\left[(x^2)\cancel{\left(\frac{\sin nx}{n}\right)}^{0} - (2x)\left(\frac{-\cos nx}{n^2}\right) + 2\cancel{\left(\frac{-\sin nx}{n^3}\right)}^{0}\right]$$

$$= \frac{2}{\pi n^2}\left[x \cos nx\right]_0^{2\pi}$$

$$= \frac{2}{\pi n^2}\left[2\pi \cos n2\pi - 0\right]$$

$$\frac{4\pi}{\pi n^2} = 4/n^2 \qquad \boxed{a_n = 4/n^2}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} x^2 \sin nx \cdot dx$$

$$= \frac{1}{\pi}\left[x^2\left(\frac{-\cos nx}{n}\right) - 2x\cdot\cancel{\left(\frac{-\sin nx}{n^2}\right)}^{0} + 2\left(\frac{\cos nx}{n^3}\right)\right]_0^{2\pi}$$

$$\frac{1}{\pi}\left[\frac{-1}{n}\left(x^2 \cos nx\right) + \frac{2}{n^3}\left(\cos nx\right)\right]_0^{2\pi}$$

$$\frac{1}{\pi}\left[\frac{-1}{n}\left(4\pi^2\right) + \cancel{\frac{2}{n^3}(1-1)}^{0}\right]$$

$$\boxed{b_n = -\frac{4\pi}{n}}$$

$$f(x) = 4/3\,\pi^2 + 4\sum_{n=1}^{\infty} \frac{1}{n^2}\cos nx - 4\pi\sum_{n=1}^{\infty} \frac{1}{n}\sin nx$$

$$= \frac{4}{3}\pi^2 + 4\sum_{n=1}^{\infty} \frac{1}{n^2}\cos nx - 4\sum_{n=1}^{\infty} \frac{\pi}{n}\sin nx \;//$$

5. Find the forier series for $f(x) = x + x^2$ in $(0, 2\pi)$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos nx + \sum_{n=1}^{\infty} b_n \sin nx.$$

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} (x + x^2)\cdot dx$$

$$\frac{1}{\pi}\left[\frac{x^2}{2} + \frac{x^3}{3}\right]_0^{2\pi}$$

$$\frac{1}{\pi}\left(\frac{4\pi^2}{2} + \frac{8\pi^3}{3}\right) - 0$$

$$a_0 = 2\pi + \frac{8\pi^2}{3}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} (x+x^2)\cdot \cos nx \cdot dx$$

$$\frac{1}{\pi}\left[(x+x^2)\left(\frac{\sin nx}{n}\right) - (1+2x)\left(\frac{-\cos nx}{n^2}\right)\right.$$

$$+ 2\left(\;\right)$$

$$= \frac{1}{\pi}\left[(1+2x)\left[\frac{\cos nx}{n^2}\right]\right]_0^{2\pi}$$

$$= \frac{1}{\pi n^2}\left\{\frac{1+4\pi}{n^2}\left[1\right] - \frac{1}{n^2}\right.$$

$$\cancel{1} + 4\pi - \cancel{1} = \frac{4}{n^2} \;//$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} (x+x^2)\sin nx \, dx$$

$$= \frac{1}{\pi}\left[(x+x^2)\left(\frac{-\cos nx}{n}\right) - (1+2x)\cancel{\left(\frac{-\sin nx}{n^2}\right)}^{0} + 2\left(\frac{\cos nx}{n^3}\right)\right]$$

$$= \frac{1}{\pi}\left[\frac{-1}{n}\left[(x+x^2)(\cos nx)\right] + \frac{2}{n^3}(\cos nx)\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[\frac{-1}{n}\left[2\pi + 4\pi^2\right] - 0\right] + \cancel{\frac{2}{n^3}(1-1)}^{0}$$

$$= \frac{-1}{\pi n}\left[2\pi + 4\pi^2\right]$$

$$= -\frac{[2+4\pi]}{n}$$

• Obtain the FS of $f(x) = x(2\pi - x)$ in $(0, 2\pi)$ 11

Period $= 2\pi$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\pi + \sum_{n=1}^{\infty} b_n \sin n\pi \rightarrow (1)$$

$$\left.\begin{aligned} a_0 &= \frac{1}{\pi}\int_0^{2\pi} f(x) \cdot dx \\ a_n &= \frac{1}{\pi}\int_0^{2\pi} f(x) \cdot \cos nx \cdot dx \\ b_n &= \frac{1}{\pi}\int_0^{2\pi} f(x) \cdot \sin nx \cdot dx \end{aligned}\right\} \rightarrow (2)$$

* (2) $\Rightarrow$ $a_0 = \frac{1}{\pi}\int_0^{2\pi} x(2\pi - x) \cdot dx$

$$\frac{1}{\pi}\int_0^{2\pi} (2\pi x - x^2) \cdot dx$$

$$= \frac{1}{\pi}\left[2\pi\left[\frac{x^2}{2}\right]_0^{2\pi} - \left[\frac{x^3}{3}\right]_0^{2\pi}\right]$$

$$\frac{1}{\pi}\left[\pi(4\pi^2) - \frac{1}{3}[8\pi^3]\right]$$

$$\frac{\pi^3}{\pi}\left[4 - \frac{8}{3}\right]$$

$$\pi^2 \quad \frac{12 - 8}{3}$$

$$a_0 = \frac{4\pi^2}{3}$$

* $a_n = \frac{1}{\pi}\int_0^{2\pi} (2\pi x - x^2) \cos nx \cdot dx$

$$= \frac{1}{\pi}\int_0^{2\pi} \cancel{2\pi x \cos nx \cdot dx} - \int_0^{2\pi} \cancel{x^2 \cos nx \cdot dx}$$

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$$\cancel{\frac{1}{\pi}\left[2\pi\left[x\left(\frac{\sin nx}{n}\right) - (1)\left[\frac{-\cos nx}{n^2}\right]\right]\right]^{2\pi}}$$

Date
Page

$$\frac{1}{\pi}\int_0^{2\pi}(2\pi x - x^2)\cos nx \cdot dx .$$

$$= \frac{1}{\pi}\left[(2\pi x - x^2)\left(\frac{\sin nx}{n}\right)^{\to 0} - (2\pi - 2x)\left(\frac{-\cos nx}{n^2}\right) + -2\left(\frac{-\sin nx}{n^3}\right)^{\to 0}\right]_0^{2\pi}$$

$$\frac{1}{n^2\pi}\left[(2\pi - 2x)\cos nx\right]_0^{2\pi}$$

$$\frac{1}{\pi n^2}\left[(2\pi - 4\pi)\cos 2n\pi - (2\pi\cos 0)\right]$$

$$\frac{1}{\pi n^2}\left[-2\pi - 2\pi\right]$$

$$\frac{-4\pi}{\pi n^2} = \frac{-4}{n^2} . \quad \boxed{a_n = \frac{-4}{n^2}}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x) \cdot \sin nx \cdot dx .$$

$$1/\pi\int_0^{2\pi}(2\pi x - x^2)\sin nx \cdot dx .$$

$$\frac{1}{\pi}\left[(2\pi x - x^2)\left(\frac{-\cos nx}{n}\right) - (2\pi - 2x)\left(\frac{-\sin nx}{n^2}\right)^{\to 0} + (-2)\left(\frac{\cos nx}{n^3}\right)\right]_0^{2\pi}$$

$$\frac{1}{\pi}\left[(2\pi x - x^2)\left(\frac{-\cos nx}{n}\right) - (2)\left(\frac{\cos nx}{n^3}\right)\right]_0^{2\pi}$$

$$\frac{1}{\pi}\left[\frac{-1}{n}\left[(2\pi x - x^2)\cos nx\right]_0^{2\pi} - \frac{2}{n^3}(\cos nx)_0^{2\pi}\right].$$

$$\frac{1}{\pi}\left[\frac{-1}{n}[0-0] - \frac{2}{n^3}[1-1]\right]$$

$$= \frac{1}{\pi}[0] = 0 //$$

$$f(x) = \frac{4\pi^2}{2\times 3} + \sum_{n=1}^{\infty}\left(\frac{-4}{n^2}\right)\cos nx + 0$$

$$\boxed{f(x) = \frac{2\pi^2}{3} - 4\sum_{n=1}^{\infty}\left(\frac{1}{n^2}\right)\cos nx}$$

* obtain the FS $x + x^2$. $(-\pi, \pi)$.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \cos n\pi x + \sum_{n=1} \sin n\pi x$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cdot dx$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cdot\cos nx\cdot dx$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cdot\sin nx\cdot dx$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi}(x + x^2)\cdot dx$$

$$\frac{1}{\pi}\left[\frac{x^2}{2} + \frac{x^3}{3}\right]_{-\pi}^{\pi}$$

$$\frac{1}{\pi}\left[\frac{1}{2}(\pi^2 - \pi^2) + \frac{1}{3}(\pi^3 + \pi^3)\right]$$

$$\frac{2\pi^3}{3\pi} = 2/3\,\pi^2$$

$$\boxed{a_0 = \frac{2}{3}\pi^2}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cdot\cos nx\cdot dx$$

$$\frac{1}{\pi}\int_{-\pi}^{\pi}(x + x^2)\cos nx\cdot dx$$

$$= \frac{1}{\pi}\left[(x+x^2)\left(\cancel{\frac{\sin nx}{n}}\right)^0 - (1+2x)\left(-\frac{\cos nx}{n^2}\right) + 2\left(-\cancel{\frac{\sin nx}{n^3}}\right)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{1}{n^2}(1+2x)\cos nx\right]_{-\pi}^{\pi}$$

$$\frac{1}{\pi n^2}\left[(1+2\pi)\cos n\pi - (1-2\pi)\cos(-n)\pi\right]$$

$$(1+2\pi)(-1)^n - (1-2\pi)(-1)^n$$

$$(-1)^n(1+2\pi - 1 + 2\pi)$$

$$\frac{1}{\pi n^2}\left[(-1)^n 4\pi\right]$$

$$\boxed{a_n = \frac{4(-1)^n}{n^2}}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cdot\sin nx \cdot dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi}(x+x^2)\sin nx \cdot dx$$

$$\frac{1}{\pi}\left[(x+x^2)\left(-\frac{\cos nx}{n}\right) - (1+2x)\cancel{\left(-\frac{\sin nx}{n^2}\right)}^0 + 2\left(\frac{\cos nx}{n^3}\right)\right]_{-\pi}^{\pi}$$

$$\frac{1}{\pi}\left[-\frac{1}{n}(x+x^2)(\cos nx) + \frac{2}{n^3}(\cos nx)\right]_{-\pi}^{\pi}$$

$$\frac{1}{\pi}\left[\left[\frac{-1}{n}(\pi+\pi^2)(-1)^n + \frac{2}{n^3}(-1)^n\right] - \left[\frac{-1}{n}(-\pi+\pi^2)(-1)^n + \frac{2}{n^3}(-1)^n\right]\right].$$

$$\frac{-1}{n\pi}\left[(-1)^n(\pi+\pi^2)-(-\pi+\pi^2)\right].$$

$$\frac{-1}{n\pi}\left[(-1)^n\left[\pi+\cancel{\pi^2}+\pi-\cancel{\pi^2}\right]\right]$$

$$\frac{-1}{n\cancel{\pi}}(2\cancel{\pi})(-1)^n.$$

$$b_n = \frac{-2(-1)^n}{n}.$$

$$f(x) = \frac{\pi^2}{3}+4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx - 2\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin nx$$

ii) obtain F.S of $f(x) = \frac{\pi - x}{2}$ in $0 \le x \le 2\pi$ $(0, 2\pi)$

$$f(x) = \frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos nx + \sum_{n=1}^{\infty}b_n\sin nx.$$

$$a_0 = \frac{1}{\pi}\int_0^{2\pi}\frac{\pi - x}{2}.\,dx.$$

$$\frac{1}{2\pi}\int_0^{2\pi}(\pi - x)\,dx.$$

$$\frac{1}{2\pi}\left[\pi x - \frac{x^2}{2}\right]_0^{2\pi}$$

$$\frac{1}{2\pi}\left(2\pi^2 - 2\pi^2\right)$$

$$a_0 = 0.$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi}\frac{\pi - x}{2}\cos nx\,.\,dx.$$

$$\frac{1}{2\pi}\int(\pi - x)\cos nx.$$

$$(\pi - x)\cancel{\left(\frac{\sin nx}{n}\right)}^{0} + 1\left(\frac{-\cos nx}{n^2}\right)$$

$$\frac{1}{2\pi n^2}\left[\cos nx\right]_0^{2\pi}$$

$$-\left(\cos 2n\pi - \cos 0\right).$$

$$\frac{1}{2\pi n^2} \times (+1 - 1).$$

$$a_n = 0.$$

$$b_n = \frac{1}{2\pi}\int_0^{2\pi} (\pi - x)\cdot \sin nx \cdot dx.$$

$$= \frac{1}{2\pi}\left[(\pi - x)\left(\frac{-\cos nx}{n}\right) + 1\cancel{\left(\frac{-\sin nx}{n^2}\right)}^{0}\right]_0^{2\pi}$$

$$= \frac{-1}{2\pi n}\left[(\pi - x)(\cos nx)\right]_0^{2\pi}$$

$$\frac{-1}{2\pi n}\left[\left((-\pi)\cdot \cos 2n\pi - \pi \cos 0\right)\right].$$

$$\frac{1}{2\pi n}(-\pi - \pi)$$

$$\frac{-1}{2\pi n} \cdot 2\pi \quad b_n = -1/n$$

put $x = \pi/2$

$$\frac{\pi - x}{2} = \frac{2}{\pi}\sin \frac{\pi}{2} x$$

$$\pi - x = \frac{1}{\pi}$$

$$\pi^2 - x = 0$$

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{n}\sin nx$$

$$\frac{\pi - x}{2} = \sum_{n=1}^{\infty} \frac{1}{n} \sin nx.$$

$$\frac{\pi - x}{2} = \sin x + \frac{\sin 2x}{2} + \frac{\sin 3x}{3} + \cdots$$

$n = 1, 2, 3 \ldots \quad x = \pi/2$

$$\frac{\pi - \pi/2}{2} = 1 \sin \pi/2 \qquad \pi/4 = 1$$

$$\frac{2\pi - \pi}{\ldots} = \sin \pi/2 + \frac{1}{2} \cancel{\sin \pi}$$

obtain the FS expansion of

$$f(x) = \begin{cases} -\pi, & -\pi \le x \le 0 \\ x & 0 \le x \le \pi \end{cases}$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx.$$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cdot dx.$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} (-\pi)\,dx + \int_{0}^{\pi} x \cdot dx\right].$$

$$\frac{1}{\pi}\left[-[\pi x]_{-\pi}^{0} + \left[\frac{x^2}{2}\right]_0^{\pi}\right]$$

$$\frac{1}{\pi}\left[-\pi^2 + \frac{\pi^2}{2}\right].$$

$$a_0 = -\frac{\cancel{3}\pi}{2}.$$

$$a_n = \frac{1}{\pi}\left[\int_{-\pi}^{0} (-\pi)\cos nx + \int_{0}^{\pi} x \cos nx\right]$$

$$\frac{1}{\pi}\left[\cancel{\frac{-\pi}{n}\left[\sin nx\right]_{-\pi}^{0}} + \left[(x)\left(\cancel{\frac{\sin nx}{n}}\right) - 1\left(-\frac{\cos nx}{n^2}\right)\right]_0^{\pi}\right]$$

$$\frac{1}{\pi}\left[\frac{1}{n^2}(\cos nx)\right]_0^{\pi}$$

$$\frac{1}{\pi}\left[\frac{1}{n^2}(\cos n\pi - \cos 0)\right]$$

$$a_n = \frac{1}{n^2\pi}\left[(-1)^n - 1\right] \text{ //}$$

$$b_n = \frac{1}{\pi}\left[\int_{-\pi}^{0} (-\pi)\sin nx \cdot dx + \int_{0}^{\pi} (x)\sin nx \cdot dx\right].$$

$$\frac{1}{\pi}\left[(-\pi)\int_{-\pi}^{0}\sin nx\cdot dx+\int_{0}^{\pi}(x)\sin nx\cdot dx\right]$$

$$\frac{1}{\pi}\left[(-\pi)\left[-\frac{\cos nx}{n}\right]_{-\pi}^{0}+\left[x\left(-\frac{\cos nx}{n}\right)-1\left(-\frac{\sin nx}{n^2}\right)\right]_{0}^{\pi}\right]$$

$$\cancel{\frac{1}{\pi}\left[\frac{\pi}{n}\left[(-1)^n+1\right]\right.}$$

$$\frac{1}{\pi}\left[\frac{\pi}{n}\left[1-(-1)^n\right]+\left[-\frac{1}{n}\left(\pi(-1)^n\right)\right]\right]$$

$$\frac{1}{n}\left[1-(-1)^n-(-1)^n\right]$$

$$b_n=\frac{1}{n}\left[1-2(-1)^n\right]$$

2^n+2^n

2^{n+n}

$$f(x)=\cancel{\frac{1}{2\pi n^2}\left[(-1)^n-1\right]}+$$

$$f(x)=\frac{-\pi}{4}+\frac{1}{\pi}\sum_{n=0}^{\infty}\frac{\left[(-1)^n-1\right]}{n^2}\cos nx+\sum_{n=0}^{\infty}\frac{1}{n}\left[1-2(-1)^n\right]\sin nx.$$

Obtain the FS expansion of

$$f(x)=\begin{cases}1+\frac{2x}{\pi} & -\pi\le x\le 0\\ 1-\frac{2x}{\pi} & 0\le x\le\pi\end{cases}$$

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos nx+\sum_{n=1}^{\infty}b_n\sin nx$$

$$a_0=\frac{1}{\pi}\left[\int_{-\pi}^{0}\left(1+\frac{2x}{\pi}\right)dx+\int_{0}^{\pi}\left(1-\frac{2x}{\pi}\right)dx\right]$$

$$a_0=\frac{1}{\pi}\left[\left[x+\frac{x^2}{\pi}\right]_{-\pi}^{0}+\left(x-\frac{x^2}{\pi}\right)_{0}^{\pi}\right]$$

$$\frac{1}{\pi}\left[(0+\pi)+\frac{1}{\pi}\left[0-\pi^2\right]+(\pi-0)-\frac{1}{\pi}(\pi^2-0)\right]$$

$$\boxed{a_0 = 0}$$

$$a_n = \frac{1}{\pi}\left[\int_{-\pi}^{0}\left(1+\frac{2x}{\pi}\right)\cos nx \cdot dx + \int_{0}^{\pi}\left(1-\frac{2x}{\pi}\right)\cos nx \, dx\right]$$

$$= \frac{1}{\pi}\left[\left[\cancel{\left(1+\frac{2x}{\pi}\right)\left(\frac{\sin nx}{n}\right)}^{0} - \left(\frac{2}{\pi}\right)\left(-\frac{\cos nx}{n^2}\right)\right] + \right.$$

$$\left.\left[\cancel{\left(1-\frac{2x}{\pi}\right)\left(\frac{\sin nx}{n}\right)}^{0} + \frac{2}{\pi}\left(-\frac{\cos nx}{n^2}\right)\right]\right]$$

$$\frac{1}{\pi}\left[\frac{2}{\pi n^2}\left(\cos nx\right)_{-\pi}^{0} - \frac{2}{\pi n^2}\left(\cos nx\right)_{0}^{\pi}\right].$$

$$\frac{2}{\pi}\left[\frac{1}{\pi n^2}\left[1-(-1)^n\right] - \frac{1}{\pi n^2}\left[(-1)^n - 1\right]\right]$$

$$\frac{2}{\pi^2 n^2}\left[\cancel{1} - (-1)^n - (-1)^n + \cancel{1}\right]$$

$$\frac{2}{\pi^2 n^2}\left[2(-1)^n\right]$$

$$a_n = \frac{4(-1)^n}{\pi^2 n^2} \,//$$

$$b_n = \frac{1}{\pi}\left[\int_{-\pi}^{0}\left(1+\frac{2x}{\pi}\right)\sin nx \, dx + \int_{0}^{\pi}\left(1-\frac{2x}{\pi}\right)\sin nx \cdot dx\right]$$

$$= \frac{1}{\pi}\left[\left[\left(1+\frac{2x}{\pi}\right)\left(-\frac{\cos nx}{n}\right) - 2\cancel{\left(-\frac{\sin nx}{n^2}\right)}^{0}\right]_{-\pi}^{0} + \right.$$

$$\left.\left[\left(1-\frac{2x}{\pi}\right)\left(-\frac{\cos nx}{n}\right) + \frac{2}{\pi}\cancel{\left(-\frac{\sin nx}{n^2}\right)}^{0}\right]_{0}^{\pi}\right]$$

$$= \frac{1}{\pi}\left[\left(1+\frac{2x}{\pi}\right)\frac{-1}{n}\left(\cos nx\right)\right]_{-\pi}^{0} + \left(1-\frac{2x}{\pi}\right)\frac{-1}{n}\left(\cos nx\right)\Big]_{0}^{\pi}$$

$= \frac{-1}{n\pi}\left[(1 - \cos 0) - (-1)(-1)^n\right] + \left[-1(-1)^n - 1\right]$

$\frac{-1}{n\pi}\left(\cancel{1} + \cancel{(-1)^n}\right) + -\cancel{(-1)^n} - \cancel{1}\Big]$

$\left[1 - (-1\cos n\pi) + (-\cos n\pi) - 1\right]$

$\boxed{b_n = 0}$

$f(x) = \frac{2(-1)^n}{\pi^2 (n^2)}$

obtain the $F(s)$ for $f(x) = 2x - x^2$ in $(0, 3)$
$(c, c+2l)$

$f(x) = 2x - x^2$

to find the value of l we use

$2l = UL - LL$

$2l = 3 - 0$

$2l = 3$

$l = 3/2$

$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$

$a_0 = \frac{1}{l}\int_0^3 f(x)\,dx$

$= \frac{2}{3}\int_0^3 (2x - x^2)\,dx$

$\frac{2}{3}\left[x^2 - \frac{x^3}{3}\right]_0^3 = [9 - 0] - \frac{1}{\cancel{3}}\left[2\cancel{7}^{9} - 0\right]$

$\boxed{a_0 = 0}$

$a_n = \frac{2}{3}\int_0^3 (2x - x^2)\cos\left(\frac{n\pi x}{l}\right)dx$

$$\frac{2}{3}\left[(2x-x^2)\left(\frac{\sin(2/3\, n\pi x)}{\frac{2n\pi}{3}}\right)^{\cancel{0}} - (2-2x)\left(\frac{-\cos(2/3\, n\pi x)}{\left(\frac{2n\pi}{3}\right)^2}\right)\right.$$

$$\left. - \left(\frac{-\sin(2/3\, n\pi x)}{\left(\frac{2n\pi}{3}\right)^3}\right)^{\cancel{0}\,0}\right].$$

$$+\frac{2}{3}\left[\frac{3\cdot 9}{4n^2\pi^2}\times(2-2x)(\cos 2/3\, n\pi x)\right]_0^3.$$

$$\frac{6}{4n^2\pi^2}\left[(2-2x)(\cos 2/3\, n\pi x)\right]_0^3$$

$$\left[(-4)(\cos 2/3\, 3n\pi) - 2(1).\right]$$

$$\frac{3}{2}\frac{\cancel{6}}{4n^2\pi^2}\left[(-4) - (2)\right] \; \cancel{-6} = -6$$

$$\frac{-\cancel{18}\,9}{2n^2\pi^2} = \frac{-9}{n^2\pi^2}$$

$$\boxed{a_n = \frac{-9}{n^2\pi^2}}$$

$$b_n = 2/3\int_0^3 f(x)\cdot \sin\left(\frac{n\pi x}{l}\right)\cdot dx$$

$$= 2/3\int_0^3 (2x-x^2)\cdot \sin\left(\frac{2n\pi x}{3}\right)\cdot dx$$

$$= 2/3\left[(2x-x^2)\left[\frac{-\cos\left(\frac{2n\pi x}{3}\right)}{\left(\frac{2n\pi}{3}\right)}\right] - (2-2x)\left[\frac{-\sin\left(\frac{2n\pi x}{3}\right)}{\left(\frac{2n\pi}{3}\right)^2}\right]^{\cancel{x}\,0}\right.$$

$$+ (-2)\left[\frac{\cos\left(\frac{2n\pi x}{3}\right)}{\left(\frac{2n\pi}{3}\right)^3}\right]\Bigg]_0^3$$

Date / Page

$$\frac{2}{3}\left[(2x-x^2)\left(\frac{-3}{2}n\pi\right)\cos\left(\frac{2n\pi x}{3}\right) - \frac{\cancel{2}\times 27}{4\cancel{8}n^3\pi^3}\cos\left(\frac{2n\pi x}{3}\right)\right.$$

$$\frac{2}{3}\left[\left(\frac{-3}{2}\,n\pi\right)(6-9)\cos(2n\pi) - 0\right] - \frac{27}{4n^3\pi^3}\left[\;\right]^0$$

$$\cdot \cos 2n\pi - \cos 0\;\Big]^0$$

$$= -\cancel{2}/\cancel{3}\left[\frac{\cancel{3}}{\cancel{2}\,n\pi}(-3)\right]$$

$$= \frac{3}{n\pi}$$

$$\boxed{b_n = 3/n\pi}$$

$$f(x) = \frac{-9}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\cos\left(\frac{n\pi x}{l}\right) + \frac{3}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\left(\frac{n\pi x}{l}\right)$$

where $l = 3/2$

Expand $f(x) = \begin{cases}\pi x & 0 \le x \le 1\\ \pi(2-x) & 1 \le x \le 2\end{cases}$ as F.S.

$$f(x) = \frac{a_0}{2}\sum_{n=1}^{\infty} a_n\cos\left(\frac{n\pi x}{l}\right) + \sum_{n=1}^{\infty} b_n\sin\left(\frac{n\pi x}{l}\right)$$

$$2l = 2 - 0$$

$$\boxed{l = 1}$$

$$a_0 = \int_0^1 \pi x\,.dx + \int_1^2 \pi(2-x)\,.dx$$

23

$$a_0 = \pi\left[\frac{x^2}{2}\right]_0^1 + \pi\left[2x - \frac{x^2}{2}\right]_1^2$$

$$\frac{\pi}{2} + \pi\left[2 - 1/2\right]$$

$$\cancel{\frac{\pi}{2}} + \cancel{\frac{3\pi}{2}} \qquad \cancel{\frac{4\pi}{2}} = 2\pi$$

$$\cancel{\boxed{a_0 = 2\pi}}$$

$$\cancel{\pi/2} + \pi\left[(4-2) + (2 - 1/2)\right]$$

$$4 - 2 + 2 - 1/2$$

$$4 - 1/2 = 7/2$$

$$\pi/2 + \pi\left[4 - 2 - 2 + 1/2\right]$$

$$\boxed{a_0 = \pi}$$

$$a_n = \left[\int_0^1 \pi x \cos(n\pi x) \cdot dx + \int_1^2 \pi(2-x)\cos(n\pi x) \cdot dx\right]$$

$$= \pi\left[x\left(\frac{\sin n\pi x}{n\pi}\right)^{\to 0} - 1\left(\frac{-\cos n\pi x}{n^2\pi^2}\right)\right]_0^1 +$$

$$\left[(2-x)\left(\frac{\sin n\pi x}{n\pi}\right)^{\to 0} - (-1)\left(\frac{-\cos n\pi x}{n^2\pi^2}\right)\right]_1^2$$

$$\pi\left[\frac{\cos n\pi x}{n^2\pi^2}\right]_0^1 - \frac{1}{n^2\pi^2}\left[\cos n\pi x\right]_1^2$$

$$\frac{\frac{1}{\pi}}{n^2\pi^2}\left[\cos n\pi - \cos 0 - \cos 2n\pi + \cos n\pi\right]$$

$$2\cos n\pi - 1 - 1$$

$$\frac{1}{n^2\pi^2}\left[2\cos n\pi - 2\right]$$

$$a_n = \left[\frac{2}{n^2\pi}\left[(-1)^n - 1\right]\right]$$

$$b_n = \left[\int_0^1 \pi x \sin(n\pi x).dx + \int_1^2 \pi(2-x)\sin(n\pi x).dx\right]$$

$$\pi\left[x\left(\frac{-\cos(n\pi x)}{n\pi}\right) - 1\left(\cancel{\frac{-\sin(n\pi x)}{n^2\pi^2}}\right)\right]_0^1 + \left[(2-x)\left(\frac{-\cos(n\pi x)}{n\pi}\right) - 1\;\cancel{\frac{\sin(n\pi x)}{n\pi}}\right]$$

$$\pi\left[\cancel{x\cos}\ \frac{-1}{n\pi}x\cos n\pi x\right]_0^1 + \left[\frac{-1}{n\pi}(2-x)\cos n\pi x\right]_1^2$$

$$\frac{-\pi}{n\pi}\left[\cos \cancel{n}\pi - (1)\cos \cancel{n}\pi\right]$$

$$\boxed{b_n = 0}$$

$$f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{2}{n^2\pi}\left[(-1)^n - 1\right]$$

Obtain the Fourier series Expansion of $f(x) = e^{-ax}$ in $(-\pi, \pi)$

period $= 2\pi$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx .$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} e^{-ax} \, dx .$$

$$= \left[\frac{e^{-ax}}{-a}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{-\pi a} \cdot \left[e^{-\pi a} - e^{\pi a}\right] .$$

$$a_0 = \frac{2}{\pi a}\left[\sinh a\pi\right]$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} e^{-ax} \cos nx \cdot dx .$$

$$= \frac{e^{-ax}}{a^2+n^2}\left[-a\cos nx + \cancel{n \sin nx}^{\,0}\right]_{-\pi}^{\pi}$$

$$\frac{1}{\pi}\frac{-a}{(a^2+n^2)}\left[e^{-a\pi}\cos n\pi - e^{a\pi}\cos n\pi\right]$$

$$\frac{1}{\pi}\frac{a\cos n\pi}{(a^2+n^2)}\left[e^{a\pi} - e^{-a\pi}\right] .$$

$$\frac{a(-1)^n}{\pi(a^2+n^2)} \, e^{a\pi} - e^{-a\pi}$$

$$a_n = \frac{2a(-1)^n}{\pi(a^2+n^2)} \sinh a\pi \quad //$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} e^{-ax} \sin nx \cdot dx .$$

$$= \frac{1}{\pi}\left[\frac{e^{-ax}}{a^2+n^2}\left[-a\cancel{\sin nx}^{\,0} - n\cos nx\right]\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{-n e^{-ax}}{a^2+n^2} \cos nx\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{-n}{a^2+n^2}\left(e^{-a\pi} \cos n\pi - e^{a\pi} \cos n\pi\right)\right.$$

$$\frac{1}{\pi}\left[\frac{-n \cos n\pi}{a^2+n^2}\left[e^{-a\pi} - e^{a\pi}\right]\right]$$

$$\frac{1}{\pi}\left[\frac{(-1)^n}{a^2+n^2}\left(e^{a\pi} - e^{-a\pi}\right)\right]$$

$$b_n = \frac{1}{\pi}\frac{(-1)^n}{(a^2+n^2)} \cancel{\cosh a\pi}\, 2\sinh a\pi$$

$$f(x) = \frac{1}{a\pi}[\sinh a\pi] + \sum_{n=1}^{\infty} \frac{2a}{\pi} \frac{(-1)^n}{(a^2+n^2)}[\sinh a\pi] +$$

$$\sum_{n=1}^{\infty} \frac{2}{\pi} \cdot \frac{(-1)^n}{(a^2+n^2)} \sinh a\pi$$

Obtain the FS of $f(x) = e^{-x}$ $\quad 0 < x < 2$.

