Principal Stresses

* The value of normal & shear stresses at a point depends on the plane under consideration

* The angle of inclination is also a factor in determining the value of normal & shear stresses.

* In practical engg. application everyone is interested in find the max. normal(σ) & max shear stress, so that safety is insured.

→ To determine principal stresses & principal plane:

"The plane in which shear stress is zero & normal stress is max. Such a plane is called as principal plane, & that max. normal stress is called principal stress."

$$\frac{\partial \sigma_N}{\partial l} = 0 \qquad \frac{\partial \sigma_N}{\partial m} = 0$$

$$\sigma_N = \sigma_x l^2 + \sigma_y m^2 + \sigma_z n^2 + 2\tau_{xy} lm + 2\tau_{yz} mn + 2\tau_{zx} nl \text{ --- } (1)$$

$$\frac{\partial \sigma_N}{\partial l} = 0 ;$$

$$0 = 2\sigma_x l + 0 + \sigma_z 2n \frac{\partial n}{\partial l} + 2\tau_{xy} m + 2\tau_{yz} m \frac{\partial n}{\partial l} + 2\tau_{zx}\left[n + l\frac{\partial n}{\partial l}\right] \text{ --- } (2)$$

$$\boxed{\begin{aligned} l^2 + m^2 + n^2 &= 1 \\ n^2 &= 1 - l^2 - m^2 \end{aligned}}$$

Diff. (3) w.r.t 'x' $2n\frac{\partial n}{\partial l} = 0 - 2l$

$$\boxed{\frac{\partial n}{\partial l} = -l/n}$$

Re-arranging the eqn (2) ÷ by 2

$$\sigma_x l + \tau_{xy} m + \tau_{zx} n + \frac{\partial n}{\partial l}\left(\sigma_z n + \tau_{zy} m + \tau_{zx} l\right) = 0$$

$$\bar{X} + \bar{Z}(-l/n) = 0$$

$$\bar{X} = \bar{Z}(l/n)$$

$$\boxed{\frac{\bar{X}}{l} = \frac{\bar{Z}}{n}}$$

llly diff ① w.r.t 'm' & 'l'

$$\frac{\partial \sigma_n}{\partial m} \Rightarrow \frac{\bar{Y}}{m} = \frac{\bar{X}}{l}$$

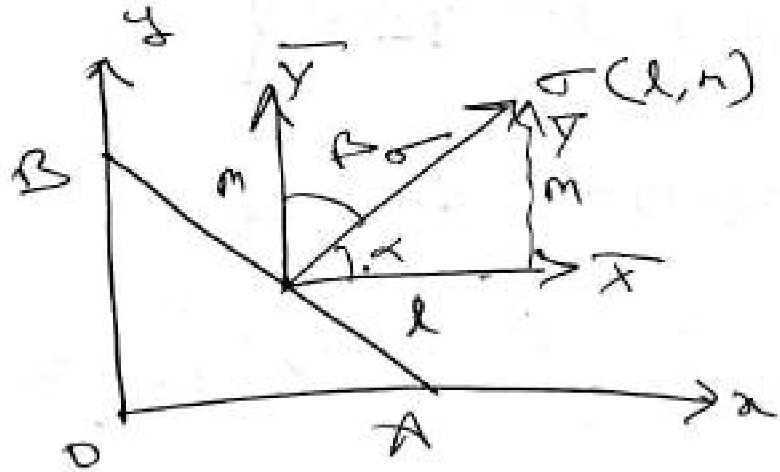
$$\frac{\partial \sigma_n}{\partial n} \Rightarrow \frac{\bar{Z}}{n} = \frac{\bar{Y}}{m}$$

∴ for 3D element $\boxed{\frac{\bar{X}}{l} = \frac{\bar{Y}}{m} = \frac{\bar{Z}}{n}}$

$$\cos\alpha = l = \frac{\bar{X}}{\sigma} \Rightarrow \frac{\bar{X}}{l} = \sigma$$

$$\cos\beta = m = \frac{\bar{Y}}{\sigma} \Rightarrow \frac{\bar{Y}}{m} = \sigma$$

$$\cos\gamma = n = \frac{\bar{Z}}{\sigma} \Rightarrow \frac{\bar{Z}}{n} = \sigma$$



$$\frac{\bar{X}}{l} = \frac{\bar{Y}}{m} = \frac{\bar{Z}}{n} = \sigma \quad \text{— ⑤}$$

$\boxed{\bar{X} = \sigma l}$ $\boxed{\bar{Y} = \sigma m}$ $\boxed{\bar{Z} = \sigma n}$

$$\bar{X} = \sigma_x l + \tau_{xy} m + \tau_{xz} n$$

$$\sigma \cdot l = \sigma_x l + \tau_{xy} m + \tau_{xz} n$$

1) $(\sigma_x - \sigma) l + \tau_{xy} m + \tau_{xz} n = 0$

2) $\tau_{xy} l + (\sigma_y - \sigma) m + \tau_{yz} n = 0$

3) $\tau_{xz} l + \tau_{yz} m + (\sigma_z - \sigma) n = 0$

$$\begin{bmatrix} \sigma_x - \sigma & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y - \sigma & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z - \sigma \end{bmatrix} \begin{bmatrix} l \\ m \\ n \end{bmatrix} = 0$$

$$\Rightarrow (\sigma_x - \sigma)\left[(\sigma_y - \sigma)(\sigma_z - \sigma) - \tau_{yz}\tau_{zy}\right]$$

$$- \tau_{xy}\left[\tau_{yx}(\sigma_z - \sigma) - \tau_{xz}\tau_{zx}\right]$$

$$+ \tau_{xz}\left[\tau_{zy}\tau_{yx} - (\sigma_y - \sigma)\tau_{zx}\right] = 0$$

$$\Rightarrow \left\{ \sigma^3 + \sigma^2(-\sigma_x - \sigma_y - \sigma_z) + \sigma(\sigma_x\sigma_y + \sigma_y\sigma_z + \sigma_z\sigma_x) - \sigma_x\sigma_y\sigma_z \right\}$$

$$+ (\sigma_x - \sigma)\tau_{yz}^2 + \tau_{xy}^2(\sigma_z - \sigma) - 2\tau_{xy}\tau_{xz}\tau_{yz}$$

$$+ \tau_{zy}^2(\sigma_y - \sigma) = 0$$

$$\Rightarrow \sigma^3 - \sigma^2(\sigma_x + \sigma_y + \sigma_z)$$

$$+ \sigma(\sigma_x\sigma_y + \sigma_y\sigma_z + \sigma_z\sigma_x - \tau_{xy}^2 - \tau_{yz}^2 - \tau_{zx}^2)$$

$$+ (\sigma_x\sigma_y\sigma_z + 2\tau_{xy}\tau_{yz}\tau_{zx} - \sigma_x\tau_{yz}^2 - \sigma_y\tau_{xz}^2 - \sigma_z\tau_{xy}^2)$$

$$\Rightarrow \boxed{\sigma^3 - \sigma^2 I_1 + \sigma I_2 - I_3 = 0} \quad — 6$$

The above eqn is called as stress characteristic relation

I_1, I_2 & I_3 are called 1st, 2nd & 3rd stress invariants

And these are independent of co-ordinates axes (x, y, z)

→ To determine direction cosines

$$\begin{bmatrix} \sigma_x - \sigma_i & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y - \sigma_i & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z - \sigma_i \end{bmatrix} \begin{bmatrix} l_i \\ m_i \\ n_i \end{bmatrix} = 0$$

$i = 1, 2, 3$

$$A_i = \begin{bmatrix} \sigma_y - \sigma_i & \tau_{yz} \\ \tau_{zy} & \sigma_z - \sigma_i \end{bmatrix}$$

$$B_i = \begin{bmatrix} \sigma_x - \sigma_i & \tau_{xz} \\ \tau_{zx} & \sigma_z - \sigma_i \end{bmatrix}$$

$$C_i = \begin{bmatrix} \sigma_x - \sigma_i & \tau_{xy} \\ \tau_{yx} & \sigma_y - \sigma_i \end{bmatrix}$$

Solution for above

$$\frac{l_i}{A_i} = \frac{m_i}{B_i} = \frac{n_i}{C_i} = K$$

$$K = \frac{\sqrt{l_i^2 + m_i^2 + n_i^2}}{\sqrt{A_i^2 + B_i^2 + C_i^2}}$$

$$K = \frac{1}{\sqrt{A_i^2 + B_i^2 + C_i^2}}$$

Direction cosines

$$l_i = A_i \cdot K = \frac{A_i}{\sqrt{A_i^2 + B_i^2 + C_i^2}}$$

Octahedral

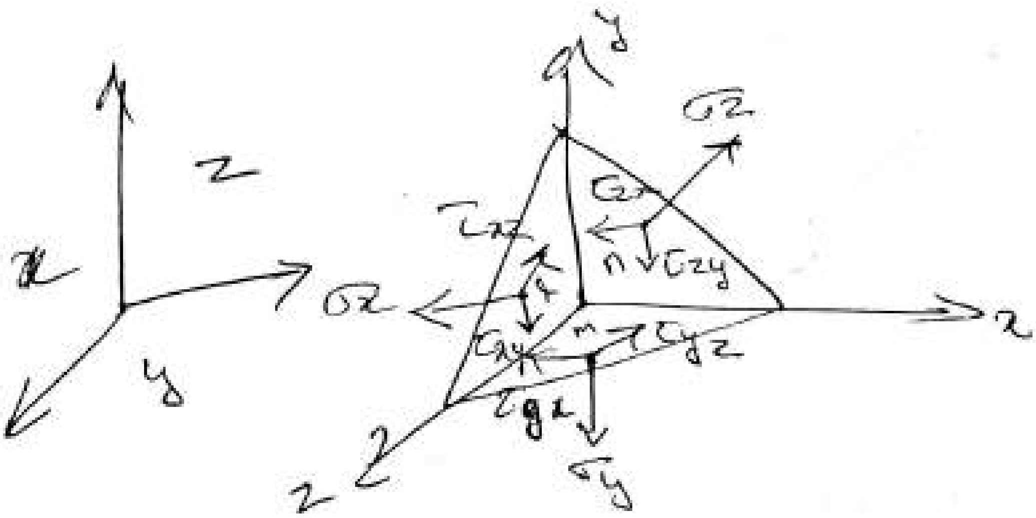
To derive relation for surface force

(or)

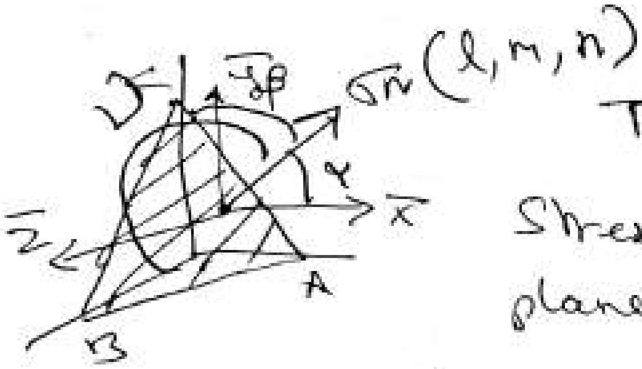
relations for boundary condition

(or)

Cauchy's boundary condn



Given



To find out stress in a given plane

$\bar{X}, \bar{Y}, \bar{Z} \rightarrow$ Normals (should be given)

(They should give a plane + its normal)

$$\begin{bmatrix} \bar{X} \\ \bar{Y} \\ \bar{Z} \end{bmatrix} = \begin{bmatrix} \sigma_x & \tau_{yx} & \tau_{zx} \\ \tau_{yx} & \sigma_y & \tau_{zy} \\ \tau_{xz} & \tau_{yz} & \sigma_z \end{bmatrix} \begin{bmatrix} l \\ m \\ n \end{bmatrix}$$

Gauss Th.

$\rightarrow \boxed{\sigma_R^2 = \sigma_N^2 + \tau^2}$

→ For tetrahedron to be in equilibrium

$\Sigma F_x = 0,\ \Sigma F_y = 0,\ \Sigma F_z = 0$

$\therefore \cancel{\bar{X} A = \sigma_x A l +}$

$\Sigma F_x = 0$

$\rightarrow \therefore \bar{X} A - (\tau_{yx} A) n - (\tau_{zx} A) m - (\sigma_x A) l = 0$

$\bar{X} A = (\tau_{yx} A) n + (\tau_{zx} A) m + (\sigma_x A) l$

$\cancel{\bar{X} = \tau_{xy}} \quad \boxed{\bar{X} = \sigma_x l + \tau_{xy} m + \tau_{xz} n}$ — (1)

$\rightarrow \Sigma F_y = 0$

$\bar{Y} A - (\tau_{xy} \cdot A) l - (\tau_{zy} \cdot A) n - (\sigma_y \cdot A) m = 0$

$\bar{Y} A = (\tau_{xy} \cdot A) l + (\tau_{zy} \cdot A) m + (\sigma_y \cdot A) n$

$\boxed{\bar{Y} = \sigma_y m + \tau_{yx} l + \tau_{yz} n}$ — (2)

$\rightarrow \Sigma F_z = 0$

$\bar{Z} A - (\sigma_z A) n - (\tau_{xz} A) l - (\tau_{yz} A) m$

$\boxed{\bar{Z} = (\tau_{zx}) l + (\tau_{zy}) m + (\sigma_z) n}$ — (3)

→ To determine normal stress

$\Sigma F = 0$, summing the forces acting on the plane = 0

$$\sigma_N \cancel{A} - \bar{X} \cancel{A} l - \bar{Y} \cancel{A} m - \bar{Z} \cancel{A} n = 0 \quad (A \to 0)$$

$$\sigma_N = (\bar{X}) l + (\bar{Y}) m + (\bar{Z}) n$$

$$\Rightarrow \sigma_N = (\sigma_x l + \tau_{xy} m + \tau_{xz} n) l$$
$$+ (\tau_{yx} l + \sigma_y m + \tau_{yz} n) m$$
$$+ (\tau_{zx} l + \tau_{zy} m + \sigma_z n) n = 0$$

$$\therefore \sigma_N = \sigma_x l^2 + \sigma_y m^2 + \sigma_z n^2 + 2\tau_{xy} lm + 2\tau_{yz} mn + 2\tau_{zx} nl$$

for 2d

$$\sigma_n = \sigma_x l^2 + \sigma_y m^2 + 2\tau_{xy} lm$$

1) Determine normal stress & shear stress. Also obtain for the following stress materials Direction cosines for shear stress $\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$

$$\tau_{ij} = \begin{bmatrix} 18 & 0 & 24 \\ 0 & -50 & 0 \\ 24 & 0 & 32 \end{bmatrix} \text{MPa}$$

∴ normal stress

$$\rightarrow \sigma_{n} = \sigma_x l^2 + \sigma_y m^2 + \sigma_z n^2 + 2\tau_{xy} lm + 2\tau_{yz} mn + 2\tau_{zx} nl$$

$$= 18\left(\frac{1}{3}\right) + (-50)\left(\frac{1}{3}\right) + 32\left(\frac{1}{3}\right) + 2(0) + 2(0) + 2 \times 24 \times \frac{1}{3}$$

$$\boxed{\sigma_n = 16 \text{ MPa}}$$

$$\rightarrow \bar{X} = \sigma_x l + \tau_{xy} m + \tau_{xz} n$$

$$= 18 \times \frac{1}{\sqrt{3}} + 0 + 24 \times \frac{1}{\sqrt{3}}$$

$$\boxed{\bar{X} = 24.248 \text{ MPa}}$$

$$\rightarrow \bar{Y} = \tau_{yx} l + \sigma_y m + \tau_{yz} n$$

$$= 0 - 50 \times \frac{1}{\sqrt{3}} + 0$$

$$\boxed{\bar{Y} = -28.687 \text{ MPa}}$$

$$\rightarrow \bar{Z} = \tau_{zx} l + \tau_{zy} m + \sigma_z n$$

$$= 24 \times \frac{1}{\sqrt{3}} + 0 + 32 \times \frac{1}{\sqrt{3}}$$

$$\boxed{\bar{Z} = 32.33 \text{ MPa}}$$

W.K.T $\sigma_R = \sqrt{\bar{x}^2 + \bar{y}^2 + \bar{z}^2}$

$= \cancel{\sqrt{49}}$

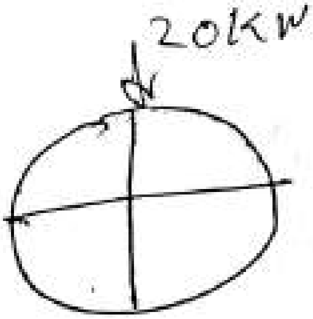
$= 49.66 \text{ MPa}$

$\sigma_R^2 = \sigma_N^2 + \tau^2$

$\tau^2 = \sigma_R^2 - \sigma_N^2$

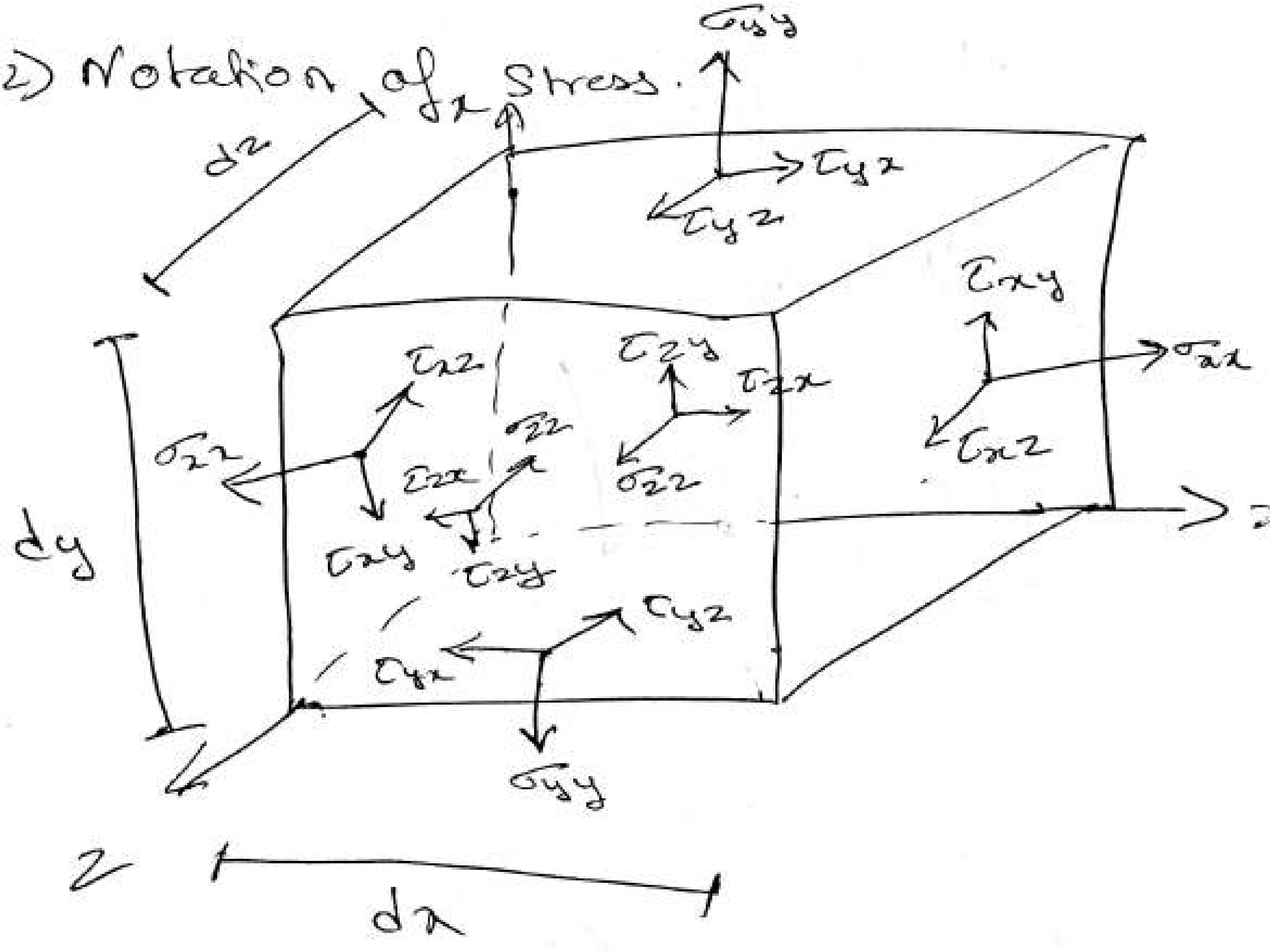
$\tau = 47.016 \text{ MPa}$

$$\sigma = \frac{20\times10^3}{20} = \underline{1\,kN/mm^2}$$



$$\sigma = \frac{20\times10^3}{5} = 4\,kN/mm^2$$

2) Notation of Stress.



6 – Normal Stresses

12 – Shear Stresses

Octahedral Stresses

→ "The maximum shear stress acting on an octahedral plane is called octahedral stress"

→ "A plane that is equally inclined to all the three principal axis is called octahedral pl."

→ Also the octahedral plane is free from normal stress

W.K.T $\sigma_R^2 = \sigma_{n_{oct}}^2 + \tau_{oct}^2$

$$\tau_{oct}^2 = \sigma_R^2 - \sigma_{n_{oct}}^2 \quad \text{--- (1)}$$

Also, $\sigma_n = \bar{X} l + \bar{Y} m + \bar{Z} n$

$= \sigma_1 l^2 + \sigma_2 m^2 + \sigma_3 n^2$

$\sigma_{n_{oct}} = \sigma_1 \left(\frac{1}{\sqrt{3}}\right)^2 + \sigma_2 \left(\frac{1}{\sqrt{3}}\right)^2 + \sigma_3 \left(\frac{1}{\sqrt{3}}\right)^2$

$\bar{X} = \sigma_1 l$, $\bar{Y} = \sigma_2 m$, $\bar{Z} = \sigma_3 n$ — Principle stress derivation

(or)

$$\boxed{\sigma_{n_{oct}} = \frac{\sigma_1 + \sigma_2 + \sigma_3}{3}} \quad \text{--- (2)}$$

Also, $\sigma_R^2 = \bar{X}^2 + \bar{Y}^2 + \bar{Z}^2$

$= (\sigma_1 l)^2 + (\sigma_2 m)^2 + (\sigma_3 n)^2$

$= \sigma_1^2 \left(\frac{1}{\sqrt{3}}\right)^2 + \sigma_2^2 \left(\frac{1}{\sqrt{3}}\right)^2 + \sigma_3 \left(\frac{1}{\sqrt{3}}\right)^2$

$$\boxed{\sigma_R^2 = \frac{\sigma_1^2 + \sigma_2^2 + \sigma_3^2}{3}} \quad \text{--- (3)}$$

put (3) + (2) in (1) ; $\tau_{oct}^2 = \sigma_R^2 - \sigma_{n\,oct}^2$

$$\tau_{oct}^2 = \underbrace{\frac{\sigma_1^2+\sigma_2^2+\sigma_3^2}{3}}_{(3)} - \underbrace{\left[\frac{(\sigma_1+\sigma_2+\sigma_3)}{3}\right]^2}_{(2)}$$

$$= \frac{3\sigma_1^2+3\sigma_2^2+3\sigma_3^2-\sigma_1^2-\sigma_2^2-\sigma_3^2-2\sigma_1\sigma_2-2\sigma_2\sigma_3-2\sigma_3\sigma_1}{9}$$

* [use: $(a+b+c)^2 = a^2+b^2+c^2+2ab+2bc+2ca$]

$$= \frac{2\sigma_1^2+2\sigma_2^2+2\sigma_3^2-2\sigma_1\sigma_2-2\sigma_2\sigma_3-2\sigma_3\sigma_1}{9}$$

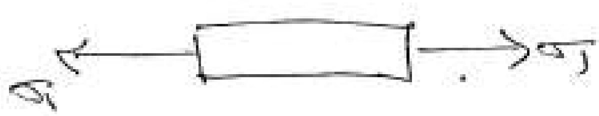
$\cancel{\tau_{oct} = \frac{1}{3}}$

$$\tau_{oct} = \frac{1}{3}\sqrt{2(\sigma_1^2+\sigma_2^2+\sigma_3^2)-2(\sigma_1\sigma_2+\sigma_2\sigma_3+\sigma_3\sigma_1)}$$

$$= \frac{1}{3}\sqrt{(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2}$$

$$9\tau_{oct} = (\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2$$

Case 1: For uniaxial load



$\sigma_1 \neq 0 ; \sigma_2 = \sigma_3 = 0$

$\cancel{\tau_{oct}^2 = \sigma_1^2 + \sigma_2^2}$

$\tau_{oct}^2 = \sigma_1^2 + \sigma_1^2 \Rightarrow \tau_{oct}^2 = \frac{2}{9}\sigma_1^2$

$\Rightarrow \boxed{\tau_{oct} = \frac{\sqrt{2}}{3}\sigma_1}$

Case 2: In terms of Stress Invariants

$$\tau_{oct} = \frac{1}{3}\sqrt{2(\sigma_1+\sigma_2+\sigma_3)^2 - 6(\sigma_1\sigma_2+\sigma_2\sigma_3+\sigma_3\sigma_1)}$$

W.K.T
$$\sigma_{ij} = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix}$$

$$I_1 = \sigma_1 + \sigma_2 + \sigma_3$$

$$I_2 = \sigma_x\sigma_y + \sigma_y\sigma_z + \sigma_z\sigma_x - \cancel{\tau_{xy}^2} - \cancel{\tau_{yz}^2} - \cancel{\tau_{zx}^2}$$

$$= \sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1$$

$$\Rightarrow \tau_{oct} = \frac{1}{3}\sqrt{2(\sigma_1+\sigma_2+\sigma_3)^2 - 6(\sigma_1\sigma_2+\sigma_2\sigma_3+\sigma_3\sigma_1)}$$

$$\tau_{oct} = \frac{1}{3}\sqrt{2I_1^2 - 6I_2}$$

→ Sphericd/volumetric/Dilational Stresses
(or) Hydrastatic stress
If stress acting on elements produced some change in volume with no distortion of the element, then it is called hydrastatic stress.

→ Deviatric stress (or) pure shear Stress
If the stress produces distortion only & no change in the volume of the element is called a (deviatric stress)
(or)
(pure stress)

Formulas

→ Principal Stresses

1) → $I_1 = \sigma_x + \sigma_y + \sigma_z$ [Stress Invariants]

→ $I_2 = \sigma_x\sigma_y + \sigma_y\sigma_z + \sigma_z\sigma_x - \tau_{xy}^2 - \tau_{yz}^2 - \tau_{zx}^2$

→ $I_3 = \begin{bmatrix} \sigma_x & \tau_{yz} & \tau_{zx} \\ \tau_{xy} & \sigma_y & \tau_{zy} \\ \tau_{xz} & \tau_{yz} & \sigma_z \end{bmatrix}$

$= \sigma_x\sigma_y\sigma_z + 2\tau_{xy}\tau_{yz}\tau_{zx} - \sigma_x\tau_{yz}^2 - \sigma_y\tau_{zx}^2 - \sigma_z\tau_{xy}^2$

2) Direction Cosines (l_i, m_i, n_i)

→ $l_i = A_i \cdot K = \dfrac{A_i}{\sqrt{A_i^2 + B_i^2 + C_i^2}}$

→ $m_i = B_i \cdot K = \dfrac{B_i}{\sqrt{A_i^2 + B_i^2 + C_i^2}}$

→ $n_i = C_i \cdot K = \dfrac{C_i}{\sqrt{A_i^2 + B_i^2 + C_i^2}}$

where, → $K = \dfrac{1}{\sqrt{A_i^2 + B_i^2 + C_i^2}}$

→ $A_i = \begin{bmatrix} \sigma_y - \sigma_i & \tau_{yz} \\ \tau_{zy} & \sigma_z - \sigma_i \end{bmatrix}$

→ $B_i = -\begin{bmatrix} \sigma_x - \sigma_i & \tau_{xz} \\ \tau_{zx} & \sigma_z - \sigma_i \end{bmatrix}$

→ $C_i = \begin{bmatrix} \sigma_x - \sigma_i & \tau_{xy} \\ \tau_{yx} & \sigma_y - \sigma_i \end{bmatrix}$

[Note: $\dfrac{l_i}{A_i} = \dfrac{m_i}{B_i} = \dfrac{n_i}{C_i} = K$]

$$\rightarrow \begin{bmatrix} \sigma_x - \sigma_i & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y - \sigma_i & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z - \sigma_i \end{bmatrix} \begin{bmatrix} l_i \\ m_i \\ n_i \end{bmatrix} = 0$$

3) Normal stresses ($\bar{X}, \bar{Y}, \bar{Z}$)

$\rightarrow \bar{X} = \sigma_x l + \tau_{xy} m + \tau_{xz} n$

$\rightarrow \bar{Y} = \tau_{yx} l + \sigma_y m + \tau_{yz} n$

$\rightarrow \bar{Z} = \tau_{zx} l + \tau_{zy} m + \sigma_z n$

4) Resultant Normal stress (σ_N)

$\rightarrow \sigma_N = \bar{X} l + \bar{Y} m + \bar{Z} n$

5) $\sigma_R^2 = \sigma_N^2 + \tau^2$

6) $\sigma^3 - \sigma^2 I_1 + \sigma I_2 - I_3 = 0$

- Formulas

→ Octahedral Stresses

1) $$\tau_{oct} = \frac{1}{3}\sqrt{(\sigma_x - \sigma_y)^2 + (\sigma_y - \sigma_z)^2 + (\sigma_z - \sigma_x)^2 + 6(\tau_{xy}^2 + \tau_{yz}^2 + \tau_{zx}^2)}$$

2) $$\sigma_{n_{oct}} = \frac{\sigma_1 + \sigma_2 + \sigma_3}{3}$$

3) $$\tau_{oct} = \frac{1}{3}\sqrt{2I_1^2 - 6I_2}$$

→ Principal Stresses (Problems)

1) June/July 2016 (10)

For the following state of stress. Determine the magnitude of principal stresses & direction of maximum principal stress.

$$[\sigma_{ij}] = \begin{bmatrix} 30 & -25 & 20 \\ -25 & 40 & 30 \\ 20 & 30 & -20 \end{bmatrix} \text{MPa.}$$

Sol: → $I_1 = 50\,\text{MPa}$

$I_2 = -2125\,(\text{MPa})$

$I_3 = -84500\,\text{MPa}$

→ $\sigma^3 - 50\sigma^2 - 2125\sigma - (-84500) = 0$

$\sigma_1 = 62.345\,\text{MPa}$

$\sigma_2 = 31.156\,\text{MPa}$

$\sigma_3 = -43.5\,\text{MPa}$

$$\rightarrow \begin{bmatrix} 30-62.345 & -25 & 20 \\ -25 & 40-62.345 & 30 \\ 20 & 30 & -20-62.345 \end{bmatrix} \begin{Bmatrix} l \\ m \\ n \end{Bmatrix} = 0$$

→ $A_1 = 939.999$

$A_2 = -1458.62$

$A_3 = -303.1$

→ $\sqrt{A_1^2 + A_2^2 + A_3^2} = \sqrt{\Sigma A^2} = 1761.545$

$$K = \frac{1}{\sqrt{\Sigma A^2}}$$

$\rightarrow l = \frac{A_1}{\sqrt{\Sigma A_0^2}} = 0.5336 = l_i \cdot k$

$\rightarrow m = \frac{A_2}{\sqrt{\Sigma A_0}} = -0.8375 = m_i \cdot k$

$\rightarrow n = \frac{A_3}{\sqrt{\Sigma A}} = -0.1740 = n_i \cdot k$

2) Determine the magnitude $(\sigma_1 \sigma_2 \sigma_3)$ and direction of principal stresses for the given stress tensor.

$$\sigma_{ij} = \begin{bmatrix} 0 & 10 & -10 \\ 10 & 0 & 20 \\ -10 & 20 & 0 \end{bmatrix} \text{ MPa.}$$

$\rightarrow I_1 = 0 \text{ MPa}$

$I_2 = \sigma_x\sigma_y + \sigma_y\sigma_z + \sigma_z\sigma_x - \tau_{xy}^2 - \tau_{yz}^2 - \tau_{zx}^2$
$= -600 \text{ MPa}^2$

$I_3 = \begin{vmatrix} & & \\ & & \end{vmatrix} = -4000 \text{ MPa}^3$

$\rightarrow \sigma^3 - 600\sigma + 4000 = 0$

$\sigma_1 = 20 \text{ MPa}, \sigma_2 = 7.32 \text{ MPa}, \sigma_3 = -27.32 \text{ MPa}$

$\rightarrow \sigma_1 = 20 \text{ MPa}, \quad \begin{bmatrix} 20 & 10 & -10 \\ 10 & 0 & 20 \\ -10 & 20 & 0 \end{bmatrix} \begin{Bmatrix} l_1 \\ m_1 \\ n_1 \end{Bmatrix} = 0$

$\rightarrow$ Minors

$A_q = \begin{vmatrix} -20 & 20 \\ 20 & -20 \end{vmatrix} = 0$

$B_q = \begin{vmatrix} 10 & 20 \\ -10 & -20 \end{vmatrix} = 0$

$$C_1 = \begin{vmatrix} 10 & -20 \\ -10 & 20 \end{vmatrix} = 0$$

$$\rightarrow K = \frac{1}{\sqrt{A^2 + B^2 + C^2}}$$

$$\rightarrow l_1 = A_1 \cdot K = 0$$

$$m_1 = B_1 \cdot K = 0$$

$$c_1 = C_1 \cdot K = 0$$

$$\rightarrow \sigma_2 = 7.32 \text{ MPa}$$

$$\begin{bmatrix} -7.32 & 20 \\ 20 & -7.32 \end{bmatrix} \begin{Bmatrix} l_2 \\ m_2 \\ n_2 \end{Bmatrix} = 0$$

Minors are

$$A_2 = \begin{vmatrix} -7.32 & 20 \\ 20 & 7.32 \end{vmatrix} = -346.417$$

$$B_2 = -\begin{vmatrix} 10 & 20 \\ -10 & -7.32 \end{vmatrix} = -126.8$$

$$C_2 = \begin{vmatrix} 10 & -7.32 \\ -10 & 20 \end{vmatrix} = 126.8$$

$$\rightarrow K = \frac{1}{\sqrt{A_2^2 + B_2^2 + C_2^2}} = \cancel{390.078}$$

$$\rightarrow l_2 = A_2 \cdot K = 0.888$$

$m_2 = B_2 \cdot K = -0.325$

$n_2 = C_2 \cdot K = 0.325 = 128$

$\rightarrow \sigma_3 = -27.32 \text{ MPa}$

$$\begin{bmatrix} 27.32 & 10 & 10 \\ 10 & 27.32 & 20 \\ -10 & 20 & 27.32 \end{bmatrix} \begin{Bmatrix} l_3 \\ m_3 \\ n_3 \end{Bmatrix} = 0$$

$$A_3 = \begin{vmatrix} 27.32 & 20 \\ 20 & 27.32 \end{vmatrix} = 346.38$$

$$B_3 = -\begin{vmatrix} 10 & 20 \\ -10 & 27.32 \end{vmatrix} = -473.2$$

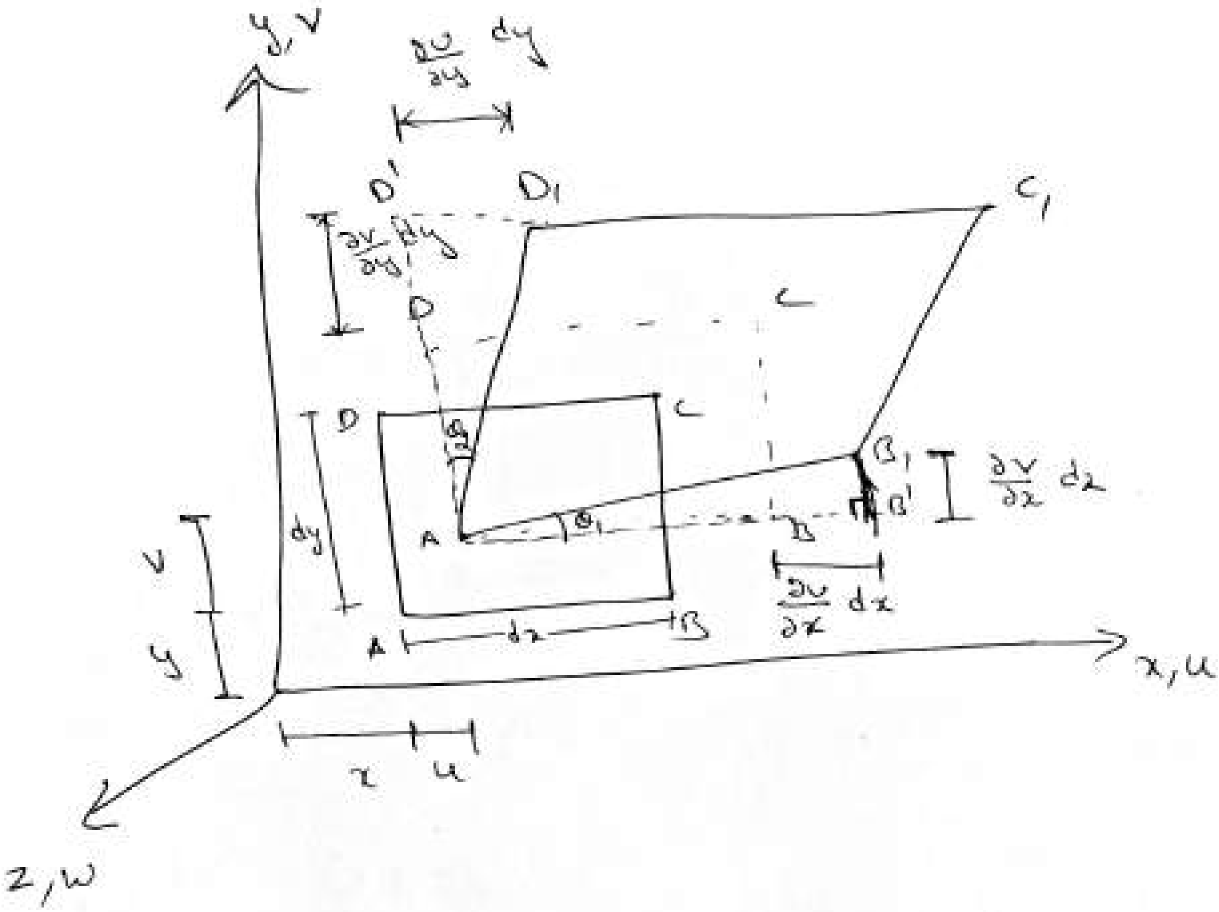
$$C_3 = \begin{vmatrix} 10 & 27.32 \\ -10 & 20 \end{vmatrix} = 473.2$$

$l_3 = A_3 \cdot K = 0.4597$

$m_3 = B_3 \cdot K = -0.6279$

$n_3 = C_3 \cdot K = 0.6279$

To determine strain components



→ After strain, the part. will undergo disp. u + v at x + y dir.