Period $= 2$ $\quad 2l = 2 - 0$ $\quad l = 1$.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) + \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{l}\right) b_n.$$

$$a_0 = \frac{1}{l}\int_0^2 f(x)\, dx$$

$$= \int_0^2 e^{-x}\, dx$$

$$= \left[\frac{e^{-x}}{-1}\right]_0^2$$

$$= -1\left[e^{-2} - e^{0}\right]$$

$$-1\left[e^{-2} - 1\right]$$

$$a_0 = \left[1 - e^{-2}\right]$$

$$a_n = \int_0^2 e^{-x} \cos n\pi x \cdot dx .$$

$$= \frac{e^{-x}}{1+(n\pi)^2}\left[-\cos n\pi x + n\pi \cancel{\sin n\pi \pi}^{0}\right]_0^2$$

$$= \frac{e^{-x}}{1+n^2\pi^2}\left[-\cos n\pi x\right]_0^2$$

$$= e^{-2}\left[\cos 2n\pi\right] - \left[\cos 0\right].$$

$$a_n = \frac{-1}{1+n^2\pi^2}\left[e^{-2}-1\right].$$

$$b_n = \int_0^2 e^{-x} \sin n\pi x \cdot dx .$$

$$= \frac{e^{-x}}{1+(n\pi)^2}\left[-1\cancel{\sin n\pi x}^{0} - n\pi \cos n\pi x\right]_0^2$$

$$= \frac{-n\pi e^{-x}}{1+n^2\pi^2}\left[\cos n\pi x\right]_0^2$$

$$\frac{-n\pi}{1+n^2\pi^2}\left[e^{-2}\cos 2n\pi - 1\right]$$

$$-\left[e^{-2}-1\right]$$

$$b_n = \frac{n\pi}{1+n^2\pi^2}\left[1-e^{-2}\right]$$

$$f(x) = \frac{1-e^{-2}}{2} + \sum_{n=1}^{\infty} \frac{-1}{1+n^2\pi^2}\left[e^{-2}-1\right] + \sum_{n=1}^{\infty} \frac{n\pi}{1+n^2\pi^2}\left[1-e^{-2}\right]$$

obtain the FS of ~~×~~ & hence deduce the

$$f(x) = \begin{cases} 2-x & 0 \le x \le 4 \\ x-6 & 4 \le x \le 8 \end{cases}$$

series

$$\frac{\pi^2}{8} = \sum \frac{1}{(2n-1)^2}$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

$2l = 8 - 0$

$\boxed{l = 4}$

$$a_0 = \frac{1}{l}\int f(x)\,dx$$

$$a_0 = \frac{1}{4}\left[\int_0^4 (2-x)\,dx + \int_4^8 (x-6)\,dx\right]$$

$$= \frac{1}{4}\left[2x - \frac{x^2}{2}\right]_0^4 + \left[\frac{x^2}{2} - 6x\right]_4^8$$

$$8 - \frac{16}{8} + \frac{64}{8}$$

$$\left(\frac{64}{2} - 48\right) - \left(\frac{16}{2} - 24\right)$$

$\boxed{a_0 = 0}$

$$a_n = \frac{1}{l}\left[\int_0^4 (2-x)\cos\frac{n\pi x}{4}\,dx + \int_4^8 (x-6)\cos\frac{n\pi x}{4}\,dx\right]$$

$$= \frac{1}{4}\left[(2-x)\frac{\sin\left(\frac{n\pi x}{4}\right)}{\left(\frac{n\pi}{4}\right)} - (-1)\left[\frac{-\cos\left(\frac{n\pi x}{4}\right)}{\left(\frac{n\pi}{4}\right)^2}\right]\right]$$

$$\frac{16}{n^2\pi^2}\left[(x-6)\left[\frac{\sin\left(\frac{n\pi x}{4}\right)}{(n\pi/4)}\right] - 1\left[\frac{-\cos\left(\frac{n\pi x}{4}\right)}{\left(\frac{n\pi}{4}\right)^2}\right]\right]$$

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$$\frac{1}{4}\left[-\frac{16}{n^2\pi^2}\cos\left(\frac{n\pi x}{4}\right)\right]_0^4+\left[\frac{16}{n^2\pi^2}\cos\frac{n\pi x}{4}\right]_4^8$$

$$\frac{16}{4n^2\pi^2}\left\{\left[-\cos\frac{n\pi x}{4}\right]_0^4+\left[\cos\left(\frac{n\pi x}{4}\right)\right]_4^8\right.$$

$$\frac{4}{n^2\pi^2}\left[-[\cos n\pi-\cos 0]+[\cos 2n\pi-\cos n\pi]\right].$$

$$\cancel{[1-(-1)^n]+[1-(-1)^n}$$

$$[\cos 0-\cos n\pi+\cos 2n\pi-\cos n\pi]$$

$$[\cos 0+\cos 2n\pi-2\cos n\pi]$$

$$2-2(-1)^n$$

$$a_n=\frac{8}{n^2\pi^2}[1-(-1)^n].$$

$$b_n=\frac{1}{4}\left\{\int_0^4(2-x)\sin\left(\frac{n\pi x}{4}\right).dx+\int_4^8(x-6)\sin\left(\frac{n\pi x}{4}\right).dx.\right.$$

$$=\frac{1}{4}\left[(2-x)\left[\frac{-\cos\left(\frac{n\pi x}{4}\right)}{\frac{n\pi}{4}}\right]_0^4+-(-1)\left[\frac{-\sin\left(\frac{n\pi x}{4}\right)}{\frac{n^2\pi^2}{16}}\right]^{0}+\right.$$

$$(x-6)\left[\frac{-\cos\left(\frac{n\pi x}{4}\right)}{\frac{n\pi}{4}}\right]-1\left[\frac{-\sin\left(\frac{n\pi x}{4}\right)}{\frac{n\pi}{4}}\right]^{0}$$

$$\frac{1}{4}\left[(2-x)\left[\frac{4}{n\pi}\cos\left(\frac{n\pi x}{4}\right)\right]_0^4+(x-6)\left(\frac{4}{n\pi}\right)\left[\cos\left(\frac{n\pi x}{4}\right)\right]_0^8\right.$$

$$\frac{-1}{n\pi}\left[(2-x)\cos\left(\frac{n\pi x}{4}\right)\right]_0^4 + \left[(x-6)\cdot\cos\left(\frac{n\pi x}{4}\right)\right]_4^8$$

$$\frac{-1}{n\pi}\left[-\cancel{2\cdot\cos n\pi} - 2\cos 0\right] + \left[2\cos(2n\pi) + \cancel{2\cos n\pi}\right]$$

$$-\frac{2}{n\pi}\left[\cos(2n\pi) - \cos 0\right]$$

$$\frac{2}{n\pi}\left[1-1\right]$$

$$= \frac{2}{n\pi}[0] = 0$$

$$f(x) = \sum_{n=1}^{\infty} \frac{8}{n^2\pi^2}\left[1-(-1)^n\right]\cos\left(\frac{n\pi x}{4}\right).$$

$$f(x) = \frac{-8}{\pi^2}\sum_{n=1}^{\infty}\left[\frac{1}{n^2}\left[1-(-1)^n\right]\cos\left(\frac{n\pi x}{4}\right)\right] \rightarrow (1)$$

Note :- To deduce the given series, we put x = lower limit or upper limit or the mid point of the given interval.

Put $x=0$ in (1).

$$f(0) = \frac{-8}{\pi^2}\sum_{n=1}^{\infty}\left[\frac{1}{n^2}\left[1-(-1)^n\right]\right]$$

from given eq^n $f(x) = 2-x$, $0 \le x \le 4$.

$f(0) = 2-0 = 2$.

$$\therefore 2 = \frac{8}{\pi^2}\sum\frac{1}{n^2}\left(1-(-1)^n\right)$$

$$\frac{\pi^2\times 2}{8} = \sum_{n=1}\frac{1}{n^2}\left[1-(-1)^n\right].$$

Put $n = 1, 2, 3 \ldots$

$$\frac{\pi^2\times 2}{8} = \frac{1}{1^2}\times 2 + \frac{1}{(2)^2}(0) + \frac{1}{(3)^2}(2) + \frac{1}{(4)^2}(0) + \frac{1}{(5)^2}\times 2 \ldots$$

$$\frac{\pi^2\times \cancel{2}}{8} = \cancel{2}\left[\frac{1}{1^2} + 1/3^2 + 1/5^2 \ldots\right]$$

$$\therefore \frac{\pi^2}{8} = \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}$$

Fourier series of even and odd function in $(-\pi, \pi)$

A fn^n $f(x)$ is said to be even function if

$$f(-x) = f(x)$$

eg:- all even powers of polynomial.

$x^2, x^4, \cos x, |x|$

A fun^n $f(x)$ is said to be odd function if

$$f(-x) = -f(x)$$

eg:- $\sin(-x) = -\sin x$. $-(f(x))$

x, x^3, x^5

FS of odd function in $(-\pi, \pi)$

If $f(x)$ is odd in $(-\pi, \pi)$ then the fourier co-efficients are given by

$$a_0 = 0$$

$$a_n = 0$$

$$b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx \cdot dx.$$

$$\int_{-a}^{a} f(x) \cdot dx = 2 \int_0^{a} f(x)$$

FS of even funcion in $(-\pi, \pi)$

If $f(x)$ is even fun^n, then the Fourier coefficients are given by

$$a_0 = 2/\pi \int_0^{\pi} f(x) \cdot dx.$$

$a_n = 2/\pi \int_0^\pi f(x) \cos nx \cdot dx$.

$b_n = 0$

$FS = f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$

1. Obtain the Fourier Series of $f(x) = |x|$ in $(-\pi, \pi)$

$|x|$ is even function.

$f(-x) = |-x| = |x| = f(x)$.

$f(x)$ is even funtion.

$\therefore \; b_n = 0$.

$a_0 = \frac{2}{\pi} \int_0^\pi f(x) \cdot dx$

$|x|$ in $0 - \pi$ is $+\pi$

$= \frac{2}{\pi} \int_0^\pi |x| \cdot dx$

$= \frac{2}{\pi} \int_0^\pi x \cdot dx$

$\frac{2}{\pi} \left[\frac{x^2}{2}\right]_0^\pi = \pi$

$\boxed{a_0 = \pi}$

$a_n = \frac{2}{\pi} \int_0^\pi f(x) \cdot \cos nx \cdot dx$

$= \int_0^\pi x \cdot \cos nx \cdot dx$

$\frac{2}{\pi} \left[x \left[\cancel{\frac{\sin nx}{n}}\right]^0 - 1 \left[-\frac{\cos nx}{n^2}\right] \right]_0^\pi$

$\frac{2}{\pi} \left[\frac{\cos n\pi}{n^2}\right]_0^\pi = \frac{2}{n^2\pi} \left[\cos n\pi - \cos 0\right]$

$\frac{2}{n^2\pi} \left[(-1)^n - 1\right]$

$a_n = \frac{2}{n^2\pi} \left[(-1)^n - 1\right]$

$\therefore$ F S

$$|x| = \frac{\pi}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{1}{n^2}\left[(-1)^n - 1\right]\cos nx$$

2. Obtain the Fo S Expansion of $f(x) = x\cos x$ in $(-\pi, \pi)$

$$f(-x) = -x\cos(-x)$$
$$= -x\cos x \quad \text{odd function}$$

$$a_0 = 0$$
$$a_n = 0$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} x\cos x \cdot \sin nx \longrightarrow (1)$$

$$= \frac{2}{\pi}\frac{1}{2}\int_0^{\pi} x\left[\sin(nx+x) + \sin(nx-x)\right].dx$$

$$= \frac{1}{\pi}\left[\int_0^{\pi} x\sin(n+1)x\,.dx + \int_0^{\pi} x\sin(n-1)x\,.dx\right]$$

$$\frac{1}{\pi}\left[\left[(x)\left[\frac{-\cos(n+1)x}{(n+1)}\right] - (1)\left(\frac{-\cancel{\sin(n+1)x}}{(n+1)^2}\right)\right]_0^{\pi}\right.$$
$$\left. + x\left(\frac{-\cos(n-1)x}{(n-1)}\right) - (1)\left[\frac{-\cancel{\sin(n-1)x}}{(n-1)^2}\right]\right]_0^{\pi}$$

$$\frac{1}{\pi}\left[\frac{-1}{(n+1)}\left(x\cos(n+1)x\right)_0^{\pi} - \frac{1}{(n-1)}\left(x\cos(n-1)x\right)_0^{\pi}\right]$$

$$-\frac{1}{\pi}\left[\frac{1}{(n+1)}\left\{(\pi\cos(n+1)\pi) - 0\right\} \mp \frac{1}{(n-1)}\left\{\pi\cos(n-1)\pi - 0\right\}\right]$$

$$-\frac{\pi}{\pi}\left[\frac{1}{(n+1)}(-1)^{n+1} \mp \frac{1}{(n-1)}(-1)^{(n-1)}\right].$$

$$= -\left[\frac{(-1)^{n-1}}{n-1} + \frac{(-1)^{n+1}}{n+1}\right]$$

$$(-1)^n\left[\frac{-1}{(n-1)} - \frac{(-1)}{(n+1)}\right]$$

$$(-1)\left[\frac{1}{(n-1)} + \frac{1}{(n+1)}\right]$$

$$(-1)\left[\frac{(n+1)+(n-1)}{n^2-1}\right]$$

$$(-1)\frac{\cancel{n^2-n+n+n-1}}{(n^2-1)} \qquad n+\cancel{1}+n-\cancel{1} = 2n$$

$$\boxed{b_n = (-1)^n \cdot \frac{2n}{n^2-1}} \quad , n \neq 1$$

For $n=1$, we take eq^n ① & sub $n=1$.

$$b_n = \frac{2}{\pi}\int_0^{\pi} x\cos x \sin nx \cdot dx$$

$$b_1 = \frac{2}{\pi}\int_0^{\pi} x\cos x \sin x \cdot dx$$

$$\frac{2}{\pi}\int_0^{\pi} x\,\frac{\sin 2x}{2} \cdot dx$$

$$\frac{1}{\pi}\left[x\left[\frac{-\cos 2x}{2}\right] - (1)\left[\cancel{\frac{-\sin 2x}{4}}\right]^{0}\right]_0^{\pi}$$

$$\frac{-1}{2\pi}\left[x\cos 2x\right]_0^{\pi}$$

$$\frac{-1}{2\pi}\left[\pi\cos 2\pi\right] - \cancel{\cos 0}$$

$$\boxed{b_1 = -1/2}$$

$$FS \cdot b_1\sin x + \sum_{n=1}^{\infty} b_n \sin nx .$$

$$= -\tfrac{1}{2}\sin x + 2\sum \frac{(-1)^n n}{n^2-1}\sin nx .$$

3. Find the F.S Expansion of $f(x) = x \sin x$ in $(-\pi, \pi)$

$\because f(-x) = -x \sin(-x) = x \sin x$

$\therefore b_n = 0$.

$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x) \cdot dx$.

$= \frac{2}{\pi}\int_0^{\pi} x \sin x \cdot dx$.

$= \left[x \cdot (-\cos x) - (1)(-\cancel{\sin x}^{\,0})\right]_0^{\pi}$

$= \frac{-2}{\pi}\left[\pi \cos\pi - 0\right]$

$= \frac{-2}{\pi}[-1]$

$\boxed{a_0 = 2}$

$a_n = \frac{2}{\pi}\int_0^{\pi} f(x) \cos nx \cdot dx$

$= \frac{2}{\pi}\int_0^{\pi} x \sin x \cdot \cos nx \cdot dx$

$= \frac{2}{\pi}\int_0^{\pi} \frac{x}{2}\left[\sin(x+nx) + \sin(x-nx)\right] \cdot dx$

$= \frac{1}{\pi}\left[\int_0^{\pi} x \sin(n+1)x - \int_0^{\pi} x \sin(n-1)x \cdot dx\right.$

$\frac{1}{\pi}\left[x\left(\frac{-\cos(n+1)x}{(n+1)}\right) - 1\left(\frac{-\cancel{\sin(n+1)x}^{\,0}}{(1+n)^2}\right)\right]_0^{\pi} -$

$\left[x\left(\frac{-\cos(n-1)x}{(n-1)}\right) - 1\left(\frac{-\cancel{\sin(n-1)x}^{\,0}}{(1-n)^2}\right)\right]_0^{\pi}$

$$= \frac{-1}{\pi}\left[\frac{1}{1+n}\left(x\cdot \cos(n+1)x\right)_0^{\pi} - \frac{1}{1-n}\left(x \cos\left(\frac{n-1}{1-n}\right)x\right)_0^{\pi}\right]$$

$$= \frac{-\pi}{\pi}\left[\frac{1}{1+n}\left(\cos\left(\frac{n+1}{1+n}\right)\pi - \frac{1}{1-n}\cos\left(\frac{n-1}{1-n}\right)\pi\right]$$

$$= \left[\frac{(-1)^{n+1}}{(n+1)} - \frac{(-1)^{(n-1)}}{(n-1)}\right]$$

$$= (-1)^n \frac{(-1)}{1+n} + \frac{(-1)}{}$$

$$= (-1)^n\left[\frac{(-1)^{n+1}}{n+1} - \frac{(-1)^{(n-1)}}{n-1}\right]$$

$$= -(-1)^n\left[\frac{(-1)}{(n+1)} - \frac{(-1)}{(n-1)}\right]$$

$$= -(-1)^n\left[\frac{1}{(n+1)} + \frac{1}{(n-1)}\right]$$

$$(-1)^n\left[\frac{(n-1)+(n+1)}{(n^2+1)}\right]$$

$$(-1)^n\left[\frac{2n}{(n^2+1)}\right]$$

$$= \frac{1}{\pi}\left[(x)\left(-\frac{\cos(n+1)x}{n+1}\right) - (1)\left(-\frac{\sin(n+1)x}{(n+1)^2}\right)^{\to 0} - (x)\left(-\frac{\cos(n-1)x}{(n-1)}\right) - (1)\left(-\frac{\sin(n-1)}{(n-1)^2}\right)^{\to 0}\right]$$

$$= \frac{1}{\pi}\left[-\frac{1}{n+1}\left(x\cos(n+1)x\right) + \frac{1}{n-1}\left(x\cos(n-1)x\right)\right]$$

$$= \frac{1}{\pi}\left[-\frac{1}{n+1}\{\pi\cos(n+1)\pi - 0\} + \frac{1}{n-1}\{\pi\cos(n-1)\pi - 0\}\right]$$

$$= \frac{\pi}{\pi}\left[-\frac{1}{n+1}(-1)^{n+1} + \frac{1}{n-1}(-1)^{n-1}\right]$$

$$= \left[\frac{1\cdot(-1)^{n-1}}{(n-1)} - \frac{1\,(-1)^{n+1}}{(n+1)}\right]$$

$$= (-1)^n\left[\frac{1}{(n-1)} - \frac{1}{(n+1)}\right]$$

$$(-1)^n\left[\frac{1}{n+1} - \frac{1}{n-1}\right]$$

$$n+1-n+1$$

$$-2$$

$$a_n = (-1)^n\left[\frac{(n+1) - n+1}{n^2-1}\right]$$

$$a_n = \frac{(-1)^n 2}{n^2-1}$$

$$(-1)^n\left[\frac{(-1)}{n-1} - \frac{(-1)}{n+1}\right]$$

$$\frac{-1}{(n-1)} + \frac{1}{n+1}$$

$$\frac{-n-1+n-1}{n^2-1}$$

$$= \frac{-2(-1)^n}{n^2-1}$$

$$a_1 = \frac{2}{\pi}\int_0^\pi x\sin x\cos x\,dx$$

$$= \frac{2}{\pi}\int_0^\pi \frac{x\sin 2x}{2}\,dx$$

$$\boxed{a_1 = -1/2}$$

F.S. is

$$x\sin x = \frac{2}{2} + \left(-\tfrac{1}{2}\right)\cos x - 2\sum \frac{(-1)^n}{(n^2-1)}\cos nx$$

Half range series :- $(0,\pi)$ or $(0,l)$

If $f(x)$ is define

1. Fourier sine half range series :- If $f(x)$ is defined in $(0,\pi)$ then the fourier sine half range series is given by

$$f(x) = \sum b_n \sin x.$$

where

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x).\sin nx \,.dx$$

In $(0,l)$, the sine half range is given by

$$f(x) = \sum b_n \sin\left(\frac{n\pi x}{l}\right)$$

Where $b_n = \frac{2}{l}\int_0^{l} f(x)\sin\left(\frac{n\pi x}{l}\right)\cdot dx.$

Fourier cosine half Range Series :-

in $(0,\pi)$

$$f(x) = \frac{a_0}{2} + \sum a_n \cos nx$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx.$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x).\cos nx.dx$$

in $(0,l)$

$$f(x) = \frac{a_0}{2} + \sum a_n \cos\left(\frac{n\pi x}{l}\right)$$

$$a_0 = \frac{2}{l}\int_0^{l} f(x).dx$$

$$a_n = \frac{2}{l}\int_0^{l} f(x)\cos\left(\frac{n\pi x}{l}\right).dx$$

Note: If finding the half range series for the range $(0, l)$ we take the upper limit of the given interval as the value of 'l'.

1. obtain half range cosine series for the function $f(x) = \sin x$ in $0 \le x \le \pi$

We've half range cosine series

$$f(x) = \frac{a_0}{2} + \sum a_n \cos nx.$$

$$a_0 = \frac{2}{\pi}\int_0^\pi f(x) \cdot dx$$

$$= \frac{2}{\pi}\int_0^\pi \sin x \cdot dx$$

$$\frac{2}{\pi}\left[-\cos x\right]_0^\pi \cdot dx$$

$$\frac{-2}{\pi}\left[\cos\pi - \cos 0\right]$$

$$\frac{-2}{\pi}\left[(-1) - 1\right]$$

$(a-\pi)$

$$\frac{(-2)-2}{\pi} = \frac{4}{\pi}$$

$$a_n = \frac{2}{\pi}\int_0^\pi f(x) \cdot \cos nx \cdot dx$$

$$= \frac{2}{\pi}\int_0^\pi \sin x \cdot \cos nx \cdot dx.$$

$$\frac{2}{\pi}\cdot\frac{1}{2}\int_0^\pi \left[\sin(nx + x) + \sin(x - nx)\right] dx$$

$$\frac{1}{\pi}\left[\int_0^\pi \sin(n+1)x \cdot dx - \int_0^\pi \sin(n-1)x \cdot dx\right.$$

$$\frac{1}{\pi}\left[\left[\frac{-\cos(n+1)x}{(n+1)}\right]_0^\pi - \left[\frac{-\cos(n-1)x}{(n-1)}\right]_0^\pi\right]$$

$$= \frac{1}{\pi}\left[\frac{-1}{n+1}\left(\cos(n+1)x\right)_0^{\pi}\right] + \frac{1}{n-1}\left[\cos(n-1)x\right]_0^{\pi}\Big]$$

$$= \frac{-1}{(n+1)}\left[\cos(n+1)\pi - 1\right] + \frac{1}{n-1}\left[\cos(n-1)\pi - 1\right]$$

$$= \frac{1}{\pi}\left[\frac{-1}{n+1}\left[(-1)^{n+1} - 1\right] + \frac{1}{n-1}\left[(-1)^{n-1} - 1\right]\right]$$

$$= \frac{1}{\pi}\left[\frac{1}{n+1} - \frac{1}{n-1} - \frac{1}{n+1}(-1)^{n+1} + \frac{(-1)^{n-1}}{n-1}\right]$$

$$= \frac{1}{\pi}\left[\frac{-2}{n^2-1} - (-1)^n\left[\frac{-1}{n+1} + \frac{1}{n-1}\right]\right]$$

$$= \frac{1}{\pi}\left[\frac{-2}{n^2-1} - (-1)^n\left[\frac{2}{n^2-1}\right]\right]$$

$$a_n = \frac{-2}{\pi(n^2-1)}\left[1 + (-1)^n\right] \quad //$$

half range Cosine Series is

$$f(x) = \frac{2\pi}{\pi} - \frac{2}{\pi}\sum \frac{1}{(n^2-1)}\left[1+(-1)^n\right]\cos nx \quad //$$

Q: Obtain the half range cosine series for the fn $f(x) = (x-1)^2$ in $0 \le x \le 1$. Hence deduce the series $\sum 1/n^2 = \frac{\pi^2}{6}$

$\boxed{l = 1}$

$$f(x) = \frac{a_0}{2} + \sum a_n \cos\left(\frac{n\pi x}{l}\right)$$

$$a_0 = \frac{2}{1}\int_0^1 (x-1)^2\, dx$$

$$= 2\left[\frac{(x-1)^3}{3}\right]_0^1$$

$$2\int_0^1 (x^2 - 2x + 1)\, dx$$

$$= 2\left[\frac{x^3}{3} - \frac{2x^2}{2} + x\right]_0^1$$

$$\frac{1}{3} - \frac{1}{2} + 1$$

$$= 2/3$$

$f(\frac{x-1}{2})$

$$a_n = 2\int_0^1 f(x)\cos\frac{n\pi x}{l}\,.dx$$

$$= 2\int_0^1 (x-1)^2 . \cos n\pi x\,.dx.$$

$$= 2\left[\cancel{(x-1)^2\left[\frac{\sin n\pi x}{(n\pi)}\right]}^{0} - 2(x-1)\left[\frac{-\cos n\pi x}{(n\pi)^2}\right] + 2\cancel{\left[\frac{-\sin n\pi x}{(n\pi)^3}\right]}^{0}\right]$$

$$2.\left[\frac{2(x-1)}{n^2\pi^2}\left[\cos n\pi x\right]\right]_0^1$$

$$= \frac{4}{n^2\pi^2}\left[0 - (-1)(1)\right]$$

$$a_n = \frac{4}{n^2\pi^2}$$

$$f(x) = \frac{1}{3} + \frac{4}{\pi^2}\sum \frac{1}{n^2}\cos n\pi x.$$

$$(x-1)^2 = \frac{1}{3} + \frac{4}{\pi^2}\sum \frac{1}{n^2}\cos n\pi x.$$

put $x = 0$.

$$1 = \frac{1}{3} + \frac{4}{\pi^2}\sum \frac{1}{n^2}\;1$$

$$\sum \frac{1}{n^2} = 1 - \frac{1}{3} - \frac{4}{\pi^2}$$

$$\sum \frac{1}{n^2} = \frac{\pi^2}{6}$$

$$\boxed{\frac{\pi^2}{6} = \sum \frac{1}{n^2}}$$

3. $f(x) = \begin{cases} \frac{1}{4} - x, & 0 \le x \le \frac{1}{2} \\ x - \frac{3}{4}, & \frac{1}{2} \le x \le 1 \end{cases}$

$l = 1$

obtain sine half range series.

$$f(x) = \sum b_n \sin n\pi x$$

$$b_n = 2\left[\int_0^{1/2} \left(\tfrac{1}{4} - x\right) \sin n\pi x \, dx + \int_{1/2}^{1} \left(x - \tfrac{3}{4}\right) \sin n\pi x \, dx\right]$$

$$\cancel{2\left[\frac{(\frac{1}{4}-x)}{n\pi}\left(-\cos n\pi x\right)\right]_0^{1/2} + 2\left[\frac{(x-\frac{3}{4})}{n\pi}\left[-\cos n\pi x\right]_{1/2}^{1}\right]_{1/2}}$$

$$\cancel{2\left[\frac{-(\frac{1}{4}-x)}{n\pi}\left[\cos \tfrac{1}{2} n\pi + \cos\right.\right.}$$

$$2\left[\left(\tfrac{1}{4}-x\right)\left(-\frac{\cos n\pi x}{n\pi}\right) - (-1)\left(-\frac{\sin n\pi x}{(n\pi)^2}\right)\right]_0^{1/2} +$$

$$\left[\left(x-\tfrac{3}{4}\right)\left(-\frac{\cos n\pi x}{n\pi}\right) - (1)\left(-\frac{\sin n\pi x}{(n\pi)^2}\right)\right]_{1/2}^{1}$$

$$2\left[-\frac{1}{n\pi}\left[\boxed{\cancel{-\tfrac{1}{4}\cos \frac{n\pi}{2}}} - \tfrac{1}{4}\right] - \frac{1}{n^2\pi^2}\left[\sin \frac{n\pi}{2} - 0\right] + -\right.$$

$$\left.\frac{1}{n\pi}\left[\tfrac{1}{4}\cos n\pi + \boxed{\cancel{\tfrac{1}{4}\cos \frac{n\pi}{2}}}\right] + \frac{1}{n^2\pi^2}\left[0 - \sin \frac{n\pi}{2}\right]\right]$$

$$= 2\left[\frac{-1}{n\pi}\left(-\tfrac{1}{4} + \tfrac{1}{4}\cos n\pi\right) + \frac{1}{n^2\pi^2}\left[-\frac{\sin n\pi}{2} - \sin \frac{n\pi}{2}\right]\right]$$

$$= 2\left[\frac{-1}{4n\pi}\left((-1)^n - 1\right) - \frac{2}{n^2\pi^2}\sin \frac{n\pi}{2}\right]$$

$$f(x) = \sum 2\left[\frac{-1}{4n\pi}\left((-1)^n - 1\right) - \frac{2}{n^2\pi^2}\sin \frac{n\pi}{2}\right] //$$

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4. Find the half range cosine series for

$$f(x) = \begin{cases} kx & 0 \le x \le l/2 \\ k(l-x) & l/2 \le x \le l. \end{cases}$$

$\boxed{l = l.}$

$b_n = 0$

$$a_0 = \frac{2}{l}\left[\int_0^{l/2} kx \cdot dx + \int_{l/2}^{l} k(l-x) \cdot dx\right]$$

$$= \frac{2}{l}\left[k\left[\frac{x^2}{2}\right]_0^{l/2} + k\left[lx - \frac{x^2}{2}\right]_{l/2}^{l}\right]$$

$$\frac{2}{l}\left[\frac{k}{2}\left[\frac{l^2}{4}\right] + k\left[\left(l^2 - \frac{l^2}{2}\right) - \left(\frac{l^2}{2} + \frac{l^2}{8}\right)\right]\right]$$

$$\frac{2}{l}\left[\frac{kl^2}{8} + \frac{kl^2}{8}\right]$$

$$\frac{2}{l} \quad \frac{2kl^2}{8\,4} \qquad \boxed{a_0 = \frac{kl}{2}}$$

$$a_n = \frac{2}{l}\left[\int_0^{l/2} kx \cos\left(\frac{n\pi x}{l}\right) \cdot dx + \int_{l/2}^{l} k(l-x)\cos\left(\frac{n\pi x}{l}\right) \cdot dx\right]$$

$$= \frac{2}{l}\left[k(x)\left[\frac{\sin\left(\frac{n\pi x}{l}\right)}{\left(\frac{n\pi}{l}\right)}\right] - 1\left[\frac{-\cos n\pi x/l}{\frac{n^2\pi^2}{l^2}}\right]\right]_0^{l/2} + k(l-x)\left[\frac{\sin\left(\frac{n\pi x}{l}\right)}{\frac{n\pi}{l}}\right]$$

$$-k(-1)\left[\frac{-\cos\left(\frac{n\pi x}{l}\right)}{\frac{n^2\pi^2}{l^2}}\right]_{l/2}^{l \cdot 0}$$

$$= \frac{2}{l}\left[\frac{kl}{n\pi}\,\frac{l}{2}\left[\sin\left(\frac{n\pi\, l/2}{l}\right)\right] + \frac{kl^2}{n^2\pi^2}\left[\cos\frac{n\pi}{2} - 1\right] + \right.$$

$K[0 - \frac{(l-l/2)l}{n\pi}[\sin n\pi/2] - [\frac{l^2}{n^2\pi^2}[\cos n\pi - \cos n\pi/2]]$

$\frac{2K}{l}[\frac{l}{n\pi}\{\cancel{\frac{l}{2}\sin\frac{n\pi}{2}}\}^0 + \frac{l^2}{n^2\pi^2}\{\cos n\pi/2 - 1\} + \frac{l}{n\pi}$

$\frac{l}{n\pi}[0 - \cancel{l/2 \sin n\pi/2}\} - \frac{l^2}{n^2\pi^2}[\cos n\pi - \cos n\pi/2]]$

$\frac{2K}{l}[\frac{l^2}{n^2\pi^2}[\cos n\pi/2 - 1 - \cos n\pi + \cos n\pi/2]]$

$a_n = \frac{2Kl}{n^2\pi^2}[2\cos n\pi/2 - 1 - (-1)^n]$

FCS is

$f(x) = \frac{Kl}{4} + \frac{2Kl}{\pi^2}\sum \frac{1}{n^2}[2\cos n\pi/2 - 1 - (-1)^n]\cos\frac{n\pi x}{l}$

5. Express $f(x)$ as a fourier cosine series in $(0,\pi)$ if $f(x) = \begin{cases} x & 0 \le x \le \pi/2 \\ \pi - x & \pi/2 \le x \le \pi \end{cases}$ $\boxed{l = \pi}$

$f(x) = \frac{a_0}{2} + \sum a_n \cos nx.$

$a_0 = \frac{2}{\pi}\left[\int_0^{\pi/2} x \, dx + \int_{\pi/2}^{\pi} \pi - x \, dx\right]$

$$= \frac{2}{\pi}\left[\frac{x^2}{2}\right]_0^{\pi/2} + \left[\pi x - \frac{x^2}{2}\right]_{\pi/2}^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{\pi^2}{8} + \pi^2 - \frac{\pi^2}{2} - \frac{\pi^2}{2} + \frac{\pi^2}{8}\right].$$

$$\frac{2}{\pi}\left[\frac{2\pi^2}{8}\right] = \pi/2$$

$$\boxed{a_0 = \pi/2}$$

$$a_n = \frac{2}{\pi}\left[\int_0^{\pi/2} x \cdot \cos nx \cdot dx + \int_{\pi/2}^{\pi} (\pi - x)\cos nx \cdot dx\right]$$

$$= \frac{2}{\pi}\left[\left[(x)\left(\frac{\sin nx}{n}\right) - 1\left(-\frac{\cos nx}{n^2}\right)\right]_0^{\pi/2} + \left[(\pi - x)\left(\frac{\sin nx}{n}\right) - (-1)\left(\frac{-\cos nx}{n^2}\right)\right]_{\pi/2}^{\pi}\right]$$

$$= \frac{2}{\pi}\left[\left\{\frac{\pi}{2n}\cancel{\sin n\pi/2} - 0\right\} + \frac{1}{n^2}\left\{\cos n\pi/2 - 1\right\} + \left\{0 - \left\{\frac{\pi}{2n}\cancel{\sin n\pi/2}\right\}\right\} - \left\{\frac{1}{n^2}\left[\cos n\pi - \cos n\pi/2\right]\right]\right.$$

$$= \frac{2}{\pi}\left[\frac{1}{n^2}\left[\cos n\pi/2 - 1 - \cos n\pi + \cos n\pi/2\right]\right]$$

$$\frac{2}{\pi n^2}\left[2\cos n\pi/2 - 1 - (-1)^n\right].$$

$$a_n = \frac{2}{\pi n^2}\left[2\cos n\pi/2 - 1 - (-1)^n\right]$$

$$f(x) = \pi/4 + \frac{2}{\pi}\sum \frac{1}{n^2}\left[2\cos n\pi/2 - 1 - (-1)^n\right]\cos nx.$$

Harmonic Analysis :-

The process of finding the first few sine & cosine terms in the fourier expansion of $f(x)$ is known as harmonic analysis.

We've :

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

Assigning the values of n as $1, 2, 3 \ldots$ and rearranging we get

$$f(x) = \frac{a_0}{2} + (a_1 \cos x + b_1 \sin x) + (a_2 \cos 2x + b_2 \sin 2x) + (a_3 \cos 3x + b_3 \sin 3x).$$

where the terms $\frac{a_0}{2}$ is known as constant term, $a_1 \cos x + b_1 \sin x$ is known as first harmonic, $a_2 \cos 2x + b_2 \sin 2x$ is 2^{nd} harmonic and $a_3 \cos 3x + b_3 \sin 3x$ is 3^{rd} harmonic.

the amplitude of first harmonic is given by

$$= \sqrt{a_1^2 + b_1^2}$$

$$2^{nd} \text{ harmonic} = \sqrt{a_2^2 + b_2^2}.$$

If 'N' number of values of 'x' and 'y' are given, then the fourier co-efficients a_i's & b_i's are given by

$$a_0 = \frac{2}{N}\Sigma y$$

$$a_1 = \frac{2}{N}\Sigma y \cos x. \qquad b_1 = \frac{2}{N}\Sigma y \sin x.$$

$$a_2 = \frac{2}{N}\Sigma y \cos 2x. \qquad b_2 = \frac{2}{N}\Sigma y \sin 2x.$$

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Note :- the given values of x are always return in degrees. and fix the calculator in degree mode.

1. Find the constant term, First harmonic & Second harmonic for the following data

x	0	60°	120°	180°	240°	300°	360°
y	7.9	7.2	3.6	0.5	0.9	6.8	7.9

$a_0 = \frac{2}{N}\Sigma y$ $\boxed{N = 6.}$

$a_1 = \frac{2}{N}\Sigma y\cos x$ $a_2 = \frac{2}{N}\Sigma y \cos 2x$.

$b_1 = \frac{2}{N}\Sigma y \sin x$ $b_2 = \frac{2}{N}\Sigma y \sin 2x$.

Note :- Since both the end values are given, we ignore 1 end value
ie 360°

x	y	$y\cos x$	$y\sin x$	$y\cos 2x$	$y\sin 2x$.
0	7.9	7.9	0	7.9	0.
60	7.2	3.6	6.23	-3.6	6.23
120	3.6	-1.8	3.117	-1.8	-3.11
180	0.5	-0.5	0	0.5	0
240	0.9	-0.45	-0.77	-0.45	0.77
300	6.8	3.4.	-5.88	-3.4.	-5.889
	26.9.	12.15	2.697	-0.85	-1.9989

$a_0 = \frac{2}{\cancel{6}\,3}\Sigma y = \frac{26.9}{3} = 8.9667$

$a_1 = \frac{1}{3} \times 12.15 = 4.05$

Date ______
Page ______

$a_2 = \frac{1}{3} \times -0.85 = -0.2833$

$b_1 = \frac{1}{3} \times 2.697 = 0.899$.

$b_2 = \frac{1}{3} \times -1.9989 = -0.6663$

constant term $= \frac{8.962}{2} = \frac{a_0}{2} = 4.483$

First harmonic $= a_1 \cos x + b_1 \sin x$

$= 4.05 \cos x + 0.899 \sin x$.

2^{nd} harmonic $= a_2 \cos 2x + b_2 \sin 2x$.

$= -0.283 \cos 2x + -0.666 \sin 2x$.