→ Consider 2D case, ABCD → $AB_1C_1D_1$

→ The change in length along x-dir is (AB' − AB)

$$\epsilon_x = \frac{AB' - AB}{AB}$$

$$AB \cdot \epsilon_x = AB' - AB$$

$$AB \cdot \epsilon_x + AB = AB'$$

$$(1 + \epsilon_x) AB = AB' \quad \text{——— (1)}$$

→ From fig

$$AB_1^2 = AB'^2 + B'B_1^2$$

$$= \left(dx + \frac{\partial u}{\partial x} dx\right)^2 + \left(\frac{\partial v}{\partial x} dx\right)^2$$

$$= dx^2 + \cancel{\left(\frac{\partial u}{\partial x} dx\right)^2} + 2\frac{\partial u}{\partial x} dx^2 + \cancel{\left(\frac{\partial v}{\partial x} dx\right)^2}$$

$$= dx^2 + 2\frac{\partial u}{\partial x} dx^2$$

*$(AB' = AB_1)$

$$(AB')^2 = \left(1 + 2\frac{\partial u}{\partial x}\right) dx^2 \quad \text{——— (2)}$$

Equate (1) & (2)

$$(AB')^2 = \left(1 + 2\frac{\partial u}{\partial x}\right) dx^2 \quad \&$$

$$(AB')^2 = (1+\epsilon_x)^2 (AB)^2$$

$$\left(1 + 2\frac{\partial u}{\partial x}\right) \cancel{dx^2} = (1+\epsilon_x)^2 \cancel{(dx)^2}$$

$$\cancel{1} + \cancel{2}\frac{\partial u}{\partial x} = \cancel{1} + \cancel{2}\epsilon_x + \cancel{\epsilon_x^2}$$

$$\boxed{\epsilon_x = \frac{\partial u}{\partial x}} \quad \text{lly} \quad \epsilon_y = \frac{\partial v}{\partial y}, \quad \epsilon_z = \frac{\partial w}{\partial x}$$

→ To det. shear strain (γ)

The shear strain at a pt. is defined as the change in the value of the angle b/w 2 elements, originally $\parallel^{le}$ to the x & y axis.

γ_{xy} at 'A' is the change in the angle b/w AB & AD.

$$\Rightarrow \gamma = \theta_1 + \theta_2 \quad \Rightarrow \gamma_{xy} = \tan\theta_1 + \tan\theta_2$$

$\Rightarrow$ ~~$\gamma_{xy} =$~~ $\tan\theta_1 = \frac{opp}{adj} = \frac{\frac{\partial v}{\partial x} dx}{\left(dx + \frac{\partial u}{\partial x} dx\right)}$

$$\tan\theta_1 = \frac{\frac{\partial v}{\partial x}\cancel{dx}}{\cancel{dx} + \cancel{\frac{\partial u}{\partial x}dx}}, \quad \tan\theta_2 = \frac{\frac{\partial u}{\partial y}\cancel{dy}}{\cancel{dy} + \cancel{\frac{\partial v}{\partial y}}dy}$$

$$\boxed{\gamma_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}}$$

$$\text{IIly } \gamma_{yz} = \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} \quad \text{IIly } \gamma_{zx} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}$$

$\gamma_{xy}, \gamma_{yz}, \gamma_{zx}$ – Shear strain components

Prepared by : Gökhan Karagöz
26.10.2009
Lecture note-8

Generalized Hook's Law

Stres-Strain Relation

Generalized Hooke's Law

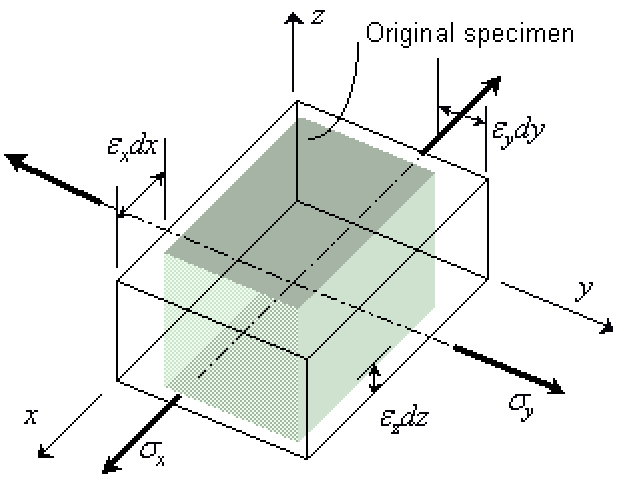
The generalized Hooke's Law can be used to predict the deformations caused in a given material by an arbitrary combination of stresses.

The linear relationship between stress and strain applies for $0 \le \sigma \le \sigma_{Yield}$

$$\varepsilon_x = \frac{\sigma_x}{E} - \nu\frac{\sigma_y}{E} - \nu\frac{\sigma_z}{E}$$

$$\varepsilon_y = -\nu\frac{\sigma_x}{E} + \frac{\sigma_y}{E} - \nu\frac{\sigma_z}{E}$$

$$\varepsilon_z = -\nu\frac{\sigma_x}{E} - \nu\frac{\sigma_y}{E} + \frac{\sigma_z}{E}$$



where: E is the Young's Modulus
n is the Poisson Ratio

The generalized Hooke's Law also reveals that strain can exist without stress. For example, if the member is experiencing a load in the y-direction (which in turn causes a stress in the y-direction), the Hooke's Law shows that strain in the x-direction does not equal to zero. This is because as material is being pulled outward by the y-plane, the material in the x-plane moves inward to fill in the space once occupied, just like an elastic band becomes thinner as you try to pull it apart. In this situation, the x-plane does not have any external force acting on them but they experience a change in length. Therefore, it is valid to say that strain exist without stress in the x-plane.

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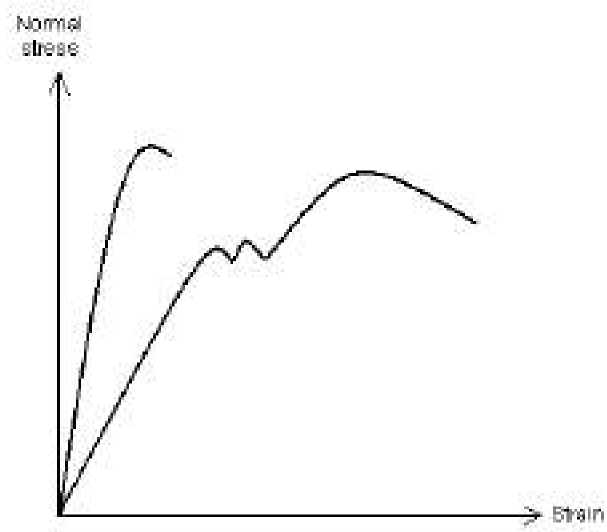
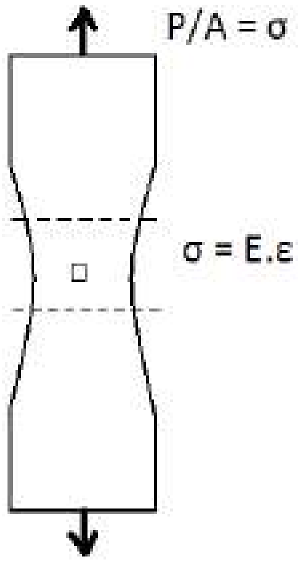
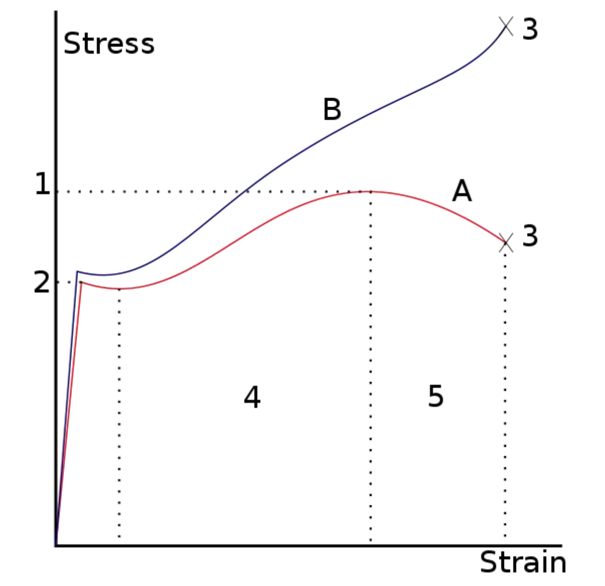
http://www.engineering.com/Library/ArticlesPage/tabid/85/articleType/A rticleView/articleId/208/Generalized-Hookes-Law.aspx

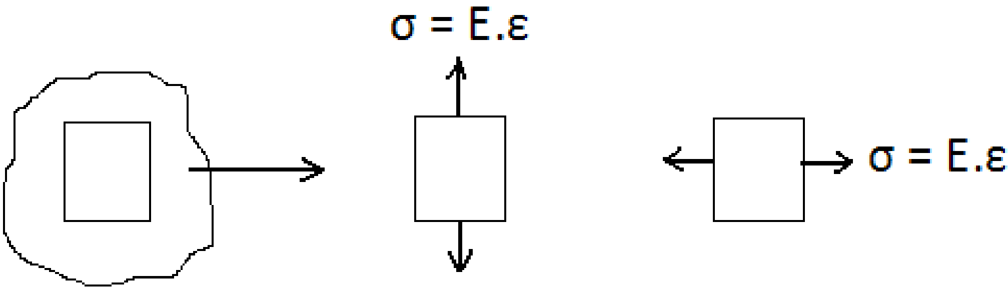


Stress-strain diagram

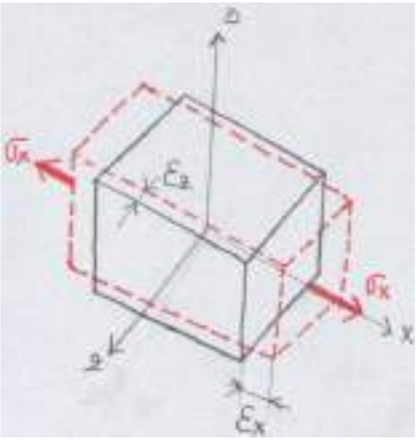


	Stress–strain curve for low-carbon steel. Hooke's law is only valid for the portion of the curve between the origin and the yield point(2). 1. Ultimate strength 2. Yield strength-corresponds to yield point. 3. Rupture 4. Strain hardening region 5. Necking region. A: Apparent stress (F/A0) B: True stress (F/A)

- We need to connect all six components of stres to six components of strain.
- Restrict to linearly elastic-small strains.
- An isotropic materials whose properties are independent of orientation.



Consider an elment on which there is only one component of normal stres acting.



$$\sigma_y = \sigma_z = \tau_{xy} = \tau_{yz} = 0$$

$$\varepsilon_x = \frac{1}{E}\sigma_x$$

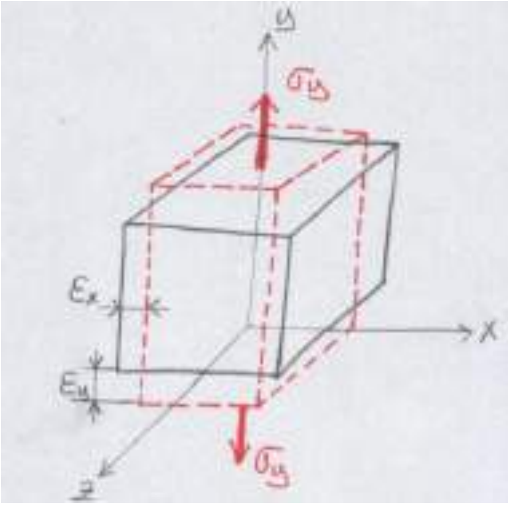
In addition to normal strain there is a lateral contraction

$\varepsilon_y = \varepsilon_z = -\nu\,\varepsilon_y = -\nu\,.\,\sigma_x/E$

There is no shear strain due to normal stres in isotropic materials.

$\gamma_{xy} = \gamma_{yz} = \gamma_{xz} = 0$ *(γ= gamma)*

- Now σ_y is applied



$\varepsilon_y = 1/E \,.\, \sigma_y$ because of isotropy

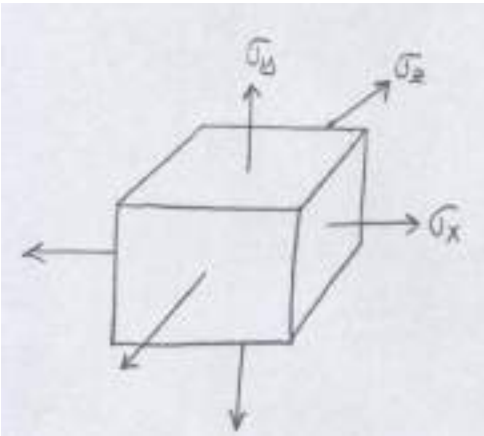
$\varepsilon_x = \varepsilon_z = -\nu\, \varepsilon_y = -\nu.\sigma_y/E$

Similar result for loading in the z direction

$$\varepsilon_z = \frac{\sigma_z}{E}$$

$\varepsilon_x = \varepsilon_y = -\nu\, \varepsilon_z = -\nu.\sigma_z/E$

$\varepsilon_x = \sigma_x/E - \nu/E \,.\, \gamma_y - \nu/E \,.\, \sigma_x$



* normal strains

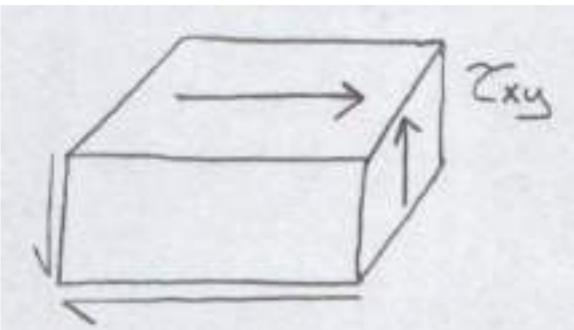
$\varepsilon_x = 1/E\, (\, \sigma_x - \nu(\sigma_y + \sigma_z)\,)$

$\varepsilon_y = 1/E\, (\, \sigma_y - \nu(\sigma_x + \sigma_z)\,)$

$\varepsilon_z = 1/E\, (\, \sigma_z - \nu(\sigma_x + \sigma_y)\,)$

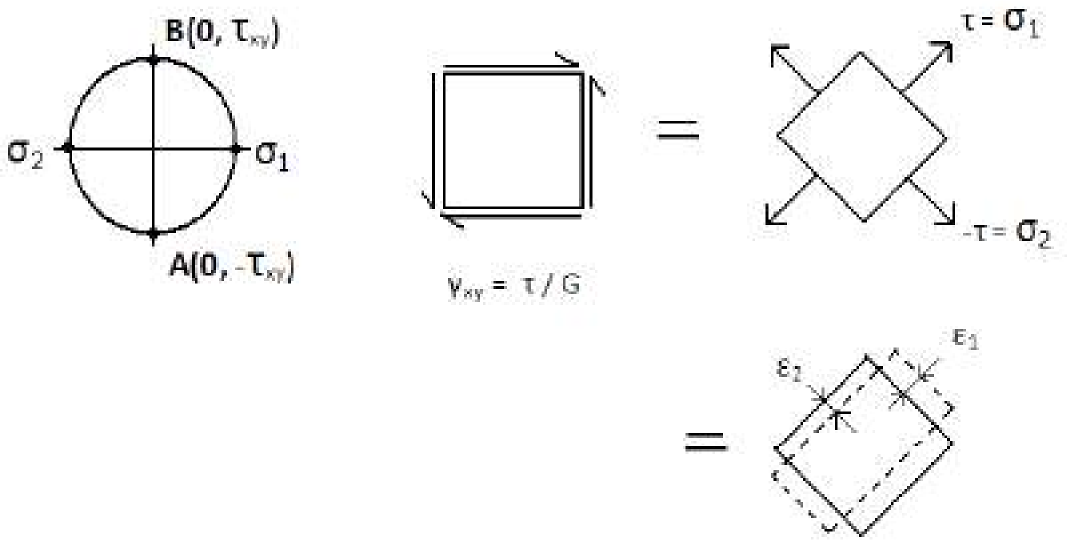
Shear Stres

Each shear stres component produces only its corresponding shear strain component.