$f(x) = 4.483 + 4.05 \cos x + 0.899 \sin x + -0.283 \cos 2x$
$- 0.666 \sin 2x + \ldots$

2. The displacement y of a part of mechanism is tabulated with corresponding angular moment x° of the crank.
express y as a fourier series neglecting the harmonics above second.

x: 0 30 60 90 120 150 180 210, 240, 270, 300, 3

x:	0	30	60	90	120	150	180	210	240	270	300	330
y:	1.8	1.1	0.3	0.16	1.5	1.3	2.16	1.25	1.3	1.52	1.76	2

$a_0 = \frac{2}{N} \Sigma y$.

~~y: 1.8, 1.1~~

$a_1 = \frac{2}{N} \Sigma y \cos x$

$a_2 = \frac{2}{N} \Sigma y \cos 2x$

$b_1 = \frac{2}{N} \Sigma y \sin x$.

$b_2 = \frac{2}{N} \Sigma y \sin 2x$.

x	y	$y\cos x$	$y\sin x$	$y\cos 2x$	$y\sin 2x$
0	1.8	1.8	0	1.8	0.
30	1.1	0.9526	0.55	0.55	0.9526
60	0.3	0.15	0.2598	-0.15	0.2598
90	0.16	0	0.16	-0.16	0
120	1.5	-0.75	1.2990	-0.75	-1.2990
150	1.3	-1.1258	0.65	0.65	-1.1258
180	2.16	-2.16	0	2.16	0
210	1.25	-1.0825	-0.625	0.625	1.0825
240	1.3	-0.65	-1.1258	-0.65	1.1258
270	1.55	0	-1.55	-1.55	0
300	1.76	0.88	-1.5242	-0.88	-1.5242
330	2.	1.7320	-1	1	-1.7320.
	16.13	-0.2537	-2.9062	2.645	-2.2603

$a_0 = \frac{1}{6}\Sigma y = \frac{16.13}{6} = 2.688$

$a_1 = \frac{1}{6}\Sigma y\cos x = \frac{-0.2537}{6} = -0.0422$

$a_2 = \frac{1}{6}\Sigma y\cos 2x = \frac{2.645}{6} = 0.4408$

$b_1 = \frac{1}{6}\Sigma y\sin x = \frac{-2.9062}{6} = -0.4943$

$b_2 = \frac{1}{6}\Sigma y\sin 2x = \frac{-2.2603}{6} = -0.3767$

$f(x) = \frac{a_0}{2} + [a_1\cos x + b_1\sin x] + [a_2\cos 2x + b_2\sin 2x]$

$= 1.344 + [-0.0422\cos x - 0.4943\sin x] +$

$[0.4408\cos 2x - 0.3767\sin 2x]$

3. Express y as a fourier series upto 2nd harmonic, & hence find amplitude of 1st & 2nd harmonic

x:	0	$\pi/6$	$\pi/3$	$\pi/2$	$4\pi/6$	$5\pi/6$	π	$7\pi/6$	$8\pi/6$	$3\pi/2$	$10\pi/6$	$11\pi/6$	2π
y:	1.6	1.8	2.2	2.4	2.8	3	2.7	2.5	2.6	2.8	2.9	3	1.6

x	y	y cos x	y sin x	y cos 2x	y sin 2x
0	1.6	1.6	0	1.6	0
30	1.8	1.5588	0.9	0.9	1.5588
60	2.2	1.1	1.9053	-1.1 ~~1.9053~~	1.9053
90	2.4	0	2.4	-2.4	0
120	2.8	-1.4	2.4249	-1.4	-2.4249
150	3	-2.5981	1.5	1.5	-2.5981
180	2.7	-2.7	0	2.7	0
210	2.5	-2.1651	-1.25	1.25	2.1651
240	2.6	-1.3	-2.2517	-1.3	2.2517
270	2.8	0	-2.8	-2.8	0
300	2.9	1.45	-2.5115	-1.45	-2.5115
330	3	2.5981	-1.5	1.5	-2.5981
	30.3	-1.8563	-1.183	-1	-2.2552

$a_0 = \frac{1}{6}\Sigma y = \frac{30.3}{6} = 5.05$

$a_1 = \frac{1}{6}\Sigma y\cos x = \frac{-1.8563}{6} = -0.3093$

$a_2 = \frac{1}{6}\Sigma y\cos 2x = \frac{-1}{6} = -0.1666$

$b_1 = \frac{1}{6}\Sigma y\sin x = \frac{-1.183}{6} = -0.1971$

$b_2 = \frac{1}{6}\Sigma y\sin 2x = \frac{-2.2552}{6} = -0.3758$

$f(x) = 2.525 + [-0.3093\cos x - 0.1971\sin x] +$

$[-0.166\cos 2x - 0.3758\sin 2x]$

4. obtain the constant term, first & 2^{nd} harmonic for the following data.

x: 0 2 4 6 8 10 12

y: 9 18.2 24.4 27.8 27.5 22 9

x y $y\cos x$ $y\sin x$ $y\cos 2x$ $y\sin 2x$.

we convert the given values of x into degrees by using the relation $\theta = \frac{\pi x}{l}$.

$$\text{where } 2l = UL - LL$$

$$2l = 12 - 0$$

$$\boxed{l = 6} \quad \therefore \theta = \frac{\pi x}{6}.$$

at $x = 0$, $\theta = 0$

at $x = 2$, $\frac{\pi 2}{6_3} = \pi/3 = 60°$

$\therefore x =$ 0. 60 120 180 240 300 360

x	y	$y\cos\theta$	$y\sin\theta$	$y\cos 2\theta$	$y\sin 2\theta$
0	9	9	0	9	0
60	18.2	9.1	15.7616	−9.1	15.7616
120	24.4	−12.2	21.1310	−12.2	−21.1310
180	27.8	−27.8	0	27.8	0
240	27.5	−13.75	−23.8156	−13.75	23.8156
300	22	11	−19.0525	−11	−19.0525
	128.9	−24.65	−5.9755	−9.25	−0.6063

$$a_0 = \frac{2}{N}\Sigma y = \frac{128.9}{3} = 42.967$$

$$a_1 = \frac{2}{N}\Sigma y\cos x = \frac{-24.65}{3} = -8.217$$

$$a_2 = \frac{2}{N}\Sigma y\cos 2x = \frac{-9.25}{3} = -3.083$$

$b_1 = \frac{2}{N} \Sigma y \sin x = \frac{-5.9755}{3} = -1.991$

$b_2 = \frac{2}{N} \Sigma y \sin 2x = \frac{-0.6063}{3} = -0.2021$

$f(x) = 21.483 + [-8.217\cos x - 1.991\sin x] + [-3.083 \cos 2x - 0.2021 \sin 2x]$

$= 21.483 - [8.217\cos x + 1.991\sin x] - [3.083\cos 2x + 0.2021 \sin 2x]$.

constant term $= \frac{a_0}{2} = 21.483$.

1^{st} harmonic $= -8.217\cos x - 1.991\sin x$.

2^{nd} harmonic $= -3.083\cos 2x - 0.2021 \sin 2x$.

5. The following table gives the variations of a periodic current 'A' over a period 'T'

t:	0	$T/6$	$T/3$	$T/2$	$2T/3$	$5T/6$	T.
A:	1.98	1.30	1.05	1.30	-0.88	-0.25	1.98.

Show that there is a constant part of 0.75 A in the current 'A' & obtain the amplitude of 1^{st} harmonic

$2l = UL - LL$

$2l = T - 0$

$\boxed{l = T/2}$.

$x \quad y$.

$\therefore \theta = \frac{\pi t}{l} \Rightarrow \frac{\pi t}{T/2} \Rightarrow \frac{2\pi t}{T}$.

at $t = 0$, $\theta = 0$.

at $t = T/6$, $\theta = \frac{2\pi\, T/6}{T} = 2\pi/6 = 60°$

x	y	$y\cos\theta$	$y\sin\theta$
0	1.98	1.98	0
60	1.30	0.65	1.126
120	1.05	-0.525	0.9093
180	1.30	-1.3	0
240	-0.88	0.44	0.762
300	-0.25	-0.125	0.217
	4.5	1.12	~~2.812~~ 3.014

$a_0 = \frac{4.5}{3} = 1.5$

$a_1 = \frac{1.12}{3} = 0.373$

$b_1 = \frac{\cancel{2.812}\ 3.014}{3} = \cancel{0.937}\ 1.005$

Constant part $= \frac{a_0}{2} = \frac{1.5}{2} = 0.75$ A.

Amplitude $= \sqrt{a_1^2 + b_1^2}$

$= \sqrt{(0.373)^2 + (\cancel{0.937}\ 1.005)^2}$

Amplitude $= \underline{1.072}$

6. Express the following data in Fourier series upto second harmonics

x : 0 1 2 3 4 5

y : 1.2 1.6 1.8 2 2.2 2

to calculate the value of 'l', we include the last value of 'x', ie $x = 6$.

x	y	$y\cos x$	$y\sin x$	$y\cos 2x$	$y\sin 2x$
0	1.2	1.2	0	1.2	0
60	1.6	0.8	1.3856	-0.8	1.3856
120	1.8	-0.9	1.559	-0.9	-1.559
180	2	-2	0	2	0
240	2.2	-1.1	-1.905	-1.1	1.905
300	2	1	-1.732	-1	-1.732
	10.8	-1	-0.6924	-0.6	0.

$a_0 = \frac{10.8}{3} = 3.6$

$a_1 = \frac{-1}{3} = -0.333$

$a_2 = \frac{-0.6}{3} = -0.2$

$b_1 = \frac{-0.6924}{3} = -0.2308$

$b_2 = 0/3 = 0.$

$$f(x) = 1.8 + (0.233\cos x + 0.230\sin x)$$
$$- (0.28\cos 2x + 0) \,//$$

MODULE - 2

FOURIER TRANSFORMS :-

The Fourier transfer of $F(x)$ is defined as

$$F\{f(x)\} = \int_{-\infty}^{\infty} f(x) e^{iux} \cdot dx = F(u)$$

The inverse Fourier transform is defined as

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(u) e^{-iux} \cdot du.$$

Fourier cosine Transform.

$$F_c\{f(x)\} = \int_0^{\infty} f(x) \cos ux \cdot dx = F_c(u)$$

Inverse Fourier cosine transform.

$$f(x) = 2/\pi \int_0^{\infty} F_c(u) \cos ux \cdot du.$$

Fourier sine transform.

$$F_s\{f(x)\} = \int_0^{\infty} f(x) \sin ux \cdot dx = F_s(u).$$

Inverse Fourier sine transform.

$$f(x) = 2/\pi \int_0^{\infty} F_s(u) \sin ux \cdot du.$$

Note :-

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} \qquad e^{i\theta} = \cos\theta + i\sin\theta.$$

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}.$$

$|x| \leq 1$

$\Rightarrow -1 \leq x \leq 1$

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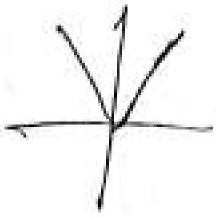
$y = |x| = \begin{cases} -x & -\infty \le x \le 0 \\ x & 0 \le x \le \infty \end{cases}$

x	-3	-2	-1	0	1	2	3
y	-(-3)	-(-2)	-(-1)	0	1	2	3

3



* $\int e^{ax} \cos bx \, dx = \frac{e^{ax}}{a^2+b^2}[a\cos bx + b\sin bx]$

* $\int e^{ax} \sin bx \, dx = \frac{e^{ax}}{a^2+b^2}[a\sin bx - b\cos bx]$

1. Find F.T of $f(x)$

$f(x) = \begin{cases} 1 & \text{for } |x| < 1 \\ 0 & \text{for } |x| > 1 \end{cases}$

Hence evaluate $\int_0^\infty \frac{\sin x}{x} dx$.

We've $F\{f(x)\} = \int_{-\infty}^{\infty} f(x) e^{iux} dx = F(u)$

$f(x) = \begin{cases} 1 & \text{for } -1 < x < 1 \\ 0 & \text{for } |x| > 1 \end{cases}$

$F\{f(x)\} = \int_{-1}^{1} (1) e^{iux} \cdot dx$

$= \left[\frac{e^{iux}}{iu}\right]_{-1}^{1}$

$= \frac{1}{iu}\left[e^{iu} - e^{-iu}\right]$

$$= \frac{1}{iu}\left[2i \sin u\right] \qquad \left\{ e^{i\theta} - e^{-i\theta} = 2i\sin\theta \right.$$

$$= \frac{2\sin u}{u} = F(u)$$

To evaluate the given integral we take inverse Fourier transform.

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(u)\, e^{-iux}\, du.$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{2\sin u}{u} \cdot e^{-iux} \cdot du.$$

put $x = 0$

$$f(0) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sin u}{u}\, e^{0} \cdot du.$$

$$1 = \frac{1}{\pi}\, 2\int_{0}^{\infty} \frac{\sin u}{u} \cdot du$$

$$\int_{0}^{\infty} \frac{\sin u}{u} \cdot du = \pi/2.$$

$u \to x$

$$\int_{0}^{\infty} \frac{\sin x}{x} \cdot dx = \pi/2.$$

2. $f(x) = \begin{cases} 1-x^2 & |x| \le 1 \\ 0 & |x| > 1 \end{cases}$

Hence evaluate $\displaystyle\int_{0}^{\infty} \frac{x\cos x - \sin x}{x^3} \cdot \cos\frac{x}{2} \cdot dx.$

$F\{f(x)\} = \int_{-\infty}^{\infty} f(x)\, e^{iux}\, dx = F(u)$

$F\{f(x)\} = \int_{-1}^{1} (1-x^2)\, e^{iux}\, dx$

$= \Big[\cancel{(1-x^2)\left(\frac{e^{iux}}{iu}\right)} \to 0 - (-2x)\left(\frac{e^{iux}}{(iu)^2}\right)$

$+ (-2)\left(\frac{e^{iux}}{(iu)^3}\right) \Big]_{-1}^{1}$ $\quad i^2 = -1$

$= (0-0) - \frac{2}{u^2}\left[e^{iu} + e^{-iu}\right]$ $\quad \frac{1}{i^3} = \frac{1}{i^2 \times i} = \frac{-1}{i} = \frac{-i}{i^2} = \frac{+i}{+1}$

$= \frac{-2i}{u^3}\left[e^{iu} - e^{-iu}\right]$

$= \frac{-2}{u^2}\left[2\cos u\right] - \frac{2i}{u^3}\left[2i\sin u\right]$

$F\{f(x)\} = \frac{-4\cos u}{u^2} + \frac{4\sin u}{u^3}$

$F\{f(x)\} = 4\left[\frac{-u\cos u + \sin u}{u^3}\right] = F(u)$

To evaluate the given integral we use inverse Fourier transform given by $f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(u)\, e^{-iux}\, du$

$= \frac{4}{2\pi}\int_{-\infty}^{\infty} \frac{-u\cos u + \sin u}{u^3}\, e^{-iux}\, du$

$$\left\{\begin{array}{l} e^{i\theta} = \cos\theta + i\sin\theta \\ e^{i\theta/2} = \cos\theta/2 + i\sin\theta/2 \end{array}\right\}$$

put $x = 1/2$.

$$f(1/2) = \frac{12}{2\pi}\int_{-\infty}^{\infty} \frac{-u\cos u + \sin u}{u^3} e^{-iu/2} \cdot du$$

$$= 3/4 = \frac{2}{\pi}\int_{-\infty}^{\infty} \frac{-u\cos u + \sin u}{u^3} \times e^{-iu/2} \cdot du \qquad \left\{\begin{array}{l} f(x) = 1 - x^2 \\ f(1/2) = 1 - 1/4 \\ \quad = 3/4 \end{array}\right.$$

$$\frac{3\pi}{16} = \int_0^{\infty} \frac{-u\cos u + \sin u}{u^3} \times e^{-iu/2} \cdot du.$$

$$\frac{3\pi}{16} = \int_0^{\infty} \frac{-u\cos u + \sin u}{u^3} \cdot \left(\cos\frac{u}{2} - i\sin\frac{u}{2}\right) du$$

equating the real part on B.S

$$\frac{3\pi}{16} = \int_0^{\infty} \frac{-u\cos u + \sin u}{u^3} \cos\left(\frac{u}{2}\right) \cdot du.$$

put $u = x$.

$$\frac{-3\pi}{16} = \int_0^{\infty} \frac{x\cos x - \sin x}{x^3} \cos x/2 \; dx \,!.$$

3: Obtain Fourier cosine transform of $\frac{1}{1+x^2}$

$$F_c(u) = \int_0^{\infty} f(x) \cos ux \cdot dx$$

$$F_c(u) = I = \int_0^{\infty} \frac{1}{1+x^2} \cdot \cos ux \cdot dx \rightarrow (1)$$

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Differentiating w.r.t to u. treating x as constant we get.

$$\frac{dI}{du} = \int_0^\infty \frac{1}{1+x^2} (-\sin ux)(x) \cdot dx$$

$$= -\int_0^\infty \frac{x \cdot \sin ux}{1+x^2}\, dx$$

X and ÷ by x on R.H.S.

$$= -\int \frac{x^2}{x(1+x^2)} \sin ux \cdot dx$$

$$= -\int \frac{1+x^2-1}{x(1+x^2)} \sin ux \cdot dx$$ + and − by 1

$$= -\int \frac{(1+x^2)}{x(1+x^2)} \sin ux \cdot dx + \int \frac{\sin ux}{x(1+x^2)} \cdot dx.$$

$$= -\int_0^\infty \frac{\sin ux}{x} dx + \int \frac{\sin ux}{x(1+x^2)} \cdot dx.$$

$$= -\pi/2 + \int \frac{\sin ux}{x(1+x^2)} \cdot dx$$ $\pi/2 \rightarrow$ known value.

again diff w.r.t. u. treatin x as cons

$$\frac{d^2I}{du^2} = -0 + \int_0^\infty \frac{\cancel{x} \cos ux}{\cancel{x}(1+x^2)} \cdot dx$$

$$\frac{d^2I}{du^2} = I \left\{ \text{by } \{1\} \right\}$$

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$$\frac{d^2I}{du^2} - I = 0.$$

$(D^2-1)I = 0.$ $\quad D = d/du.$

$m^2 - 1 = 0$

$m = \pm 1$

$$\boxed{I = c_1 e^{u} + c_2 e^{-u}}$$

4. Find the Fourier cosine transform of e^{-x^2}

$$F_c(u) = \int_0^\infty f(x) \cos ux \, dx.$$

$$F_c(u) = I = \int_0^\infty e^{-x^2} \cos ux \, . dx \longrightarrow (1)$$

differentiating w.r.t u, under the integral sign treating x as constant.

$$\frac{dI}{du} = -\int_0^\infty e^{-x^2} \sin ux \, . x \, . dx$$

$$= -1/2 \int_0^\infty 2x e^{-x^2} . \sin ux \, . dx$$

$$= 1/2 \int_0^\infty (-2x e^{-x^2}) \sin ux \, dx.$$

Integrate by parts.

$$\frac{dI}{du} = \frac{1}{2}\left[\sin ux (e^{-x^2}) - \int (e^{-x^2}) \cos ux \, . u dx \right]_0^\infty$$

Consider $\int (-2x e^{-x^2}) . dx$

Put $-x^2 = t$

$-2x . dx = dt$

$= \int e^{t} dt = e^t = e^{-x^2}$

$$= -1/2 \left[uI \right] \quad \text{from } (1).$$

$e^{-\infty} = 0$

$\sin 0 = 0$

$$\frac{dI}{du} = -\frac{uI}{2}.$$

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$$\int \frac{dI}{I} = -\int \frac{u\,du}{2}.$$

$$\log I = -\frac{1}{2}\,\frac{u^2}{2} + \log c.$$

$$\log \frac{I}{c} = \frac{-u^2}{4}.$$

$$\frac{I}{c} = e^{-\frac{u^2}{4}}$$

$$\boxed{I = c e^{-u^2/4}}$$

1. find the Fourier sine transform of $e^{-|x|}$ Hence S.T

$$\int_0^\infty \frac{x \sin mx}{1+x^2} \cdot dx = \frac{\pi e^{-m}}{2}$$

$f(x) = e^{-|x|}$

$$F_s(u) = \int_0^\infty f(x) \cdot \sin ux \cdot dx$$

$$= \int_0^\infty e^{-|x|} \sin ux \cdot dx$$

$$= \int_0^\infty e^{-x} \cdot \sin ux \cdot dx$$

$$= \frac{e^{-x}}{1^2+u^2} \left[-\sin ux - u \cos ux \right]_0^\infty \quad a=-1, b=u$$

$$= \frac{-1}{1+u^2} [0 - u]$$

$e^{-\infty} = 0$

$$F_s(u) = \frac{u}{1+u^2}$$

using inverse sine transform

$$f(x) = \frac{2}{\pi} \int_0^\infty F_s(u) \sin ux \, du$$

$$= \frac{2}{\pi} \int_0^\infty \frac{u}{1+u^2} \cdot \sin ux \cdot du$$

$x \to m$

$$f(m) = \frac{2}{\pi} \int_0^\infty \frac{u}{1+u^2} \sin um \cdot du$$

$u \to x$

$$e^{-m} = \frac{2}{\pi} \int_0^\infty \frac{x}{1+x^2} \cdot \sin mx \, dx$$

$$\therefore \frac{\pi e^{-m}}{2} = \int_0^\infty \frac{x \sin mx}{1+x^2} \cdot dx$$

2. Find the Fourier sine transform of $\frac{e^{-ax}}{x}$.

$$F_s(u) = \int_0^\infty f(x) \cdot \sin ux \cdot dx \qquad f(x) = \frac{e^{-ax}}{x}.$$

$$= \int_0^\infty \frac{e^{-ax}}{x} \cdot \sin ux \cdot dx.$$

Differentiating w.r.t u under the integral sign using libtnz rule

$$\frac{d}{du} F_s(u) = \int_0^\infty \frac{e^{-ax}}{\cancel{x}} \cos ux \cdot (\cancel{x})\, dx.$$

$$= \int_0^\infty e^{-ax} \cdot \cos ux \cdot dx.$$

$$= \frac{e^{-ax}}{a^2+u^2} \left[-a\cos ux + u \sin ux \right].$$

$a \to -a, \; b \to u.$

$$= \frac{1}{a^2+u^2} \left[e^{-ax} \cdot (-a\cos ux + u \sin ux) \right]_0^\infty$$

$$= \frac{1}{a^2+u^2} \left[0 - (-a + 0) \right].$$

$$\frac{d}{du} f_s(u) = \frac{a}{a^2+u^2} \;//$$

Integrating B.S w.r.t. u.

$$F_s(u) = \alpha \cdot \frac{1}{\alpha} \tan^{-1} u/a + c.$$

$$F_s(u) = \tan^{-1} u/a + c.$$

3. Find the inverse Fourier sine transform of $F_s(u) = \frac{e^{-au}}{u}$.

$$f(x) = \frac{2}{\pi}\int_0^\infty F_s(u) \sin ux \cdot du$$

$$= \frac{2}{\pi}\int_0^\infty \frac{e^{-au}}{u} \cdot \sin ux \; du$$

Diff w.r.t x: treating u as constant

$$\frac{d}{dx} F_s(u) = \frac{2}{\pi}\int_0^\infty \frac{e^{-au}}{\cancel{u}} \cdot \cos ux \, \cancel{(u)} \cdot du$$

$$= \frac{2}{\pi}\int_0^\infty e^{-au} \cos ux \cdot du$$

$a = -a, \quad b = x.$

$$\frac{2}{\pi}\left[\frac{e^{-au}}{a^2+x^2}\left[-a\cos ux + x \sin ux\right]\right]_0^\infty$$

$$= \frac{2}{\pi}\left[0 - \left[-a + 0\right]\right]$$

$$= \frac{2}{\pi}\frac{a}{a^2+x^2}$$

$$\frac{d}{dx} F_s(u) = \frac{2}{\pi}\frac{a}{a^2+x^2}$$

Integrating w.r.t. x.

$$F_s(u) = \frac{2}{\pi}\,\cancel{a}\,\frac{1}{\cancel{a}}\tan^{-1}\frac{x}{a}$$

$$F_s(u) = \frac{2}{\pi}\tan^{-1} x/a \;//$$

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Z-transform

If $x(n)$ is a sequence, then z transform of $x(n)$ is defined as

$$Z[x(n)] = \sum_{n=0}^{\infty} x(n) \cdot z^{-n} = X(z)$$

z-transform of some standard functions.

① $$z(a^n) = \sum_{n=0}^{\infty} a^n \cdot z^{-n}$$

$$= \sum_{n=0}^{\infty} \left(\frac{a}{z}\right)^n \quad n = \{1, 2, 3 \ldots$$

$$= 1 + \frac{a}{z} + \frac{a^2}{z^2} + \frac{a^3}{z^3} + \ldots$$

G.P. $\{1, r, r^2, r^3 \ldots$
$= \frac{1}{1-r}$

$$= \frac{1}{1 - \frac{a}{z}}$$

$$= \frac{1}{\frac{z-a}{z}}$$

$$z(a^n) = \frac{z}{z-a}$$

$$z(1) = \frac{z}{z-1}.$$

2. S.T. $Z(n^p) = -z \frac{d}{dz} Z(n^{p-1})$ where p is a +ve integer

We've,

$$Z(n^p) = \sum_{n=0}^{\infty} n^p z^{-n} \rightarrow ①$$

$$= \sum n \, n^{p-1} z \; z^{-(n+1)}$$

$$Z(n^p) = z \sum n \, n^{p-1} \cdot z^{-(n+1)} \rightarrow ②$$

conside $Z(n^{p-1})$

$$= \sum_{n=0}^{\infty} n^{p-1} z^{-n}$$

differenciating w.r.t z.

$$\frac{d}{dz} Z(n^{p-1}) = \sum n^{p-1} (-n) z^{-n-1}$$

$$= -\sum n \, n^{p-1} z^{-(n+1)} \rightarrow ③$$

using ③ in ② $\Rightarrow$

$$\boxed{Z(n^p) = -z \frac{d}{dz} Z(n^{p-1})}$$

3. Find the z-transform of n, n^2, n^3

$$Z(n^p) = -z \frac{d}{dz} Z(n^{p-1})$$

① $Z(n) = -z \frac{d}{dz} Z(n^{1-1})$

$$= -z \frac{d}{dz} Z(1)$$

$$= -z \frac{d}{dz} \left[\frac{z}{z-1} \right]$$

$$= -z \left[\frac{(z-1) - z}{(z-1)^2} \right]$$

$$= -z \; \frac{\cancel{z}-1-\cancel{z}}{}$$

$$-z\frac{[\cancel{z}-1-\cancel{z}]}{(z-1)^2}$$

$$\boxed{Z(n) = \frac{z}{(z-1)^2}}$$

2. $Z(n^2) = -z\frac{d}{dz} Z(n^{2-1})$

$$= -z\frac{d}{dz} Z(n)$$

$$= -z\frac{d}{dz}\left[\frac{z}{(z-1)^2}\right]$$

$$= -z\,\frac{(z-1)^2 - z\,2(z-1)}{(z-1)^4}$$

$$= -z\left[\frac{(z-1) - 2z}{(z-1)^3}\right]$$

$$= -z\left[\frac{-1-z}{(z-1)^3}\right]$$

$$\boxed{Z(n^2) = \frac{z^2+z}{(z-1)^3}}$$

3. $Z(n^3) = -z\frac{d}{dz} Z(n^2)$

$$= -z\frac{d}{dz}\left[\frac{z^2+z}{(z-1)^3}\right]$$

$$= -z\,\frac{(z-1)^3(2z+1) - (z^2+z)3(z-1)^2}{(z-1)^6}$$

$$= -z\,\frac{(z-1)(2z+1) - 3(z^2+z)}{(z-1)^4}$$

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$$= -z\left[\frac{z-1+2z^2-2z-3z-3z^2}{(z-1)^4}\right].$$

$$= -z\left[\frac{-z^2-4z-1}{(z-1)^4}\right]$$

$$Z(n^3) = \left[\frac{z^3+4z^2+z}{(z-1)^4}\right]$$

4. find the Z transform of $\cos n\theta$ & $\sin n\theta$

consider Z-transform of $(e^{-in\theta})$

$$= Z(e^{-in\theta}) = Z(e^{-i\theta})^n.$$

$$= Z(\cos n\theta - i\sin n\theta) = \frac{z}{z-e^{-i\theta}}$$

$\times$ & $\div$ by $z-e^{i\theta}$

$$= \frac{z(z-e^{i\theta})}{(z-e^{-i\theta})(z-e^{i\theta})}$$

$$= \frac{z[z-(\cos\theta+i\sin\theta)]}{z^2-ze^{i\theta}-ze^{-i\theta}+e^{i\theta}e^{-i\theta}}$$

$$= \frac{z[z-\cos\theta-i\sin\theta]}{z^2-z[e^{i\theta}+e^{-i\theta}]+1}$$

$$= \frac{z(z-\cos\theta)-iz\sin\theta}{z^2-2z\cos\theta+1}$$

~~If~~ · Z - Transforms

If $x(n)$ is a sequence where $n = 0, 1, 2, \ldots$ then z-transform of $x(n)$ is defined as

$$Z[x(n)] = \sum x(n) z^{-n}.$$

$Z(\cos n\theta - i \sin n\theta) =$

$$\frac{z(z-\cos\theta)}{z^2 - 2z\cos\theta + 1} - \frac{i z \sin\theta}{z^2 - 2z\cos\theta + 1}.$$

equating the real and imaginary part

$$\boxed{Z(\cos n\theta) = \frac{z(z-\cos\theta)}{z^2 - 2z\cos\theta + 1}}$$

$$\boxed{Z(\sin n\theta) = \frac{z \sin\theta}{z^2 - 2z\cos\theta + 1}}$$

Damping Rule

If $Z\{x(n)\} = X(z)$ then

$Z\{a^{-n} x(n)\} = X(az)$

and $Z\{a^n x(n)\} = X(z/a)$

1. Find the Z transform of na^n

$Z(na^n) = Z(a^n n)$

$= Z(n)_{z \to z/a}$

$$= \left[\frac{z}{(z-1)^2}\right]_{z \to z/a}$$

$$= \left[\frac{z/a}{(z/a - 1)^2}\right]$$

$$= \frac{z/a}{\frac{(z-a)^2}{a^2}}$$

$$\boxed{Z(na^n) = \frac{az}{(z-a)^2}}$$

lly $\quad Z(a^n n^2) = \dfrac{az^2 + a^2 z}{(z-a)^3}$

$$\boxed{Z(a^n \cos n\theta) = \frac{z(z - a\cos\theta)}{z^2 - 2az\cos\theta + a^2}}$$

$$Z(a^n \sin n\theta) = \frac{az\sin\theta}{z^2 - 2az\cos\theta + a^2}$$

$x(n)$	$X(z)$
a^n	$z/z-a$
1	$z/z-1$
$\delta(n)$	1
n	$\frac{z}{(z-1)^2}$
n^2	$\frac{z(z+1)}{(z-1)^3}$
$\cos n\theta$	$\frac{z(z-\cos\theta)}{z^2-2z\cos\theta+1}$
$\sin n\theta$	$\frac{z\sin\theta}{z^2-2z\cos\theta+1}$

1. Find the Z transform of $Z[(n+1)^2]$

$$= Z[(n+1)^2]$$

$$= Z[n^2+1+2n]$$

$$= Z(n^2) + Z(1) + 2Z(n)$$

$$= \frac{z(z+1)}{(z-1)^3} + \frac{z}{z-1} + 2\frac{z}{(z-1)^2}$$

$$= \frac{z(z+1) + z(z-1)^2 + 2z(z-1)}{(z-1)^3} \qquad = \frac{z^3+z^2}{(z-1)^3}$$

$$= \frac{z^2+z+z^3+\cancel{z}-\cancel{2z^2}+\cancel{2z^2}-2z}{(z-1)^3}$$

2. Find the $Z\left[\sin\left(\frac{n\pi}{4}+\alpha\right)\right]$

$$= Z\left[\sin\frac{n\pi}{4}\cos\alpha + \cos\frac{n\pi}{4}\sin\alpha\right]$$

$$= \cos\alpha\, Z\left[\sin\frac{n\pi}{4}\right] + \sin\alpha\, Z\left[\cos\frac{n\pi}{4}\right]$$

$$\begin{cases} Z(\cos n\theta) = \dfrac{z(z-\cos\theta)}{z^2-2z\cos\theta+1} \\ Z(\sin n\theta) = \dfrac{z\sin\theta}{z^2-2z\cos\theta+1} \end{cases}$$

$$= \cos\alpha \cdot \frac{z\,\cancel{(z-\cos)}\sin\pi/4}{z^2-2z\cos\pi/4+1} + \sin\alpha \cdot \frac{z(z-\cos\pi/4)}{z^2-2z\cos\pi/4+1}$$

$$= \cos\alpha + \frac{z\frac{1}{\sqrt{2}}}{z^2+2z\frac{1}{\sqrt{2}}+1} + \sin\alpha\left[\frac{z(z-1/\sqrt{2})}{z^2-2z\frac{1}{\sqrt{2}}+1}\right]$$

$$\frac{z}{\sqrt{2}}\cdot\left[\frac{\cos\alpha+\sin\alpha(\sqrt{2}z-1)}{z^2-\sqrt{2}z+1}\right] \quad //$$

3. Find the $Z\left[\cosh\left(\frac{n\pi}{2}+\alpha\right)\right]$

$$= Z\left[\frac{e^{(n\pi/2+\alpha)}+e^{-(n\pi/2+\alpha)}}{2}\right] \qquad \frac{e^{\theta}+e^{-\theta}}{2}$$

$\frac{1}{2} Z \left[e^{\alpha}(e^{i\pi/2})^n + e^{-\alpha}(e^{-i\pi/2})^n \right]$

$= \cancel{\#} \frac{1}{2} \cdot e^{\alpha} Z\left[e^{i\pi/2}\right]^n + \frac{1}{2} e^{-\alpha} Z\left[e^{-i\pi/2}\right]^n.$

$= \frac{1}{2} e^{\alpha} \frac{z}{z - e^{i\pi/2}} + \frac{1}{2} e^{-\alpha} \frac{z}{z - e^{-i\pi/2}}$ //

4. Find the $Z[e^{bn} n^2]$

$\left\{ \begin{array}{l} \text{if } Z\{x(n)\} = X(z) \\ Z(a^{-n}x(n)) = X(az) \end{array} \right\}$

$Z[e^{bn} n^2]$

$Z[(e^{-b})^{-n} n^2]$

$x(n) = n^2$

$Z[x(n)] = \frac{z^2+z}{(z-1)^3}$

$\therefore Z[e^{bn} n^2] = \left[\frac{z^2+z}{(z-1)^3}\right]_{z \to e^{-b}z}$

$= \frac{[e^{-b}z]^2 + e^{-b}z}{(e^{-b}z - 1)^3}$

$= \frac{\frac{z^2}{e^{2b}} + \frac{z}{e^{b}}}{(z - e^{b})^3 / e^{3b}}$

$\cancel{e^2} \quad \frac{z^2 + ze^b}{e^{2b}} \Big/ (z - e^b)^3 / e^{3b}$

$$\frac{e^{b}[z^{2}+ze^{b}]}{(z-e^{b})^{3}}$$

5. $Z[e^{-ak}\cos bk]$

$k \to n$.

$$Z[e^{-an}\cos bn] = Z[(e^{a})^{-n}\cos bn]$$

$$x(n) = \cos bn$$

$$X(z) = \frac{z(z-\cos b)}{z^{2}-2z\cos b+1}$$

$$\therefore Z[e^{-bn}\cos bn] = \left[\frac{z(z-\cos b)}{z^{2}-2z(\cos b)+1}\right]$$

$$= \frac{e^{a}z(e^{a}z-\cos b)}{(e^{a}z)^{2}-2e^{a}z(\cos b)+1}$$

$U(z)$

$F(z)^{n}$

Inverse z - transforms by partial fractions

1. Find the inverse z - transform of

$$x(z) = \frac{z}{(z-2)(z-3)}.$$

$$\frac{X(z)}{z} = \frac{1}{(z-2)(z-3)} \rightarrow (1)$$

$$= \frac{A}{(z-2)} + \frac{B}{(z-3)}$$

$$\frac{1}{(z-2)(z-3)} = \frac{A(z-3)+B(z-2)}{(z-2)(z-3)}$$

Put $z=2$ | $z=3$

$\boxed{A=-1}$ | $\boxed{B=1}$

(1) ⇒

$$\frac{X(z)}{z} = \frac{-1}{(z-2)} + \frac{1}{z-3}$$

$$\therefore X(z) = -\frac{z}{z-2} + \frac{z}{z-3}$$

taking z^{-1} on B.S

$$\therefore z^{-1}(X(z)) = -z^{-1}\left[\frac{z}{z-2}\right] + z^{-1}\left[\frac{z}{z-3}\right]$$

$$\boxed{x(n) = -2^n + 3^n}$$

2. Find the inverse z - transform of

$$\frac{z^2}{(z-1)(z-1/2)}$$

$$X(z) = \frac{z^2}{(z-1)(z-1/2)}$$

$$\frac{X(z)}{z} = \frac{z}{(z-1)(z-1/2)} \rightarrow (1)$$

$$\frac{z}{(z-1)(z-1/2)} = \frac{A}{z-1} + \frac{B}{z-1/2}.$$

$$z = A(z-1/2) + B(z-1)$$

put $z=1 \Rightarrow A$

$$1 = A(1-1/2) \Rightarrow \boxed{A=2}$$

put $z = 1/2$.

$$1/2 = B(1/2-1)$$

$$\therefore \boxed{B=-1}$$

$$(1) \Rightarrow \frac{X(z)}{z} = \frac{2}{z-1} - \frac{1}{z-1/2}.$$

$$X(z) = \frac{2z}{z-1} - \frac{z}{z-1/2}.$$

$$z^{-1}[X(z)] = 2z^{-1}\left[\frac{z}{z-1}\right] - z^{-1}\left[\frac{z}{z-1/2}\right]$$

$$\boxed{x(n) = 2(1) - (1/2)^n}$$

3. $X(z) = \frac{z^3}{(z-3)(z-2)^2} \rightarrow$ Improper

$$\frac{X(z)}{z} = \frac{z^2}{(z-3)(z-2)^2} \rightarrow \text{proper}$$

$$\frac{z^2}{(z-3)(z-2)^2} = \frac{A}{z-3} + \frac{B}{z-2} + \frac{C}{(z-2)^2}$$

$$\frac{z^2}{(z-3)(z-2)^2} = \frac{A(z-2)^2 + B(z-3)(z-2) + C(z-3)}{(z-3)(z-2)^2}$$

$$z^2 = A(z-2)^2 + B(z-3)(z-2) + C(z-3).$$

put $z=2$, $4 = -C$ $\therefore \boxed{C=-4}$

Put $z=3$

$\boxed{9 = A}$

Put $z=0$.