$\gamma_{xy} = \tau_{xy}/G$ *(G: shear modulus)*

Relationship Between G, E and V



$$\varepsilon_1 = \frac{\sigma_1}{E} - \frac{\upsilon}{E}\sigma_2$$

$$\varepsilon_2 = \frac{\sigma_2}{E} - \frac{\upsilon}{E}\sigma_1 = \tau\frac{(1+\upsilon)}{E}$$

$$\frac{\gamma_{xy}}{2} = \frac{\varepsilon_1 - \varepsilon_2}{2} = \frac{2(1+\upsilon)\tau}{E}$$

$$G = \frac{E}{2(1+\upsilon)}$$

Just 2 independent elastic constant

ε_{xx}	ε_{yy}	ε_{zz}		$\varepsilon_{xy} = \gamma_{xy}/2$	$\varepsilon_{xz} = \gamma_{xz}/2$	$\varepsilon_{yz} = \gamma_{yz}/2$
ε_{11}	ε_{22}	ε_{33}		ε_{12}	ε_{13}	ε_{23}

$$\begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \varepsilon_{12} \\ \varepsilon_{13} \\ \varepsilon_{23} \end{bmatrix} = \left[\begin{array}{ccc|ccc} 1/E & -\nu/E & -\nu/E & & & \\ -\nu/E & 1/E & -\nu/E & & & \\ -\nu/E & -\nu/E & 1/E & & & \\ \hline & & & 1/2G & & \\ & & & & 1/2G & \\ & & & & & 1/2G \end{array}\right] = \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{13} \\ \sigma_{23} \end{bmatrix}$$

Hooke's Law in Compliance Form

By convention, the 9 elastic constants in orthotropic constitutive equations are comprised of 3 Young's modulii E_x, E_y, E_z, the 3 Poisson's ratios ν_{yz}, ν_{zx}, ν_{xy}, and the 3 shear modulii G_{yz}, G_{zx}, G_{xy}.

The **compliance matrix** takes the form,

$$\begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \varepsilon_{yz} \\ \varepsilon_{zx} \\ \varepsilon_{xy} \end{bmatrix} = \begin{bmatrix} \frac{1}{E_x} & -\frac{\nu_{yx}}{E_y} & -\frac{\nu_{zx}}{E_z} & 0 & 0 & 0 \\ -\frac{\nu_{xy}}{E_x} & \frac{1}{E_y} & -\frac{\nu_{zy}}{E_z} & 0 & 0 & 0 \\ -\frac{\nu_{xz}}{E_x} & -\frac{\nu_{yz}}{E_y} & \frac{1}{E_z} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2G_{yz}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2G_{zx}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2G_{xy}} \end{bmatrix} \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{yz} \\ \sigma_{zx} \\ \sigma_{xy} \end{bmatrix}$$

where $\frac{\nu_{yz}}{E_y} = \frac{\nu_{zy}}{E_z}, \ \frac{\nu_{zx}}{E_z} = \frac{\nu_{xz}}{E_x}, \ \frac{\nu_{xy}}{E_x} = \frac{\nu_{yx}}{E_y}$.

Note that, in orthotropic materials, there is no interaction between the normal stresses σ_x, σ_y, σ_z and the shear strains ε_{yz}, ε_{zx}, ε_{xy}

The factor 1/2 multiplying the shear modulii in the compliance matrix results from the difference between shear strain and engineering shear strain, where
$\gamma_{xy} = \varepsilon_{xy} + \varepsilon_{yx} = 2\varepsilon_{xy}$, etc.

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{13} \\ \sigma_{23} \end{bmatrix} = \left[\begin{array}{ccc|ccc} 2G+\lambda & \lambda & \lambda & & & \\ \lambda & 2G+\lambda & \lambda & & & \\ \lambda & \lambda & 2G+\lambda & & & \\ \hline & & & 2G & 0 & 0 \\ & & & 0 & 2G & 0 \\ & & & 0 & 0 & 2G \end{array}\right] = \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \varepsilon_{12} \\ \varepsilon_{13} \\ \varepsilon_{23} \end{bmatrix}$$

$\sigma_{11} = (2G+\lambda).\varepsilon_{11} + \lambda.(\varepsilon_{22} + \varepsilon_{33})$

$\sigma_{11} = 2G.\varepsilon_{11} + \lambda.(\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33})$

$\sigma_{22} = (2G+\lambda).\varepsilon_{22} + \lambda.(\varepsilon_{11} + \varepsilon_{33})$

$\sigma_{22} = 2G.\varepsilon_{22} + \lambda.(\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33})$

$\sigma_{33} = (2G+\lambda).\varepsilon_{33} + \lambda.(\varepsilon_{11} + \varepsilon_{22})$

$\sigma_{33} = 2G.\varepsilon_{33} + \lambda.(\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33})$

where

$$\lambda = \frac{\upsilon E}{(1+\upsilon)(1-2\upsilon)} \qquad G = \frac{E}{2(1+\upsilon)}$$

$$\sigma_{ij} = 2G\varepsilon_{ij} + \lambda\delta_{ij}\varepsilon_{kk}$$ i,j= 1,2,3......

i=1 j=2

$$\sigma_{12} = 2G\varepsilon_{12} + \lambda\delta_{12}(\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33})$$ *here (δ=0)*

i=1 j=1

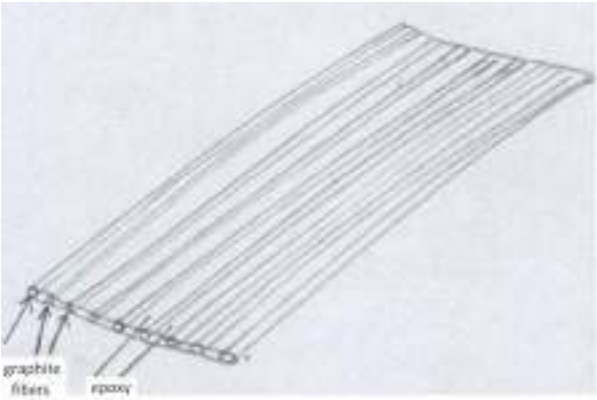
$$\sigma_{11} = 2G\varepsilon_{11} + \lambda\delta_{11}(\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33})$$ *here (δ=1)*

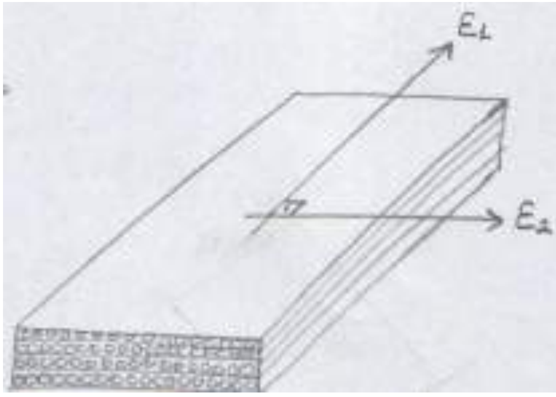
$$\sigma_{11} = 2G\varepsilon_{11} + \lambda(\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33})$$

Materials with different properties in different directions are called **anisotropic**.

Exp :

Composites materials





$$
\begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \hline \varepsilon_{12} \\ \varepsilon_{13} \\ \varepsilon_{23} \end{bmatrix}
=
\left[\begin{array}{ccc|ccc}
C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\
C_{21} & C_{22} & C_{23} & C_{24} & \cdots & \\
C_{31} & C_{32} & C_{33} & C_{34} & \cdots & \\
\hline
\vdots & \vdots & & & & \\
& & & & &
\end{array}\right]
=
\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \hline \sigma_{12} \\ \sigma_{13} \\ \sigma_{23} \end{bmatrix}
$$

If there are axes of symmetry in 3 perpendicular directions, material is called **ORTHOTROPIC** materials.

An **orthotropic material** has two or three mutually orthogonal two-fold axes of rotational symmetry so that its mechanical properties are, in general, different along the directions of each of the axes. Orthotropic materials are thus **anisotropic**; their properties depend on the direction in which they are measured. An **isotropic material**, in contrast, has the same properties in every direction.

One common example of an orthotropic material with two axes of symmetry would be a polymer reinforced by parallel glass or graphite fibers. The strength and stiffness of such a composite material will usually be greater in a direction parallel to the fibers than in the transverse direction. Another example would be a biological membrane, in which the properties in the plane of the membrane will be different from those in the perpendicular direction. Such materials are sometimes called transverse isotropic.

A familiar example of an orthotropic material with three mutually perpendicular axes is wood, in which the properties (such as strength and stiffness) along its grain and in each of the two perpendicular directions are different. Hankinson's equation provides a means to quantify the difference in strength in different directions. Another example is a metal which has been rolled to form a sheet; the properties in the rolling direction and each of the two transverse directions will be different due to the anisotropic structure that develops during rolling.

It is important to keep in mind that a material which is anisotropic on one length scale may be isotropic on another (usually larger) length scale. For instance, most metals are polycrystalline with very small grains. Each of the individual grains may be anisotropic, but if the material as a whole comprises many randomly oriented grains, then its measured mechanical properties will be an average of the properties over all possible orientations of the individual grains.

Generalized Hooke's Law (Anisotropic Form)

Cauchy generalized Hooke's law to three dimensional elastic bodies and stated that the 6 components of stress are linearly related to the 6 components of strain.

The stress-strain relationship written in matrix form, where the 6 components of stress and strain are organized into column vectors, is,

$$\begin{bmatrix}\varepsilon_{xx}\\ \varepsilon_{yy}\\ \varepsilon_{zz}\\ \varepsilon_{yz}\\ \varepsilon_{zx}\\ \varepsilon_{xy}\end{bmatrix}=\begin{bmatrix}S_{11}&S_{12}&S_{13}&S_{14}&S_{15}&S_{16}\\ S_{21}&S_{22}&S_{23}&S_{24}&S_{25}&S_{26}\\ S_{31}&S_{32}&S_{33}&S_{34}&S_{35}&S_{36}\\ S_{41}&S_{42}&S_{43}&S_{44}&S_{45}&S_{46}\\ S_{51}&S_{52}&S_{53}&S_{54}&S_{55}&S_{56}\\ S_{61}&S_{62}&S_{63}&S_{64}&S_{65}&S_{66}\end{bmatrix}\begin{bmatrix}\sigma_{xx}\\ \sigma_{yy}\\ \sigma_{zz}\\ \sigma_{yz}\\ \sigma_{zx}\\ \sigma_{xy}\end{bmatrix}, \quad \varepsilon = \mathbf{S}\cdot\sigma$$

or,

$$\begin{bmatrix}\sigma_{xx}\\ \sigma_{yy}\\ \sigma_{zz}\\ \sigma_{yz}\\ \sigma_{zx}\\ \sigma_{xy}\end{bmatrix}=\begin{bmatrix}C_{11}&C_{12}&C_{13}&C_{14}&C_{15}&C_{16}\\ C_{21}&C_{22}&C_{23}&C_{24}&C_{25}&C_{26}\\ C_{31}&C_{32}&C_{33}&C_{34}&C_{35}&C_{36}\\ C_{41}&C_{42}&C_{43}&C_{44}&C_{45}&C_{46}\\ C_{51}&C_{52}&C_{53}&C_{54}&C_{55}&C_{56}\\ C_{61}&C_{62}&C_{63}&C_{64}&C_{65}&C_{66}\end{bmatrix}\begin{bmatrix}\varepsilon_{xx}\\ \varepsilon_{yy}\\ \varepsilon_{zz}\\ \varepsilon_{yz}\\ \varepsilon_{zx}\\ \varepsilon_{xy}\end{bmatrix}, \quad \sigma = \mathbf{C}\cdot\varepsilon$$

where **C** is the **stiffness matrix**, **S** is the **compliance matrix**, and **S** = $\mathbf{C}^{-1}$.

In general, stress-strain relationships such as these are known as **constitutive relations**.

In general, there are 36 stiffness matrix components. However, it can be shown that conservative materials possess a strain energy density function and as a result, the stiffness and compliance matrices are symmetric. Therefore, only 21 stiffness components are actually independent in Hooke's law. The vast majority of engineering materials are conservative.

Please note that the **stiffness** matrix is traditionally represented by the symbol **C**, while **S** is reserved for the **compliance** matrix. This convention may seem backwards, but perception is not always reality. For instance, Americans hardly ever use their feet to play (American) football.

http://www.efunda.com/formulae/solid_mechanics/mat_mechanics/hooke.cfm

15 Governing Equation

1-) Equations of equilibrium (3)

$$\frac{\partial\sigma_{11}}{\partial x_1}+\frac{\partial\sigma_{12}}{\partial x_2}+\frac{\partial\sigma_{13}}{\partial x_3}+B_1=0 \qquad i=1$$
$$\frac{\partial\sigma_{21}}{\partial x_1}+\frac{\partial\sigma_{22}}{\partial x_2}+\frac{\partial\sigma_{23}}{\partial x_3}+B_2=0 \qquad i=2$$
$$\frac{\partial\sigma_{31}}{\partial x_1}+\frac{\partial\sigma_{32}}{\partial x_2}+\frac{\partial\sigma_{33}}{\partial x_3}+B_3=0 \qquad i=3$$

$$\frac{\partial\sigma_{ij}}{\partial x_j}+B_i$$

2-) Strain Displacement Equations (6)

$$\varepsilon_{11}=\frac{\partial U_1}{\partial x_1} \qquad \varepsilon_{22}=\frac{\partial U_2}{\partial x_2} \qquad \varepsilon_{33}=\frac{\partial U_3}{\partial x_3}$$

$$\varepsilon_{12}=\frac{1}{2}\left(\frac{\partial U_1}{\partial x_2}+\frac{\partial U_2}{\partial x_1}\right) \qquad \varepsilon_{13}=\frac{1}{2}\left(\frac{\partial U_1}{\partial x_3}+\frac{\partial U_3}{\partial x_1}\right) \qquad \varepsilon_{23}=\frac{1}{2}\left(\frac{\partial U_2}{\partial x_3}+\frac{\partial U_3}{\partial x_2}\right)$$

2-D Strain Compatibility

$$\frac{\partial^2 \varepsilon_{11}}{\partial x_1^2} + \frac{\partial^2 \varepsilon_{22}}{\partial x_2^2} = 2\frac{\partial^2 \varepsilon_{12}}{\partial x_1 \partial x_2}$$

3-) Generalized Hook's Law-Stress-Strain (6)

$$\varepsilon_{11} = \frac{1}{E}\left(\sigma_{11} - \upsilon(\sigma_{22} - \sigma_{33})\right) \qquad \varepsilon_{12} = \frac{1}{2G}\sigma_{12}$$

$$\varepsilon_{22} = \frac{1}{E}\left(\sigma_{22} - \upsilon(\sigma_{11} - \sigma_{33})\right) \qquad \varepsilon_{13} = \frac{1}{2G}\sigma_{13}$$

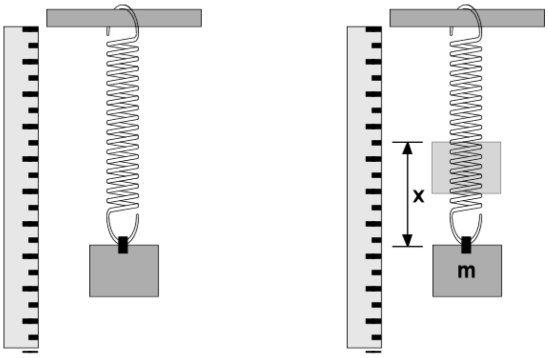
$$\varepsilon_{33} = \frac{1}{E}\left(\sigma_{33} - \upsilon(\sigma_{11} - \sigma_{22})\right) \qquad \varepsilon_{23} = \frac{1}{2G}\sigma_{23}$$

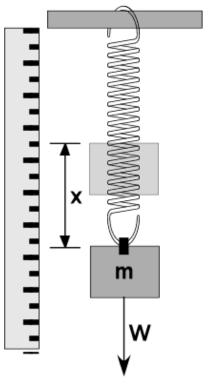
Question :
I have a spring, ruler,3 known masses, and 1 unknown mass. How would I find the unknown mass using these materials? Is it possible to solve using Hooke's Law? It would be very helpful if you guys can provide some equations or include any diagrams. Also how would I derive the needed equations from a graph?
Answer:
Hook's Law says "the restoring force of the $\boxed{\text{spring}}$ is proportional to the extension or compression of the spring from its equilibrium." In formula form its F=-kx (the negative indicates that the force is in the opposite direction from the extension, x).

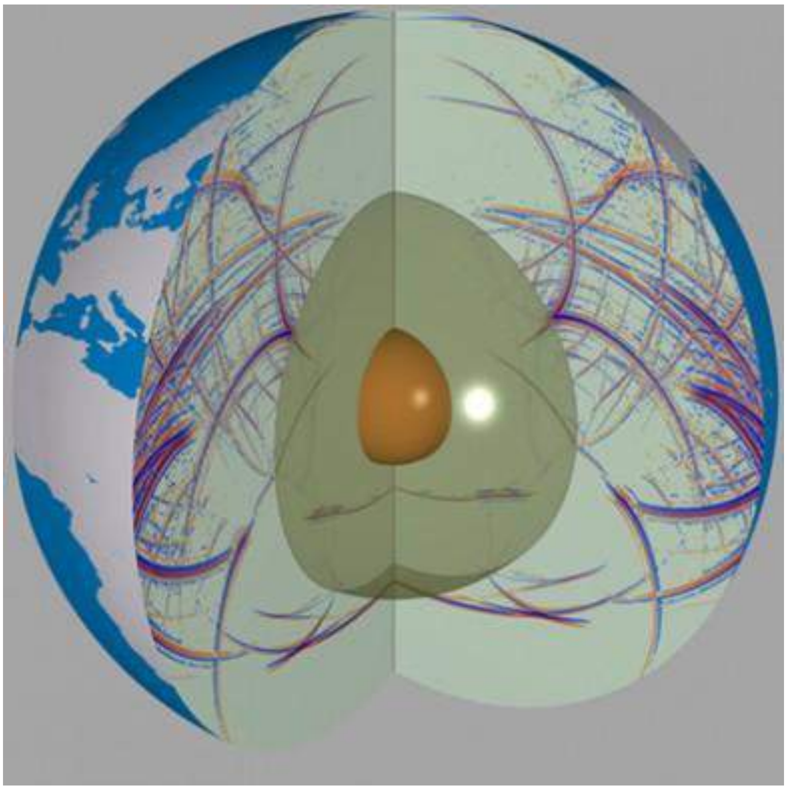
So, for every spring, there is a constant, k. Use your known masses and find how much of an "x" they will get on your spring. Now you have three sets of F and x. How are they related? Through "k".

Find k. Now you have k and you can measure the x of the unknown mass to get its weight (F).

Graphically: think "slope."







Hooke's law for isotropic continua, elastic wave equation, reflection and refraction methods for imaging the Earth's internal structure, plane waves in an infinite medium and interaction with boundaries, body wave seismology, inversion of travel-time curves, generalized ray theory, crustal seismology, surface waves and earthquake source studies

Strain Analysis

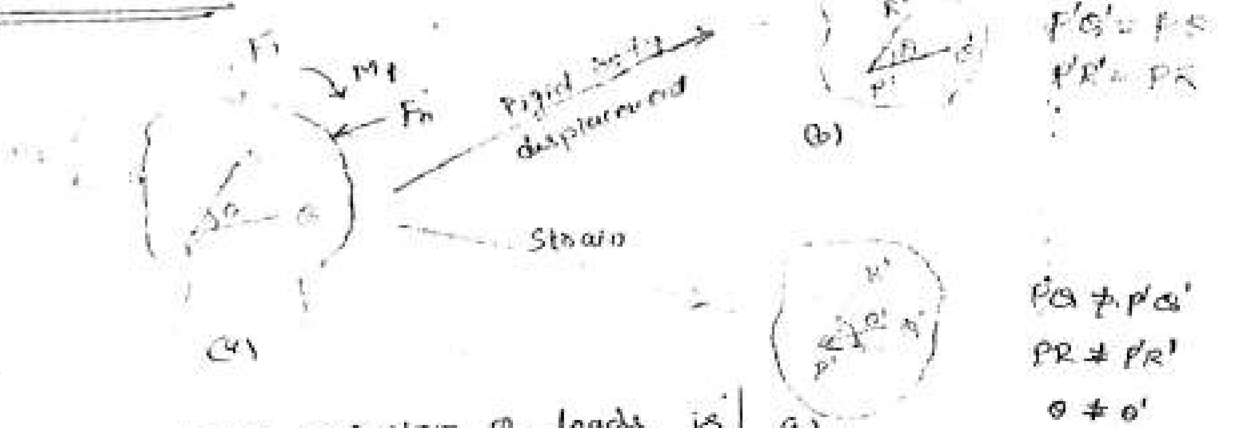
Strain :- Strain is a quantity used to provide a measure of the intensity of a deformation [deformation / unit length]. Just as stress which is used to provide measure of intensity of an internal force.

Similar to 2 types of stresses, normal stress σ and shearing stress τ the same classification can be used for strain (1) normal strain (ϵ) (2) Shearing strain (γ)

(1) Is used to provide a measure of the elongation (or) contraction of an arbitrary line segments in a body during deformations.

(2) Is used to provide a measure of angular distortion [change in angle b/w the two lines that are orthogonal in the undeformed state]. The deformation (or) strain may be the result of a change in temp, (or) of a stress (or) of other physical phenomena such as grain growth (or) shrinkage.

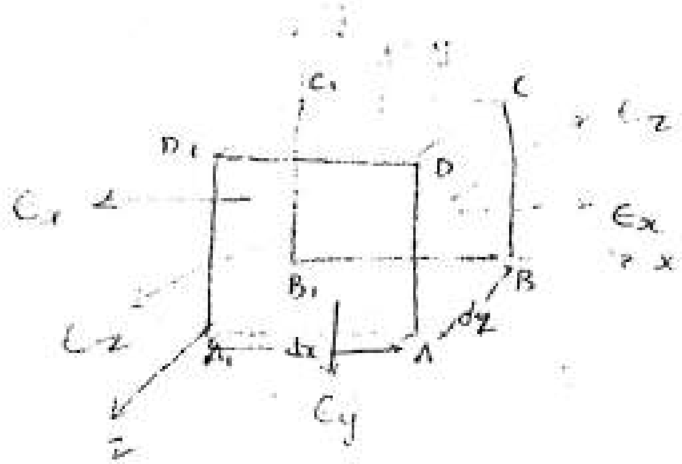
Interaction :-

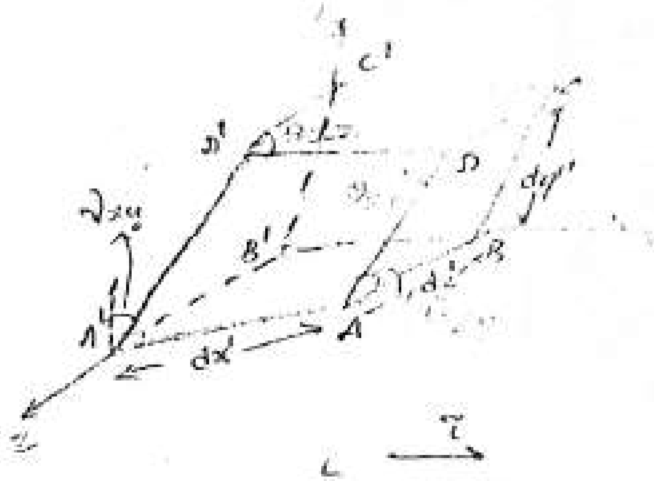


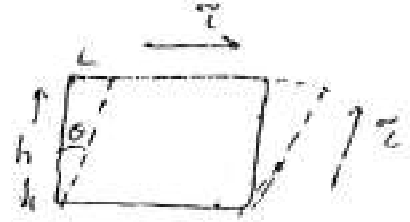
When a system of loads is applied to a machine elements (or) structural element individual points of the body generally move. This movement of a point w.r.to some convinent refference system of axis is known as a displacement.

In some instances displacement are associated with translation & rotation of a body as a whole (fig 1).

Strain at a generally loaded arbitrary body.







$\tan\theta = l/h = \gamma$

$\theta \to \gamma$

$$\text{Strain tensor} = \begin{bmatrix} \epsilon_x & \gamma_{xy} & \gamma_{xz} \\ \gamma_{yx} & \epsilon_y & \gamma_{yz} \\ \gamma_{zx} & \gamma_{zy} & \epsilon_z \end{bmatrix}$$

$$\text{Strain} = \frac{\text{change in length}}{\text{original length}} \quad \epsilon_x = \frac{dx' - dx}{dx}$$

$$\text{Ily} \quad \epsilon_y = \frac{dy' - dy}{dy} \qquad \epsilon_z = \frac{dz' - dz}{dz}$$

$$\gamma_{xy} = \frac{\pi}{2} - \theta_{xy}$$

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NOTE FOR 2-D

$$\sigma_x' = \sigma_x l^2 + \sigma_y m^2 + 2\tau_{xy} lm$$

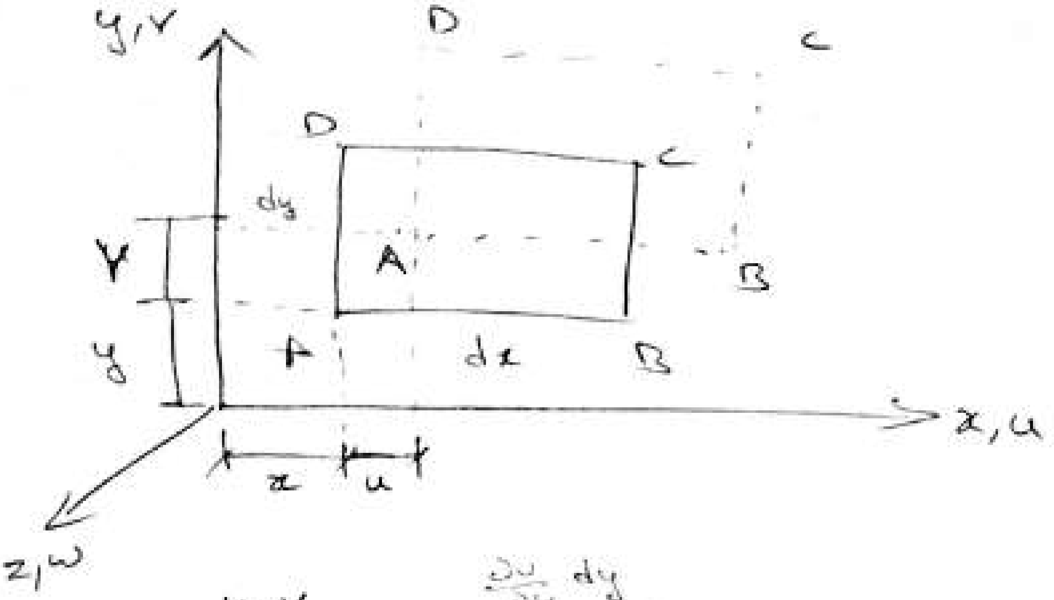
$$\epsilon_x' = \epsilon_x l^2 + \epsilon_y m^2 + \gamma_{xy} lm$$

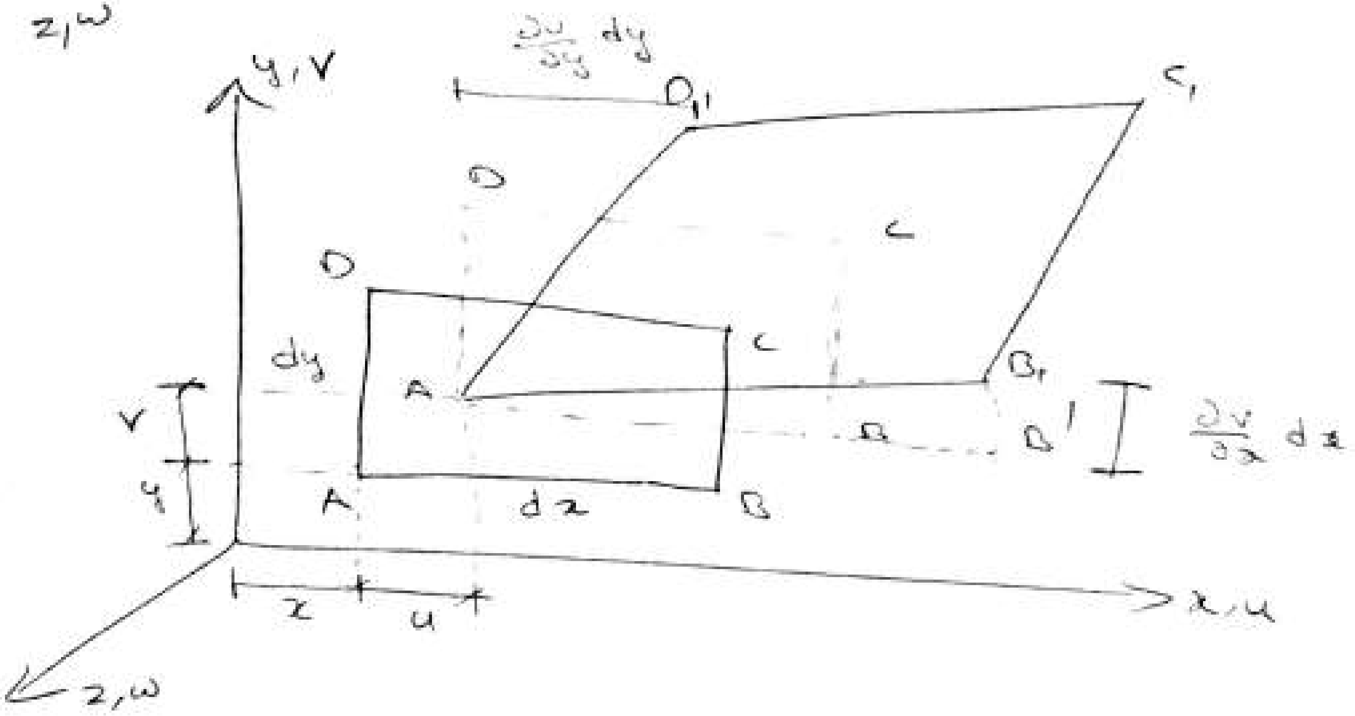
$$2\tau_{xy} = \gamma_{xy}$$

$$\boxed{\tau_{xy} = \frac{\gamma_{xy}}{2}}$$

$$\therefore \quad \epsilon_{ij} = \begin{bmatrix} \epsilon_x & \gamma_{xy}/2 & \gamma_{xz}/2 \\ \gamma_{yx}/2 & \epsilon_y & \gamma_{yz}/2 \\ \gamma_{zx}/2 & \gamma_{zy}/2 & \epsilon_z \end{bmatrix}$$

→ To determine Strain Components





→ Strain Components → Normal Strain Components $[\epsilon_x, \epsilon_y, \epsilon_z]$
→ Shear Strain Components $[\gamma_{xy}, \gamma_{yz} + \gamma_{zx}]$

Formulas Mod 2 (Compatibility condition)

1) Strain components in terms of displacement components

$$\epsilon_x = \frac{\partial u}{\partial x} \qquad \gamma_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}$$

$$\epsilon_y = \frac{\partial v}{\partial y} \qquad \gamma_{xz} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}$$

$$\epsilon_z = \frac{\partial w}{\partial z} \qquad \gamma_{zy} = \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}$$

2) Replace displacement components with linear strain components ($\epsilon_x, \epsilon_y + \epsilon_z$)

$$\gamma_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \longrightarrow \frac{\partial^2(\gamma_{xy})}{\partial y\,\partial x} = \frac{\partial^2 \epsilon_y}{\partial x^2} + \frac{\partial^2 \epsilon_x}{\partial y^2}$$

$$\gamma_{yz} = \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} \longrightarrow \frac{\partial^2(\gamma_{yz})}{\partial y\,\partial z} = \frac{\partial^2}{\partial y^2}\epsilon_z + \frac{\partial^2}{\partial z^2}\epsilon_y$$

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3) Replace displacement component with shear strain component

$$\cancel{2\frac{\partial^2}{\partial x\,\partial y}(\epsilon_z) = \frac{\partial}{\partial z}\left[\frac{\partial \gamma_{zx}}{\partial y} + \frac{\partial \gamma_{yz}}{\partial x} - \frac{\partial \gamma_{xy}}{\partial z}\right]}$$

$$2\frac{\partial^2 \epsilon_x}{\partial y\,\partial z} = \frac{\partial}{\partial x}\left[\frac{\partial \gamma_{xy}}{\partial z} + \frac{\partial \gamma_{zx}}{\partial y} - \frac{\partial \gamma_{yz}}{\partial x}\right]$$

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Strain Analysis

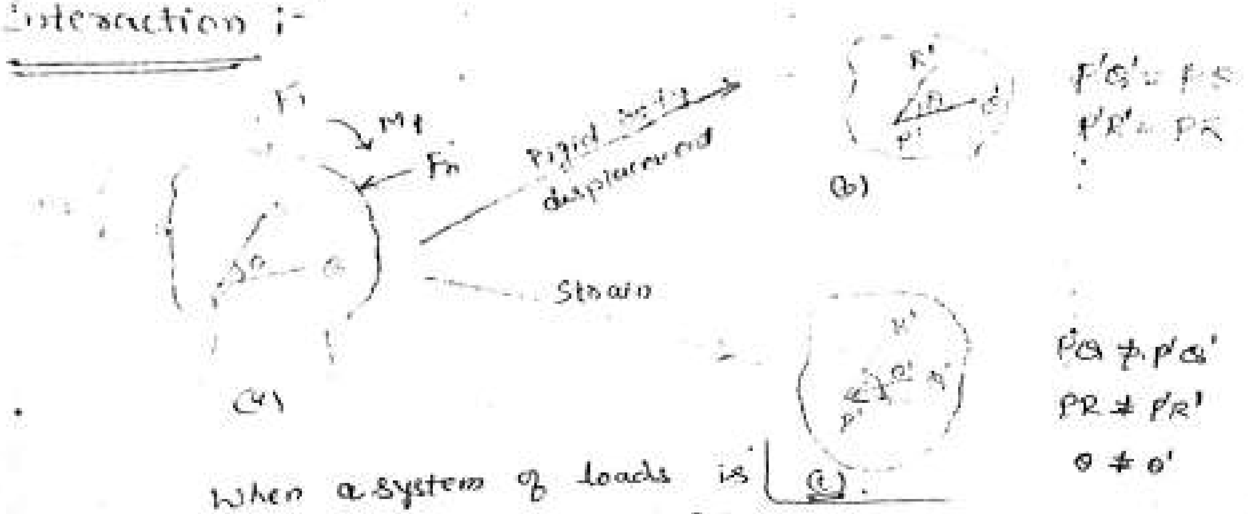
Strain :- Strain is a quantity used to provide a measure of the intensity of a deformation [deformation / unit length]. Just as stress which is used to provide measure of intensity of an internal force.

Similar to 2 types of stresses, normal stress σ and shearing stress τ the same classification can be used for strain (1) normal strain (ϵ) (2) Shearing strain (γ)

(1) Is used to provide a measure of the elongation (or) contraction of an arbitrary line segments in a body during deformations.

(2) Is used to provide a measure of angular distortion [change in angle b/w the two lines that are orthogonal in the undeformed state]. The deformation (or) strain may be the result of a change in temp, (or) of a stress (or) of other physical phenomena such as grain growth (or) shrinkage.