$0 = 4A + 6B - 3C$

$0 = 4(9) + 6B - 3(-4)$

$0 = 6B + 48$

$6B = -48$

$\boxed{B=-8}$

$$\frac{X(z)}{z} = \frac{z^2}{(z-3)(z-2)^2} = \frac{9z}{z-3} - \frac{8z}{z-2} - \frac{4}{(z-2)^2}$$

$$z^{-1}(X(z)) = 9\,z^{-1}\left(\frac{z}{z-3}\right) - 8\,z^{-1}\left(\frac{z}{z-2}\right) - 2\,z^{-1}\left(\frac{2z}{(z-2)^2}\right)$$

$$x(n) = 9(3^n) - 8(2)^n - 2n2^n$$

4. $\dfrac{18z^2}{(2z-1)(4z+1)}$

$$\frac{X(z)}{z} = \frac{18z}{(2z-1)(4z+1)}$$

$$\frac{18z}{(2z-1)(4z+1)} = \frac{A}{(2z-1)} + \frac{B}{(4z+1)}$$

$$18z = A(4z+1) + B(2z-1)$$

Put $z = -1/4$

$$\therefore \boxed{B = 3}$$

Put $z = 1/2$

$$9 = 3A \quad \therefore \boxed{A = 3}$$

$$\frac{X(z)}{z} = \left(\frac{3}{2z-1}\right) + \frac{3}{(4z+1)}$$

$$x(n) = \frac{3}{2}\left[\frac{z}{z-1/2}\right] + \frac{3}{4}\left[\frac{z}{z+1/4}\right]$$

$$x(n) = \frac{3}{2}\left(1/2\right)^n + 3/4\left(-1/4\right)^n$$

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Solve the differential eq^n

$$y_{n+2} - 5y_{n+1} + 6y(n) = u(n), \quad y(0)=0$$
$$y(1)=1$$

$$z(y_{n+2}) - 5z(y_{n+1}) + 6z(y_n) = \frac{z}{z-1}$$

$$z^2X(z) - z^2y(0) - zy(1) - 5[zX(z) - zy(0)] + 6\cdot X(z) = \frac{z}{z-1}$$

$$z^2X(z) - z - 5zX(z) + 6X(z) = \frac{z}{z-1}$$

$$z^2X(z) - 5zX(z) + 6X(z) = \frac{z}{z-1} + z$$

$$X(z)[z^2 - 5z + 6] = \frac{z}{z-1} + z$$

$$X(z) = \frac{z}{(z-1)(z-2)(z-3)} + \frac{z}{(z-2)(z-3)}$$

$$\frac{X(z)}{z} = \frac{\cancel{1} + z - \cancel{1}}{(z-1)(z-2)(z-3)}$$

$$\frac{z}{(z-1)(z-2)(z-3)} = \frac{A}{(z-1)} + \frac{B}{(z-2)} + \frac{C}{(z-3)}$$

$$z = A(z-2)(z-3) + B(z-1)(z-3) + C(z-1)(z-2)$$

$z = 2$	$z = 1$	$z = 3$
$2 = B(-1)$	$1 = A(-2)(-2)$	$3 = C(2)$
$B = -2$	$A = 1/2$	$C = 3/2$

$$\frac{X(z)}{z} = \frac{1}{2(z-1)} + \frac{-2}{(z-2)} + \frac{3}{2(z-3)}$$

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$$y(n) = \frac{1}{2} - 2(2)^n + \frac{3}{2}(3)^n$$

$$= 1/2 - (2)^{n+1} + \frac{3^{n+1}}{2}.$$

$y_{n+2} - 4y_n = 0.$ $y_0 = 0,\ y_1 = 2.$

$Z(y_{n+2}) - 4Z(y_n) = 0$

$z^2 x(z) - z^2 y(0) - z y(1) - 4x(z) = 0.$

$z^2 x(z) - 2z - 4x(z) = 0.$

$z^2 x(z) - 4x(z) = 2z.$

$x(z)[z^2 - 4] = 2z$

$$x(z) = \frac{2z}{(z+2)(z-2)}$$

$$\frac{x(z)}{z} = \frac{2}{(z+2)(z-2)}$$

$2 = A(z-2) + B(z+2).$

$z = 2$	$z = -2$
$2 = B(4)$	$2 = A(-4).$
$B = 1/2.$	$A = -1/2.$

$$\frac{x(z)}{z} = -\frac{1}{2}\left[\frac{z}{z+2}\right] + \frac{1}{2}\left[\frac{z}{z-2}\right]$$

$y(n) = -1/2(-2)^n + 1/2(2)^n$

$= -(-2)^{n-1} + 2^{n-1}$ //

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1. Find the fourier transform of $f(x) = \begin{cases} 1 & \text{for } |x| < 1 \\ 0 & \text{for } |x| > 1 \end{cases}$

hence evaluate $\int_0^\infty \frac{\sin x}{x} \cdot dx$.

we've

$$F\{f(x)\} = \int_{-\infty}^{\infty} f(x) \cdot e^{iux} \cdot dx = F(u)$$

$$f(x) = \begin{cases} 1 \\ \end{cases}$$

done at the starting of fourier transform

2. Find the fourier transform of $f(x) = \begin{cases} 1-|x| & \text{for } |x| \leq 1 \\ 0 & \text{for } |x| > 1 \end{cases}$

and hence evaluate $\int_0^\infty \frac{\sin^2 t}{t^2} \cdot dt$.

$$F\{f(x)\} = \int_{-\infty}^{\infty} f(x) \cdot e^{iux} \cdot dx = F(u) \longrightarrow (1)$$

The given function is

$$f(x) = \begin{cases} 1-(-x) & \text{for } -1 \leq x \leq 0 \\ 1-(+x) & \text{for } 0 \leq x \leq 1 \\ 0 & \text{for } |x| \geq 1 \end{cases}$$

③ in ① ⟹

$$F\{f(x)\} = \int_{-1}^{0}(1+x)e^{iux}\,dx + \int_{0}^{1}(1-x)e^{iux}\,dx.$$

$$= \left[(1+x)\frac{e^{iax}}{iu} - (1)\frac{e^{iux}}{i^2u^2}\right]_{-1}^{0} +$$

$$\left[(1-x)\frac{e^{iux}}{iu} - (-1)\frac{e^{iux}}{(iu)^2}\right]_{0}^{1}$$

$$= \left\{\frac{1}{iu}\left[1-0\right] - \frac{1}{i^2u^2}\left[1-e^{-iu}\right]\right\} +$$

$$\left\{\frac{1}{iu}\left[0-1\right] + \frac{1}{u^2}\left[e^{iu}-1\right]\right\}$$

$$= \frac{-1}{u^2}\left[-1+e^{-iu} + e^{+iu} - 1\right]$$

$$= \frac{-1}{u^2}\left[e^{-iu} + e^{+iu} - 2\right]$$

$$= \frac{-1}{u^2}\left[2\cos u - 2\right]$$

$$= \frac{-2}{u^2}\left[\cos u - 1\right]$$

$$= \frac{2}{u^2}\left[1 - \cos u\right]$$

$$= \frac{2}{u^2}\left[2\sin^2\frac{u}{2}\right]$$

$$F(u) = \frac{4}{u^2}\sin^2 u/2$$

consider the inverse fourier transform given by

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(u) \cdot e^{-iux} \cdot du$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\frac{4}{u^2}\sin^2 u/2\right) \cdot e^{-iux} \cdot du.$$

Put $x = 0$

$$f(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{4}{u^2}\sin^2 u/2 \cdot du$$

$F(x) = \begin{cases} 1-|x| & -1 \le x \le 1 \\ 0 & |x| > 1 \end{cases}$

$f(0) \Rightarrow$
$1-|x|$
$1-0 = 1$

$\frac{4}{u^2} = \frac{1}{\left(\frac{u}{2}\right)^2}$

Put $u/2 = t$

$du = 2\,dt$

$$1 = \frac{1}{\cancel{2}\pi}\int_{-\infty}^{\infty}\frac{\sin^2 t}{t^2} \cdot \cancel{2}\,dt$$

$$\pi = 2 \times \int_{0}^{\infty}\frac{\sin^2 t}{t^2} \cdot dt$$

$$\frac{\pi}{2} = \int_{0}^{\infty}\frac{\sin^2 t}{t^2} \cdot dt.$$

2. Find the fourier transform of $e^{-|x|}$

$$F\{f(x)\} = \int_{-\infty}^{\infty} f(x) \cdot e^{iux}\,dx.$$

$$= \int_{-\infty}^{\infty} e^{-|x|} \cdot e^{iux} \cdot dx.$$

$$= \int_{-\infty}^{0} e^{-(-x)} e^{iux} \cdot dx + \int_{0}^{\infty} e^{-(x)} e^{iux} \cdot dx$$

$$= \int_{-\infty}^{0} e^{x} \cdot e^{iux} \cdot dx + \int_{0}^{\infty} e^{-x} \cdot e^{iux} \cdot dx.$$

$$= \int_{-\infty}^{0} e^{(1+iu)x} \cdot dx + \int_{0}^{\infty} e^{-(1-iu)x} \cdot dx.$$

$$= \left.\frac{e^{(1+iu)x}}{1+iu}\right]_{-\infty}^{0} + \left.\frac{e^{-(1-iu)}}{-(1-iu)}\right]_{0}^{\infty}$$

$$= \frac{1}{1+iu}[1-0] - \frac{1}{1-iu}[0-1]$$

$$= \frac{1}{1+iu} + \frac{1}{1-iu}$$

$$= \frac{2}{1+u^2}$$

$$f(x) = \cancel{x}e^{-|x|}$$

8. Find the fourier sine and cosine transform of $F(x) = \begin{cases} x & 0 < x < 2 \\ 0 & \text{elsewhere.} \end{cases}$

$$F_c(u) = \int_{0}^{\infty} f(x) \cos ux \cdot dx$$

$$= \int_{0}^{2} x \cos ux \cdot dx.$$

$$= \left[(x)\left(\frac{\sin ux}{u}\right) - (1)\left(-\frac{\cos ux}{u^2}\right)\right]_0^2$$

$$= \left[\frac{2\sin 2u}{u} - 0\right] - \frac{1}{u^2}\left[\cos 2u - 1\right]$$

$$= \frac{2u\sin 2u + \cos 2u - 1}{u^2}$$

$$F_c(u) = \frac{2u\sin 2u - 2\sin^2 u}{u^2}$$

$$F_c(u) = \int_0^\infty f(x)\sin ux \cdot dx$$

$$= \int_0^2 x\sin ux \cdot dx$$

$$= \left[x\left[\frac{-\cos ux}{u}\right] - (1)\left[\frac{-\sin ux}{u^2}\right]\right]_0^2$$

$$= \left[\frac{-2\cos u^2}{u} - 0\right] + \frac{1}{u^2}\left[\sin 2u - 0\right]$$

$$= \frac{1}{u}\left[-2\cos 2u\right] + \frac{1}{u^2}\left[\sin 2u - 0\right]$$

$$F_s(u) = \frac{-2u\cos 2u + \sin 2u}{u^2}$$

4: Find the fourier cosine transform of the function

$$F(x) = \begin{cases} 4x & 0 < x < 1 \\ 4 - x & 1 < x < 4 \\ 0 & x > 4 \end{cases}$$

$$F_c(u) = \int_0^{\infty} f(x) \cos ux \, dx.$$

$$= \int_0^1 (4x) \cos ux \cdot dx + \int_1^4 (4-x) \cos ux \cdot dx$$

$$= 4\left[(x)\left(\frac{\sin ux}{u}\right) - 1\left(-\frac{\cos ux}{u^2}\right)\right]_0^1 +$$

$$\left[(4-x)\left(\frac{\sin ux}{u}\right) - (-1)\left(-\frac{\cos ux}{u^2}\right)\right]_1^4$$

$$= 4\left[\frac{1}{u}(\sin u - 0) + \frac{1}{u^2}(\cos u - 1)\right] +$$

$$= \left[\frac{1}{u}(0 - 3\sin 3u) - \frac{1}{u^2}(\cos 4u - \cos u)\right]$$

$$= \frac{4u \sin u + \cos u - 1}{u^2} + \frac{-3u \sin 3u - \cos 4u + \cos u}{u^2}$$

MODULE - 3

Date
Page

-: STASTICAL AND NUMERICAL METHODS :-

Curve fitting by the method of least square

* To fit a straight line of the form $y = ax + b$.

$$y = ax + b \rightarrow (1).$$

* Find the normal equations as .

$$\left.\begin{aligned} \Sigma y &= a\Sigma x + nb \\ \Sigma xy &= a\Sigma x^2 + b\Sigma x \end{aligned}\right\} \text{ for } (1)$$

$\Sigma b = b\Sigma 1 \ldots\ldots n$.

→ Solving the above normal eq^n's we get the values of a & b .

→ Substituting the values of a & b in 1 , we get the bet fit straight line.

1. Fit a straight line for the following data by the method of least square . of the form $y = ax + b$.

x :	1	2	3	4	5
y :	4	5	7	8	10.

$$y = ax + b \rightarrow (1).$$

Normal equation .

$$\left.\begin{aligned} \Sigma y &= a\Sigma x + nb \\ \Sigma xy &= a\Sigma x^2 + b\Sigma x \end{aligned}\right\} \rightarrow (2)$$

x	y	xy	x^2
1	4	4	1
2	5	10	4
3	7	21	9
4	8	32	16
5	10	50	25
15	34	117	55

$34 = 15a + 5b$

$117 = 55a + 15b$

$a = 3/2 = \underline{1.5}$

$b = \underline{0.3}$ → (3)

Sub (3) in (1)

$y = ax + b$

$$\boxed{y = 1.5x + 0.3}$$

2. Fit a straight line of the form, $y = a + bx$ for the following data, by the method least squares

x : 0 0.2 0.4 0.6 0.8 1

y : 0.2326 0.3242 0.4248 0.6242 0.8226 0.9886.

$y = a + bx$. $n = 6$

$\Sigma y = na + b\Sigma x$.

$\Sigma xy = a\Sigma x + b\Sigma x^2$

x	y	xy	x^2
0	0.2326	0	0
0.2	0.3242	0.0648	0.04
0.4	0.4248	0.1699	~~0.0288~~ 0.16
0.6	0.6242	0.3745	0.36
0.8	0.8226	0.6580	0.64
1	0.9886	0.9886	1
3	3.417	2.2558	2.2

$3.417 = 6a + 3b$.

$2.2558 = 3a + 2.2b$

$a = 0.1785$

$b = 0.7818$

$$\boxed{y = 0.1785 + 0.7818x}$$

3. Fit a straight line of the form $y = A + Bx$

x : -3 -2 -1 0 1 2 3

y : 2 8 12 14 18 22 28.

& hence find y at $x = 4$.

$y = A + Bx$

$\Sigma y = na + b\Sigma x$

$\Sigma xy = a\Sigma x + b\Sigma x^2$

x	y	xy	x^2
-3	2	-6	9
-2	8	-16	4
-1	12	-12	1
0	14	0	0
1	18	18	1
2	22	44	4
3	28	84	9
0	104	112	28

$104 = 7a + 0b$

$112 = 0 + 28b$

$a = 14.85$

$b = 4$

$y = 14.85 + 4x$

at $x = 4$

$y = 14.85 + 16$

$\boxed{y = 30.85}$

To Fit a parabola :-

To fit a parabola of the form $y = ax^2 + bx + c$.

* $y = ax^2 + bx + c \longrightarrow (1)$
* Normal eq^{-n}

$\Sigma y = a\Sigma x^2 + b\Sigma x + nc \longrightarrow$ direct Σ

$\Sigma xy = a\Sigma x^3 + b\Sigma x^2 + c\Sigma x \longrightarrow \times$ by x & Σ

$\Sigma x^2y = a\Sigma x^4 + b\Sigma x^3 + c\Sigma x^2 \longrightarrow \times$ by x^2 & Σ.

Solving the above normal equations we get the values of a, b & c & hence substituting these values in 1 we get the required parabolas.

1. $y = ax^2 + bx + c$ for the following data & hence find the value of y at $x = 12$

x :	1	2	3	4	5	6	7	8
y :	2	4	6	8	4	2	1	6

$n = 8$

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II. $y = ax^2 + bx + c$.

$\Sigma y = a\Sigma x^2 + b\Sigma x + nc$.

$\Sigma xy = a\Sigma x^3 + b\Sigma x^2 + c\Sigma x$.

$\Sigma x^2y = a\Sigma x^4 + b\Sigma x^3 + c\Sigma x^2$

x	y	x^2	x^3	x^4	xy	x^2y
1	2	1	1	1	2	2
2	4	4	8	16	8	16
3	6	9	27	81	18	54
4	8	16	64	256	32	128
5	4	25	125	625	20	100
6	2	36	216	1296	12	72
7	1	49	343	2401	7	49
8	6	64	512	4096	48	384
36	33	204	1296	8772	147	805

$33 = 204a + 36b + 8c$

$147 = 1296a + 204b + 36c$

$805 = 8772a + 1296b + 204c$

$a = -0.1369$, $b = 1.196$, $c = 2.2321$.

$\therefore y = -0.1369x^2 + 1.1964x + 2.2321$.

at $x = 12$.

$\boxed{y = -3.1247}$

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2. Fit a parabola of the form
$y = a + bx + cx^2$

x	1	2	3	4	5	6	7	8	9	10	11	12
y	18	16	15	14	12	15	18	19	20	21	22	23

$y = a + bx + cx^2$

$\Sigma y = na + b\Sigma x + c\Sigma x^2$

$\Sigma xy = a\Sigma x + b\Sigma x^2 + c\Sigma x^3$

$\Sigma x^2 y = a\Sigma x^2 + b\Sigma x^3 + c\Sigma x^4$.

x	y	x^2	x^3	x^4	xy	x^2y
1	18	1	1	1	18	18
2	16	4	8	16	32	64
3	15	9	27	81	45	135
4	14	16	64	256	56	224
5	12	25	125	625	60	300
6	15	36	216	1296	90	540
7	18	49	~~49~~ 343	2401	126	882
8	19	64	512	4096	152	1216
9	20	81	729	6561	180	1620
10	21	100	1000	10000	210	2100
11	22	121	1331	14641	242	2662
12	23	144	1728	20736	276	3312
78	213	650	6084	60710	1487	~~10073~~ 13073

$213 = 12a + 78b + 650c$.

$1487 = 78a + 650b + 6084c$

$13073 = 650a + 6084b + 60710c$.

$a = 9.9745$, $b = 1.2065$, $c = -0.01237$

$y = 9.9745 + 1.2065x - 0.01237x^2$ //

(iii) To fit an exponential curve $y = ae^{bx}$

$y = ae^{bx} \longrightarrow (1)$

taking log on Bs.

$\log y = \log(a \cdot e^{bx})$.

$\log y = \log a + \log e^{bx}$.

$\log y = \log a + bx$

$\boxed{Y = A + bx} \longrightarrow (2)$

Where

$\log y = Y$

$\log a = A$ $\boxed{\therefore a = e^{A}}$

Now we write normal eq^{n} for eq^{n} (2).

$$\left.\begin{array}{l} \Sigma Y = n\cancel{\Sigma}A + b\Sigma x \\ \Sigma xY = A\Sigma x + b\Sigma x^2 \end{array}\right\} \ldots\ldots$$

Solving the above normal eq^{n}s we get the values of 'A' & 'b' & hence using ~~Acce~~ $\boxed{a = e^{A}}$ we get 'a'

Substitution the values of a, & b in (1) we get the best fit exponential curve.

1. Fit an exponential curve ~~for~~ of the form $y = ae^{bx}$ for the following data.

x :	2	4	6	8	10
y :	0.6	0.686	0.783	0.845	0.969

$y = ae^{bx}$

$\log y = \log a + \log e^{bx}$

$\log y = \log a + bx$

$Y = A + bx$ where $\boxed{a = e^{A}}$

$$\left.\begin{aligned} \Sigma Y &= nA + b\Sigma x \\ \Sigma xY &= A\Sigma x + b\Sigma x^2 \end{aligned}\right\}$$

x	y	$Y = \log y$	xY	x^2
2	0.6	−0.5108	−1.0216	4
4	0.686	−0.3768	−1.5072	16
6	0.783	−0.2446	−1.4676	36
8	0.848	−0.1648	−1.3184	64
10	0.969	−0.0314	−0.314	100
30	3.886	−1.3284	−5.6288	220

$\cancel{3.886}\ -1.3284 = 5A + 30b$

$-5.6288 = 30A + 220b$

~~A = 5.1169, b = −0.7236~~

$\therefore$ ~~$a = e^{A}$~~ $= e^{5.1189}$

~~$a = 167.1814$~~

$\therefore Y = \quad A = -0.6169 \quad \therefore a = e^{A} = 0.539$

$\boxed{b = 0.0585} \quad \boxed{a = 0.539}$

$\therefore y = ae^{bx} \Rightarrow 0.539e^{0.058x}$

2. Fit an exponential curve of the curve $y = \lambda e^{\mu x}$ for the following data.

x:	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8
y:	2	3	4	6	8	12	14	15

$y = \lambda e^{\mu x}$

$\log y = \log \lambda + \mu x$

$Y = A + \mu x$

$\Sigma Y = nA + \mu \Sigma x$ where $a = e^A$

$\Sigma xY = A \Sigma x + \mu \Sigma x^2$

x	y	$Y = \log y$	x^2	xY	xY
0.1	2	0.6931	0.01	0.0301	0.069
0.2	3	1.0986	0.04	0.0954	0.22
0.3	4	1.3862	0.09	0.1806	0.416
0.4	6	1.7917	0.16	0.3112	0.717
0.5	8	2.0794	0.25	0.4515	1.04
0.6	12	2.4849	0.36	0.6475	1.49
0.7	14	2.6390	0.49	0.8022	1.847
0.8	15	2.7080	0.64	0.9408	2.166
3.6	64	14.8804	2.04		7.965

$14.889 = 8A + 3.6\mu$

$7.965 = 3.6A + 2.04\mu$

$A = 0.502$, $\mu = 3.019$

$\lambda = e^A = 1.652$

$y = 1.652 e^{3.019x}$

3. The voltage 'v' across a capacitor at time 't' seconds is given by a following table. Using the principle least squares, fit a curve of the form $v = ae^{kt}$ to the data

t	0	2	4	6	8
v	150	63	28	12	5.6

$$v = ae^{kt}$$

$$\log v = \log a + kt$$

$$V = A + kt \rightarrow (1)$$

$$\Sigma V = nA + k\Sigma t \quad \text{where } V = y$$

& $k = x$.

$$\Sigma tV = A\Sigma t + k\Sigma t^2$$

t	v	$V = \log v$	tV	t^2
0	150	5.010	0	0
2	63	4.1431	8.2862	4
4	28	3.3322	13.3288	16
6	12	2.4849	14.9094	36
8	5.6	1.7227	13.7821	64
20	258.6	16.6929	50.3065	120

$$16.6929 = 5A + 20k \qquad A = 4.9850$$

$$50.3065 = 20A + 120k \qquad k = -0.4116$$

$$\therefore a = e^{A} = e^{4.9850}$$

$$\boxed{a = 146.20}$$

$$\therefore V = 146.20e^{-0.4116t}$$

Date ______
Page ______

NOTE :-

Mean for raw data.

✓ ① $\bar{x} = \frac{\Sigma x}{n}$

Mean for grouped frequency data

② $\bar{x} = \frac{\Sigma f x}{N}$, $N = \Sigma f$

Variance denoted

③ $\sigma^2 = \frac{\Sigma (x - \bar{x})^2}{n}$ ✓

(or)

$\sigma^2 = \Sigma \frac{x^2}{n} - (\bar{x})^2$ ✓

- Standard deviation

$S.D = \sqrt{Variance}$

$SD = \sqrt{\sigma^2} = \sigma$

Correlation :- If two variables x and y are such that, if one variable increase or decrease, then the other variable also increase or decrease, then the variables are said to be correlated.

The correlation is measured throug the co-efficient known as karl-pearson's co-efficient denoted by 'r'

$\boxed{r = \frac{\Sigma XY}{n \, \sigma_x . \sigma_y}}$ where $X = x - \bar{x}$

$Y = y - \bar{y}$

$\bar{x}^2 = \frac{\Sigma x^2}{n} - (\bar{x})^2$

"Alternative formula for correlation co-efficient "r".

Show that $r = \dfrac{\sigma_x^2 + \sigma_y^2 - \sigma_{x-y}^2}{2\sigma_x \sigma_y}$ with usual notation.

Proof :-

let $z = x - y$. $\rightarrow$ ①

takin Σ & dividing by 'n', we get

$$\frac{\Sigma z}{n} = \frac{\Sigma x}{n} - \frac{\Sigma y}{n}$$

$$\bar{z} = \bar{x} - \bar{y} \rightarrow ②$$

consider ① − ②

$$z - \bar{z} = x - y - \bar{x} + \bar{y}$$

$$z - \bar{z} = (x - \bar{x}) - (y - \bar{y})$$

Squaring B.S, diving by n & taking Σ

$$= (z - \bar{z})^2 = [(x - \bar{x}) - (y - \bar{y})]^2$$

$$= \frac{(z - \bar{z})^2}{n} = \frac{(x - \bar{x})^2}{n} + \frac{(y - \bar{y})^2}{n} - \frac{2(x - \bar{x})(y - \bar{y})}{n}$$

$$= \frac{\Sigma(z - \bar{z})^2}{n} = \frac{\Sigma(x - \bar{x})^2}{n} + \frac{\Sigma(y - \bar{y})^2}{n} - \frac{\Sigma 2(x - \bar{x})(y - \bar{y})}{n}$$

$$\left\{ r = \frac{\Sigma XY}{n \sigma_x \sigma_y} \right\}$$

$\therefore$ $\sigma_z^2 = \sigma_x^2 + \sigma_y^2 - 2\sigma_x \sigma_y r$.

$$2\sigma_x \sigma_y r = \sigma_x^2 + \sigma_y^2 - \sigma_z^2.$$

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$$\therefore \boxed{r = \frac{\sigma_x^2 + \sigma_y^2 - \sigma_{x-y}^2}{2\,\sigma_x\,\sigma_y}}$$

1. Show that the value of correlation co-efficient does not exceed unity. or
S.T $-1 \le r \le 1$

consider $S = \frac{1}{2n}\sum\left[\frac{x}{\sigma_x} + \frac{y}{\sigma_y}\right]^2$ and.

$S' = \frac{1}{2n}\sum\left[\frac{x}{\sigma_x} - \frac{y}{\sigma_y}\right]^2$

Consider $S = \frac{1}{2n}\sum\left[\frac{x}{\sigma_x} + \frac{y}{\sigma_y}\right]^2$

From the above expression, It is clear that $\boxed{S \ge 0}$

ie. $\frac{1}{2n}\sum\left[\frac{x}{\sigma_x} + \frac{y}{\sigma_y}\right]^2 \ge 0$.

$= \frac{1}{2n}\sum\left[\frac{x^2}{\sigma_x^2} + \frac{y^2}{\sigma_y^2} + \frac{2xy}{\sigma_x\sigma_y}\right]$.

$= \frac{1}{2}\left[\frac{1}{\sigma_x^2}\sum\frac{x^2}{n} + \frac{1}{\sigma_y^2}\sum\frac{y^2}{n} + \frac{2}{\sigma_x\sigma_y}\sum xy\right] \ge 0$

$= \frac{1}{2}\left[\frac{1}{\cancel{\sigma_x^2}}\cancel{\sigma_x^2} + \frac{1}{\sigma_y^2}\cdot\sigma_y^2 + 2r\right] \ge 0$

$= \frac{1}{2}\left[1 + 1 + 2r\right] \ge 0$

$\frac{1}{\cancel{2}}\left[\cancel{2} + \cancel{2}r\right] \ge 0$

$\boxed{(1+r) \ge 0}$

$\boxed{r \ge -1}$ or $\boxed{-1 \le r} \rightarrow (1)$

Similarly by taking $S' = \frac{1}{2n}\sum\left[\frac{x}{\sigma_x} - \frac{y}{\sigma_y}\right]^2$
we can prove that

$\boxed{r \le 1} \rightarrow (2)$

$\therefore$ From ① and ② we get

$-1 \leq r \leq 1$

② Find the correlation co-efficient for the following data

x : 1 2 3 4 5 6

y : 2 3 5 7 8 9

$$r = \frac{\sigma_x^2 + \sigma_y^2 - \sigma_{x-y}^2}{2\sigma_x \sigma_y} \rightarrow ①$$

$$\sigma_x^2 = \frac{\Sigma x^2}{n} - (\bar{x})^2 \rightarrow ②$$

$$\bar{x} = \frac{\Sigma x}{n}$$

x	y	x^2	y^2	$(x-y)^2$	$x-y$
1	2	1	4	1	-1
2	3	4	9	1	-1
3	5	9	25	4	-2
4	7	16	49	9	-3
5	8	25	64	9	-3
6	9	36	81	9	-3
21	34	91	232	33	

$\bar{x} = \frac{\Sigma x}{n} = \frac{21}{6} = 3.5$

$\bar{y} = \frac{\Sigma y}{n} = \frac{34}{6} = 5.661$

$\sigma_x^2 = \frac{\Sigma x^2}{n} - (\bar{x})^2$

$= \frac{91}{6} - (3.5)^2$

$= 2.91$

$\sigma_y^2 = \frac{\Sigma y^2}{n} - (\bar{y})^2$

$= \frac{232}{6} - (5.66)^2$

$= \cancel{6.63}\ 6.63$

$\sigma_{(x-y)}^2 = \Sigma \frac{(x-y)^2}{n} - (\overline{x-y})^2$

$= \frac{36}{6} - (-2.16)^2$

$\sigma_{(x-y)}^2 = 0.83$

Date ____
Page ____

Subs all in ①

$$r = \frac{2.91 + 6.63 - 0.83}{2\sqrt{2.91} \times \sqrt{6.63}}$$

$$= \underline{0.99}$$

Here are x and y are fully correlated

③ Find the correlation co-efficient for the following data.

x:	2	4	5	7	9	10
y:	9	8	6	5	4	3

$$r = \frac{\sigma_x^2 + \sigma_y^2 - \sigma_{x-y}^2}{2\sigma_x \sigma_y} \rightarrow ①$$

$$\sigma_x^2 = \frac{\Sigma x^2}{n} - (\bar{x})^2 \rightarrow ②$$

$$\bar{x} = \frac{\Sigma x}{n}$$

x	y	x^2	y^2	$(x-y)^2$
2	9	4	81	49
4	8	16	64	16
5	6	25	36	1
7	5	49	25	4
9	4	81	16	25
10	3	100	9	49
37	35	275	231	144

$$\bar{x} = \frac{\Sigma x}{n} = \frac{37}{6} = \underline{6.16}$$

$$\bar{y} = \frac{\Sigma y}{n} = \frac{35}{6} = \underline{5.833}$$

$$\sigma_x^2 = \frac{\Sigma x^2}{n} - (\bar{x})^2$$

$$= \frac{275}{6} - (6.16)^2$$

$$= \underline{7.88}$$

$$\sigma_y^2 = \frac{\Sigma y^2}{n} - (\bar{y})^2$$

$$= \frac{231}{6} - (5.83)^2$$

$$= \underline{4.51}$$

$$\sigma_{(x-y)}^2 = \frac{\Sigma (x-y)^2}{n} - (\overline{x-y})^2$$

$$= \frac{144}{6} - (0.33)^2 = 23.89.$$

Here x and y are -vely Corelated.