Interaction :-



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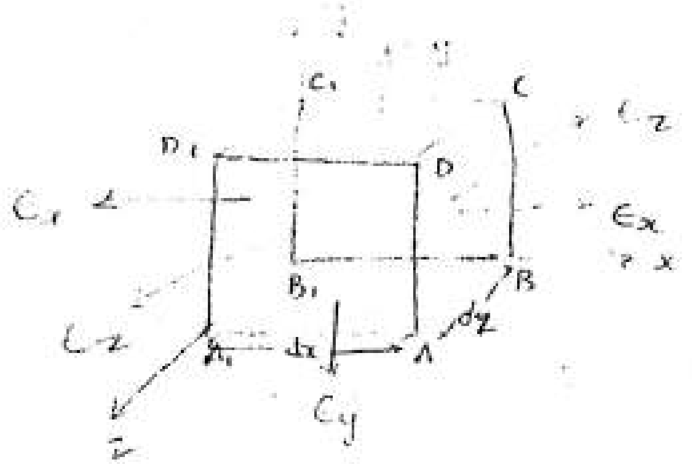

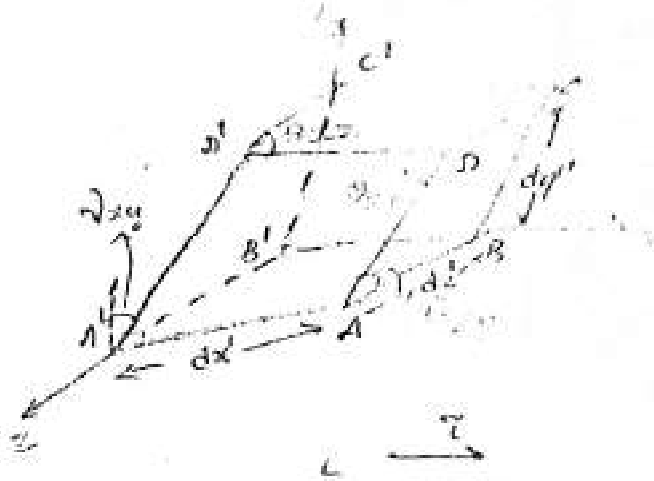

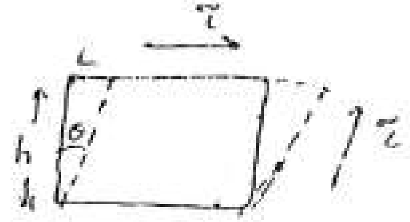

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Mod 2

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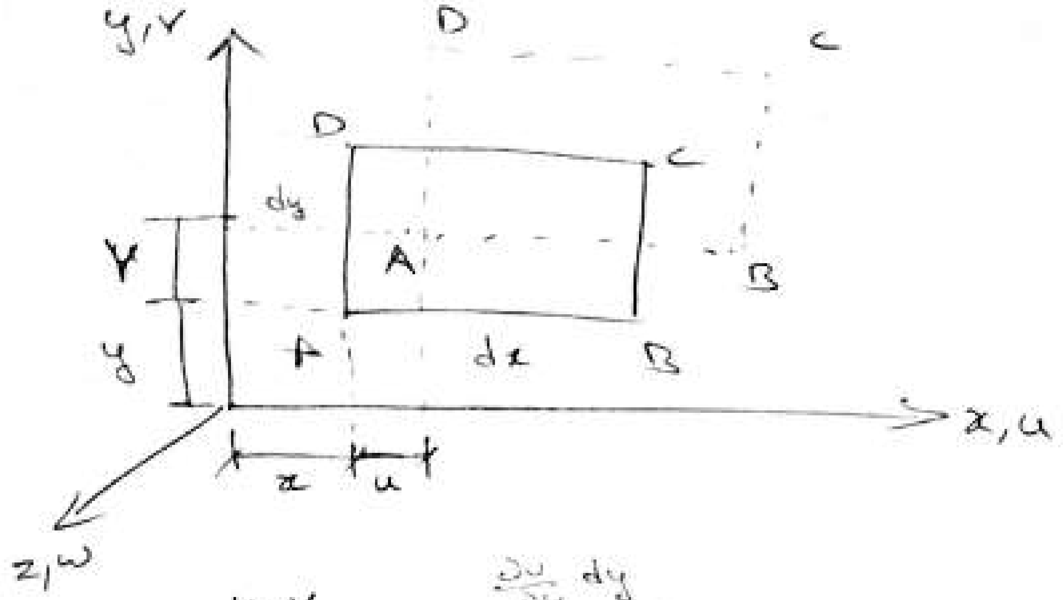

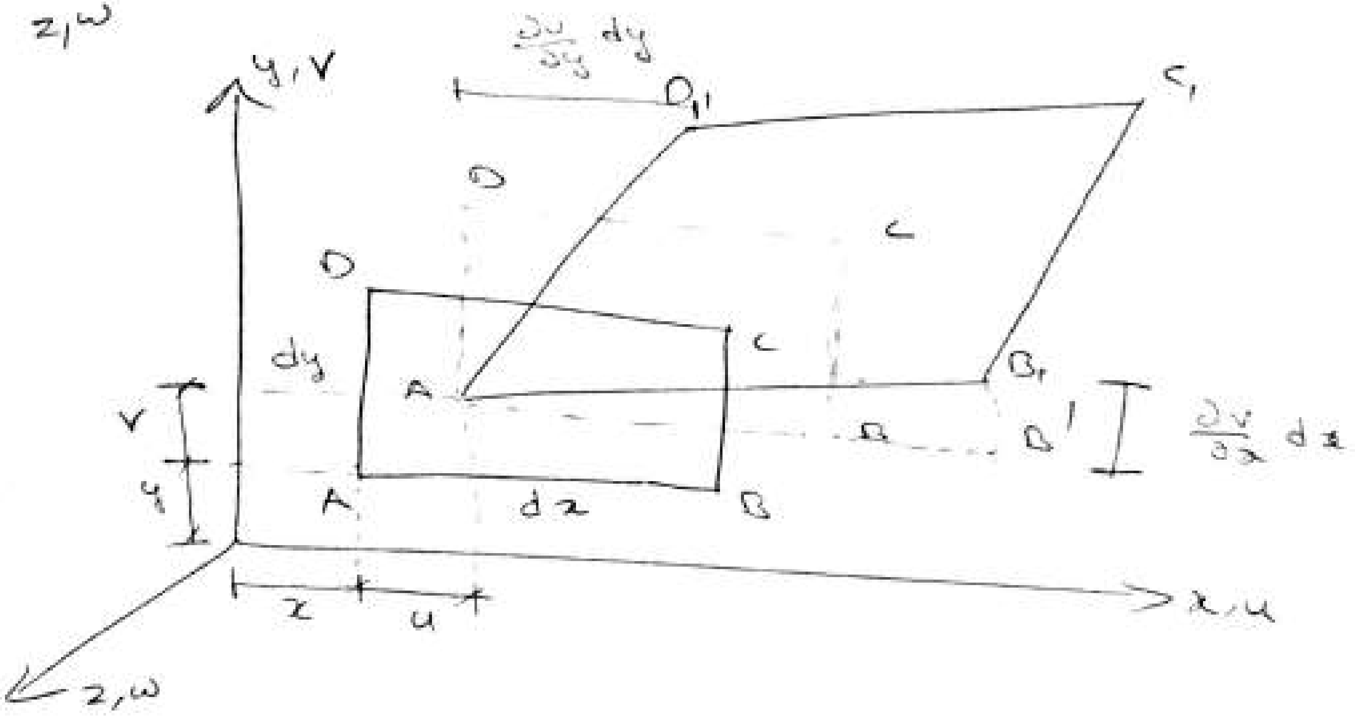

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$$\gamma_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \longrightarrow \frac{\partial^2(\gamma_{xy})}{\partial y \partial x} = \frac{\partial^2 \epsilon_y}{\partial x^2} + \frac{\partial^2 \epsilon_x}{\partial y^2}$$

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$$\gamma_{zx} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \longrightarrow \frac{\partial^2(\gamma_{xz})}{\partial x \partial z} = \frac{\partial^2}{\partial z^2}\epsilon_x + \frac{\partial}{\partial x^2}\epsilon_z$$

3) Replace displacement component with shear strain components

$$\cancel{2\frac{\partial^2(\epsilon_z)}{\partial x \partial y} = \frac{\partial}{\partial z}\left[\frac{\partial \gamma_{zx}}{\partial y} + \frac{\partial \gamma_{yz}}{\partial x} - \frac{\partial \gamma_{xy}}{\partial z}\right]}$$

$$2\frac{\partial^2 \epsilon_x}{\partial y \partial z} = \frac{\partial}{\partial x}\left[\frac{\partial \gamma_{xy}}{\partial z} + \frac{\partial \gamma_{zx}}{\partial y} - \frac{\partial \gamma_{yz}}{\partial x}\right]$$

$$2\frac{\partial^2}{\partial z \partial x}\epsilon_y = \frac{\partial}{\partial y}\left[\frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{xy}}{\partial z} - \frac{\partial \gamma_{zx}}{\partial y}\right]$$

$$2\frac{\partial^2}{\partial x \partial y}\epsilon_z = \frac{\partial}{\partial z}\left[\frac{\partial \gamma_{zx}}{\partial y} + \frac{\partial \gamma_{yz}}{\partial x} - \frac{\partial \gamma_{xy}}{\partial z}\right]$$

Compatibility Condition/Continuity Eqn

"* Compatibility means, a displacement under the load should be continuous"

→ Strain Components

$$\epsilon_x = \frac{\partial u}{\partial x} \qquad \gamma_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}$$

$$\epsilon_y = \frac{\partial v}{\partial y} \qquad \gamma_{yz} = \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}$$

$$\epsilon_z = \frac{\partial w}{\partial z} \qquad \gamma_{zx} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}$$

Six strain components expressed in terms of 3 displacement (u, v, w) components.

→ Derive the set of compatibility eq'n for a given stress field.

W.k.t $\gamma_{xy} = \dfrac{\partial v}{\partial x} + \dfrac{\partial u}{\partial y}$

Diff. w.r.t x & y

$$\Rightarrow \frac{\partial \gamma_{xy}}{\partial x} = \frac{\partial^2 v}{\partial x^2} + \frac{\partial u}{\partial x \partial y} \qquad (\text{w.r.t. } x)$$

Diff w.r.t y

$$\Rightarrow \frac{\partial \gamma_{xy}}{\partial x \partial y} = \frac{\partial^2}{\partial x^2}\left(\frac{\partial v}{\partial y}\right) + \frac{\partial}{\partial y^2}\left(\frac{\partial u}{\partial x}\right)$$

$$\left.\begin{aligned}
&\boxed{\frac{\partial^2 \gamma_{xy}}{\partial x \partial y} = \frac{\partial^2}{\partial x^2}\epsilon_y + \frac{\partial^2}{\partial y^2}\epsilon_x} \\
&\text{llly } \frac{\partial^2 \gamma_{yz}}{\partial y \partial z} = \frac{\partial^2}{\partial y^2}(\epsilon_z) + \frac{\partial}{\partial z^2}(\epsilon_y) \\
&\frac{\partial^2 \gamma_{zx}}{\partial z \partial x} = \frac{\partial^2}{\partial z^2}(\epsilon_x) + \frac{\partial}{\partial x^2}(\epsilon_z)
\end{aligned}\right\} \quad (1)$$

Now to replace displacement component (u,v,w) with shear strain components ($\gamma_{xy}, \gamma_{yz}, \gamma_{zx}$)

$\rightarrow$ $$\gamma_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}$$

(Diff. w.r.t x & z)

$$\frac{\partial \gamma_{xy}}{\partial x} = \frac{\partial}{\partial x}\left[\frac{\partial v}{\partial x}\right] + \frac{\partial}{\partial x}\left[\frac{\partial u}{\partial y}\right] \rightarrow (\text{w.r.t } 'x')$$

$$\frac{\partial^2 \gamma_{xy}}{\partial x \partial z} = \frac{\partial^2}{\partial x \partial z}\left[\frac{\partial v}{\partial x}\right] + \frac{\partial^2}{\partial z \partial x}\left[\frac{\partial u}{\partial y}\right] \rightarrow (\text{w.r.t } 'z')$$

$$\frac{\partial^2 \gamma_{xy}}{\partial x \partial z} = \frac{\partial^2}{\partial x \partial z}\left[\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right] \quad \text{— (2)}$$

$\rightarrow$ $$\gamma_{xz} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \quad (\text{diff. w.r.t } y + x)$$

$$\frac{\partial(\gamma_{zx})}{\partial x} = \frac{\partial}{\partial x}\left(\frac{\partial w}{\partial x}\right) + \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial z}\right) \rightarrow (\text{w.r.t } x)$$

$$\frac{\partial^2(\gamma_{zx})}{\partial x \partial y} = \frac{\partial^2}{\partial x \partial y}\left[\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right] \quad \text{— (3)}$$

$\rightarrow$ Adding eqn (2) + (3)

$$\frac{\partial^2}{\partial x \partial z}(\gamma_{xy}) + \frac{\partial^2(\gamma_{zx})}{\partial x \partial y} = \frac{\partial^2}{\partial x \partial z}\left[\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right] + \frac{\partial^2}{\partial x \partial y}\left[\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right]$$

$$= \frac{\partial^2}{\partial x \partial z}\left[\frac{\partial v}{\partial x}\right] + \frac{\partial^2}{\partial x \partial z}\left[\frac{\partial u}{\partial y}\right] + \frac{\partial^2}{\partial x \partial y}\left[\frac{\partial w}{\partial x}\right] + \frac{\partial^2}{\partial x \partial y}\left[\frac{\partial u}{\partial z}\right]$$

$$= \frac{\partial^2}{\partial x^2}\left[\frac{\partial v}{\partial z}\right] + \frac{\partial^2}{\partial y \partial z}\left[\frac{\partial u}{\partial x}\right] + \frac{\partial^2}{\partial x^2}\left[\frac{\partial w}{\partial y}\right] + \frac{\partial^2}{\partial y \partial z}\left[\frac{\partial u}{\partial x}\right]$$

$$= \frac{\partial^2}{\partial x^2}\left[\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right] + \frac{\partial^2}{\partial y \partial z}\left[\epsilon_x + \epsilon_x\right]$$

$$\frac{\partial^2 \gamma_{zy}}{\partial x^2} + \frac{2\,\partial^2 \epsilon_x}{\partial y \partial z}$$

$$\Rightarrow 2\frac{\partial^2 \epsilon_x}{\partial y \partial z} = \frac{\partial^2(\gamma_{xy})}{\partial x \partial z} + \frac{\partial^2(\gamma_{zx})}{\partial y \partial x} - \frac{\partial^2}{\partial x^2}(\gamma_{zy})$$

$$\Rightarrow 2\frac{\partial^2 \epsilon_x}{\partial y \partial z} = \frac{\partial}{\partial x}\left[\frac{\partial \gamma_{xy}}{\partial z} + \frac{\partial \gamma_{zx}}{\partial y} - \frac{\partial \gamma_{zy}}{\partial x}\right]$$

IIIy

$$2\frac{\partial^2 \epsilon_y}{\partial z \partial x} = \frac{\partial}{\partial y}\left[\frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{xy}}{\partial z} - \frac{\partial \gamma_{zx}}{\partial y}\right] \quad \text{--- (4)}$$

$$2\frac{\partial^2 \epsilon_z}{\partial x \partial y} = \frac{\partial}{\partial z}\left[\frac{\partial \gamma_{zx}}{\partial y} + \frac{\partial \gamma_{yz}}{\partial x} - \frac{\partial \gamma_{xy}}{\partial z}\right]$$

→ Problem (Compatibility)

1. Given the foll. system of strain; $\epsilon_x = 5 + x^2 + y^2 + x^4 + y^4$, $\epsilon_y = 6 + 3x^2 + 3y^2 + x^4 + y^4$, $\gamma_{xy} = 10 + 4xy(x^2 + y^2 + 2)$, $\epsilon_z = \gamma_{yz} = \gamma_{zx} = 0$

Det. wether the above strain field is possible. If it is possible, determine disp. components in terms of x & y. Assuming $u = v = 0$ at the origine

Sol:- [LHS = RHS = 0] ; $\frac{\partial^2 \gamma_{xy}}{\partial x \partial y} = \frac{\partial^2 \epsilon_x}{\partial y^2} + \frac{\partial^2 \epsilon_y}{\partial x^2}$

$$\Rightarrow \frac{\partial^2}{\partial x \partial y}\left[10 + 4x^3y + 4xy^3 + 8xy\right]$$

$$= \frac{\partial^2}{\partial y^2}\left[5 + x^2 + y^2 + x^4 + y^4\right] + \frac{\partial^2}{\partial x^2}\left[6 + 3x^2 + 3y^2 + x^4 + y^4\right]$$

$$\Rightarrow 12x^2 + 12y^2 + 8 = 2 + 12y^2 + 6 + 12x^2$$

$$\boxed{LHS = RHS} \quad \text{--- (1)}$$

∴ The system is compatible.

$$\rightarrow \frac{\partial^2(\gamma_{yz})}{\partial y \partial x} = \frac{\partial^2 \epsilon_y}{\partial z^2} + \frac{\partial^2 \epsilon_x}{\partial y^2} \Rightarrow \boxed{0=0} \text{ — } (2)$$

$$\rightarrow \frac{\partial^2(\gamma_{xz})}{\partial x \partial z} = \frac{\partial^2 \epsilon_x}{\partial z^2} + \frac{\partial^2 \epsilon_z}{\partial x^2} \Rightarrow \boxed{0=0} \text{ — } (3)$$

—x—

For shear; $\rightarrow 2\frac{\partial^2 \epsilon_x}{\partial y \partial z} = \frac{\partial}{\partial x}\left[-\frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{xz}}{\partial y} + \frac{\partial \gamma_{xy}}{\partial z}\right]$

$$\Rightarrow 2\frac{\partial^2}{\partial y \partial z}[5+x^2+y^2+x^4+y^4] = \frac{\partial}{\partial x}\left[\frac{-\partial}{\partial x}[0] + \frac{\partial}{\partial y}[0] + \frac{\partial}{\partial z}(10+4x^3y+4y^3x+8xy\right.$$

$$\boxed{0=0} \Rightarrow LHS = RHS \text{ — } (4)$$

$$\rightarrow 2\frac{\partial^2 \epsilon_y}{\partial x \partial z} = \frac{\partial}{\partial y}\left[-\frac{\partial \gamma_{xz}}{\partial y} + \frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{xy}}{\partial z}\right]$$

$$2\frac{\partial^2}{\partial x \partial z}[6+3x^2+3y^2+x^4+y^4] = \frac{\partial}{\partial y}\left[\frac{-\partial}{\partial y}(0) + \frac{\partial}{\partial x}(0) + \frac{\partial}{\partial z}(10+4x^3y+4y^3x+8xy)\right]$$

$$\boxed{0=0} \Rightarrow LHS = RHS \text{ — } (5)$$

$$\rightarrow 2\frac{\partial^2 \epsilon_z}{\partial y \partial x} = \frac{\partial}{\partial z}\left[-\frac{\partial \gamma_{xy}}{\partial z} + \frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{xz}}{\partial y}\right]$$

$$\boxed{0=0} \Rightarrow LHS = RHS \text{ — } (6)$$

—x—

$\therefore$ The Given Strain field is possible or Compatible

—x—

w.r.t $\epsilon_x = \frac{\partial u}{\partial x}$, $\epsilon_y = \frac{\partial u}{\partial y}$, ~~[To find~~

[To find disp comp. (u, v) in terms of x & y]

$\therefore \epsilon_x = \frac{\partial u}{\partial x} \Rightarrow \int 5 + x^2 + y^2 + x^4 + y^4 = \int \frac{\partial u}{\partial x}$

(Integrating w.r.t dx)

$$\Rightarrow 5x + \frac{x^3}{3} + y^2x + \frac{x^5}{5} + xy^4 + C_1 = u \text{ — } (7)$$

$\therefore \epsilon_y = \frac{\partial v}{\partial y} \Rightarrow \int 6 + 3x^2 + 3y^2 + x^4 + y^4 = \int \frac{\partial v}{\partial y}$

$$\Rightarrow 6y + 3x^2y + \frac{3y^3}{3} + x^4y + \frac{y^5}{5} + C_2 = v \text{ — } (8)$$

$\therefore$ $u = 5x + \frac{x^3}{3} + xy^2 + \frac{x^5}{5} + xy^4$

$v = 6y + 3x^2y + \frac{3y^3}{3} + x^4y + \frac{y^5}{5}$

$\therefore x = y = 0$ & $u = v = 0$

at ...

Problem (2): An Elastic body under the action of external forces as displacement field given by ~~$u=(x^2+y)i+(x^2+2y)k$~~

$u = (x^2+y)i + (3+z)j + (x^2+2y)k$.

$\quad u \qquad\qquad v \qquad\qquad w \qquad\qquad x\ y\ z$

Determine the principle strain at $(3,1,-2)$ & direction of the min. pricipal strain.

Sol:- $\epsilon_x = \frac{\partial u}{\partial x} = 2x$, $\gamma_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} = 0+1=1$

$\epsilon_y = \frac{\partial v}{\partial y} = 0$, $\gamma_{yz} = \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} = 2+1=3$

$\epsilon_z = \frac{\partial w}{\partial z} = 0$, $\gamma_{xz} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} = 2x = 2(3) = 6$

→ strain tensor : $\epsilon_{ij} = \begin{bmatrix} \epsilon_x & \gamma_{xy}/2 & \gamma_{xz}/2 \\ \gamma_{yx}/2 & \epsilon_y & \gamma_{yz}/2 \\ \gamma_{zx}/2 & \gamma_{zy}/2 & \epsilon_z \end{bmatrix} = \begin{bmatrix} 6 & 1/2 & 3 \\ 1/2 & 0 & 3/2 \\ 3 & 3/2 & 0 \end{bmatrix}$

→ To det. principle strain

$I_1 = \epsilon_x + \epsilon_y + \epsilon_z = \underline{\underline{6}}$, $I_2 = \begin{bmatrix} 0 & 3/2 \\ 3/2 & 0 \end{bmatrix} + \begin{bmatrix} 6 & 3 \\ 3 & 0 \end{bmatrix} + \begin{bmatrix} 6 & 1/2 \\ 1/2 & 0 \end{bmatrix}$

$I_3 = \begin{vmatrix} 6 & 1/2 & 3 \\ 1/2 & 0 & 3/2 \\ 3 & 3/2 & 0 \end{vmatrix} = \underline{\underline{-9}}$ $\qquad = -\frac{9}{4} - 9 + 0 - \frac{1}{4} = \underline{\underline{-23/2}}$

→ $\epsilon^3 - \epsilon^2 I_1 + \epsilon I_2 - I_3 = 0 \Rightarrow \epsilon^3 - 6\epsilon^2 - \frac{23}{2}\epsilon + 9 = 0$

$\Rightarrow \boxed{\epsilon_1 = 7.391}$ $\boxed{\epsilon_2 = 0.608}$ $\boxed{\epsilon_3 = -2}$ ✓

→ Direction cosine for min principle strain

$\epsilon_{ij} = \begin{bmatrix} 6-(-2) & 1/2 & 3 \\ 1/2 & (0+2) & 3/2 \\ 3 & 3/2 & (0+2) \end{bmatrix} = \begin{bmatrix} 8 & 0.5 & 3 \\ 0.5 & 2 & 1.5 \\ 3 & 1.5 & 2 \end{bmatrix}$

$A_1 = \begin{bmatrix} 2 & 1.5 \\ 1.5 & 2 \end{bmatrix} \Rightarrow 4 - 1.5^2 = 1.75$

$B_1 = -\begin{bmatrix} 8 & 3 \\ 3 & 2 \end{bmatrix} \Rightarrow 16 - 9 = 7$

$C_1 = \begin{bmatrix} 8 & 0.5 \\ 0.5 & 2 \end{bmatrix} \Rightarrow 16 - 0.5^2 = 15.73$

$$K = \frac{1}{\sqrt{A_1^2+B_1^2+C_1^2}} = \frac{1}{\sqrt{1.75^2+7^2+15.73^2}} \Rightarrow K = 0.0577$$

$l_1 = A_1 K = 1.75 \times 0.0577 \Rightarrow l_1 = 0.101$

$m_1 = B_1 K = 7 \times 0.0577 \Rightarrow m_1 = 0.4037$

$n_1 = C_1 K = 15.73 \times 0.0577 \Rightarrow n_1 = 0.907$

$l^2+m^2+n^2 = \underline{\underline{0.997}}$

→ Def. Octahedral stress - strain

- Normal octahedral strain $\epsilon_{n_{oct}} = \frac{\epsilon_1+\epsilon_2+\epsilon_3}{3}$

$$= \frac{7.391+0.608-2}{3} = \underline{\underline{1.99}}$$

- $\frac{\gamma_{oct}}{2} = \frac{1}{3}\sqrt{2I_1^2-6I_2} = \frac{1}{3}\sqrt{2(6)^2+6(23/2)} =$

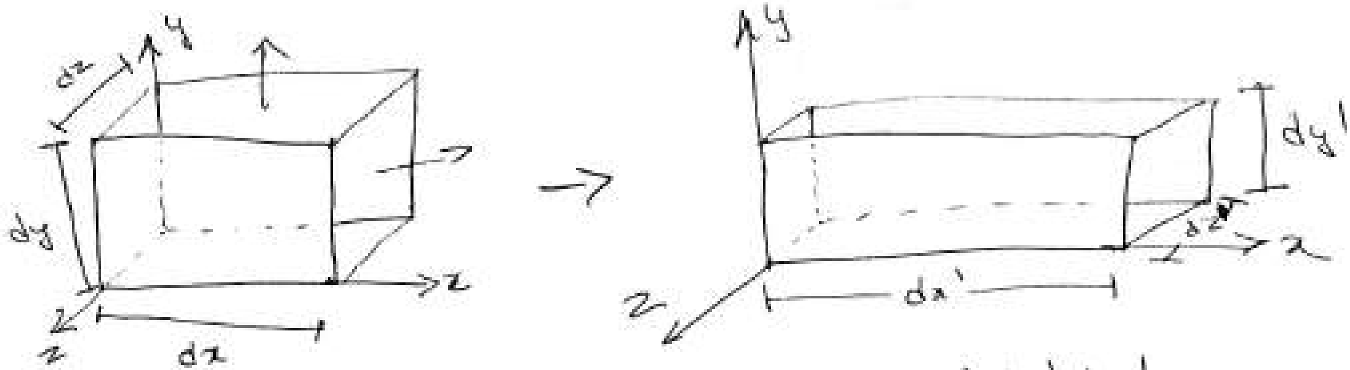
→ Def. deviatoric + spherical strains

- Spherical strains $= \begin{bmatrix} \epsilon_m & 0 & 0 \\ 0 & \epsilon_m & 0 \\ 0 & 0 & \epsilon_m \end{bmatrix}$

- $\epsilon_m = \frac{\epsilon_x+\epsilon_y+\epsilon_z}{3}$

- Deviatoric strains $= \begin{bmatrix} \epsilon_x-\epsilon_m & \gamma_{xy}/2 & \gamma_{xz}/2 \\ \gamma_{yx}/2 & \epsilon_y-\epsilon_m & \gamma_{yz}/2 \\ \gamma_{zx}/2 & \gamma_{zy}/2 & \epsilon_z-\epsilon_m \end{bmatrix}$

→ Cubical dialation / Vol. change / Volumetric strain



Vol = dxdydz
(Initial vol.)

vol. = dx' dy' dz'
(final vol.)

→ $\epsilon_x = \frac{dx' - dx}{dx}$ ⇒ $dx' = (1+\epsilon_x)\,dx$

lly $dy' = (1+\epsilon_y)\,dy$

$dz' = (1+\epsilon_z)\,dz$

⇒ Final vol. $= (1+\epsilon_x)dx\,(1+\epsilon_y)dy\,(1+\epsilon_z)dz$

$= dx\,dy\,dz\,(1+\epsilon_x\epsilon_y+\epsilon_x+\epsilon_y)(1+\epsilon_z)$

$= dx\,dy\,dz\,(1+\epsilon_x\epsilon_y+\epsilon_x+\epsilon_y+\epsilon_z+\epsilon_x\epsilon_y\epsilon_z+\epsilon_x\epsilon_z+\epsilon_y\epsilon_z)$

*(Neglecting 2nd & 3rd order terms)

$dV' = dx\,dy\,dz\,(1+\epsilon_x+\epsilon_y+\epsilon_z)$ —— (1)

Now, Change in vol. = Final vol. − initial vol.

$\underline{\delta V} = dV' - dV = dx\,dy\,dz\,(1+\epsilon_x+\epsilon_y+\epsilon_z) - dx\,dy\,dz$

⇒ $dx\,dy\,dz\,(\epsilon_x+\epsilon_y+\epsilon_z)$

∴ $\boxed{\delta V = dx\,dy\,dz\ \Delta}$

"Δ is called dilational strain"

"The dilational strain is defined as the change in volume per unit initial volume"

i.e., $\Delta = \frac{\delta V}{dx\,dy\,dz} = \frac{\text{Change in vol.}}{\text{Initial vol.}}$

Multiple cell closed ...

Module 3 Page (1)

→ To determine Axisymmetrical stresses

A function which satisfies differential eqn of equillibrium is called Airy's stress function

Note: In absence of body forces

∴ While solving plane problem, the stress components σ_x, σ_y + τ_{xy} must satisfy differential eqn of equilibrium.

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \cancel{X}^{0} = 0 \quad \text{——} (1)$$

$$\frac{\partial \tau_{yx}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \cancel{Y}^{0} = 0 \quad \text{——} (2)$$

The system (eqn (1) + (2)) is non-homogeneous (indeterminant - A variable with no value assigned to it, like the variable in a polynomial) And its general soln may be expressed as the sum of the general solution of homogenous system

(or)

If we add solution of eqn (1) + eqn (2), we will get the "General Solution"

→ X & Y are body forces

→ To find general Solution of ① + ② Eqn

$$\frac{\partial \sigma_x}{\partial x} = -\frac{\partial \tau_{xy}}{\partial y} = \frac{\partial}{\partial y}(-\tau_{xy}) \quad \text{— from Eqn ①}$$

Now, According to differential calculus there exist certain fxn $A(x, y)$

$$\Rightarrow \sigma_x = \frac{\partial A}{\partial y} \quad \text{— ③}$$

$$\Rightarrow -\tau_{xy} = \frac{\partial A}{\partial x} \quad \text{— ④}$$

$||^{ly}$ from Eqn ②, there is another fxn $B(x, y)$

$$\cancel{\frac{\partial \sigma_y}{\partial}} \quad \sigma_y = \frac{\partial B}{\partial x} \quad \text{— ⑤}$$

$$-\tau_{xy} = \frac{\partial B}{\partial y} \quad \text{— ⑥}$$

Adding ④ + ⑥, $\frac{\partial A}{\partial x} = \frac{\partial B}{\partial y}$

*above eq'n ensures the existance of still another fxn $\phi(x, y)$, so that

$$A = \frac{\partial \phi}{\partial y} \quad \text{— ⑦}$$

&

$$B = \frac{\partial \phi}{\partial x} \quad \text{— ⑧}$$

Substitute → Eqn (7) in (3)
→ Eqn (8) in (5)

$$\Rightarrow \sigma_x = \frac{\partial}{\partial y}\left(\frac{\partial \phi}{\partial y}\right) = \frac{\partial^2 \phi}{\partial y^2}$$

$$\boxed{\sigma_x = \frac{\partial^2 \phi}{\partial y^2}} \qquad \boxed{\sigma_y = \frac{\partial^2 \phi}{\partial x^2}}$$

$$\Rightarrow \tau_{xy} = -\frac{\partial}{\partial x}\left(\frac{\partial \phi}{\partial y}\right)$$

$$\boxed{\tau_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y}}$$

$\left.\begin{matrix}\sigma_x \\ \sigma_y \\ \tau_{xy}\end{matrix}\right\}$ Stress components of 2D (or) **Airy's Stress fxns**

→ To determine Bi-harmonic Eqn in Cartesian co-ordinates for plane stress + plane strain

w.k.T Airy's stress fxn

$$\sigma_x = \frac{\partial^2 \phi}{\partial y^2}, \quad \sigma_y = \frac{\partial^2 \phi}{\partial x^2}, \quad \tau_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y}$$

where ϕ = is the fxn of (x, y)

Also the eqn of equilibrium for 2D element is given by

$$\left.\begin{aligned} \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = 0 \quad &— (1) \\ \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} = 0 \quad &— (2) \end{aligned}\right\}$$

neglection body forces (X + Y)

subs value of σ_x, σ_y + τ_{xy} in (1) + (2)

$$\rightarrow \frac{\partial}{\partial x}\left(\frac{\partial^2 \phi}{\partial y^2}\right) + \frac{\partial}{\partial y}\left(-\frac{\partial^2 \phi}{\partial x \partial y}\right) = 0$$

$$\frac{\partial^3 \phi}{\partial x \partial y^2} - \frac{\partial^3 \phi}{\partial x \partial y^2} = 0$$

$$\boxed{0 = 0}$$

$$\rightarrow \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} = 0$$

$$\frac{\partial}{\partial x}\left(-\frac{\partial^2 \phi}{\partial x \partial y}\right) + \frac{\partial}{\partial y}\left(\frac{\partial^2 \phi}{\partial x^2}\right) = 0$$

$$\boxed{0 = 0}$$

$\therefore$ W.K.T $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)(\sigma_x + \sigma_y) = 0$

subs values of σ_x & σ_y

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\left(\frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial x^2}\right) = 0$$

$$\boxed{\frac{\partial^4 \phi}{\partial x^4} + 2\frac{\partial^4 \phi}{\partial x^2 \partial y^2} + \frac{\partial^4 \phi}{\partial y^4} = 0}$$

$\therefore$ finding the soln to an elasticity problem consist of finding the fn $\phi(x, y)$ which would satisfy above eqn & stress derived from it should also satisfy eq'n of equilibrium & boundary condition

Definition

→ Equation which satisfies equations of equilibrium & compatability stress condition is called biharmonic eqn.

→ Compatability Eq'n interms of stresses

Case 1: Plane Stress

w.k.T the compatability eq'n for strain is given by

$$\frac{\partial^2 \epsilon_x}{\partial y^2} + \frac{\partial^2 \epsilon_y}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y} \quad \text{--- ①}$$

Using Hooke's relation $\epsilon_x = \frac{1}{E}\left[\sigma_x - \nu(\sigma_y + \cancel{\sigma_z}^{\,0})\right]$

for 2D

$$\epsilon_x = \frac{1}{E}\left[\sigma_x - \nu\sigma_y\right]$$

IIly

$$\epsilon_y = \frac{1}{E}\left[\sigma_y - \nu\sigma_x\right]$$

Take $G = \frac{\tau_{xy}}{\gamma_{xy}}$ $\Rightarrow \gamma_{xy} = \tau_{xy} \cdot \frac{1}{G}$

$$\gamma_{xy} = \tau_{xy} \frac{2(1+\nu)}{E}$$

[MOM, derivation]

⇒ Subs above eqn in ①

$$\frac{\partial^2}{\partial y^2}\left[\frac{1}{E}(\sigma_x - \nu\sigma_y)\right] + \frac{\partial^2}{\partial x^2}\left[\frac{1}{E}(\sigma_y - \nu\sigma_x)\right] = \frac{\partial^2}{\partial x \partial y}\left[\tau_{xy}\frac{2(1+\nu)}{E}\right]$$

$$\Rightarrow \frac{1}{\cancel{E}}\left[\frac{\partial^2}{\partial y^2}(\sigma_x - \nu\sigma_y)\right] + \frac{1}{\cancel{E}}\left[\frac{\partial^2}{\partial x^2}(\sigma_y - \nu\sigma_x)\right] = \frac{2(1+\nu)}{\cancel{E}}\left[\frac{\partial^2}{\partial x \partial y}\tau_{xy}\right] \quad \text{--- 1(a)}$$

w.k.t Eqn of Equilibrium for 2D is given by

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = 0 \quad \text{--- ②}$$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} = 0 \quad \text{--- ③}$$

Now,

Differentiate Eqn ② by 'x' & ③ by 'y'

$$\frac{\partial^2 \sigma_x}{\partial x^2} + \frac{\partial^2 \tau_{xy}}{\partial x \partial y} = 0 \quad — ④ \longrightarrow \text{w.r.t } 'x'$$

$$\frac{\partial^2 \tau_{xy}}{\partial x \partial y} + \frac{\partial^2 \sigma_y}{\partial y^2} = 0 \quad — ⑤ \longrightarrow \text{w.r.t } 'y'$$

Adding ④ + ⑤

$$\frac{\partial^2 \sigma_x}{\partial x^2} + 2\frac{\partial^2 \tau_{xy}}{\partial x \partial y} + \frac{\partial^2 \sigma_y}{\partial y^2} = 0$$

$$2\frac{\partial^2 \tau_{xy}}{\partial x \partial y} = -\left[\frac{\partial^2 \sigma_x}{\partial x^2} + \frac{\partial^2 \sigma_y}{\partial y^2}\right]$$

Now, sub above eqn in 1ⓐ

$$\frac{\partial^2}{\partial y^2}(\sigma_x - \nu\sigma_y) + \frac{\partial^2}{\partial x^2}(\sigma_y - \nu\sigma_x) = -(1+\nu)\left[\frac{\partial^2 \sigma_x}{\partial x^2} + \frac{\partial^2 \sigma_y}{\partial y^2}\right]$$

$$\Rightarrow \frac{\partial^2 \sigma_x}{\partial y^2} - \cancel{\frac{\nu\partial^2 \sigma_y}{\partial y^2}} + \frac{\partial^2 \sigma_y}{\partial x^2} - \cancel{\nu\frac{\partial^2 \sigma_x}{\partial x^2}} = -\frac{\partial^2 \sigma_x}{\partial x^2} - \frac{\partial^2 \sigma_y}{\partial y^2} - \cancel{\nu\frac{\partial^2 \sigma_x}{\partial x^2}} - \cancel{\nu\frac{\partial^2 \sigma_y}{\partial y^2}}$$

$$\Rightarrow \frac{\partial^2 \sigma_x}{\partial y^2} + \frac{\partial^2 \sigma_y}{\partial x^2} + \frac{\partial^2 \sigma_x}{\partial x^2} + \frac{\partial^2 \sigma_y}{\partial y^2} = 0$$

(or)

$$\boxed{\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right](\sigma_x + \sigma_y) = 0} \longleftarrow ⑥$$

$$\nabla^2(\sigma_x + \sigma_y) = 0$$

$$\nabla^2 \phi = 0$$

Eqn ⑥ is called compatability condn for plane stress prob.