Subs all in ① we get

$$r = \frac{7.88 + 4.51 - 23.89}{2\sqrt{7.88}\sqrt{4.51}}$$

$$r = -0.96$$

5. Find the correlation co-efficient for the data.

x: 2 4 6 8 10 12.

y: -3 -2 4 6 3 7.

$$r = \frac{\sigma_x^2 + \sigma_y^2 + \sigma_{x-y}^2}{2\,\sigma_x\,\sigma_y}.$$

$$\sigma_x^2 = \frac{\Sigma x^2}{n} - (\bar{x})^2. \quad \bar{x} = \frac{\Sigma x}{n}.$$

x	y	x^2	y^2	$(x-y)^2$
2	-3	4	9	25
4	-2	16	4	36.
6	4	36	16	4.
8	6	64	36	4.
10	3	100	9	49.
12	7.	144.	49.	25
42.	15	364.	123	143

$$\bar{x} = \frac{\Sigma x}{n} = \frac{42}{6} = 7$$

$$\bar{y} = \frac{\Sigma y}{n} = \frac{15}{6} = 2.5.$$

$$\sigma_x^2 = \frac{364}{6} - (7)^2 = 11.66.$$

$$\sigma_y^2 = \frac{123}{6} - (2.5)^2 = 14.25$$

$$\sigma_{(x-y)}^2 = \frac{143}{6} - (4.5)^2 = 3.58$$

$$r = \frac{11.66 + 14.25 - 3.58}{2\sqrt{11.66}\,\sqrt{14.25}} = 0.86$$

$$\boxed{r = 0.86}$$

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6. Regression :-

It is an estimation of one independent variable in terms of the other.

If x and y are co-related, the best fit straight line in the least square sense gives a good relation b/w x and y.

The best fit straight line of the form $y = ax+b$ [x being independent variable] is called the regression line of y on x and $x = ay+b$ [y being independent variable] is called regression line of x on y.

→ The line of regression of y on x is given by

$$(y-\bar{y}) = \frac{r\sigma_y}{\sigma_x}(x-\bar{x})$$

→ The line of regression of x on y is given by

$$(x-\bar{x}) = \frac{r\sigma_x}{\sigma_y}(y-\bar{y})$$

→ The lines of regressions are also given by

$$Y = \frac{\Sigma XY}{\Sigma X^2} X = \text{line of regression of } y \text{ on } x$$

$$X = (x-\bar{x}) \quad Y = (y-\bar{y})$$

$$X = \frac{\Sigma XY}{\Sigma Y^2} Y = \text{line of reg of } x \text{ on } y.$$

→ The co-efficient of co-relation 'r' is given by

$$r = \pm\sqrt{(\text{co-efficient of } x) \times (\text{co-efficient of } y)}$$

Date ____
Page ____

1. Compute the co-efficient of co-relation & the eq^n of lines of regression for the data.

x : 1 2 3 4 5 6 7

y : 9 8 10 12 11 13 14

$\bar{x} = \frac{\Sigma x}{n}$, $\bar{y} = \frac{\Sigma y}{n}$, $\overline{x-y} = \frac{\Sigma(x-y)}{7}$

x	y	x^2	y^2	$(x-y)$	$(x-y)^2$
1	9	1	81	-8	64
2	8	4	64	-6	36
3	10	9	100	-7	49
4	12	16	144	-8	64
5	11	25	121	-6	36
6	13	36	169	-7	49
7	14	49	196	-7	49
28	77	140	875	-49	347

$\bar{x} = \frac{\Sigma x}{n} = \frac{28}{7} = 4$

$\bar{y} = \frac{\Sigma y}{n} = \frac{77}{7} = 11$

$\overline{x-y} = \frac{\Sigma(x-y)}{n} = \frac{-49}{7} = -7$

$\sigma_x^2 = \frac{\Sigma x^2}{n} - (\bar{x})^2 = \frac{140}{7} - (4)^2 = 4$

$\sigma_y^2 = \frac{\Sigma y^2}{n} - (\bar{y})^2 = \frac{875}{7} - (77)^2 = 4$

$\sigma_{(x-y)}^2 = \frac{\Sigma(x-y)^2}{n} - (\overline{x-y}) = \frac{347}{7} - (-7) = 0.57$

$$r = \frac{4+4-0.59}{2(2)(2)}$$

$$\boxed{r = 0.926}$$

$$\boxed{r = 0.93}$$

y on x is

$$(y-\bar{y}) = r\frac{\sigma_y}{\sigma_x}(x-\bar{x})$$

$$y - 11 = 0.93\left(\frac{2}{2}\right)(x-4)$$

$$y = 11 + 0.93x - 3.72$$

$$\boxed{y = 0.93x + 7.28}$$ LOR of y on x.

$$(x-\bar{x}) = r\frac{\sigma_x}{\sigma_y}(y-\bar{y})$$

$$x - 4 = 0.93(2/2)(y-11)$$

$$x = 4 + 0.93y - 10.23$$

$$\boxed{x = 0.93y - 6.23}$$ LOR of x on y

2. Obtain the LOR & hence find the co-efficient of corelation for the data.

x :	1	2	3	4	5	6	7
y :	9	8	10	12	11	13	14

<u>Note</u>:- Since first we need to find lines of regeretion & hence 'r' we use the formula

$$Y = \frac{\Sigma XY}{\Sigma X^2} X \qquad X = x - \bar{x}$$

$$Y = y - \bar{y}$$

$$X = \frac{\Sigma XY}{\Sigma Y^2} Y$$

Date ______
Page ______

x	y	x^2 (X^2)	y^2 (Y^2)	X	Y	XY
1	9	9	4	-3	-2	6
2	8	4	9	-2	-3	6
3	10	1	1	-1	-1	1
4	12	0	1	0	1	0
5	17	1	0	1	0	0
6	13	4	4	2	2	4
7	14	9	9	3	3	9
		28	28			26

$X = (x - \bar{x}) = 1 - $

$\bar{x} = \frac{\Sigma x}{n} = \frac{28}{7} = 4$

$Y = (y - \bar{y}) =$

$\bar{y} = \frac{\Sigma y}{n} = \frac{77}{7} = 11$

$\left\{ Y = \frac{\Sigma XY}{\Sigma X^2} X \right.$

$(y - 11) = \frac{26}{28} (x - 4)$

$y - 11 = 0.93 (x - 4)$

$y = 11 + 0.93x - 3.72$

$y = 0.93x + 7.28$ // LOR of Y on x.

$X = \frac{\Sigma XY}{\Sigma Y^2} Y$

$(x - 4) = \frac{26}{28} (y - 11)$

$x - 4 = 0.93 (y - 11)$

$x = 4 + 0.93y - 10.23$

$\boxed{x = 0.93y - 6.23}$ // LOR of X on Y.

$r = \sqrt{0.93 \times 0.93} = \sqrt{0.93^2} = \boxed{r = 0.93}$ //

3. The foll date gives x & y from the 2^{nd} ... lines on x calculate the age of husband corresponding 16 years age of wife.

x : 36 23 27 28 28 29 30 31 33 [illegible]

y : 29 18 20 22 27 21 29 27 29 [illegible]

x	y	$x-y$	x^2	y^2	$(x-y)^2$
36	29	7	1291	841	49
23	18	5	529	324	25
27	20	7	729	400	49
28	22	6	784	484	36
28	27	1	~~78491~~	729	1
29	21	8	841 ~~900~~ 900	441	64
30	29	1	~~900~~	841	1
31	27	4	961 ~~1089~~	729	16
33	29	4	1089	841	16
35	28	7	1225	784	49
300	250	50	9138	6414	306

$$\bar{x} = \frac{\Sigma x}{N} = \frac{300}{10} = 30$$

$$\bar{y} = \frac{250}{10} = 25$$

$$\sigma_x^2 = \frac{\Sigma x^2}{n} - (\bar{x})^2 = \frac{9138}{10} - (30)^2$$

$$= 13.8$$

$$\sigma_y^2 = \frac{\Sigma y^2}{n} - (\bar{y})^2 = \frac{6414}{10} - (25)^2 = 16.4$$

$$\sigma_{(x-y)}^2 = \frac{306}{10} - (5) = 5.6$$

$$r = \frac{\sigma_x^2 + \sigma_y^2 - \sigma_{(x-y)}^2}{2\sqrt{\sigma_x}\sqrt{\sigma_y}} = 0.817$$

Date ____
Page ____

(01)

$(y-\bar{y}) = r\frac{\sigma y}{\sigma x}(x-\bar{x})$

$(y-25) = 0.817\sqrt{\frac{16.4}{13.8}}(x-30)$

$(y-25) = 0.817(1.09)(x-30)$

$y - 25 + 0.89x - 26.70.$

$\boxed{y = 0.89x - 1.7}$

$(x-\bar{x}) = r\frac{\sigma x}{\sigma y}(y-\bar{y})$

$(x-30) = 0.817\sqrt{\frac{3.8}{16.4}}(y-25).$

$\boxed{x = 0.74y + 11.42}$

$r = \pm\sqrt{0.89 \times 0.74}.$

$\boxed{r = 0.81}$

therefore at $y = 16$

② ⟹ $0.74 \times 16 + 11.42$

$\boxed{x = 23.26}$

4. psychological test of intelligence & engineering ability were applied to student. Here is a record of ungrouped data showing intelligence ratio [IR] & engineering ratio [ER]. calculate co-efficient of corelation

Students :-	A	B	C	D	E	F	G	H	I	J
IR (x) :-	105	104	102	101	100	99	98	96	93	92
ER (y) :-	101	103	100	98	95	96	104	92	97	94

x	y	x^2	y^2	$(x-y)$
105	101	11025	10201	16
104	103	10816	10609	1
102	100	10404	10000	4
101	98	10201	9604	9
100	95	10000	~~9604~~ 9025	25
99	96	9801	~~9218~~ 9216	9
98	104	9604	10816	36
96	92	9216	8464	16
93	97	8649	9409	16
92	94	8464	8536	4
990	980	~~87979~~ 98180	~~178882~~ 96180	136

$$r = \frac{\sigma_x^2 + \sigma_y^2 - \sigma_{x-y}^2}{2\sigma_x \sigma_y}$$

$$\bar{x} = \frac{\Sigma x}{n} = \frac{990}{10} = 99.$$

$$\bar{y} = \frac{\Sigma y}{n} = \frac{980}{10} = 98.$$

$$\sigma_x^2 = \frac{\Sigma x^2}{n} - (\bar{x})^2 = \frac{98180}{10} - (99)^2 = 17$$

$$\sigma_y^2 = \frac{\Sigma y^2}{n} - (\bar{y})^2 = 14.$$

$$\sigma_{x-y}^2 = \cancel{\frac{1}{10}(136)^2} = \cancel{18495.9}$$

$$\frac{136}{10} - (1)^2 = 12.6$$

$$r = \frac{17 + 14 - 12.6}{2\sqrt{17}\sqrt{14}} = \cancel{1.44}\ \underline{0.596}$$

Date ____
Page ____

5. Find the corelation co-efficient b/w x & y for the following data & find the line of gregression of y on x.

x:	1	2	3	4	5	6	7	8	9	10.
y:	10	12	16	28	25	36	41	49	40	50

x	y	x^2	y^2	$(x-y)^2$
1	10	1	100	81
2	12	4	144	100
3	16	9	256	169.
4	28	16	784	576
5	25	25	625	400
6	36	36.	1296	900
7	41	49	1681	1156
8	49	64	2401	1681
9	40	81	1600	961
10	50.	100	2500	1600
55	307	385	11387	7624

$\bar{x} = \frac{\Sigma x}{n} = \frac{55}{10} = 5.5$

$\bar{y} = \frac{\Sigma y}{n} = \frac{307}{10} = 30.7$

$\sigma_x^2 = \frac{385}{10} - (5.5)^2 = 8.25$

$r = 0.95$

$\sigma_y^2 = \frac{11387}{10} - (30.7)^2 = 196.21$

6.

If 'θ' is the angle b/w lines of regression then show that

$$\tan\theta = \left(\frac{1-r^2}{r}\right) \frac{\sigma_x \sigma_y}{\sigma_x^2 + \sigma_y^2}$$

proof

$$(y-\bar{y}) = r\frac{\sigma_y}{\sigma_x}(x-\bar{x}) \rightarrow (1)$$

$$(x-\bar{x}) = r\frac{\sigma_x}{\sigma_y}(y-\bar{y}) \rightarrow (2)$$

$$\left\{\begin{array}{l} y = m_1 x + c. \\ y = m_2 x + d. \end{array}\right.$$

Rearranging eq^n (2)

$$(y-\bar{y}) = (x-\bar{x}) \frac{1}{r} \frac{\sigma_y}{\sigma_x} \rightarrow (3)$$

$\therefore$ the angle b/w lines of regression (1) & (3) is given by

$$\tan\theta = \frac{m_2 - m_1}{1 + m_1 m_2} \quad \text{where } m_1 = r\frac{\sigma_y}{\sigma_x}.$$

$$m_2 = \frac{1}{r}\frac{\sigma_y}{\sigma_x}.$$

$$\therefore \tan\theta = \frac{\frac{1}{r}\frac{\sigma_y}{\sigma_x} - r\frac{\sigma_y}{\sigma_x}}{1 + \cancel{r}\frac{\sigma_y}{\sigma_x} \cdot \frac{1}{\cancel{r}}\frac{\sigma_y}{\sigma_x}}$$

$$= \frac{\frac{\sigma_y}{\sigma_x}\left[\frac{1}{r} - r\right]}{1 + \frac{\sigma_y^2}{\sigma_x^2}}$$

$$= \frac{\frac{\sigma_y}{\sigma_x}\left[\frac{1-r^2}{r}\right]}{\frac{\sigma_x^2 + \sigma_y^2}{\sigma_x^2}}$$

Date ______
Page ______

$$= \frac{\sigma_y}{\sigma_x}\left[\frac{1-r^2}{r}\right] \times \frac{\sigma_x^2}{\sigma_x^2+\sigma_y^2}$$

$$\tan\theta = \left[\frac{1-r^2}{r}\right]\left[\frac{\sigma_x \sigma_y}{\sigma_x^2+\sigma_y^2}\right]$$

Note :- The mean values $\bar{x}$ & $\bar{y}$ satisfies the lines of regression.

i.e $\bar{y} = a\bar{x} + b$

$\bar{x} = c\bar{y} + d$.

The co-efficient of regression of y on x denoted by b_{yx} is given by $r\frac{\sigma_y}{\sigma_x}$.

& b_{xy} is given by $r\frac{\sigma_x}{\sigma_y}$

① In a partially destroyed the laboratory record only are available as LOR of y on x & x on y are available as $4x - 5y + 33 = 0$ & $20x - 9y = 107$ respectively calculate $\bar{x}$ & $\bar{y}$ & the co-efficient of correlatio b/w x & y.

y on x = $4x - 5y + 33 = 0$. $\longrightarrow$ ①

x on y = $20x - 9y = 107$ $\longrightarrow$ ②

Since $\bar{x}$ & $\bar{y}$ satisfies the lines of regression, we have

$20\bar{x} - 9\bar{y} = 107$.

$4\bar{x} - 5\bar{y} + 33 = 0$. $\Longleftrightarrow$ $4\bar{x} - 5\bar{y} = -33$.

$20\bar{x} - 9\bar{y} = 107$.

$$\boxed{\begin{aligned}\bar{x} &= 13 \\ \bar{y} &= 17\end{aligned}}$$

$r = \sqrt{(\text{co-eff of } x)(\text{co-eff of } y)}$

rearranging ① & ② as

$$5y = 4x + 33.$$

$$y = \frac{4}{5}x + \frac{33}{5}$$

$$y = 0.8x + 6.6.$$

$$20x = 9y + 107$$

$$x = \frac{9}{20}y + \frac{107}{20}$$

$$x = 0.45y + 5.35$$

$$r = \sqrt{0.8 \times 0.45} = 0.6$$

$$\boxed{r = 0.6}$$

2. $8x - 10y + 66 = 0$ and $40x - 18y = 214$ are the two lines of regression. Find the mean & x's & y's & the co-relation coefficient [r]. find σ_y if $\sigma_x = 3$.

* Since mean satisfies the LOR we've

$$8x - 10y + 66 = 0 \longrightarrow ①$$

$$40x - 18y = 214 \longrightarrow ②$$

$$\therefore \quad 8\bar{x} - 10\bar{y} = -66$$

$$40\bar{x} - 18\bar{y} = 214$$

$$\boxed{\begin{array}{l}\bar{x} = 13 \\ \bar{y} = 17\end{array}}$$

* rearranging ① & ② as.

$$10y = 66 + 8x.$$

$$y = \frac{66}{10} + \frac{8x}{10}. \Rightarrow \frac{8x}{10} + \frac{66}{10}.$$

$40\bar{x} - 18\bar{y} = 214$

$40x = 18\bar{y} + 214$

$x = \frac{18}{40}\bar{y} + \frac{214}{40}$

$\therefore$ LOR $r = \sqrt{\left(\frac{8}{10}\right) \cdot \left(\frac{18}{40}\right)} = 0.6$

$\boxed{r = 0.6}$

* σ_y if $\sigma_x = 3$

$b_{yx} = r \frac{\sigma_y}{\sigma_x}$

b_{yx} is co-eff of x in LOR of y on x.

$b_{yx} = \frac{8}{10} = 0.8$

$\therefore \sigma_y = \frac{b_{yx}\,\sigma_x}{r}$

$= \frac{0.8 \times 3}{0.6}$

$\boxed{\sigma_y = 4}$

3. compute $\bar{x}$ $\bar{y}$ & r from the following eq^n of regression line.

$2x + 3y + 1 = 0 \rightarrow (1)$ & $x + 6y - 4 = 0 \rightarrow (2)$

* since mean satisfies the LOR w.r.t.

$2\bar{x} + 3\bar{y} = -1$ $\qquad \bar{x} = -2$

$\bar{x} + 6\bar{y} = 4$ $\qquad \bar{y} = 1$

* rearranging (1) & (2) as.

$3y = -2x - 1$ $\qquad | \qquad x = -6y + 4$

$y = -\frac{2}{3}x - \frac{1}{3}$

$\therefore r = \sqrt{(-2/3)(-6)} = 2$

so we've interchang the co-eff of x & y.

$\Rightarrow 6y = -x + 4$

$y = \frac{-x}{6} + \frac{4}{6}$.

$\Rightarrow x = \frac{-3}{2}y - \frac{1}{2}$.

$r = \sqrt{\left(\frac{-1}{6}\right)\left(-3/2\right)}$

$\boxed{r = 0.5}$

4. Given & $r = 0.8$

	x-series	y-series
mean	18	100
S.D	14	20

Find eq^n of LOR & hence find the most probable value of y when $x = 70$.

given:

$\bar{x} = 18$, $\bar{y} = 100$, $r = 0.8$

$\sigma_x = 14$ $\sigma_y = 20$.

Find LOR.

y on x $= (y - \bar{y}) = r\frac{\sigma_y}{\sigma_x}(x - \bar{x})$.

$(y - 100) = (0.8)\frac{20}{14}(x - 18)$.

$y - 100 = 1.14(x - 18)$

$y = 1.14x - 20.57 + 100$

$\underline{y = 1.14x + 79.43}$.

x on y $= (x-\bar{x}) = r\frac{\sigma_x}{\sigma_y}(y-\bar{y})$

$$(x-14) = (0.8)\cdot\frac{14}{20}(y-100).$$

$$x - 18 = 0.56y - 56.$$

$$\underline{x = 0.56y - \cancel{40}.38}.$$

value of y when $x = 70$.

$$y = 1.14(70) + 79.43.$$

$$\boxed{y = 159.23}$$

Numerical Solution of algebraic & Transcendental eq^{-n}

I Regula Falsi Method :-

Consider the eq^{-n} $f(x) = 0$ for which the real root has to be found.

* $f(x) = 0 \rightarrow (1)$
* Now we find 2-values of x say a & b such that $f(a)$ & $f(b)$ are of opposite signs.
* Suppose $f(a)$ is >0 & $f(b) < 0$.

The first approximation to the root is given by

$$* \quad x = \frac{af(b) - bf(a)}{f(b) - f(a)}.$$

* We continue with this iterative process till we get the root to the desired degree of accuracy.

1. Find a real root of $x^3 - 2x - 5 = 0$ by regula falsi method.

A:- $x^3 - 2x - 5 = 0$

$f(x) = x^3 - 2x - 5$

put $x = 0$, $f(0) = -5 < 0$

put $x = 1$, $f(1) = -6 < 0$

put $x = 2$, $f(2) = -1 < 0$

put $x = 3$, $f(3) = 16 > 0$

$\Rightarrow$ root lie in $(2, 3)$

$\therefore$ the first approximation to the root is given by

$$x_1 = \frac{af(b) - bf(a)}{f(b) - f(a)}$$

$$= \frac{2f(3) - 3f(2)}{f(3) - f(2)}$$

$$= \frac{2(16) - 3(-1)}{16 + 1}$$

$$x_1 = \frac{32 + 3}{17} = 2.0588$$

$$\boxed{x_1 = 2.0588}$$

$$f(2.0588) = (2.0588)^2 - 2(2.0588) - 5$$

$$= -0.391$$

$\therefore$ the root lie b/w $(2.0588, 3)$

2nd approx.

$$x_2 = \frac{af(b) - bf(a)}{f(b) - f(a)}$$

$$= \frac{2.0588 \times 16 - 3(-0.391)}{16 - -0.391}$$

$$\underline{x_2 = 2.08125}$$

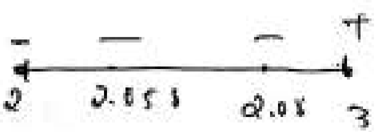
$$f(2.08125) = -0.147$$

$\therefore$ root line in $(2.0812, 3)$.

3rd approx.

$$x_3 = \frac{af(b) - bf(a)}{f(b) - f(a)}$$

$$= \frac{2.0812 \times 16 - 3(-0.147)}{16 - -0.147} = 2.0895$$



$$\therefore f(2.0895) = -0.05$$

$\therefore$ the root is $\underline{2.09}$.

2. Find the real root of $x^3 - 5x - 7 = 0$ to 3 decimal places by using regula falsi method [carry out 3 iteration]

$f(x) = x^3 - 5x - 7$.

$f(0) = -7 < 0$.

$f(1) = -11 < 0$.

$f(2) = -9 < 0$.

$f(3) = 5 > 0$.

$\therefore$ root lies b/w $(2, 3)$.

$a = 2,\ b = 3$.

* $x_1 = \dfrac{2f(3) - 3f(2)}{f(3) - f(2)}$

$x_1 = \dfrac{2 \times 5 - 3(-9)}{5 - -9}$.

$\boxed{x_1 = 2.643}$

$f(2.64) = -1.752$ root lie b/w $(2.64, 3)$

* <u>2nd approx</u>

$x_2 = \dfrac{2.643 \times 5 - 3 \times (-1.75)}{5 - -(1.73)} = 2.736$.

$f(2.736) = -0.199$.

$\therefore$ root lies b/w $(2.736, 3)$

* <u>3rd approx</u>

$x_3 = \dfrac{2.736 \cdot f(3) - 3(f(2.736))}{f(3) - f(2.736)}$

$x_3 = 2.746$.

$\therefore$ Root is 2.746.

2. Find the real root of $x^3+x^2-3x-3=0$ by regula falsi method [carry out 2- iterations]

$f(x) = x^3+x^2-3x-3$

$f(0) = -3<0$

$f(1) = -4<0$

$f(2) = 3>0$

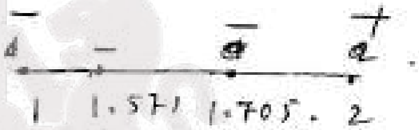
$\therefore$ root lies b/w $(1,2)$.

$a=1$, $b=2$.

* $x_1 = \dfrac{1\times f(2) - 2\times f(1)}{f(2)-f(1)}$

$= \dfrac{1\times 3 - 2\times(-4)}{3-(-4)} = 1.571$

$\therefore f(1.571) = -1.367$.



* 2nd approx

$x_2 = \dfrac{\overset{571}{\cancel{1.367}}\times 2 - 2\times(-1.367)}{2-(-1.367)}$

$= \cancel{1.623}\ 1.705$.

$f(1.705) = -0.251$

$x_3 = \dfrac{1.705\times 2 + 2\cdot\overset{0.251}{\cancel{(1.705)}}}{2+1.705} = 1.728$.

$f(1.728) = -0.04$.

$\therefore$ root is 1.728

4. Find the real roots of

$x\log_{10}x - 1.2 = 0$.

$f(x) = x\log_{10}x - 1.2$

~~$f(0) = -1.2 < 0$~~.

$f(1) = 1\log_{10}(1) - 1.2 = -1.2 < 0$.

$f(2) = 2\log_{10}(2) - 1.2 = -0.597 < 0$

$f(3) = 3\log_{10}(3) - 1.2 = 0.231 > 0$.

$\therefore$ root lies b/w $(2, 3)$.
                                          a   b

$-\quad -\quad -\quad +$

$2 \quad \cancel{0.017} \quad 2.740 \quad 3$.

2.721

$$x_1 = \frac{af(b) - bf(a)}{f(b) - f(a)}$$

$$= \frac{2(0.231) - 3(-0.597)}{0.231 + 0.597}.$$

$x_1 = 2.721$

$f(2.721) = -0.017$

$\therefore$ root lies b/w $(2.721, 3)$.

$$x_2 = \frac{2.721(0.231) - 3(-0.017)}{0.231 + 0.017} = 2.740.$$

$f(2.740) = -0.00056\,34$.

$\therefore$ root lies b/w $(2.740, 3)$,

$= x_2 = 2.740$

5. $x^3 - 3x + 4 = 0$.

$f(x) = x^3 - 3x + 4$.

$f(0) = 4 > 0$.

$f(1) = 2 > 0$

$f(2) = 6 > 0$.

$f(3) = 22 > 0$.

$f(4) = 60 > 0$.

$f(5) = 114 > 0$

$f(6) = 202 > 0$.

$f(x) = x^3 - 3x + 4$.

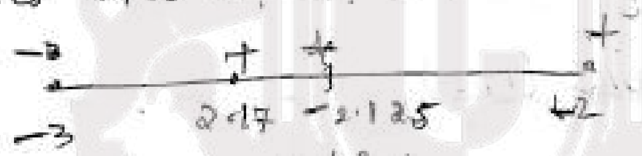
$f(0) = 4 > 0$.

$f(-1) = 6 > 0$.

$f(-2) = 2 > 0$

$f(-3) = -14 < 0$.

lies b/w $(-3, -2)$.



$$x_1 = \frac{a f(b) - b f(a)}{f(b) - f(a)}$$

$$= \frac{-3(2) - (-2)(-14)}{2 - (-14)} = -2.125$$

$f(-2.125) = 0.77$.

$\therefore$ root lies b/w $(-2.125, -2)$.
$\qquad\qquad\qquad\qquad -a \qquad -b$

$$\cancel{x_2 = \frac{-2.125(2) - (-2)(0.72)}{2 - 0.77}}$$

$= \qquad a = -3, \quad b = -2.125$

$$x_2 = \frac{-3 \times 0.779 - 2.125 \times 14}{0.779 + 14}$$

$= -2.1711$

$f(-2.1711) = 0.297.$

$$x_3 = \frac{-3 \times 0.297 - 2.17 \times 14}{0.297 + 14}$$

$x_3 = -2.187.$

$\therefore$ the root is $\underline{-2.187}$

6. Find the fourth root of 12 by regular falsi method

$x = \sqrt[4]{12}.$

$x^4 = 12.$

$x^4 - 12 = 0$

$f(x) = x^4 - 12.$

$f(0) = -12 < 0.$

$f(1) = -11 < 0$

$f(2) = 4 > 0$

$\therefore$ root lies b/w $(1, 2)$

$$x_1 = \frac{1.(4) - 2(-11)}{4 - -11} = 1.733$$

$f(1.733) = -2.980.$

$\therefore$ the root lies b/w $(1.733, 2)$

$$x_2 = \frac{1.733(4) - 2(-2.980)}{4 - -11} = \cancel{1.928}\ 1.847$$

$f(1.847) = -0.362.$

$\underline{x_3 = 1.84.}$

$\therefore$ root is 1.84//

Newton - Raphson method [N R method]

consider the eq^n $f(x)=0 \rightarrow$ ①
for which the real root has to be found

① we find two values of x : say a & b such that $f(a)$ & $f(b)$ are of opposite signs.

② take the initial approximation to the root as

$$x_0 = \frac{a+b}{2}.$$

③ The first approximation to the root is given by $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$.

④ The second approximation is given by

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$ so on.

In general we've

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \qquad n = 0, 1, 2, \ldots$$

We continue with the procedure till we get the root to the desired accuracy.

1. Find the real root of $x^3-2x-5=0$ by NR-method.

A:- $f(0) = -5 < 0$

$f(1) = -6 < 0$

$f(2) = -1 < 0$

$f(3) = 10 > 0$

$\therefore$ the root lies b/w $(2, 3)$

* $x_0 = \frac{a+b}{2} = \frac{2+3}{2} = 2.5$.

* $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$

$f(x) = x^3-2x-5$

$f'(x) = 3x^2-2$.

$x_1 = 2.5 - \frac{(2.5^3 - 2(2.5)-5)}{3(2.5)^2-2}$

$\boxed{x_1 = 2.164.}$

* $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$.

$= 2.164 - \frac{f(2.164)}{f'(2.164)}$.

$= 2.164 - \frac{[2.164^3 - 2(2.164)-5]}{[3\cdot(2.164)^2-2]}$

$x_2 = 2.097$

$$x_3 = 2.097 - \left[\frac{(2.097)^3 - 2(2.097) - 5}{[3(2.097)^2 - 2]}\right]$$

$$\boxed{x_3 = 2.094}$$

$\therefore$ The root is $\underline{2.094}$

2. $\underline{x^3 - 5x - 7 = 0}$. whose root lie b/w $(2, 3)$

$f(0) = -7 < 0$
$f(1) = -11 < 0$
$f(2) = -9 < 0$
$f(3) = 5 > 0$

$\therefore$ root lies b/w $(2, 3)$

* $x_0 = \frac{a+b}{2} = \frac{2+3}{2} = 2.5$

* $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$

$f(x) = x^3 - 5x - 7$
$f'(x) = 3x^2 - 5$

$$x_1 = 2.5 - \frac{[2.5^3 - 5(2.5) - 7]}{3[2.5]^2 - 5}$$

$$\boxed{x_1 = 2.781}$$

* $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$

$$= 2.781 - \frac{[2.781^3 - 5(2.781) - 7]}{[3(2.78)^2 - 5]}$$

$$= 2.747$$

$$x_3 = 2.747 - \frac{[2.747^3 - 5(2.747) - 7]}{[3(2.747)^2 - 5]}$$

$$x_3 = \underline{\underline{2.747}}$$

$\therefore$ The root is $\underline{\underline{2.747}}$

5. Find the real root of the eq^n $x \log_{10} x - 1.2 = 0$. ~~by using~~ which lies b/w 2 & 3 using NR method

$a = 2, \quad b = 3$

$$x_0 = \frac{a+b}{2} = \frac{2+3}{2} = 2.5$$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$f(x) = x \log_{10} x - 1.2$$

$$= \frac{x \log_e x}{\log_e 10} - 1.2$$

$\left\{ \text{since diff of } \log_e x = \frac{1}{x} \right\}$

$$\frac{1}{\log_e 10} = 0.4343$$

$$\therefore f(x) = 0.4343\, x \log_e x - 1.2$$

$$f'(x) = 0.4343 \left[x \frac{1}{x} + \log_e x \right]$$

$$= 0.4343 [1 + \log_e x]$$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$= 2.5 - \frac{0.4343(2.5)\ln(2.5) - 1}{0.4343[1 + \ln(2.5)]}$$

$x_1 = \underline{2.746}$.

$x_2 = 2.74$.

$x_3 = 2.741$.

$$x_3 = 2.704 - \frac{0.4343(2.704)\ln(2.704) - 1.2}{0.4343(1 + \ln(2.704))}.$$

$= 2.741$.

Radians

4. Find the real root of $x\sin x + \cos x = 0$ which lies near $x = \pi$ carry out the iteration upto 4 decimal places. By NR-Method.

$x_0 = \pi$

$f(x) = x\sin x + \cos x$.

$f'(x) = x\cos x + \cancel{\sin x} - \cancel{\sin x}$.

$f'(x) = x\cos x$.

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}.$$

$$= \pi - \frac{(\pi\sin\pi + \cos\pi)}{\pi\cos\pi}$$

$= 2.8232$..

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$= 2.8232 - \frac{2.8232\sin(2.8232) + \cos 2.8232}{2.8232\cos(2.8232)}.$$

$x_2 = 2.7985$

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$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

$$= 2.7985 - \frac{2.7985\sin(2.7985) + \cos(2.7985)}{2.7985\cos(2.7985)}$$

$$x_3 = 2.7983$$

$$x_4 = 2.7983$$

$\therefore$ the root is 2.7983

5. $\cos x = 3x - 1$ by NR-method

$\cos x = 3x - 1$

$\cos x - 3x + 1 = 0$

$f(x) = \cos x - 3x + 1$

$f(0) = 2 > 0$

$f(1) = -1.45 < 0$

$\therefore$ the root lies b/w $(0, 1)$

$x_0 = \frac{0+1}{2} = 0.5$

x_1

$f'(x) = -\sin x - 3$

$$x_1 = 0.5 - \frac{(\cos(0.5)) - 3(0.5) + 1}{\sin(0.5) - 3}$$

$$= 0.6085$$

$$x_2 = 0.6085 - \frac{\cos(0.6085) - 3(0.6085) + 1}{-\sin(0.6085) - 3}$$

$$x_2 = 0.6071$$

$$x_3 = 0.6071 - \frac{\cos(0.6071) - 3\cdot(0.6085) + 1}{-\sin(0.6085) - 3}$$

$x_3 = 0.60710$

$x_4 = 0.60710$

$\therefore$ the root is 0.60710

i. Find $\cos x = xe^x$ by NR - Method.

$f(x) = \cos x - xe^x = 0$.

$f'(x) = -\sin x - [xe^x + e^x]$

$= -\sin x - xe^x - e^x$.

$f(0) = 1 > 0$.

$f(1) = -2.17 < 0$.

$\therefore$ the root lies b/w $(0,1)$.

$x_0 = \frac{0+1}{2} = 0.5$.

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$= 0.5 - \frac{\cos(0.5) - (0.5)e^{(0.5)}}{-\sin(0.5) - 0.5 \cdot e^{(0.5)} - e^{0.5}}$$

$x_1 = 0.5180$

$$x_2 = 0.5180 - \frac{\cos(0.5180) - (0.5180)e^{0.5180}}{-\sin(0.5180) - 0.5180e^{0.5180} - e^{0.5180}}$$

$x_2 = 0.5168$.

$$x_3 = 0.5168 - \frac{\cos(0.516) - (0.516)e^{0.516}}{-\sin(0.516) - 0.516e^{0.516} - e^{0.516}}$$

$x_3 = 0.521$

$\therefore$ the root is 0.521

7. Find the real root of $\tan x + \tanh x = 0$ which lies b/w 2 & 3 using NR-method.

$f(x) = \tan x + \tanh x$.

$f'(x) = \sec^2 x - \operatorname{sech}^2 x \cdot \left[1/\cosh^2 x \right]$

$x_0 = \frac{2+3}{2} = 2.5$

$x_1 = 2.5 - \frac{\tan(2.5) + \tanh(2.5)}{\frac{1}{\cos^2(2.5)} - \left[\frac{1}{\cosh^2(2.5)}\right]}$

$x_1 = 2.3435$

$x_2 = 2.3435 - \frac{\tan(2.343) + \tanh(2.343)}{\frac{1}{\cos^2(2.343)} - \frac{1}{\cosh^2(2.343)}}$

$x_2 = 2.3653$.

$x_3 = 2.365 - \frac{\tan(2.365) + \tanh(2.365)}{\frac{1}{\cos^2(2.365)} - \frac{1}{\cosh^2(2.365)}}$

$x_3 = \underline{2.365}$

$\therefore$ the root is $\underline{2.365}$

8. $\tan x - x = 0$. near $x_0 = 4.5$

$x_1 = 4.5 - \frac{\tan(4.5) - 4.5}{\sec^2(4.5) - 1}$

$= 4.4936$

$x_2 = 4.4930$

$x_3 = 4.4934$

Date ______
Page ______

9. $xe^x = 2.$

$xe^x - 2 = 0$

$f'(x) = x \cdot e^x + e^x.$

$f(0) = -2 < 0.$

$f(1) = 0.718 > 0$

$\therefore$ The root lies b/w $(0, 1)$.

$x_0 = 0+1/2 = 0.5$

$$x_1 = 0.5 - \frac{0.5e^{0.5} - 2}{0.5e^{0.5} + e^{0.5}}.$$

$x_1 = 0.9753$

$x_2 = 0.863$

$x_3 = 0.8526$

$x_4 = 0.852.$ //

9. solve $x\log_{10}x$

Date 20/9/16

MODULE - 4

NUMERICAL ANALYSIS - 2

Finite differences :- If $y = f(x)$ & $y_0, y_1, y_2 \dots y_n$ are the values of y corresponding to ~~$x_0, x_1, x_2 \dots$~~ $x_0, x_0+h, x_0+2h \dots x_0+nh$ or $[x_0, x_1, x_2, x_3 \dots x_n]$.

then the forward differences are defined as

1st forward differences $\Delta y_0 = y_1 - y_0$

$\Delta y_1 = y_2 - y_1$

$\Delta y_2 = y_3 - y_2$ etc.

Second forward differences $\Delta^2 y_0 = \Delta y_1 - \Delta y_0$

$\Delta^2 y_1 = \Delta y_2 - \Delta y_1$ etc.

Forward difference table

x	y	Δ	Δ^2	...	Δ^n
x_0	y_0				
		Δy_0			
x_1	y_1		$\Delta^2 y_0$	...	
		Δy_1			$\Delta^n y_0$
x_2	y_2		$\Delta^2 y_1$	...	
		Δy_2			
x_3	y_3		⋮		
⋮	⋮	⋮	$\Delta^2 y_{n-2}$		
		Δy_{n-1}			
x_n	y_n				

The values $\Delta y_0, \Delta^2 y_0 \dots \Delta^n y_0$ are known as leading forward differences napla ∇ 2

Backward differences

1st backward difference $\nabla y_1 = y_1 - y_0$

$\nabla y_2 = y_2 - y_1$

$\nabla y_3 = y_3 - y_2$ - etc

2nd backward difference :- $\nabla^2 y_1 = \nabla y_1 - \nabla y_0$

$\nabla^2 y_2 = \nabla y_2 - \nabla y_1$ etc

backward difference table :-

x	y	∇	∇^2	...	∇^n
x_0	y_0				
		∇y_1			
x_1	y_1		$\nabla^2 y_2$		
		∇y_2			
x_2	y_2		$\nabla^2 y_3$		$\nabla^n y_n$
		∇y_3			
x_3	y_3				
$\vdots$	$\vdots$	$\vdots$	$\nabla^2 y_n$		
		∇y_n			
x_n	y_n				

Here ∇y_n, $\nabla^2 y_n$ ... $\nabla^n y_n$ are called leading backward differences.

1. construct the forward and backward differences for the value of $x = 0, 1, 2, 3, 4$ and identify leading backward & forward entries

$y = x^3 + x^2 + 1$.

$x_0 = 0, \quad y_0 = 1$

$x_1 = 1 \quad y_0 = 3$

$x_2 = 2 \quad y_0 = 13$

$x_3 = 3 \quad y_0 = 37$

$x_4 = 4 \quad y_0 = 81$

x	y	Δ	Δ^2	Δ^3	Δ^4
0	$\boxed{1}\ y_0$				
		$\boxed{2}\ \Delta y_0$			
1	3		$\boxed{8}\ \Delta^2 y_0$		
		10		$\boxed{6}\ \Delta^3 y_0$	
2	13		14		$\boxed{0}\ \Delta^4 y_0$ $\nabla^4 y_4$
		24		$\boxed{6}\ \nabla^3 y_4$	
3	37		$\boxed{20}\ \nabla^2 y_4$		
		$\boxed{44}\ \nabla y_4$			
4	$\boxed{81}\ y_4$				

<u>Interpolation</u> :- The process of finding the value of y within the given range of x is known as interpolation.

<u>Extrapolation</u> :- The process of finding the value of y outside the given range of x is known as Extrapolation.

Newton – Gregory forward interpolation formula for equal interval.

If $y_0, y_1, y_2 \dots y_n$ are the values of the unknown function $y = f(x)$ corresponding to the values $x_0, x_1, x_2, x_3, \dots x_n$ of x then the value of y for a $(x = x_0 + ph)$ corresponding value of x which lies at the beginning of the table is given by

$(x = x_0 + ph)$

$$y(x) = y_0 + \frac{p \Delta y_0}{1!} + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3(y_0) + \frac{p(p-1)(p-2)(p-3)}{4!}\Delta^4 y_0 + \dots$$

where $p = \left(\frac{x - x_0}{h}\right)$ & h = diff b/w any two consecutive values of x

Newton – Gregory backward interpolation formal for equal interval.

If $y_0, y_1, y_2 \dots y_n$ are the values of the unknown fn $y = f(x)$ corresponding to the values $x_0, x_1, x_2, \dots x_n$ of x which are equally spaced then the values the value of y for any value of $x = x_n + ph$ that lies at the end of the table is given by

$$y(x) = y_n + \frac{p \nabla y_n}{1!} + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n + \frac{p(p+1)(p+2)(p+3)}{4!}\nabla^4 y_n + \dots$$

where $p = \frac{x - x_n}{h}$.

$p = \frac{x - x_0}{h}$ $h = \frac{x - x_0}{}$

Date
Page

Q. Find $y(105)$ & $y(355)$ for the following table.

x :	100	150	200	250	300	350	400
y :	10.63	13.03	15.04	16.81	18.42	19.90	21.27

using suitable Interpolation formula.

Since $x = 105$ lies near the beginning of the table we use NFIF formula. &
$x = 355$ lies at the end of the table we use NBIF.

x	y	Δ	Δ^2	Δ^3	Δ^4	Δ^5	Δ^6
100	$\boxed{10.63}$ (y_0)	$\boxed{2.4}$ (Δy_0)	$\boxed{-0.39}$ ($\Delta^2 y_0$)	$\boxed{0.15}$	-0.07	0.02	0.02
150	13.03	2.01	-0.24	0.08	-0.05	0.04	
200	15.04	1.77	-0.16	0.03	-0.01		
250	16.81	1.61	-0.13	0.02			
300	18.42	1.48	-0.11				
350	19.90	$\boxed{1.37}$ (∇y_n)					
400	$\boxed{21.27}$ (y_n)						

We've NFIF

$$y(x) = y_0 + \frac{p\Delta y_0}{1!} + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \frac{p(p-1)(p-2)(p-3)}{4!}\Delta^4 y_0 + \cdots$$

$p = \left(\frac{x - x_0}{h}\right)$ $x = 105$ h = diff b/w any two x_0 values

$= \frac{105 - 100}{50} = 0.1$

$h = 150 - 100$
$= 50$

$$y(105) = 10.63 + \frac{0.1\,(2.40)}{1!} + \frac{0.1(-0.9)}{2}(-0.39) + \frac{(0.1)(-0.9)(-1.9)}{3\times 2}(0.15) + \frac{(0.1)(-0.9)(-1.9)(2.9)}{4\times 3\times 2} \times (-0.08)$$

$$= 10.63 + 0.24 + 0.017 + 0.0036 + -0.020$$

$$= \underline{\underline{10.891}}$$

$y(385)$ use NBIF.

$$y(x) = y_n + \frac{P\nabla}{1.} y_n + \frac{P(P+1)}{2!}\nabla y^2_n + \frac{P(P+1)(P+2)}{3!}\nabla^3 y_n + \frac{P(P+1)(P+2)(P+3)}{4!}\nabla^4 y_n + \cdots$$

$$① \Rightarrow 21.27 + \frac{(-0.9)(1.37)}{1!} + \frac{(-0.4)(0.9+1)}{2!}(-0.11) + \frac{(-0.9)(-0.9+1)(-0.9+2)(0.02)}{3!} + \frac{(-0.9)(-0.9+1)(-0.9+2)(-0.9+3)(-0.01)}{4!}$$

$$= 20.04$$

Q. Extrapolate for 25.4 for the given data

x	19	20	21	22	23
y	91	100.25	110	120.25	131

Date ____
Page ____

NBIF

x	y	Δ	Δ^2	Δ^3	Δ^4
19	91				
		9.25			
20	100.25		0.5		
		9.75		0	
21	110		0.5		0
		10.25		0	
22	120.25		$\boxed{0.5}$		
		$\boxed{10.75}$			
23	$\boxed{131}$				

$$y = y_n + \frac{p}{1!}\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n$$

$$= p = \frac{x - x_n}{h} = \frac{25.4 - 23}{1} = 2.4.$$

$$y = 131 + 2.4(10.75) + \frac{2.4(2.4+1)}{2}(0.5)$$

$$y = 158.84$$

3. Given $f(40) = 184$, $f(50) = 204$, $f(60) = 226$
$f(70) = 250$ $f(80) = 276$, $f(90) = 304$
find $f(38)$ & $f(85)$ using suitable interpolaⁿ formula.

x	y	Δ	Δ^2	Δ^3	Δ^4	Δ^5
40	$\boxed{184}$					
		$\boxed{20}$				
50	204		$\boxed{2}$			
		22		0		
60	226		2		0	
		24		0		0
70	250		2		0	
		26		0		
80	276		$\boxed{2}$			
		$\boxed{23}$				
90	$\boxed{304}$					

$$p = \frac{x - x_0}{h} = \frac{38 - 40}{10} = -0.2 .$$

$$y(x) = y_0 + \frac{p}{1!}\Delta y_0 + \frac{p(p-1)}{2!}\Delta y_0^2$$

$$= 184 + (-0.2)20 + \frac{(-0.2)(-0.2-1)}{2}2$$

$$y(x) = 180.24 .$$

Backward

$$p = \frac{x - x_n}{h} = \frac{85 - 90}{10} = -0.5$$

$$y(x) = y_n + \frac{p}{1!}\nabla y_n + \frac{p(p+1)}{2!}\nabla y_n^2$$

$$= 304 + (-0.5)(28) + \frac{(-0.5)(-0.5+1)}{2}2$$

$$= 304 - 14 - 0.25$$

$$y(x) = 289.75 \,//$$

Find the interpolating polynomial $f(x)$ satisfying $f(0) = 0$, $f(2) = 4$, $f(4) = 56$, $f(6) = 204$, $f(8) = 496$ and $f(10) = 980$ hence find $f(3)$ and $f(5)$.

$\frac{x - x_0}{h}$ $\quad x/h \quad p = 0$

x	y	Δ	Δ^2	Δ^3	Δ^4	Δ^5
0	0					
		4				
2	4		48			
		52		48		
4	56		96		0	
		148		48		0
6	204		144		0	
		292		48		
8	491		192			
		484				
10	980					

$x - x_0/h$ $\quad p = \frac{x - x_0}{h}$

$\frac{x - x_0}{h}$

$$p = \frac{x - x_0}{h} = x/2 .$$

$$y(x) \; y_0 + \frac{p\Delta y_0}{1!} + \frac{p(p-1)}{2!}\Delta^2 y_2 + \frac{p(p-1)}{}$$

Date
Page

$$y(x) = y_0 + p\Delta y_0 + \frac{p(p-1)\Delta^2 y_0}{2!} + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0$$

$$= 0 + \frac{x}{2}(4) + \frac{\frac{x}{2}\left[\frac{x}{2}-1\right](48)}{2!} + \frac{\frac{x}{2}\left(\frac{x}{2}-1\right)\left(\frac{x}{2}-2\right)48}{3!}$$

$$= 2x + (x^2-2x)6 + x(x-2)(x-4)$$

$$= 2x + 6x^2 - 12x + (x^2-2x)(x-4)$$

$$2x + 6x^2 - 12x + x^3 - 4x^2 + \ldots - 2x^2 + 8x$$

$$\cancel{x^3 + 2x^2 - 6x}$$

$$x^3 - 2x$$

$$y(3) = 3^3 - 2(3)$$

$$= 21$$

$$y(5) = 5^3 - 2(5) = 115$$

A survey release the following information as classified below.

Income perday	0-10	10-20	20-30	30-40	40-50
No. of person	20	45	115	210	115

estimate the probebel no of person the income group of (20) & ϵ, (25) we

x	y	Δ	Δ^2	Δ^3	Δ^4
10	20	45	70	25	-215
20	65	115	95	-190	
30	180	210	-95		
40	390	115			
50	505				

20

20

20

5 20

$y(25)$

The table wave no of people getting salary 10

220 = 65

$P = \frac{x - x_0}{h} = \frac{25-10}{10} = 1.5$

$y(25) = 20 + (1.5)(45) + \frac{1.5(1.5-1)}{2!} 70 + \frac{1.5(1.5-1)(1.5-2)}{3!}(-25) + \frac{1.5(1.5-1)(1.5-2)(1.5-3)}{4!}(2.5)$

$= \underline{107.14} \approx 107.$

The no of persons b/w 20-25 = 107 - 65
= 42.

use NFI.F to find y_{25} given $y_{20} = 512$,
$y(30) = 439$; $y(40) = 346$, $y(50) = 243$.

x	y	Δ	Δ^2	Δ^3
20	512			
		-73		
30	439		-20	
		-93		70
40	346		-10	
		-103		
50	243			

05

$P = \frac{x - 10}{10}$ 5/10

10 x 25 - 41

$y(35)$

$P = \frac{35-20}{10} = 1.5$

$= 512 + (1.5)(-73) + \frac{1.5(1.5-1)}{2!}(-20) + \frac{1.5(1.5-1)(1.5-2)}{3!}(70)$

$= 394.375$ //

Divided differences :-

If $f(x_0), f(x_1), f(x_2) \ldots f(x_n)$ [ie $y_0, y_1 \ldots y_n$] are the values of the unknown fn $y = f(x)$ corresponding to $x_0, x_1, x_2, \ldots x_n$, the values of x which are not necessarily equally spaced then the first divided difference is defined as

$$f(x_0, x_1) = \frac{f(x_1) - f(x_0)}{x_1 - x_0}.$$

$$f(x_1, x_2) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

The second divided difference is defined as

$f(x_0, x_1, x_2)$

$$\frac{f(x_1, x_2) - f(x_0, x_1)}{x_2 - x_0}$$

$$\text{Ily} \quad f(x_1, x_2, x_3) = \frac{f(x_2, x_3) - f(x_1, x_2)}{x_3 - x_1}$$

$\vdots$

etc.

divided difference table

x	y	1^{st} DD	2^{nd} DD		n^{th} DD
x_0	y_0				
		$f(x_0, x_1)$			
x_1	y_1		$f(x_0, x_1, x_2)$	$\ldots$	
		$f(x_1, x_2)$			
x_2	y_2		$f(x_1, x_2, x_3)$	$\ldots$	$f(x_0, x_1, \ldots x_n)$
		$\vdots$			
$\vdots$			$f(x_{n-2}, x_{n-1}, x_n)$		
		$f(x_{n-1}, x_n)$			
x_n	y_n				

Date ______
Page ______

Newtons divided differences for unequal intervals:-

If $y_0, y_1, \ldots y_n$ are the values of unknown fn $y = f(x)$ corresponding to $x_0, x_1, \ldots x_n$ & the values of x which are not necessarily equally spaced, then the value of y too any value of x is given by the newtons divided difference formula as.

$$y(x) = y_0 + (x-x_0) f(x_0, x_1) + (x-x_0)(x-x_1) f(x_0 x_1, x_2) + (x-x_0)(x-x_1)(x-x_2) f(x_0, x_1, x_2, x_3) + (x-x_0)(x-x_1)(x-x_2)(x-x_3) f(x_0, x_1, x_2, x_3, x_4) + \ldots$$

1. given $f(0) = 8$, $f(1) = 68$, $f(5) = 123$ construct the divided diff table & hence find the value of $f(2)$.

x	y	1^{st} D.D	2^{nd} D.D
0	$\boxed{8}$	$\boxed{60}$	$\boxed{-9.25}$
1	68	13.75	
5	123		

$$y(x) = y_0 + (x-x_0) f(x_0, x_1) + (x-x_0)(x-x_1) f(x_0, x_1, x_2)$$

$$= 8 + (2-0)60 + (2-0)(2-1)(-9.25)$$

$$= 109.5$$

$$\therefore y(2) = 109.5$$

2. By using NDDF, find the value of $f(8)$ & $f(15)$ from the following table.

x	y	1st DD	2nd DD	3rd DD	4th DD	5th DD
4	48					
		$\boxed{52}$				
			$\boxed{15}$			
5	100					
		97		$\boxed{1}$		
7	294		21		0	
		202		1		0
10	900		27		0	
		310		1		
11	1210		33			
		409				
13	2028					

$$y(x) = y_0 + (x-x_0) f(x_0, x_1) + (x-x_0)(x-x_1) f(x_0, x_1, x_2) + (x-x_0)(x-x_1)(x-x_2) f(x_0, x_1, x_2, x_3).$$

$$= 48 + (8-4)(52) + (8-4)(8-5)(15) + (8-4)(8-5)(8-7)(1).$$

$$\underline{y(8) = 448}$$

$$y(15) = 48 + (15-4)(52) + (15-4)(15-5)(15) + (15-4)(15-5)(15-7)(1)$$

$$\underline{y(15) = 3150}.$$

3. Evaluate $f(9)$ using NDDF given

x	y	1st DD	2nd DD	3rd DD	4th DD
5	150				
		121			
7	392		24		
		265		1	
11	1452		32		0
		457		1	
13	2366		42		
		709			
17	5202				

at $x = 9$.

$$y(9) = y_0 + (x - x_0) f(x_0, x_1) + (x - x_0)(x - x_1) f(x_0, x_1, x_2) + (x - x_0)(x - x_1)(x - x_2) f(x_0, x_1, x_2, x_3)$$

$$y(9) = 150 + (9-5)(121) + (9-5)(9-7)(24) + (9-5)(9-7)(9-11)(1).$$

$$y(9) = \underline{\underline{890}}$$

4. Determine $f(x)$ as a polynomial in x for the following data using Newtons Interpolation Formula.

x: -4 -1 0 2 5

$f(x)$: 1245 33 5 9 1335.

x	$f(x)$	1st DD	2nd DD	3rd DD	4th DD
-4	1245				
		-404			
-1	33		94		
		-28		-14	
0	5		10		3
		2		13	
2	9		88		
		442			
5	1335				

$$y(x) = y_0 + (x-x_0) f(x_0, x_1) + (x-x_1)(x-x_0) f(x_0, x_1, x_2) + (x-x_0)(x-x_1)(x-x_2) f(x_0, x_1, x_2, x_3)$$

$$= 1245 + (x+4)(-404) + (x+4)(x+1)(94) + (x+4)(x+1)(x)(-14) + (x+4)(x+1)(x)(x-2)(3).$$

$$= 1245 - 404x - 1616 + (x+4)(x+1)x [94 - 14x + 3x(x-2)]$$

$$= 1245 - 404x - 1616 + (x^2 + 5x + 4)(3x^2 - 20x + 94)$$

$$= \cancel{1245} - 371 - 404x + 3x^4 - 20x^3 + 94x^2 + 15x^3 - 100x^2 + 470x + 12x^2 - 80x + 376.$$

$$y(x) = 3x^4 - 5x^3 + 6x^2 - 14x + 5$$

Lagrange's interpolation formula :-

For unequal intervals

If $y_0, y_1, y_2 \ldots y_n$ are values of the unknown function $y = f(x)$ corresponding to the values of x which $x_0, x_1, x_2, \ldots x_n$, are not necessarily equally spaced then the value of y for any value of x is given by

$$y(x) = \frac{(x-x_1)(x-x_2) \ldots (x-x_n)}{(x_0-x_1)(x_0-x_2) \ldots (x_0-x_n)} \times y_0 + \frac{(x-x_0)(x-x_2) \ldots (x-x_n)}{(x_1-x_0)(x_1-x_2) \ldots (x_1-x_n)} \times y_1 +$$

$$\frac{(x-x_0)(x-x_1)(x-x_3)\cdots(x-x_n)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)\cdots(x_2-x_n)} \times y_2$$

$$+ \cdots\cdots$$

1. using Legranges Interpolation formula find $f(5)$ from the following table

x	1	3	4	6	9
$f(x)$	-3	9	30	132	136

$$y(5) = \frac{(x-x_1)(x-x_2)\cdots(x-x_n)}{(x_0-x_1)(x_0-x_2)\cdots(x_0-x_n)} \times y_0 + \frac{(x-x_0)(x-x_2)\cdots(x_0-x_n)}{(x_1-x_0)(x_1-x_2)\cdots(x_1-x_n)} \times y_1 + \cdots$$

$$y(5) = \frac{(5-3)(5-4)(5-6)(5-9)}{(1-3)(1-4)(1-6)(1-9)} \times (-3) + \frac{(5-1)(5-4)(5-6)(5-9)}{(3-1)(3-4)(3-6)(3-9)} \times (9)$$

$$+ \frac{(5-1)(5-3)(5-6)(5-9)}{(4-1)(4-3)(4-6)(4-9)} \times (30) + \frac{(5-1)(5-3)(5-9)(5-4)}{(6-1)(6-3)(6-9)(6-4)} \times 132$$

$$+ \frac{(5-1)(5-3)(5-6)(5-4)}{(9-1)(9-3)(9-6)(9-4)} \times 136$$

$$= -0.1 + (-4) + 32 + 46.93 + (-1.51)$$

$$= 73.32$$

2. Find $f(11)$ from the following table using LIF.

x :-	2	5	8	14
y :-	94.8	87.9	81.3	68.7

$$y = \frac{(11-5)(11-8)(11-14)}{(2-5)(2-8)(2-14)} \times 94.8 + \frac{(11-2)(11-8)(11-14)}{(5-2)(5-8)(5-14)} \times 87.9$$

$$+ \frac{(11-2)(11-5)(11-14)}{(8-2)(8-5)(8-14)} \times 81.3$$

$$+ \frac{(11-2)(11-5)(11-8)}{(14-2)(14-5)(14-8)} \times 68.7$$

$$= 23.7 - 87.9 + 121.95 + 17.175$$

$$y(11) = 74.925$$

3. Find the polynomial $f(x)$ by using LF & hence find $f(3)$ from the following table

x :-	0	1	2	5
y :	2	3	12	147

$$y(x) = \frac{(x-1)(x-2)(x-5)}{(-1)(-2)(-5)} \times (2) + \frac{(x-0)(x-2)(x-5)}{(1-0)(1-2)(1-5)} \times 3$$

$$+ \frac{(x-0)(x-1)(x-5)}{(2-0)(2-1)(2-5)} \times (12) + \frac{(x-0)(x-1)(x-2)}{(5-0)(5-1)(5-2)} \times 147$$

$$\frac{(x^2-2x-x+2)(x-5)}{-10_5} \times (2) + \frac{(x^2-2x)(x-5)}{4} \times 3$$

$$\frac{(x^2-x)(x-5)(12)}{-6} + \frac{(x^2-x)(x-2)}{60}$$

$$\frac{(x^3-2x^2-x^2+2x-5x^2+10x+5x-10)}{-5}$$

$$+\frac{(x^3-2x^2-5x^2+10x)\times 3}{4}$$

$$+\frac{(x^3-x^2-5x^2+5x).2}{1}$$

$$+\frac{x^3-x^2-2x^2+2x}{60}.$$

$\frac{1}{20}\left[-4(x^3.\right.$

$\Rightarrow (-0.2)(x-1)(x-2)(x-5)+(0.75)(x-0)$
$(x-2)(x-5)+(-2)(x)(x-1)(x-5)+$
$0.78(x)(x-1)(x-2).$

$$\frac{1}{20}\left[-4(x^3-8x^2+17x-10)+15(x^3-7x^2+10x)\right.$$
$$\left.40(x^3-6x^2+5x)+49(x^2-3x^2+2x)\right]$$

$$=\frac{1}{20}\left[20x^3+20x^2-20x+40\right]$$

$$=x^3+x^2-x+2$$

$$f(3)=27+9-3+2$$
$$=35.$$

given x: 300 304 305 307
$\log_{10}x$: 2.4771 2.4829 2.4843 2.4871

calculate the approximate value of $\log_{10}301$ upto 4 decimals.

Lagrange's Inverse Interpolation formula.

The value of for any given value of 'y' can be found using Lagrange's inverse interpolation formula given by

$$x = \frac{(y-y_1)(y-y_2)\dots(y-y_n)}{(y_0-y_1)(y_0-y_2)\dots(y_0-y_n)} x_0 +$$

$$\frac{(y-y_0)(y-y_2)\dots(y-y_n)}{(y_1-y_0)(y_1-y_2)\dots(y_1-y_n)} \times x_1 +$$

$$\frac{(y-y_0)(y-y_1)(y-y_3)\dots(y-y_n)}{(y_2-y_0)(y_2-y_1)(y_2-y_3)\dots(y_2-y_n)} \times x_2 +$$

$$- - - - - -$$

① Given

x :	2	5	9	11
y :	10	12	15	19

Find x corresponding to $y = 16$ using suitable interpolation formula.

Since we've to find the value of x & the value of x are not equally spaced we use LIIF

$x = ?$ at $y = 16$

$$x = \frac{(16-12)(16-15)(16-19)}{(10-12)(10-15)(10-19)} \times 2$$

$$+ \frac{(16-10)(16-15)(16-19)}{(12-10)(12-15)(12-19)} \times 5 +$$

$$\frac{(16-10)(16-12)(16-19)}{(15-10)(15-12)(15-19)} \times 9 +$$

$$\frac{(16-10)(16-12)(16-15)}{(19-10)(19-12)(19-15)} \times 11 =$$

$= 0.266 + (-2.14) + (10.8) + 1.04$

$x = 9.966$. at $y = 16$.

2. Apply Lagranges Formula Inversly to obtain the root of the eq^{-n} $f(x) = 0$ given that $f(30) = -30$, $f(34) = -13$ $f(38) = 3$, & $f(42) = 18$

<u>To find</u> $f(x) = y_0$

Find x at $y = 0$

x : 30 , 34 38 42

y : -30 -13 3 18

$$x = \frac{(0+\overset{13}{\cancel{30}})(0-3)(0-18)}{(-30+13)(-30-3)(-30-18)} \times (30)$$

$$+ \frac{(0+30)(0-3)(0-18)}{(\cancel{0}-13+30)(-13-3)(-13-18)} \times (34)$$

$$+ \frac{(0+30)(0+13)(0-18)}{(3+30)(3+13)(3-18)} \times (38)$$

$$+ \frac{(0+\overset{30}{\cancel{18}})(0+13)(0-3)}{(18+30)(18+13)(18-3)} \times (42)$$

$$= \overset{-0.782}{\cancel{0.347}} + 6.53 + 33.686 + (-2.20)$$

$$= \cancel{35.35} = \underline{37.226}$$

Numerical Integration :-

consider the Integral . $I = \int_a^b f(x)\cdot dx \rightarrow (1)$

for a given values of x_0; $x_1 = x_0 + h$, $x_2 = x_0 + 2h \ldots \ldots x_n = x_0 + nh$

we find the corresponding values of $y = f(x)$.

$y_0, y_1, y_2 \ldots y_n$ where n is the no of intervals and $h = \frac{b-a}{n}$.

$$\boxed{h = \frac{b-a}{n}}$$ $n \rightarrow$ no of Intervals

we've the following formulae to evaluate the integral given by (1).

Simpson's 1/3rd rule (n - should be multiple of 2).

$$I = \frac{h}{3}\left[(y_0 + y_n) + 2(y_2 + y_4 + y_6 + y_8 \ldots) + 4(y_1 + y_3 + y_5 + y_7 + \ldots)\right]$$

Simpson's 3/8th rule :-

(n - should be multiple of (3).

$$I = \frac{3h}{8}\left[(y_0 + y_n) + 2[y_3, y_6, y_9, y_{12} \ldots] + 3(y_1 + y_2 + y_4 + y_5 + y_7 + y_8 \ldots)\right]$$

Weddle's Rule :- (n - multiple of 6)

case (i) if $n = 6$

$$I = \frac{3h}{10}\left[1y_0 + 5y_1 + 1y_2 + 6y_3 + 1y_4 + 5y_5 + 1y_6\right]$$

case ii

$$I = \frac{3h}{10}\left[\begin{array}{l} 1y_0 + 5y_1 + 1y_2 + 6y_3 + 1y_4 + 5y_5 + y_6 \\ 1y_6 + 5y_7 + 1y_8 + 6y_9 + 1y_{10} + 5y_{11} + 1y_{12} \end{array}\right]$$

Note:- If the no of intervals is not given, then we take $n = 6$ for all the methods

→ The no of intervals is always even,
If the no of intervals is n, then the no of ordinates is $n+1$

1. Evaluate $\int_0^6 x^3 dx$ by simpsons $1/3^{th}$ rule $3/4^{th}$ rule & weddle's rule.

Let $I = \int_0^6 x^3 dx$

$y = x^3 \qquad n = 6$

$h = \frac{b-a}{n} = \frac{6-0}{6} = 1$

$x_0 = a, \; x_n = b.$

x:	0	1	2	3	4	5	6
$y = x^3$ y:	0_{y_0}	1_{y_1}	8_{y_2}	27_{y_3}	64_{y_4}	125_{y_5}	216_{y_6}

$x_0 = 0 \; , x_1 = x_0 + h$
$= 0 + 1 = 1$
$x_0 = 0$

* **Simpsons $1/3^{rd}$ Rule**

$$I = \frac{h}{3}\left[(y_0 + y_6) + 2[y_2 + y_4] + 4[y_1 + y_3 + y_5]\right]$$

$$= 1/3\left[(0 + 216) + 2[8 + 64] + 4[1 + 27 + 125]\right]$$

$$= 324.$$

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$$I = \frac{3h}{8}\left[(y_0+y_6)+2[y_3]+3[y_1+y_2+y_4+y_5]\right]$$

$$= \frac{3}{8}\left[(0+216)+2[27]+3[1+8+64+125]\right]$$

$$\boxed{I = 324}$$

<u>weddle's rule</u>

$$I = \frac{3h}{10}\left[1y_0+5y_1+1y_2+6y_3+1y_4+5y_5+1y_6\right]$$

$$= \frac{3}{10}\left[1\times0+5\times1+1\times8+6\times27+1\times64+5\times125+1\times216\right]$$

$$\boxed{I = 324}$$

2. Evaluate $\int_0^1 \frac{1}{1+x^2}\,dx$ by simpsons $3/8^{th}$ rule considering 7-ordinates and hence find approximate value of π

$y = \frac{1}{1+x^2} \rightarrow (1)$, $n=6$.

$h = \frac{b-a}{n} = \frac{1-0}{6} = 1/6$.

$x_0 = 0$, $x_1 = x_0+h$

x: 0, 1/6, 2/6, 3/6, 4/6, 5/6, 1

y: 1, 0.97, 0.9, 0.8, 0.69, 0.59, 0.5

$$I = \frac{3h}{8\times6}\left[(1+0.5)+2[0.8]+3[0.97+0.9+0.69+0.59]\right]$$

$$I = 0.7843 \rightarrow (2)$$

evaluating the given integral analytically we've.

$$I = \int_0^1 \frac{1}{1+x^2}\,dx$$

$$= \tan^{-1}x \Big]_0^1$$

$$= \tan^{-1}1 - \tan^{-1}0$$

$$= \pi/4 - 0 \rightarrow (3)$$

comparing (2) & (3).

$$\pi/4 = 0.784$$

$$\pi = 4 \times 0.784 = 3.14$$

3. Evaluate $\int_0^1 \frac{1}{1+x}$ by weddl's rule taking 6 equal intervals & hence find the value of $\log 2$

$y = \frac{1}{1+x}$ $\quad n = 6$

$h = \frac{1-0}{6} = 1/6$

x : 0 $\quad 1/6 \quad 2/6 \quad 3/6 \quad 4/6 \quad 5/6 \quad 1$

y : 1 $\quad 0.85 \quad 0.75 \quad 0.66 \quad 0.6 \quad 0.54 \quad 0.5$

$$I = \frac{3 \times 1}{10 \times 6}\left[1 \times 1 + 5 \times 0.85 + 1 \times 0.75 + 6 \times 0.66 + 1 \times 0.6 + 5 \times 0.54 + 1 \times 0.5\right]$$

$= \cancel{0.5788}$

$$I = 0.68 \rightarrow (2)$$

$$I = \int_0^1 \frac{1}{1+x}$$

$$= \log(1+x)\Big]_0^1$$

$$\log(1+1) - \log(0)$$

$$I = \log(2)$$

$$= 0.693 \longrightarrow (3)$$

compare (2) (3) (3)

$$\underline{\log(2) = 0.68}$$

4. Evaluate $I = \int_0^{0.6} e^{-x^2}\,dx$ by $1/3^{rd}$ Rule.

$y = e^{-x^2}$ $n = 6$

$h = \frac{0.6 - 0}{6} = 0.1$

x:	0	0.1	0.2	0.3	0.4	0.5	0.6
y:	1	0.99	0.96	0.91	0.85	0.77	0.69

$$I = \frac{0.1}{3}\left[(1 + 0.69) + 2\left[0.96 + 0.85\right) + 4\left(0.99 + 0.91 + 0.77\right)\right]$$

$$I = 0.523$$

5. calculate the approximate value of $\int_0^{\pi/2} \sin x\,dx$ by using simpson's $1/3^{rd}$ rule using 11 ordinates.

$y = \sin x$

$n = 10$

$h = \frac{b-a}{n} = \frac{\pi/2}{10} = \pi/20$

$I = \int_0^{\pi/2} \sin x \cdot dx$

$I = h/3 \left[(y_0 + y_n) + 2(y_2 + y_4 + \cdots) + 4(y_1 + y_3 + y_5 \ldots) \right]$

$x: \quad 0 \quad \pi/20 \quad \pi/10 \quad 3\pi/20 \quad 4\pi/20 \quad 5\pi/20 \quad 6\pi/20 \quad 7\pi/20$

$y: \quad 0 \quad 0.156 \quad 0.308 \quad 0.453 \quad 0.587 \quad 0.706 \quad 0.808 \quad 0.890$

$$I = \frac{\pi}{60}\left[(0+1) + 2(0.308 + 0.587 + 0.808 + 0.950) + 4(0.156 + 0.453 + 0.706 + 0.890 + 0.987)\right]$$

$I = 0.998$

$I = \int_0^{\pi/2} \sin x \cdot dx$

$= \left[-\cos x\right]_0^{\pi/2}$

$= -\cos \pi/2 + \cos 0$

$= (0 + 1) = 1$

6. Evaluate $\int_4^{5.2} \log x \cdot dx$ by weddle's rule.

no of intervals = 6.

$y = \log x$

$h = \frac{5.2 - 4}{6} = 0.2$.

x	4	4.2	4.4	4.6	4.8	5.0	5.2
y	1.386	1.435	1.481	1.526	1.568	1.609	1.648

$\frac{19}{20}$ $\frac{19.6}{20}$ $\frac{20}{20}$.

0.950 0.98 1.

$$I = \frac{3h}{10}\left[1y_0 + y_1 5 + y_2 1 + y_3 6 + y_4 1 + y_5 5 + y_6 1\right]$$

$$= \frac{3 \times 0.2}{10}\left[1.386 + 5(1.435) + 1.481 + 6 \times 1.526 + 1.568 + 5 \times 1.609 + 1.648\right]$$

$$I = \underline{1.82754}.$$

7. Evaluate $\int_0^1 \frac{x}{1+x^2}\, dx$ by Sympson's $3/8^{th}$ considering 7-ordinates.

$I = \int_0^1 \frac{x}{1+x^2}$, $n = 6$, $h = \frac{1-0}{6} = 1/6$.

x:	0	1/6	2/6	3/6	4/6	5/6	1
y:	0	0.162	0.3	0.4	0.461	0.491	0.5

$$I = \frac{3 \times 1}{8 \times 6}\left[0.5 + 2[0.4 + \cancel{0}] + 3[0.162 + 0.3 + 0.461 + 0.491]\right]$$

$\cancel{I = 0.408}$ $I = \underline{0.346}$

~~1/2~~ 9

Vectors

$\vec{a} = 2\hat{i} + 3\hat{j} + \hat{k}$

unit vector $\hat{a} = \dfrac{\vec{a}}{|\vec{a}|}$

$= \dfrac{2i + 3j + k}{\sqrt{2^2 + 3^2 + 1^2}}$

$\vec{b} = 1\hat{i} + 3\hat{j} + 2\hat{k}$

$\vec{a} \cdot \vec{b} = 2(1) + 3(3) + 1(2) \rightarrow$ Scalar.

$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & 1 \\ 1 & 3 & 2 \end{vmatrix} \rightarrow$ Vector.

Angle b/w two vectors.

$\cos\theta = \dfrac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|}$

* Divergence of a vector

$\nabla \cdot \vec{F} = \dfrac{\partial f_1}{\partial x} + \dfrac{\partial f_2}{\partial y} + \dfrac{\partial f_3}{\partial z}$

* $\text{curl}\,\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ f_1 & f_2 & f_3 \end{vmatrix}$

$\nabla \times \vec{F}$

$$\frac{y-y_1}{x-x_1} = \frac{y_2-y_1}{x_2-x_1}$$ eq^n of straight line in 2-point form.

<u>Vector integration</u>

Position vector $R = \vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ } $p(x,y,z)$

$$d\vec{r} = dx\hat{i} + dy\hat{j} + dz\hat{k}$$

The position vector of any point $p(x,y,z)$ on the curve C is given by

$$\vec{R} = \vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$d\vec{r} = dx\hat{i} + dy\hat{j} + dz\hat{k}$$

If $\vec{F} = f_1\hat{i} + f_2\hat{j} + f_3\hat{k}$ represents force acting on a particle along the curve 'c' then

→ $\int_c \vec{F}\cdot d\vec{r}$ represents total work done in moving the particle around 'C'.

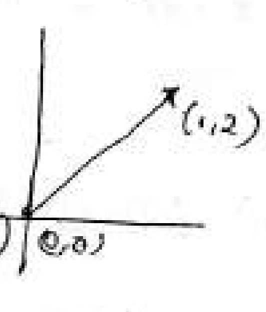
→ If $\int_c \vec{F}\cdot d\vec{r} = 0$ then the vector F is said to be irrotational.

1. If $\vec{F} = 3xy\hat{i} - y^2\hat{j}$ evaluate $\int_c \vec{F}\cdot d\vec{r}$ where c is the curve in xy plane given by $y = 2x^2$ from $(0,0)$ to $(1,2)$

<u>Ans:-</u> we've

$$\vec{F} = 3xy\hat{i} - y^2\hat{j} \longrightarrow (1)$$

$$y = 2x^2 \longrightarrow (2)$$



$$\vec{F}\cdot d\vec{r} = (3xy\hat{i} - y^2\hat{j})\cdot(dx\hat{i} + dy\hat{j})$$

$$\vec{F}\cdot d\vec{r} = 3xy\,dx - y^2\,dy \longrightarrow (3)$$

we've $y = 2x^2$
$dy = 4x\,dx$ } (4)

(4) in (3)

$$\vec{F}\cdot d\vec{r} = 3x(2x^2)\,dx$$
$$-(2x^2)^2 4x\,dx$$

$$\vec{F}\cdot d\vec{r} = (6x^3 - 16x^5)dx$$

$$\therefore \int_C \vec{F}\cdot d\vec{r} = \int_0^1 (6x^3 - 16x^5)\cdot dx$$

$$= \left[\frac{6x^4}{4} - \frac{16x^6}{6}\right]_0^1$$

$$= \left[\frac{6}{4} - \frac{16}{6}\right]$$

$$= -7/6 = -1.166$$

2. Evaluate $\int_C \vec{F}\cdot d\vec{r}$ where $F = y^2\hat{i} + 2xy\hat{j}$

and (i) c is a straight line path from the point $(0,0)$ to the point $(1,2)$

(ii) c is the parabola $y = 2x^2$.

From the pt $(0,0)$ to the pt $(1,2)$

Ans:-

$$\vec{F}\cdot d\vec{r} = (y^2\hat{i} + 2xy\hat{j})\cdot(dx\hat{i} + dy\hat{j})$$
$$y^2dx + 2xy\,dy \rightarrow (1).$$

(i) <u>Straight line</u> :- The eq^n of the line joining the points $(0,0)$ & $(1,2)$ is given by

$$\frac{y-y_1}{y_2-y_1} = \frac{x-x_1}{x_2-x_1}$$

$$= \frac{y-0}{2-0} = \frac{x-0}{1-0}$$

$$y/_2 = x$$

$$\text{or} \quad y = 2x \rightarrow (2)$$

$$dy = 2dx$$

$$\therefore (1) \Rightarrow \int_c \vec{F} \cdot d\vec{r} = \int 4x^2 dx \; 2x(2x)2\,dx$$

$$= \int 4x^2 dx + 8x^2 dx$$

$$= \int_0^1 12x^2 dx$$

$$\frac{\overset{4}{\cancel{12}} x^3}{\cancel{3}} = [4x^3]_0^1$$

$$= 4 //$$

(ii) parabola

$$y = 2x^2 \rightarrow (3)$$

$$\vec{F} \cdot d\vec{r} = y^2 dx + 2xy\,dy$$

$$dy = \cancel{2}\,4x\,dx$$

$$\therefore \int 4x^4 dx + 2x(2x^2) \cdot 4x\,dx$$

$$= \int_0^1 4x^4 dx + 16x^4 dx$$

$$= \int_0^1 20x^4 dx$$

$$= \frac{\overset{4}{\cancel{20}} x^5}{\cancel{5}} = [4x^5]_0^1$$

$$= 4 //$$

3. Find the work done in moving a particle if the Force $\vec{F}$ is $3xy\,\hat{i} - 5z\,\hat{j} + 10x\,\hat{k}$ along the curve $x = t^2+1$, $y = 2t^2$, $z = t^3$ from $t=0$ to $t=2$.

Ans:- $\vec{F} \cdot d\vec{r}$ $(3xy\,\hat{i} - 5z\,\hat{j} + 10x\,\hat{k}) \cdot (dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k})$

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$\vec{F}.d\vec{r} = 3xy\,dx - 5z\,dy + 10x\,dz$.

$x = t^2+1 \Rightarrow dx = 2t\,dt$.

$y = 2t^2 \Rightarrow dy = 4t\,dt$

$z = t^3 \Rightarrow dz = 3t^2dt$.

$= 3(t^2+1)(2t^2).2t\,dt - 5(t^3)4t\,dt +$
$10(t^2+1)\,3t^2.dt$

$= (3t^2+3)(2t^2).2t\,dt - 20t^4dt +$
$30t^4 + 30t^2.dt$.

$= (12t^5 + 12t^3)dt - 20t^4dt + (30t^4+30t^2)dt$

$\vec{F}.d\vec{r} = (12t^5 + 10t^4 + 12t^3 + 30t^2)\,dt$

$$= \int_0^2 (12t^5 + 10t^4 + 12t^3 + 30t^2).dt$$

$$\left[= \frac{\cancel{12}^{2} \times t^6}{\cancel{6}} + \frac{\cancel{10}^{2} \times t^5}{\cancel{5}} + \frac{\cancel{12}^{3} t^4}{\cancel{4}} + \frac{\cancel{30}^{10} \times t^3}{\cancel{3}} \right]_0^2$$

$$\left[2(t^6) + 2\times(t)^5 + 3(t^4) + 10\times(t^3)\right]_0^2$$

$$= \left[2(2^6) + 2\times(2)^5 + 3(2)^4 + 10(2)^3\right]$$

$= 320$ //

4. $\vec{F} = (3x^2+6y)\hat{i} - 14yz\hat{j} + 20xz^2\hat{k}$.
Evaluate $\vec{F}.d\vec{r}$ From $(0,0,0)$ to $(1,1,1)$ along the curve $x = t$, $y = t^2$ $z = t^3$

$\vec{F}.d\vec{r} = (3x^2+6y)\hat{i} - 14yz\hat{j} + 20xz^2\hat{k}$.
$\quad * (dx\hat{i} + dy\hat{j} + dz\hat{k})$

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$= (3x^2 + 6y)dx - 14yz\,dy + 20xz^2\,dz$

$3x^2dx + 6y\,dx - 14yz\,dy + 20xz^2\,dz$

$x = t$

$dx = dt$

$y = t^2$ $\quad z = t^3$

$dy = 2t \cdot dt$ $\quad dz = 3t^2 \cdot dt$

$\cancel{3t^2 dt + 6t^2 dt - 14 t^4 \cdot 2t\,dt + 20}$

$= 3t^2dt + 6t^2dt - 14t^2t^3 \cdot 2t \cdot dt + 20t(t^3)^2 \times 3t^2 \cdot dt$

$= 3t^2 \cdot dt + 6t^2 \cdot dt - 28t^6 dt + 60t^9 \cdot dt$

$\cancel{= 9t^2 \cdot dt - 32t^6 \cdot dt}$

$\cancel{= \int_0^1 (9t^2 - 32t^6) \cdot dt}$

$\cancel{= \left[\frac{9t^3}{3} - \frac{32t^7}{7}\right]_0^1}$

$\cancel{= [3 - 4.57]}$

$= \cancel{-1.571} = -4/7$

$= 9t^2 dt - 28t^6 dt + 60t^9\,dt$

$= \int_0^1 (9t^2 - 28t^6 + 60t^9)\,dt$

$= \left[\frac{9t^3}{3} - \frac{28t^7}{7} + \frac{60t^{10}}{10}\right]_0^1$

$= 3 - 4 + 6 = 5$

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5. Find the work done in moving the particle in a force field $\vec{F} = 3x^2\hat{i} + (2xz - y)\hat{j} + z\hat{k}$ along straight line from $(0,0,0)$ to $(2,1,3)$

The strigh line joining 2 points is given $(x_1 y_1 z_1)$ $(x_2 y_2 z_2)$ is given by

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}$$

$$\frac{x - 0}{2 - 0} = \frac{y - 0}{1 - 0} = \frac{z - 0}{3 - 0}$$

$$\frac{x}{2} = y/1 = z/3 = t$$ Equate it to single variable called t

$x = 2t$ $dx = 2dt$ $\left\{ x/2 = t,\ y/1 = t,\ z/3 = t \right.$

$y = t$

$z = 3t$.

$\vec{F}\cdot d\vec{r} = 3x^2\hat{i} + (2xz - y)\hat{j} + z\hat{k} \cdot dx\hat{i} + dy\hat{j} + dz\hat{k}$

$= 3x^2 dx + 2xz dy - y dy + z dz$.

$= 3(2t)^2 \cdot 2dt + 2(2t)(3t)dt - t \cdot dt$

$+ 3t \cdot 3dt$

$= 24t^2 \cdot dt + 12t^2 \cdot dt - t dt\ 9t dt$

$\vec{F}\cdot d\vec{r} = 36t^2 \cdot dt + 8t dt$

$$\int_C \vec{F}\cdot d\vec{r} = \int (36t^2 + 8t) \cdot dt$$

w.r.t t x is variying from 0 to 2.

$x = 2t$.

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at $x = 0$.
$t = 0$.
at $x = 2$
$2 = 2t$
$t = 1$.

$$= \int_0^1 (36t^2 + 8t) \cdot dt$$

$$= \left[\frac{36t^3}{3} + \frac{8t^2}{2}\right]_0^1$$

$$= [12 + 4]$$

$$= \underline{\underline{16}}.$$

6. If $F = (2y+3)\hat{i} + xz\hat{j} + (yz - x)\hat{k}$. evaluate $\int \vec{F} \cdot d\vec{r}$ where c is the curve $x = 2t^2$, $y = t$, $z = t^3$. from $(0,0,0)$ to $(2,1,1)$.

$$\vec{F} \cdot d\vec{r} = (2y+3)\hat{i} + xz\hat{j} + (yz-x)\hat{k} \cdot dx\hat{i} + dy\hat{k} + dz\hat{k}.$$

$$= 2y\,dx + 3dx + xz\,dy + yz \cdot dz - x\,dz.$$

$x = 2t^2$, $dx = 4t \cdot dt$.
$y = t$, $dy = dt$.
$z = t^3$, $dz = 3t^2 \cdot dt$

at $y = t$, $\underline{y = 0, 1}$
$t = 0$
$t = 1$

$$\therefore = 2t(4t) \cdot dt + 3(4t)\,dt + (2t^2)t^3 \cdot dt + t \cdot t^3 \cdot 3t^2 \cdot dt - 2t^2 \cdot 3t^2 \cdot dt.$$

$$= 8t^2 \cdot dt + 12t \cdot dt + 2t^5 \cdot dt + 3t^6 \cdot dt - 6t^4 \cdot dt.$$

$$= 3t^6 \cdot dt - 6t^4 \cdot dt + 8t^2 \cdot dt + 12t \cdot dt + 2t^5 \cdot dt.$$

$$= \int_0^1 (3t^6 - 6t^4 + 8t^2 + 12t + 2t^5)\, dt.$$

$$= \left[\frac{3t^7}{7} - \frac{6t^5}{5} + \frac{8t^3}{3} + \frac{12t^2}{2} + \frac{2t^6}{6}\right]_0^1$$

$288/35$ $= \frac{288}{35}$

7 If $F = x^2\hat{i} + xy\hat{j}$ evaluate $\int_C \vec{F}\cdot d\vec{r}$ From $(0,0)$ to $(1,1)$ along

(i) along the straight line $y = x$

(ii) along the parabola $y = \sqrt{x}$

$$\vec{F}\cdot d\vec{r} = (x^2\hat{i} + xy\hat{j})\cdot(dx\hat{i} + dy\hat{j}).$$

$$= x^2\, dx + xy\, dy.$$

(i) straight line

$y = x$.

$dy = dx$.

$= x^2 \cdot dx + x^2 \cdot dx$

$2x^2 \cdot dx$.

$$= 2\int_0^1 \frac{x^3}{3}$$

$$= \frac{2}{3}\left[x^3\right]_0^1 = 2/3.$$

(ii) parabola $= y = \sqrt{x}$.

$y^2 = x$

$2y \cdot dy = dx$.

$= x^2 \cdot dx + x^2 y \cdot dy$

$= y^4 \cdot 2y \cdot dy + y^2 y \cdot dy$

$2y^5 \cdot dy + y^3 \cdot dy$

$=$

$$\int_0^1 [2y^5 + y^3] \cdot dy$$

$$= \left[\frac{2y^6}{6} + \frac{y^4}{4} \right]_0^1$$

$= 7/12$

Green's theorem

if R is a closed region in the xy-plane bounded by simple closed curve 'C' and if M & N are two continous functions of x and y having continous first order partial derivatives in region R, then we have

$$\oint_C M dx + N dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx\, dy$$

2. **Stoke's theorem**: if f is the surface a simple closed curve 'C' & bounded by a simple closed curve 'C' & bounded by a continously differentiable vector $\vec{F}$ is continously differentiable vector then

$$\oint \vec{F} \cdot d\vec{r} = \iint_S \text{curl } \vec{F} \cdot \hat{n}\, ds = \iint_S (\nabla \times \vec{F}) \cdot \hat{n}\, ds$$

Where $\hat{n}\, ds = dy\,dz\, \hat{i} + dx\,dz\, \hat{j} + dy\,dx\, \hat{k}$

3. Gauss divergence theorem.

If V is the volume bounded by a surface S and $\vec{F}$ is continuously differentiable vector, then

$$\iint_S \vec{F}.\hat{n}\,ds = \iiint_V div\,\vec{F}.\,dv.$$

$$= \iiint_V \nabla.\vec{F}.\,dv.$$

① Verify Green's theorm in a plane for $\oint_C (3x^2 - 8y^2)dx + (4y - 6xy)dy$ where c is the boundry of the region enclosed by $y = \sqrt{x}$ and $y = x^2$

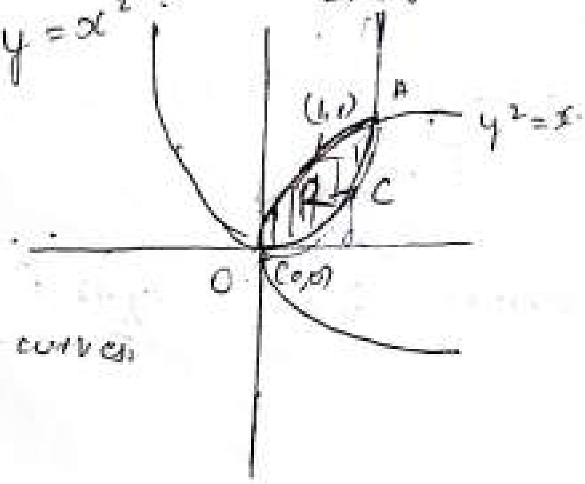
We've green's theorem as

$$\oint_C Mdx + Ndy = \iint_R \left[\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right] dx\,dy. \rightarrow ①$$

To verify the theorem, we evaluate RHS and LHS seperately and then show LHS = RHS.

given $y = \sqrt{x}$ $\quad y = x^2$

$y^2 = x$ $\quad y = x^2$.



Now we find the point of intersection by solving both the curves we've $x^2 = y$

$x = y^2$

$(y^2)^2 = y$.

$y^4 - y = 0$.

$y(y^3-1) = 0$.

$y=0,\ y=1$

at $y=0,\ x=0$.

at $y=1\ \ x=1$

$\therefore$ point of intersection $(0,0)$ and $(1,1)$

The curve C indicates two curves OA & AO

$\therefore$ consider LHS

$$\oint_C M\,dx + N\,dy = \int_{OA} M\,dx + N\,dy + \int_{AO} M\,dx + N\,dy \rightarrow (2).$$

$M = (3x^2 - 8y^2) \qquad N = 4y - 6xy$

consider

$$\int_{OA} M\,dx + N\,dy = \int_{OA} (3x^2 - 8y^2)\,dx + (4y - 6xy)\,dy \rightarrow (3)$$

Along OA, we've $x^2 = y$.

$2x\,dx = dy$.

x varies from 0 to 1 along OA.

Substitute all these in (3)

$$\int_{OA} M\,dx + N\,dy = \int_0^1 (3x^2 - 8x^4)\,dx + (4x^2 - 6x\cdot x^2)\,2x\,dx$$

$$= \int_0^1 (3x^2 - 8x^4 + 8x^3 - 12x^4)\,dx$$

$$= \int_0^1 (3x^2 - 20x^4 + 8x^3)\,dx.$$

$$\left(x^3 - 4x^5 + 2x^4\right)_0^1 = 1 - 4 + 2 = -1 \longrightarrow (5)$$

Consider

$$\int_{AO} M\,dx + N\,dy = \int_{AO} (3x^2 - 8y^2)\,dx + (4y - 6xy)\,dy$$

eq^n of AO = $y^2 = x$.

$2y\,dy = dx$.

$\therefore$ y varies from A to O as 1 – 0

$$\therefore \int_{AO} M\,dx + N\,dy = \int_1^0 (3y^4 - 8y^2)\,2y\,dy + (4y - 6y^2y)\,dy.$$

$$= \int_1^0 (6y^5 - 16y^3 + 4y - 6y^3)\,dy$$

$$= \int_1^0 (6y^5 - 22y^3 + 4y)\,dy.$$

$$= \left[y^6 - \frac{22}{4}y^4 + \frac{4y^2}{2} \right]_1^0$$

$y^6 - \cancel{22}$

$$= \left[0 - \left(1 - 11/2 + 2\right)\right] = 5/2 \rightarrow (5)$$

Substituting (4) and (5) in (2)

$$\oint_C M\,dx + N\,dy = -1 + 5/2 = 3/2 \rightarrow (6)$$

consider the R.H.S of 1

$$\iint_R \left[\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right] dy\,dx$$

$$= \iint_R \left[\partial/\partial x\,[4y - 6xy] - \partial/\partial y\,[3x^2 - 8y^2]\right] dy\,dx$$

$$= \iint [-6y - (-16y)]$$

$$= \iint 10y\;dy\,dx$$

We fix the limits using the figure.

$$= \int_0^1 \int_{x^2}^{\sqrt{x}} 10y \cdot dy\,dx$$

$$10 \int_0^1 \left[\frac{y^2}{2}\right]_{x^2}^{\sqrt{x}} \cdot dx$$

$$5 \int_0^1 \left[\sqrt{x}^2 - x^4\right] \cdot dx$$

$$5 \int_0^1 (x - x^4) \cdot dx$$

$$5\left[\frac{x^2}{2} - \frac{x^5}{5}\right]$$

$$5\left[\frac{1}{2} - 1/5\right] = 3/2 \quad \rightarrow 7$$

From 6 and 7

LHS = RHS

Hence proved.

2. Evaluate $\int (3x^2 - 8y^2)\,dx + (4y - 6xy)\,dy$ over C bounded by $x=0$, $y=0$ and $x+y=1$

Since the given integral is of the form $\int M\,dx + N\,dy$, we use green's theorem to evaluate the given integral.

ie $\int M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dy\,dx$

$x=0,\ y=0,\ x+y=1$

put $\cancel{x=0}$

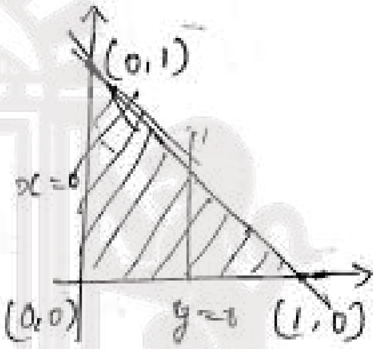
$x+y=\cancel{0}\,1$

$x=0$

$y=1$ ie $(0,1)$

& $y=0$

$x=1$ ie $(1,0)$



$M = 3x^2 - 8y^2 \qquad \frac{\partial M}{\partial y} = -16y$

$N = 4y - 6xy \qquad \frac{\partial N}{\partial x} = -6y$

$\int (3x^2 - 8y^2)\,dx + (4y - 6xy)\,dy = \iint (-6y + 16y)\,dy\,dx$

$= \int_0^1 \int_0^{1-x} (-6y + 16y)\,dy\,dx$

$\int_0^1 \left[\frac{-\overset{3}{\cancel{6}}y^2}{\cancel{2}} + \frac{\overset{8}{\cancel{16}}y^2}{\cancel{2}}\right]_0^{1-x} dx$

$\left[-3y^2 + 8y^2\right]_0^{1-x} dx$

$$\left[\frac{5\cancel{10}\, y^2}{\cancel{2}}\right]_0^{1-x}$$

$$= 5 \int [y^2]_0^{1-x}$$

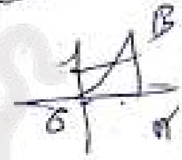
$$5 \int_0^1 (1-x)^2 \cdot dx$$

$$5 \int_0^1 (1+x^2-2x) \cdot dx .$$

$$5 \int_0^1 x + \frac{x^3}{3} - x^2 \Big]_0^1$$

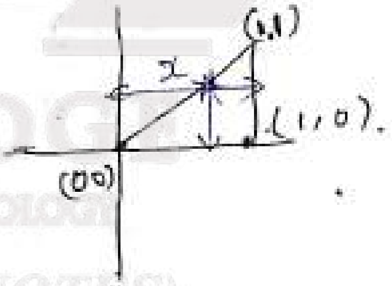
$$5\left[1 + 1/3 - 1\right] = 5/3 \,//$$

3. using green's theorem evaluate $\oint x^2 y\, dx + x^2 dy$ where c is the counter clockwise boundary describing the triangle with vertices $(0,0)$ $(1,0)$ $(1,1)$. $(y=0\ ,\ x=1,\ y=x)$



$$\int M dx + N dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dy\, dx .$$

$$\int_0^1 \int \cancel{x^2 y}\ 2x + x^2 \cdot dy\, dx$$



$(0,0)$ $(1,1)$

$$\frac{y-0}{1-0} = \frac{x-0}{1-0}$$

$$\int_0^1 \int_0^x (2x - x^2)\, dy\, dx$$

$$\int_0^1 \left[2x \cdot y - x^2 y\right]_0^x$$

$$2x^2 - x^3$$

$$\int_0^1 (2x^2 - x^3) \cdot dx .$$

$$= \left[\frac{2x^3}{3} - \frac{x^4}{4}\right]$$

$$\left[\frac{2}{3} - \frac{1}{4}\right] = \frac{5}{12}$$

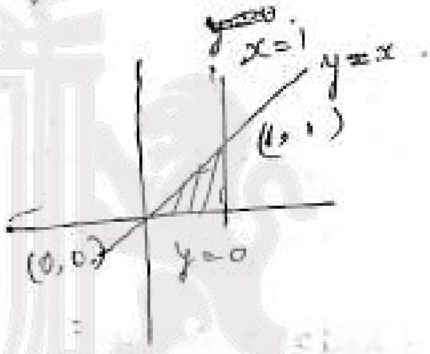
$$= \frac{5}{12}.$$

4. Evaluate $\int_c (xy - x^2)dx + x^2y\,dy$ where c is the closed curve formed by $y=0$, $x=1$ & $y=x$

Since the given integral of the form $\int M dx + N dy$ we use greens theorem to evaluate

$$= \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) . dy . dx$$

$$\int_0^1 \int_0^x \left[2xy - x\right] dy\, dx .$$



$$= \int_0^1 \left[\frac{2xy^2}{2} - xy\right]_0^x . dx$$

$$= \int_0^1 \left[x^3 - x^2\right] dx$$

$$= \left[\frac{x^4}{4} - \frac{x^3}{3}\right]_0^1$$

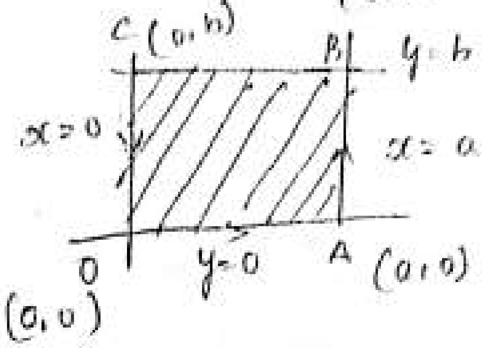
$$= \left[\frac{1}{4} - \frac{1}{3}\right] \quad \frac{3-4}{12} = -\frac{1}{12}$$

5. Verify stokes theorem for the vector $\vec{F} = x^2 + y^2\hat{i} - 2xy\hat{j}$ taken round the rectangle bounded by $x=0$, $x=a$, $y=0$, and $y=b$

We've $\vec{F} = x^2+y^2\hat{i} - 2xy\hat{j} \longrightarrow ①$

$$\oint_c \vec{F} . d\vec{r} = \iint_s \nabla \times \vec{F} . \hat{n}\, ds \longrightarrow ②$$

$\hat{n}\,ds = dy\,dz\,\hat{i} + dx\,dz\,\hat{j} + dy\,dx\,\hat{k}$



consider the LHS of (2)

$$\oint \vec{F}\cdot d\vec{r} = \left((x^2+y^2)\hat{i} - 2xy\hat{j} + 0\hat{k}\right)\cdot\left(dx\hat{i} + dy\hat{j} + dz\hat{k}\right)$$

$$= (x^2+y^2)\,dx - 2xy\,dy\,.$$

* consider the L.H.S of (2)

$$\oint_C \vec{F}\cdot d\vec{r} = \int_{OA} \vec{F}\cdot d\vec{r} + \int_{AB} \vec{F}\cdot d\vec{r} + \int_{BC} \vec{F}\cdot d\vec{r} + \int_{CO} \vec{F}\cdot d\vec{r} \quad \rightarrow (3)$$

* consider $\int_{OA} \vec{F}\cdot d\vec{r} = \int_{OA} (x^2+y^2)\,dx - 2xy\,dy\,.$

along OA : $y = 0$. $\therefore dy = 0$

$x \rightarrow 0$ to a

$$\therefore \int_{OA} \vec{F}\cdot d\vec{r} = \int (x^2+0)\,dx - 0\,.$$

$$= \int_0^a x^2\,dx = x^3/3 = a^3/3\,.$$

* Consider $\int_{AB} \vec{F}\cdot d\vec{r} = \oint$ along AB : $x = a$, $dx = 0$. $y \rightarrow 0 \rightarrow b$.

$$\therefore \int_{AB} \vec{F}\cdot d\vec{r} = \int 0 - 2(a)\,y\,dy\,.$$

$$= \left[-2a\, y^2/2\right] \quad y = 0 \to b$$

$$= -ab^2$$

* consider $\int_{BC} \vec{F} \cdot d\vec{r} =$

$y = b \quad \therefore dy = 0.$

$x \to a \text{ to } 0$

$$\int_a^0 (x^2+y^2)\, dx - 0.$$

$$\int_a^0 (x^2+b^2)\, dx$$

$$\left[\frac{x^3}{3} + b^2 x\right]_a^0$$

$$0 - \left[\frac{a^3}{3} + ab^2\right].$$

$$= -ab^2 - a^3/3$$

* Along CO.

$x = 0, \quad dx = 0.$

$y \to b \text{ to } 0.$

$$\int_b^0 -2xy\, dy.$$

$$= [0.]_b^0 = 0$$

Substitute the above integral values in ③

we get

$$\oint_C \vec{F} \cdot d\vec{r} = a^3/3 + ab^2 - a^3/3 - ab^2 + 0.$$

$= -2ab^2 = LHS \rightarrow (4)$

RHS

Now consider curl $\vec{F}$

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2+y^2 & -2xy & 0 \end{vmatrix}$$

$$\hat{i}(0-0) - \hat{j}(0-0) + \hat{k}(-2y-2y)$$

$$= -4y\hat{k}$$

Consider

$\nabla \times \vec{F} \cdot \hat{n}\, ds$

$= (-4y\hat{k}) \cdot dy\,dz\,\hat{i} + dx\,dz\,\hat{j} + dy\,dx\,\hat{k}$

$= -4y\,dy\,dx$

$$= RHS = \iint_S \nabla \times \vec{F} \cdot \hat{n}\, ds$$

$$= -\int_0^a \int_0^b 4y\,dy\,dx$$

$$= -\int_0^a \left[\frac{4y^2}{2}\right]_0^b dx$$

$$= -\int_0^a 2b^2\, dx$$

$$= -\left[2b^2 x\right]_0^a$$

$$= -2ab^2 \rightarrow (5)$$

Hence from (4) and (5) stokes theorem is verified.

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5. Verify stokes theorem for $F = y\hat{i} + z\hat{j} + x\hat{k}$ where S is the upper half of the sphere $x^2+y^2+z^2=1$ & C is its boundary.

$\vec{F} = y\hat{i} + z\hat{j} + x\hat{k}$

$$\oint_C \vec{F}.d\vec{r} = \iint_S \nabla \times \vec{F}.\hat{n}\, ds \rightarrow (2).$$

c is the circle $x^2+y^2=1$, and z is zero

S is the surface bounded by the circle C.

Consider $\vec{F}.d\vec{r} = y\hat{i} + z\hat{j} + x\hat{k} \cdot dx\hat{i} + dy\hat{j} + dz\hat{k}$

$= y\,dx + z\,dy + x\,dz$

To evaluate the integral, we consider a parametric eq^{n} of the circle.

<u>ie</u> $x = r\cos\theta$, $y = r\sin\theta$.

$r = 1$.

$\therefore x = \cos\theta$ & $y = \sin\theta$.

$dx = -\sin\theta . d\theta$

consider $\oint_C F.d\vec{r} = \int_C (y\,dx + z\,dy + x\,dz)$

$= \int y\,dx \quad \because z = 0.$

$= \int_0^{2\pi} \sin\theta\,(-\sin\theta . d\theta)$

{since c is circle θ varies from 0 to 2π}

$= -\int_0^{2\pi} \sin^2\theta . d\theta$

$$= -\int_0^{2\pi} \frac{1-\cos 2\theta}{2} \, d\theta$$

$$= -\frac{1}{2}\left[\theta - \frac{\sin 2\theta}{2}\right]_0^{2\pi}$$

$$-\frac{1}{2}\left[2\pi - \frac{1}{2}[0-0]\right]$$

$$LHS = -\pi$$

cosider RHS.

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y & z & x \end{vmatrix}$$

$$i(0-1) - \hat{j}(1-0) + \hat{k}(0-1)$$

$$= -\hat{i} - \hat{j} - \hat{k}$$

$$\hat{n}\, ds = dydz\,\hat{i} + dxdz\,\hat{k} + dxdy\,\hat{k}$$

$$\nabla \times \vec{F} \cdot \hat{n}\, ds = -dydz - dxdz - dxdy$$

Since $z=0$, $\therefore dz = 0$.

$$= -\cancel{\iint dydx} = \iint dy\,dx$$

Note (1) $\iint_S dy\,dx = \text{Area of } S$

(2) $\iiint_V dy\,dx \cdot dz = \text{volume}$

Since ~~Area~~ S is a the circle, $x^2+y^2=1$ whose area $= \pi r^2$. where $r=1 \therefore \pi$

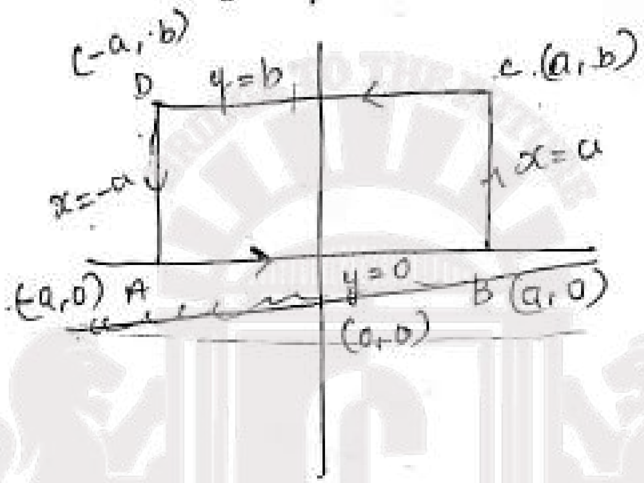
RHS $= -\pi$ Hence stockes theorem is verified.

7. Verify Stockes theorem for $\vec{F} = (x^2+y^2)\hat{i} - 2xy\hat{j}$ take round the lines rectangle bounded by the

$x = \pm a$, $y = 0$, & $y = b$.

$$\vec{F} = (x^2+y^2)\hat{i} - 2xy\hat{j} \longrightarrow (1)$$

$$\oint_C \vec{F}\cdot d\vec{r} = \iint_S \nabla \times \vec{F}\cdot \hat{n}\, ds \longrightarrow (2)$$



Consider LHS

$$\oint_C \vec{F}\cdot d\vec{r} = \left[(x^2+y^2)\hat{i} - 2xy\hat{j}\right]\cdot\left[dx\hat{i} + dy\hat{j}\right]$$

$$= (x^2+y^2)\,dx - 2xy\,dy$$

$$= \int_{AB} \vec{F}\cdot d\vec{r} + \int_{BC} \vec{F}\cdot d\vec{r} + \int_{CD} \vec{F}\cdot d\vec{r} + \int_{DA} \vec{F}\cdot d\vec{r}$$

$$* = \int_{AB} (x^2+y^2)\,dx - 2xy\,dy$$

along AB, $y = 0$ $dy = 0$

$x \rightarrow -a$ to a.

$$= \frac{1}{3}\left[x^3\right]_{-a}^{a} = \frac{2a^3}{3}$$

$$\int_{BC} \vec{F}\cdot d\vec{r} = \int_{BC} (x^2+y^2)\,dx - 2xy\,dy.$$

along BC ; $x = a$ $dx = 0$.

$y = 0$ to b

$$= \int_0^b -2x\,dy.$$

$$= -2ab^2.$$

$$\int_{CD} F\cdot dr = \int (x^2+y^2)\,dx - 2xy\,dy.$$

$y = b$, $dy = 0$

$x = a$ to $-a$

$$= \int_a^{-a} (x^2+b^2)\,dx.$$

$$= \tfrac{1}{3}\left[x^3\right]_a^{-a} + b^2\left[x\right]_a^{-a}$$

$$= \frac{-a^3 - a^3}{3} + b^2[-a-a]$$

$$= -\frac{2a^3}{3} - 2ab^2.$$

$$\int_{DA} \vec{F}\cdot dr = \int_{DA} (x^2+y^2)\,dx - 2xy\,dy.$$

along DA : $x = -a$, $dx = 0$.

$y \to b$ to 0.

$$= \int_b^0 2\,\cancel{dy}\,ay\,dy.$$

$$= a\left[y^2\right]_b^0 = -ab^2$$

$$= \frac{2a^3}{3} - ab^2 - \frac{2a^3}{3} - 2ab^2$$

$$= -4ab^2.$$

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$$RHS = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ (x^2+y^2) & -2xy & 0 \end{vmatrix}$$

$$= i(0-0) - \hat{j}(0) + \hat{k}(-2y - 2y)$$

$$\vec{F} \times \Delta = -4y\hat{k}$$

$$\hat{n}\,.ds = dy\,dz\,\hat{i} + dx\,dz\,\hat{k} + dy\,dx\,\hat{k}$$

since $z = 0$, $dz = 0$.

$$= -4y\,dy\,dx$$

$$= -\int_{-a}^{a}\int_{0}^{b} 4y\,dy\,dx$$

$$= -\int_{-a}^{a}\left[4\frac{y^2}{2}\right]_0^b dx$$

$$= \int_{-a}^{a}\left[2y^2\right]_0^b dx$$

$$-2b^2\,dx$$

$$= -2b^2\left[x\right]_{-a}^{a}$$

$$-2b^2[a+a]$$

$$-4ab^2$$

8. If $\vec{F} = (2x^2 - 3z)\hat{i} - 2xy\hat{j} - 4x\hat{k}$

$\iint_S \vec{F}\cdot\hat{n}\,ds$ using divergence theorem

where V is the region boundend by

$x=0,\ y=0,\ z=0,$ and $2x+2y+z=4$.

we've divergenis

From ~~Greenis~~ theorem $\iint_S \vec{F}\cdot\hat{n}\,ds = \iiint_V \operatorname{div}\vec{F}\,dv \rightarrow ①$

To evaluate $\iint \vec{F}\cdot\hat{n}\,ds$ we use L.H.S of divergens theorem.

$$\operatorname{div}\vec{F} = \nabla\cdot\vec{F} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}.$$

$$= \frac{\partial}{\partial x}(2x^2-3z) + \frac{\partial}{\partial y}(-2xy) + \frac{\partial}{\partial z}(-4x)$$

$$= 4x - 2x + 0$$

$$= 2x$$

$① \Rightarrow$

$$\therefore \iint \vec{F}\cdot\hat{n}\,ds = \int_0^2\int_0^{2-x}\int_0^{4-2x-2y} 2x\,dz\,dy\,dx.$$

$$= 2\iint x[z]_0^{4-2x-2y}\,dy\,dx.$$

$$= 2\int_0^2\int_0^{2-x} x\,[4-2x-2y]\,dy\,dx.$$

$$= 2\int_0^2\int_0^{2-x} (4x-2x^2-2xy)\,dy\,dx$$

$$\left[4xy - 2x^2y - xy^2\right]_0^{2-x}$$

$$= 2\int_0^2 \left[4x(2-x) - 2x^2(2-x) - x(2-x)^2\right]$$

$$\int_0^2 \left[8x - 4x^2 - 4x^2 + 2x^3 - 4x - x^3 + 2x^2\right]_0^2$$

$$\int \left[\cancel{x^3 - 6x^2 + 4x}\right]_0^2$$

$$= 8 \cancel{- 6(4) + 4(2)}$$

$$\cancel{8 - 24 +}$$

$$= \cancel{\frac{x^4}{4} - \frac{6x^3}{3} + \frac{4x^2}{2}}$$

$$= 2\int_0^2 (x^3 - 4x^2 + 4x)\, dx$$

$$= \left[\frac{x^4}{4} - \frac{4x^3}{3} + \frac{4x^2}{2}\right]_0^2$$

$$\left[\frac{16}{4} - \frac{4(8)}{3} + \frac{4(4)}{2}\right]$$

$$= 8/3$$

9. If $F = (x^2 - yz)\hat{i} + (y^2 - xz)\hat{j} + (z^2 - xy)\hat{k}$ evaluate $\iint_S \vec{F}\cdot\hat{n}\, ds$ using divergence theorem where the region is rectangular ~~piped~~ parallelopiped

i.e. $0 \le x \le a$

$0 \le y \le b$

$0 \le z \le c$

$$\iint_S \vec{F}\cdot\hat{n}\, ds = \iiint_V \operatorname{div}\vec{F}\, dv \longrightarrow (1)$$

$$\nabla\cdot\vec{F} = \frac{\partial}{\partial x}(x^2 - yz) + \frac{\partial}{\partial y}(y^2 - xz) + \frac{\partial}{\partial z}(z^2 - xy)$$

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$= 2x + 2y + 2z$

$2(x+y+z).$

$\therefore$ ① $\Rightarrow$ $= 2\int_0^a \int_0^b \int_0^c (x+y+z)\, dz\, dy\, dx.$

$= 2\int_0^a \int_0^b \left[\left(xz + yz + \frac{z^2}{2} \right) \right]_0^c dy\, dx$

$= 2\int_0^a \int_0^b \left[x(c-0) + y(c-0) + \frac{1}{2}(c^2-0) \right]$

$= 2c \int_0^a \left[xy + \frac{y^2}{2} + \frac{1}{2} c y \right]_0^b dx$

$= 2c \int_0^a \left[x(b-0) + \frac{1}{2}(b^2-0) + \frac{1}{2} c (b-0) \right]$

$= \cancel{2} bc \left[\frac{x^2}{\cancel{2}} + \frac{b}{\cancel{2}} \cdot x + \frac{c}{2} x \right]_0^a$

$= \cancel{2} \cdot bc \left[a^2 + ab + ac \right].$

$abc [a+b+c]$ //

<u>Cauchy's equation</u> :-

$x^2 y'' + 2xy' + y = 0$

$\log x = z$

$D(D-1)y + 2Dy + y = 0$

calculus of variations.

Functional. $\rightarrow I(y) = \int_{x_1}^{x_2} f(x, y, y')\,dx$

Euler's eq^n :- A necessary condition for the integral.

$$I(y) = \int_{x_1}^{x_2} f(x, y, y')\,dx$$ to be extremum is $\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0.$

or $\delta I = 0$

1. Find the extremal of the function

$$\int_{x_1}^{x_2} (y' + x^2 y'^2)\,dx.$$

$f(x, y, y') = y' + x^2 y'^2 \rightarrow (1)$

The necessary condition for the functional to be extremal is the Euler's eq^n is given by $\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0 \rightarrow (2)$

diff (1) partially w.r.t y.

$$\frac{\partial f}{\partial y} = 0. \rightarrow (3)$$

diff partially w.r.t y'. ~~= 1 + x²(2y')~~

$$\frac{\partial f}{\partial y'} = 1 + x^2(2y') = 1 + 2x^2 y' \rightarrow (4)$$

using (3) and (4) in (2).

(2) $\Rightarrow$

$$0-\frac{d}{dx}\left(1+2x^2y'\right)=0$$

Integrating the above eq^{n}

$$1+2x^2y'=K$$

$$1+2x^2\frac{dy}{dx}=K$$

$$2x^2\frac{dy}{dx}=K-1$$

$$\int 2\,dy=(K-1)\int\frac{dx}{x^2}$$

$$2y=-(K-1)\frac{1}{x}+C$$

$$2xy-Cx=(1-K)$$

$$\boxed{2yx-Cx=C_1}$$

Q- Find the function y which makes the Integral $(1+xy'+xy'^2)\,dx$ as extremum

$$\int_{x_1}^{x_2}(1+xy'+xy'^2)\,dx$$

$$f(x,y,y')=1+xy'+xy'^2$$

$$\frac{\partial f}{\partial y}=0,\quad \frac{\partial f}{\partial y'}=x+2xy'$$

Eulers eq^{n} $\quad \frac{\partial f}{\partial y}-\frac{d}{dx}\left[\frac{\partial f}{\partial y'}\right]=0.$

$$0-\frac{d}{dx}\left[x+2xy'\right]=0$$

Integrating the above eq^{n}

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$x + 2xy' = k_1$

$x[1 + 2y'] = k_1$

$\cancel{x}\left[1 + 2\frac{dy}{dx}\right] = k_1/x$

$$\int 2\,dy = \int \frac{k_1}{x}\,dx - 1$$

$$2y = [k_1 \log x - x] + k_2$$

$$y = \frac{1}{2}[k_1 \log x - x] + k_2$$

3. $$\int_{x_1}^{x_2} (y'^2 + ky^2)\,dx$$

$f(x, y, y') = y'^2 + ky^2$

$\frac{\partial f}{\partial y} = 2ky$

$\frac{\partial f}{\partial y'} \quad 2y'$

Eulers eq $\quad \frac{\partial f}{\partial y} - \frac{d}{dx}\left[\frac{\partial f}{\partial y'}\right] = 0.$

$$2ky - \frac{d}{dx}[2y'] = 0.$$

$$2ky = \frac{d}{dx}[2y']$$

$$\cancel{2}ky = \cancel{2}\frac{d}{dx}\left[\frac{dy}{dx}\right]$$

$$2ky = \frac{d^2y}{dx^2}$$

$$= \frac{d^2y}{dx^2} - ky = 0,$$

$D^2y - ky = 0$

$(D^2 - k)y = 0$

$m^2 - k = 0$

$m^2 = k$

$m = \sqrt{k}$ or 0.

i) if k is zero

$k = 0, \Rightarrow D^2y = 0.$

$$\frac{d^2y}{dx^2} = 0.$$

Integrate w.r.t x.

$$\frac{dy}{dx} = k_1$$

Integrate

$$\boxed{y = k_1 x + k_2}$$

ii) k is +ve

$k = p^2$

$\therefore (D^2 - p^2)y = 0.$

A.E $= m^2 - p^2 = 0.$

$m = \pm p$

$\therefore y = c_1 e^{px} + c_2 e^{-px}.$

iii) k is -ve, $k = -p^2$.

$(D^2 + p^2)y = 0.$

A.E $m^2 + p^2 = 0$ $\therefore m = \pm pi$

$y = c_1 \cos px + c_2 \sin px$

4. $\int_{x_1}^{x_2} (y^2 + y'^2 + 2ye^x)\, dx$

$(y^2 + y'^2 + 2ye^x) = f(x, y, y')$

$\frac{\partial f}{\partial y} = 2y + 2e^x$

$\frac{\partial f}{\partial y'} = 2y'$

Euler's eq

$\frac{\partial f}{\partial y} - \frac{d}{dx}\left[\frac{\partial f}{\partial y'}\right] = 0$

$[\cancel{2}y + \cancel{2}e^x] - \frac{d}{dx}[\cancel{2}y']$

$[y + e^x] - \frac{d^2y}{dx^2} = 0$

$-\frac{d^2y}{dx^2} + y + e^x = 0$

$\cancel{D^2 y}$

$\frac{d^2y}{dx^2} - y = e^x$

$(D^2 - 1)y = e^x$

$m^2 - 1 = 0$

$m = \pm 1$

$CF = C_1 e^x + C_2 e^{-x}$

$PI = \frac{1}{D^2 - 1} e^x$ Put $D = a$.

$= \frac{e^x}{1 \cdot 1 \cdot 0}$

$= \frac{x e^x}{2D} = \frac{x e^x}{2}$.

$y = CF + PI$

$y = c_1 e^x + c_2 e^{-x} + \frac{x e^x}{2}$.

5. $\int_{x_1}^{x_2} (x^2 y'^2 + 2y^2 + 2xy)\, dx$

$f(x, y, y') = x^2 y'^2 + 2y^2 + 2xy$.

$\frac{\partial f}{\partial y} = \cancel{x^2 2y} + 4y + 2x$.

$\frac{\partial f}{\partial y'} = 2x^2 y'$.

<u>Euler's eq</u>

$\frac{\partial f}{\partial y} - \frac{d}{dx}\left[\frac{\partial f}{\partial y'}\right] = 0$.

$[4y + 2x] - \frac{d}{dx}[2x^2 y'] = 0$

$\cancel{2}[2y + x] - \cancel{2}[x^2 y'' + y' \cancel{2}x] = 0$

$= 2y + x - [x^2 y'' + 2xy'] = 0$.

$\cancel{x^2 y'' + 2xy'}$.

$= 2y + x - x^2 y'' - 2xy' = 0$

$-x^2 y'' - 2xy' + 2y = -x$.

$x^2 y'' + 2xy' - 2y = x$

put $\log x = z$.

$x \frac{dy}{dx} = Dy$.

$x^2 \frac{d^2y}{dx^2} = D(D-1)y$

$D = d/dz$.

$= D(D-1)y + 2Dy - 2y = e^z$

$[D^2 - D + 2D - 2]y = e^z$

$[D^2 + D - 2]y = e^z$

A.E. $m^2 + m - 2 = 0$

$m = 1, -2$.

$CF = c_1 e^z + c_2 e^{-2z}$.

$PI = \frac{e^z}{(D^2 + D - 2)}$ put $D = a = 1$

$= 0$

$= \frac{z e^z}{2D + 1}$ put $D = 1$

$PI = \frac{z e^z}{3}$.

$y = CF + PI$

$y = c_1 e^z + c_2 e^{-2z} + \frac{z e^z}{3}$.

where $z = \log x$.

1. Find the extremal of the fun^n

$$\int_{x_0}^{x_1} \frac{y'^2}{x^3} \, dx.$$

Let $I = \int_{x_0}^{x} \frac{y_1^2}{x^3} \, dx \rightarrow (1)$

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0 \rightarrow (2)$$

$$f(x, y, y') = \frac{y'^2}{x^3} \rightarrow (3).$$

$$= 0 - \frac{d}{dx}\left[\frac{2y'}{x^3}\right] = 0.$$

$$= -2\left[\frac{x^3 y'' - y' 3x^2}{x^6}\right].$$

$$= -\frac{2}{x^4}\left[x y'' - 3y'\right] = 0.$$

$$= \quad xy'' - 3y' = 0.$$

$$xy'' = 3y'$$

$$\frac{y''}{y'} = \frac{3}{x}.$$

Integrating B.S.

$$\int \frac{y''}{y'} \, dy = \int 3/x \, dx \qquad \int \frac{f'(x)}{f(x)} = \log f$$

$$= \log y' = 3 \log x + \log c.$$

$$\log y' = \log c x^3$$

$$\frac{dy}{dx} = c x^3$$

$$y = \frac{c x^4}{4} + c_1$$

$$dy = c x^3 dx.$$

$$\int dy = c \int x^3 dx$$

8. Extremise the function $\int_{x_1} (y^2 + y'^2 + 2ye^x)$.

$f(x, y, y') = y^2 + y'^2 + 2ye^x$.

Euler's formula

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left[\frac{\partial f}{\partial y'}\right] = 0.$$

$$\left[2y + 2e^x - \frac{d}{dx}[2y']\right] = 0.$$

$$2y + 2e^x - 2\frac{d}{dx}\left[\frac{dy}{dx}\right] = 0$$

$$2[y + e^x - y''] = 0$$

$$y'' - y = e^x.$$

$$[D^2 - 1]y = e^x$$

$$m^2 - 1 = 0$$

$$m = \pm 1$$

$$CF = c_1 e^x + c_2 e^{-x}.$$

$$PI = \frac{e^x}{D^2 - 1} \quad \text{Put } D = 1$$

$$\frac{xe^x}{2D} = \frac{xe^x}{2}.$$

$$y = c_1 e^x + c_2 e^{-x} + \frac{xe^x}{2}.$$

9. ~~Th~~ Find the curve on which the functional

$I = \int_0^1 (y'^2 + 12xy)\,dx$ with

$y(0)=0$ & $y(1)=1$ can be extremised.

$f(x, y, y') = y'^2 + 12xy$

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left[\frac{\partial f}{\partial y'}\right] = 0 \longrightarrow (1)$$

$$12x - \frac{d}{dx}[2y'] = 0$$

$$12x - 2y'' = 0$$

$$\therefore 6x - y'' = 0$$

$$y'' = 6x$$

$$\frac{d^2y}{dx^2} = 6x$$

Integrating twice

$$\int \frac{d^2y}{dx^2} = \int 6x$$

$$\int \frac{dy}{dx} = \int \frac{6x^2}{2} + c_1$$

$$y = \cancel{3}\frac{x^3}{\cancel{3}} + c_1x + c_2$$

$$y = x^3 + c_1x + c_2 \longrightarrow (2)$$

$y(0) = 0$

$$0 = 0 + c_1(0) + c_2$$

$$\boxed{c_2 = 0}$$

$y(1) = 1$

$$1 = 1 + c_1 + c_2$$

$$\boxed{c_1 = 0} \quad \therefore \boxed{y = x^3}$$

10. Find the curve on which the functional

$$I = \int_0^{\pi/2} (y'^2 - y^2 + 2xy)$$ can be extremised

$y(0) = 0, \quad y(\pi/2) = 0.$

Sol:- $f(x, y, y') = y'^2 - y^2 + 2xy$

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left[\frac{\partial f}{\partial y'}\right] = 0.$$

$$-2y + 2x - \frac{d}{dx}[2y'] = 0.$$

$$-y + x - y'' = 0$$

$$y'' + y = x.$$

$$(D^2+1)y = x.$$

$$m^2 + 1 = 0$$

$$m = \pm i$$

$$CF = c_1 \cos x + c_2 \sin x$$

$$PI = \frac{1}{f(D)}$$

$$1+D^2 \,\Big)\, x \,\Big(\, x$$
$$x + 0$$
$$0$$

$$PI = x.$$

$$y = CF + PI$$

$$y = c_1 \cos x + c_2 \sin x + x.$$

$$y(0) = 0 \Rightarrow 0 = c_1 \qquad \boxed{c_1 = 0}$$

$$y(\pi/2) = 0 \Rightarrow 0 = c_2 + \pi/2$$

$$\boxed{c_1 = 0} \qquad \boxed{c_2 = -\pi/2}$$

$$\therefore y = -\frac{\pi}{2}\sin x + x.$$

11. $\int_0^{\pi} (y'^2 - y^2 + 4y\cos x)\,dx$ with $y(0)=0,\ y'(\pi)=0$.

$f(x,y,y') = y'^2 - y^2 + 4y\cos x$

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left[\frac{\partial f}{\partial y'}\right] = 0.$$

$$-2y + 4\cos x - \frac{d}{dx}[2y'] = 0.$$

$$-y + 2\cos x - y'' = 0.$$

$$y'' + y = 2\cos x.$$

$$[D^2+1]y = 2\cos x.$$

$m^2+1=0$.

$\therefore m = \pm i$

$CF = c_1\cos x + c_2\sin x$.

$\frac{2\cos x}{D^2+1}$ put $D^2 \to -a^2$.

$$\frac{2\cos x}{-1+1} = \cos x$$

$$= \frac{2x\cos x}{2D} = x\int \cos x \cdot dx$$
$$= x\sin x.$$

$$y = c_1\cos x + c_2\sin x + x\sin x.$$

$y(0)=0 \Rightarrow 0 = c_1 \quad \therefore \boxed{c_1 = 0}$

$y'(\pi)=0 \Rightarrow 0 =$

$$y' = -c_1\sin x + c_2\cos x + x\cos x + \sin x$$

$y'(\pi) = 0 \Rightarrow 0 = 0 + (-c_2) + (-\pi)$

$$\boxed{c_2 = -\pi}$$

$\therefore\ y = -\pi\sin x + x\sin x$ //

12 Solve the variation problem

$\delta \int_0^1 (x+y+y'^2)\,dx = 0$ under the condition

$y(0)=1$, $y(1)=2$.

The given integral is of the form $\delta I = 0$ which is same as Euler's eqn

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left[\frac{\partial f}{\partial y'}\right] = 0$$

$$= 0 - \frac{d}{dx}\left[2y'\right] = 0$$

$$2\frac{d^2y}{dx^2} = 0$$

$$\cancel{2y'' = 0}$$

$$2\frac{dy}{dx} = c_1$$

$$y = c_1 x + c_2$$

$$y = c_1 x + c_2$$

$y(0)=1 \Rightarrow \boxed{1 = c_2}$

$y(1) = 2 \Rightarrow 2 = c_1 + c_2$

$$\boxed{c_1 = 1}$$

$$\therefore \underline{y = x} + 1$$

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Eulers Equation :-

~~A nec~~

Show that the necessary condition for $I = \int_{x_1}^{x_2} f(x, y, y')$ to be extremum is,

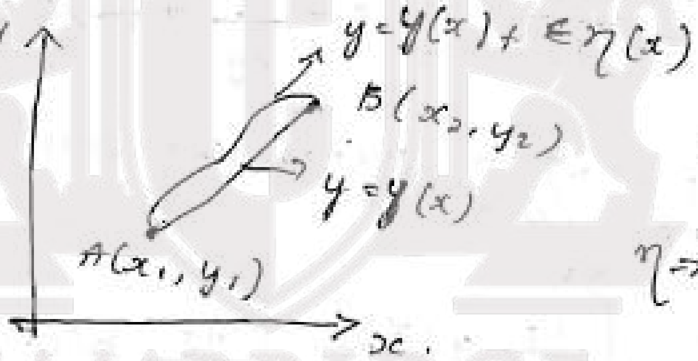
$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left[\frac{\partial f}{\partial y'}\right] = 0 \quad (or)$$

derive the eulers equation in the form

$$= \frac{\partial f}{\partial y} - \frac{d}{dx}\left[\frac{\partial f}{\partial y'}\right] = 0 .$$

wkt

$$J = \int_{x_1}^{x_2} f(x, y, y_1) \longrightarrow (1)$$



$\epsilon \to$ ~~Etta~~ Epcelon

$\eta \to$ Etta

→ Let $y = y(x)$ be the line joining the points $A(x_1, y_1)$ and $B(x_2, y_2)$ which makes I as extremum $\longrightarrow (2)$

→ consider the function $y = y(x) + \epsilon\eta(x)$ in the neighbourhood of $y = y(x)$

at A and B $\quad \eta(x_1) = 0 \;,\; \eta(x_2) = 0 \longrightarrow (3)$

$$y' = y'(x) + \epsilon\eta'(x) \longrightarrow (4)$$

The Functional I will be extremum for $\epsilon = 0$

For I to be ~~maximum~~ extrimum $\quad \frac{dI}{d\epsilon} = 0$

we've $\quad I = \int_{x_1}^{x_2} f(x, y, y') \rightarrow (1)$

differentiatin (1) w.r.t ϵ under the integral sign using Liebintz's Rule.

$$\frac{dI}{d\epsilon} = \int_{x_1}^{x_2} \left(\frac{\partial f}{\partial x} \frac{\partial x}{\partial \epsilon} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial \epsilon} + \frac{\partial f}{\partial y'} \cdot \frac{\partial y'}{\partial \epsilon} \right) \rightarrow (5)$$

Since x is independ of ϵ we've

$$\frac{\partial x}{\partial \epsilon} = 0 \quad , \frac{\partial y}{\partial \epsilon} = \partial \eta(x) \quad , \quad \frac{\partial y'}{\partial \epsilon} = \eta'(x)$$

{using (2)} $\qquad$ {using (4)}

$$(5) \Rightarrow \frac{dI}{d\epsilon} = \int_{x_1}^{x_2} \left(0 + \frac{\partial f}{\partial y} \eta(x) + \frac{\partial f}{\partial y'} [\eta'(x)] \right) dx$$

Integrating by parts the second element on the R.H.S

$$\frac{dI}{dt} = \int_{x_1}^{x_2} \frac{\partial f}{\partial y} \eta(x) + \int_{x_1}^{x_2} \frac{\partial f}{\partial y'} \cdot \eta'(x)\, dx .$$

$$= \int_{x_1}^{x_2} \frac{\partial f}{\partial y} \eta(x) + \cancel{\frac{\partial f}{\partial y'} \eta(x) \Big|_{x_1}^{x_2}} - \int_{x_1}^{x_2} \eta(x) \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)$$

$\rightarrow 0 \quad \because \eta(x_1) = \eta(x_2) = 0$

$$\frac{dI}{dt} = \int_{x_1}^{x_2} \left[\frac{\partial f}{\partial y} - \frac{d}{dx}\left[\frac{\partial f}{\partial y'}\right] \right] \eta(x)\, dx = 0$$

In order to have $\frac{dI}{d\epsilon} = 0$. we must have

$\frac{\partial f}{\partial y} - \frac{d}{dx}\left[\frac{\partial f}{\partial y'}\right]$. must be 0, which is the euler's equation

Geodesis [shortest distance]

A Geodesis on the surface is a curve along which the distance b/w any two point on the surface is minimum

Prove that, the shortest distance b/w two points in a plane is along the straight line joining them

(or)

P.T. The Geodesis on a Plane are Straight lines.

we know that, the derivative of arc length is given by

$$\frac{ds}{dx} = \sqrt{1+\left(\frac{dy}{dx}\right)^2}$$

$\therefore$ length of the arc $s = \int_{x_1}^{x_2} \sqrt{1+y'^2}\, dx$

now we've to extremise [minimise] the functional $I = s = \int_{x_1}^{x_2} \sqrt{1+y'^2}\, dx$

$\therefore f(x,y,y') = \sqrt{1+y'^2} \rightarrow (1)$

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left[\frac{\partial f}{\partial y'}\right] = 0 \rightarrow (2)$$

$$\therefore \frac{\partial f}{\partial y} = 0, \quad \frac{\partial f}{\partial y'} = \frac{1}{2\sqrt{1+y'^2}} \cdot 2y'$$

(2) $\Rightarrow$

$$0 - \frac{d}{dx}\left[\frac{y'}{\sqrt{1+y'^2}}\right] = 0.$$

$$= \frac{(\sqrt{1+y'^2})y'' - y'\frac{1}{2\sqrt{1+y'^2}}2y'}{(1+y'^2)^2} = 0.$$

$$= \frac{(1+y'^2)y'' - y'^2 y''}{\sqrt{1+y'^2}} = 0.$$

$$y'' + y'^2 y'' - y'^2 y'' = 0.$$

$$y'' = 0.$$

$$\frac{d^2y}{dx^2} = 0 \Rightarrow \int \frac{d^2y}{dx^2} = 0 \Rightarrow \frac{dy}{dx} = c_1$$

$$\Rightarrow \int \frac{dy}{dx} = \int c_1\, dx \Rightarrow$$

$$\boxed{y = c_1 x + c_2}$$

(or)

$$0 - \frac{d}{dx}\left[\frac{y'}{\sqrt{1-y'^2}}\right] = 0$$

on integration.

$$\frac{y'}{\sqrt{1-y'^2}} = c$$

$$y' = c(\sqrt{1-y'^2})$$

sq on B.S

$$y'^2 = c^2(1-y'^2).$$

$$y'^2 = c^2 - c^2y'^2$$

$$y'^2 + cy'^2 = c^2$$

$$y'^2(1+c) = c^2$$

$$y'^2 = \frac{c^2}{1+c}.$$

$$y' = \frac{c}{\sqrt{1+c}}.$$

$$\int y' = \int \frac{c}{\sqrt{1+c}}.$$

$$y = \frac{c}{\sqrt{1+c}} x + c_2.$$

$$y = cx + c_2$$

2. Find the curve passing to the pt. (x_1, y_1) & (x_2, y_2) which when rooted about x axis given minimum surface area.

(or)

PT. catenary is a which is rotated about ~~the~~ a line ~~extremum~~ generate a surface is minimum area.

The surface area is given by $\int_{x_1}^{x_2} 2\pi y \, ds$.

$\rightarrow$ Surface area $= \int_{x_1}^{x_2} 2\pi y \, ds$.

$$= \int_{x_1}^{x_2} 2\pi y \frac{ds}{dx} \cdot dx.$$

$$I = \int_{x_1}^{x_2} 2\pi y \sqrt{1+y'^2} \cdot dx.$$

Note :- ignoring the constant -2π.

$$f(x, y, y') = y\sqrt{1+y'^2} \rightarrow (1)$$

Note :- if f does not contain x, explicitely than ~~that~~ take euler eqn

~~f - f~~

$$f - y'\frac{\partial f}{\partial y'} = c \rightarrow (2)$$

$$y\sqrt{1+y'^2} - y'\left[y\frac{1}{\cancel{2}\sqrt{1+y'^2}}\cancel{2}y'\right] = c$$

$$\frac{y(1+y'^2) - yy'^2}{\sqrt{1+y'^2}} = c$$

$$\frac{y}{\sqrt{1+y'^2}} = c$$

$$y = c\sqrt{1+y'^2}$$

SqBs

$$y^2 = c^2(1+y'^2)$$

$$y^2 = c^2 + c^2y'^2$$

$$y^2 - c^2 = c^2y'^2$$

$$\frac{y^2-c^2}{c^2} = y'^2$$

$$y' = \frac{\sqrt{y^2-c^2}}{c}$$

$$\frac{dy}{dx} = \frac{\sqrt{y^2-c^2}}{c}$$

$$= \int\frac{dy}{\sqrt{y^2-c^2}} = \int\frac{dx}{c}$$

$$\Rightarrow \cosh^{-1}(y/c) = x/c + c_1$$

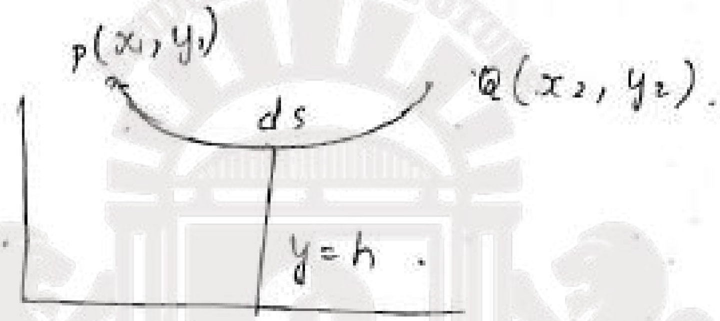
$$\therefore y = c \times \cosh\left(\frac{x+a}{c}\right)$$

$$y/c = \cosh\left(\frac{x+a}{c}\right)$$

A hanging cable hangs freely under gravity between two fixed points. ST. The shape of the cable is catinary.

Let $P(x_1, y_1)$ and $Q(x_2, y_2)$ be two fixed points of the hanging cabel, conside an elementary length ds..

Let ρ be the density of cable so that ρds is the mass of the element.



If g is accelarating due to gravity then the potential energy (mgh) is given by

$PE = \rho ds g y$.

$\therefore$ The total potential energy is given by

$$T = \int_{x_1}^{x_2} \rho g y \cdot \frac{ds}{dx} \, dx$$ x and divide by dx.

$$T = \int_{x_1}^{x_2} \rho g y \sqrt{1+y'^2} \cdot dx.$$

Ignoring the constant ρg:

$$f(x, y, y') = y\sqrt{1+y'^2} \rightarrow ①.$$

$$\boxed{f - y' \frac{\partial f}{\partial y'} = c} \rightarrow ②$$

$$y\sqrt{1-y^2} - y'\left[y \frac{1}{\cancel{2}\sqrt{1+y'^2}} \cancel{2} y'\right] = c.$$

$$\frac{y(1+y'^2) - y y'^2}{\sqrt{1+y'^2}} = c$$

$$\frac{y}{\sqrt{1+y'^2}} = c$$

$$y = c\sqrt{1+y'^2}$$

$$y^2 = c^2(1+y'^2)$$

$$y^2 = c^2 + c^2 y'^2$$

$$y^2 - c^2 = c^2 y'^2$$

$$\frac{y^2 - c^2}{c^2} = y'^2$$

$$y' = \frac{\sqrt{y^2-c^2}}{c}$$

$$\frac{dy}{dx} = \frac{\sqrt{y^2-c^2}}{c}$$

$$\int \frac{dy}{\sqrt{y^2-c^2}} = \int \frac{dx}{c}$$

$$\therefore \cosh^{-1}(y/c) = x/c + c_1 \qquad c \times c_1 = a$$

$$y/c = \cosh\left(\frac{x+a}{c}\right)$$

$$\therefore y = c \cosh\left(\frac{x+a}{c}\right)$$