Case ii) Plane Strain

$$\frac{\partial^2 \epsilon_x}{\partial y^2} + \frac{\partial^2 \epsilon_y}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y} \quad \text{—— (1)}$$

$$\checkmark \epsilon_x = \frac{(1+\nu)}{E}\left[(1-\nu)\sigma_x - \nu\sigma_y\right]$$

$$\checkmark \epsilon_y = \frac{(1+\nu)}{E}\left[(1-\nu)\sigma_y - \nu\sigma_x\right]$$

$$G = \frac{\tau_{xy}}{\gamma_{xy}} \Rightarrow \gamma_{xy} = \tau_{xy}\frac{1}{G}$$

$$\checkmark \gamma_{xy} = \tau_{xy}\frac{2(1+\nu)}{E}$$

Subs. all the values in Eqn (1) $\left[\frac{\partial^2 \epsilon_x}{\partial y^2} + \frac{\partial^2 \epsilon_y}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y}\right]$

$$\Rightarrow \frac{\partial^2}{\partial y^2}\left[\frac{1+\nu}{E}\left[(1-\nu)\sigma_x - \nu\sigma_y\right]\right] + \frac{\partial^2}{\partial x^2}\left[\frac{1+\nu}{E}\left[(1-\nu)\sigma_y - \nu\sigma_x\right]\right]$$

$$= \frac{\partial^2}{\partial x \partial y}\left[\tau_{xy}\frac{2(1+\nu)}{E}\right]$$

$$\Rightarrow \cancel{\frac{1+\nu}{E}}\left[\frac{\partial^2}{\partial y^2}\left[(1-\nu)\sigma_x - \nu\sigma_y\right]\right] + \cancel{\frac{1+\nu}{E}}\left[\frac{\partial^2}{\partial x^2}\left[(1-\nu)\sigma_y - \nu\sigma_x\right]\right]$$

$$= 2\cancel{\frac{(1+\nu)}{E}}\frac{\partial^2 \tau_{xy}}{\partial x \partial y}$$

$$\Rightarrow \frac{\partial^2}{\partial y^2}\left[(1-\nu)\sigma_x - \nu\sigma_y\right] + \frac{\partial^2}{\partial x^2}\left[(1-\nu)\sigma_y - \nu\sigma_x\right]$$

$$= 2\frac{\partial^2 \tau_{xy}}{\partial x \partial y} \quad \text{—— 1(a)}$$

w.k.T Eqn of equilibrium for 2D

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = 0 \longrightarrow (2)$$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} = 0 \longrightarrow (3)$$

Diff. (2) w.r.t 'x' & (3) w.r.t 'y'

$$\frac{\partial^2 \sigma_x}{\partial x^2} + \frac{\partial \tau_{xy}}{\partial x \partial y} = 0 \longrightarrow (4) \longrightarrow \text{w.r.t 'x'}$$

$$\frac{\partial^2 \tau_{xy}}{\partial x \partial y} + \frac{\partial^2 \sigma_y}{\partial y^2} = 0 \longrightarrow (5) \longrightarrow \text{w.r.t 'y'}$$

Adding (4) + (5);

$$\frac{\partial^2 \sigma_x}{\partial x^2} + \frac{\partial^2 \sigma_y}{\partial y^2} + 2\frac{\partial^2 \tau_{xy}}{\partial x \partial y} = 0$$

$$2\frac{\partial^2 \tau_{xy}}{\partial x \partial y} = -\left[\frac{\partial^2 \sigma_x}{\partial x^2} + \frac{\partial^2 \sigma_y}{\partial y^2}\right]$$

Now, subs. Above value in 1(a)

$$\frac{\partial^2}{\partial y^2}\left[(1-\nu)\sigma_x - \nu\sigma_y\right] + \frac{\partial^2}{\partial x^2}\left[(1-\nu)\sigma_y - \nu\sigma_x\right] = -\left[\frac{\partial^2 \sigma_x}{\partial x^2} + \frac{\partial^2 \sigma_y}{\partial y^2}\right]$$

Page 5

$$\Rightarrow (1-\nu)\frac{\partial^2 \sigma_x}{\partial y^2} - \nu \frac{\partial^2 \sigma_y}{\partial y^2} + (1-\nu)\frac{\partial^2 \sigma_y}{\partial x^2} - \nu \frac{\partial^2 \sigma_x}{\partial x^2} = -\frac{\partial^2 \sigma_x}{\partial x^2} - \frac{\partial^2 \sigma_y}{\partial y^2}$$

$$\Rightarrow (1-\nu)\left[\frac{\partial^2 \sigma_x}{\partial y^2} + \frac{\partial^2 \sigma_y}{\partial x^2}\right] + \frac{\partial^2 \sigma_y}{\partial y^2}(1-\nu) + \frac{\partial^2 \sigma_x}{\partial x^2}(1-\nu) = 0$$

$$\Rightarrow (1-\nu)\left[\frac{\partial^2 \sigma_x}{\partial y^2} + \frac{\partial^2 \sigma_y}{\partial x^2} + \frac{\partial^2 \sigma_y}{\partial y^2} + \frac{\partial^2 \sigma_x}{\partial x^2}\right] = 0$$

Q. What is a stress function? Outline the method of solving 2D problems of elasticity by the use of stress function.

FORMULATION OF ELASTICITY PROBLEMS → Valliappan

① find stress function for the following polynomials

(i) $\phi = \frac{c}{2}y^2$ (ii) $\phi = -cxy$ (iii) $\phi = \frac{c}{2}y^3$

(i) $\phi = \frac{c}{2}y^2$

The biharmonic is given by

$$\left[\frac{\partial^4}{\partial x^4} + \frac{\partial^4}{\partial y^4} + \frac{2\partial^4}{\partial x^2 \partial y^2}\right]\phi = 0 \quad \text{—— (1)}$$

$$\frac{\partial^4 \phi}{\partial x^4} = \frac{\partial^4}{\partial x^4}\left(\frac{c}{2}y^2\right) = 0$$

$$\frac{\partial^4}{\partial y^4}\left(\frac{c}{2}y^2\right) = 0$$

$$\frac{\partial^4 \phi}{\partial x^2 \partial y^2} = 0$$

LHS = RHS

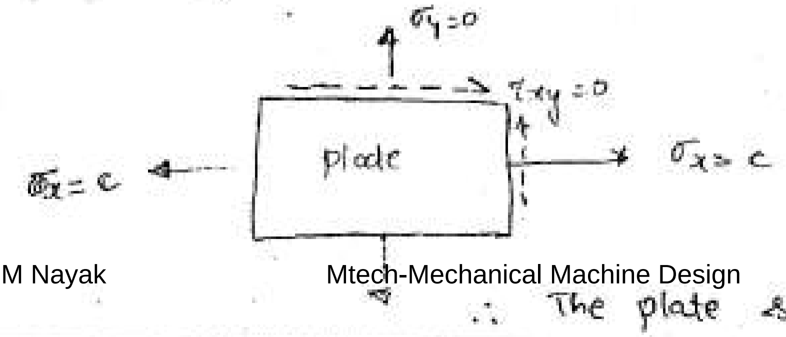
Hence given function ~~as a~~ polynomial ~~as is a~~ ~~have~~ have stress function

Stress function

$$\sigma_x = \frac{\partial^2 \phi}{\partial y^2} = \frac{\partial^2}{\partial y^2}\left(\frac{c}{2}y^2\right) = \frac{\partial}{\partial y}(cy) = c$$

$$\sigma_y = \frac{\partial^2 \phi}{\partial x^2} = \frac{\partial^2}{\partial x^2}\left(\frac{c}{2}y^2\right) = 0$$

$$\tau_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y} = -\frac{\partial^2}{\partial x \partial y}\left[\frac{c}{2}y^2\right] = 0$$



∴ The plate subjected to pure tension

(ii) $\phi = -Cxy$

$$\left[\frac{\partial^4}{\partial x^4} + \frac{\partial^4}{\partial y^4} + \frac{2\partial^4}{\partial x^2 \partial y^2}\right]\phi = 0$$

$$\frac{\partial^4}{\partial x^4}(-Cxy) + \frac{\partial^4}{\partial y^4}(-Cxy) + \frac{2}{\partial x^2 \partial y^2}(-Cxy) = 0$$

$$0 = 0$$

LHS = RHS

$$\sigma_x = \frac{\partial^2 \phi}{\partial y^2} = \frac{\partial^2}{\partial y^2}(-Cxy) = 0$$

$$\sigma_y = \frac{\partial^2 \phi}{\partial x^2} = \frac{\partial^2}{\partial \phi^2}(-Cxy) = 0$$

$$\tau_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y} = -\frac{\partial^2}{\partial x \partial y}(-Cxy) = -\frac{\partial}{\partial x}(-Cx) = +C$$

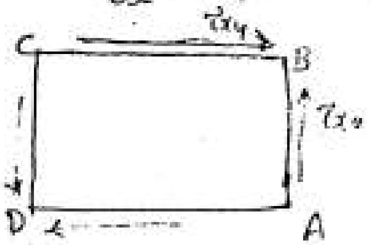


Plate is subjected to pure shear

(iii) $\phi = \frac{C}{2}y^3$

$$\frac{\partial^4}{\partial x^4}\left(\frac{C}{2}y^3\right) + \frac{\partial^4}{\partial y^4}\left(\frac{C}{2}y^3\right) + \frac{2\partial^4}{\partial x^2 \partial y^2}\left(\frac{C}{2}y^3\right) = 0$$

$$0 = 0$$

LHS = RHS

$$\sigma_x = \frac{\partial^2}{\partial y^2}\left(\frac{C}{6}y^3\right) = \frac{\partial}{\partial y}\left(\frac{3C}{6}y^2\right) = \frac{3C}{6} \times 2y = Cy \quad \text{—— (1)}$$

$$\sigma_y = \frac{\partial^2}{\partial x^2}\left(\frac{C}{6}y^3\right) = 0 \qquad \tau_{xy} = -\frac{\partial^2}{\partial x \partial y}\left(\frac{C}{6}y^3\right) = 0$$

Moment = Force x distance

$$= \sigma_x dA \times y \quad —②$$

$$= (cy)\,dA\,y$$

$$= cy^2 dA$$

$\sigma = \dfrac{\text{force}}{\text{Area}}$

force = $\sigma \times$ Area

$= \sigma_x dA$

∴ Total moment $M = \int_{-c}^{c} cy^2 dA$

$$= c\int_{-c}^{c} y^2 dA$$

where $I = \int_{-c}^{c} y^2 dA$

Area moment of inertia.

$$M = cI$$

$$c = \frac{M}{I}$$

Sub value of 'c' in ①

$$\sigma_x = \frac{M}{I}y$$

$$\boxed{\frac{\sigma}{y} = \frac{M}{I}}$$

Given polynomial represents a member subjected to bending.

Assignment

① $\phi = a_1x + b_1y$ ② $\phi = \frac{a_2}{2}x^2 + b_2xy + c_2y^2/2$

③ using following polynomial as an Airy's stressfunction.
$\phi = c_1x^3 + c_2x^2y + c_3xy^2 + c_4y^3$. Evaluate the constants so that you have the soln to beam under pure bending. take the lengths of beam 'L' & width = h.

④ For the given stress function (i) $\phi = Ax^2$ (ii) $\phi = Bxy$ (iii) $\phi = Cy^3$
(iv) $\phi = \frac{A}{2}x^2 + Bxy + C/2\,y^2$

(1) Problems

Investigate what problem of plane stress is solved by the stress fxn $\phi = \frac{3F}{4C}\left(xy - \frac{xy^3}{3c^2}\right) + \frac{P}{2}y^2$

Sol:- The biharmonic function is given by

$$\frac{\partial^4\phi}{\partial x^4} + \frac{\partial^4\phi}{\partial y^4} + \frac{2\partial^4\phi}{\partial x^2\partial y^2} = 0$$

(1) $$\frac{\partial^4}{\partial x^4}\left[\frac{3F}{4C}\left(xy - \frac{xy^3}{3c^2}\right) + \frac{P}{2}y^2\right] = 0$$

(2) $$\frac{\partial^4\phi}{\partial y^4} = \frac{\partial^4}{\partial y^4}\left[\frac{3F}{4C}\left(xy - \frac{xy^3}{3c^2}\right) + \frac{P}{2}y^2\right] = 0$$

(3) $$\frac{\partial^4\phi}{\partial x^2\partial y^2} = \frac{\partial^4}{\partial x^2\partial y^2}\left[\frac{3F}{4C}\left(xy - \frac{xy^3}{3c^2}\right) + \frac{P}{2}y^2\right] = 0$$

---

(1) $$\frac{\partial^2\phi}{\partial y^2} = \frac{\partial^2}{\partial y^2}\left[\frac{3F}{4C}\left(xy - \frac{xy^3}{3c^2}\right) + \frac{P}{2}y^2\right] = \sigma_x$$

$$= \frac{\partial}{\partial y}\left[\frac{3F}{4C}\left(x - \frac{3xy^2}{3c^2}\right) + \frac{2Py}{2}\right] \text{ — w.r.t } y$$

$$= \frac{3F}{4C}\left(\frac{-2xy}{c^2}\right) + P = \text{ — w.r.t } y$$

$$= \frac{-3F}{2c^3}xy + P \text{ ——— (1)}$$

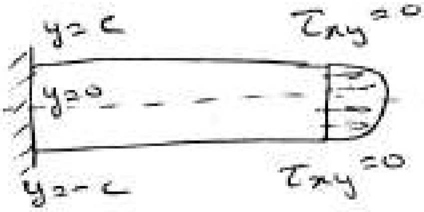
(2) $$\frac{\partial^2\phi}{\partial x^2} = \frac{\partial^2}{\partial x}\left[\frac{3F}{4C}\left(y - \frac{y^3}{3c^2}\right) + 0\right] = 0 \text{ — (2)}$$

$$= \sigma_y$$

(3) $$\tau_{xy} = \frac{-\partial^2\phi}{\partial x\partial y} = \frac{-\partial}{\partial x}\left[\frac{3F}{4C}\left(x - \frac{xy^2}{c^2}\right) + Py\right]$$

$$= \frac{-3F}{4C}\left(1 - \frac{y^2}{c^2}\right) \Rightarrow \boxed{\tau_{xy} = \frac{3F}{4C}\left(\frac{y^2}{c^2} - 1\right)} \text{ — (3)}$$

Apply bc



$\tau_{xy} = \frac{3F}{4C}\left(\frac{y^2}{c^2} - 1\right)$

→ Apply B.C to ①, $\sigma_x = -\frac{3F}{2c^3}xy + P$ —— ①

at $y = c$ $\quad \sigma_x = -\frac{3Fx}{2c^2} + P$

$y = 0 \quad \sigma_x = P$

$y = -c \quad \sigma_x = \frac{3F}{2c^3} + P$

* Consider the region include $y = \pm c$ on the x side positive, As $\sigma_y = 0$ it may be a problem under bending. As $\sigma_y = 0$ it may be a problem under bending also it may be cantilever beam.

Apply BC to ②

At $y = c \quad \tau_{xy} = \frac{3F}{4C}\left[\frac{c^2}{c^2} - 1\right] \Rightarrow \tau_{xy} = 0$

$y = 0 \quad \tau_{xy} = -\frac{3F}{4C}$

$y = -c \quad \tau_{xy} = 0$

→ Max shear = 1.5 times of Stress Stress.

$\tau_{xy} = -\frac{3}{2}\,\frac{F}{2C}$

$= -\frac{3}{2}\cdot\frac{F}{A}$

$\boxed{\tau = -1.5\sigma}$

Page 3

∴ For cantilever beam at middle fiber shear stress is max. (at $y=0$) & at ends shear stress is zero ($y=\pm c$). This condⁿ also satisfies the condⁿ for cantilever beam.

Max. shear stress = 1.5 times of stress.

To find Shear force

Shear force = Shear stress × Area

$$= -\frac{3F}{4c}\left(1-\frac{y^2}{c^2}\right) \times dy.1$$

Area = dy × 1

$$= -\frac{3F}{4c}\int_{-c}^{c}\left(1-\frac{y^2}{c^2}\right)dy$$

$$= -\frac{3F}{4c}\left[y-\frac{y^3}{3c^2}\right]_{-c}^{c}$$

By property of integral

$$\int_{-c}^{c} = 2\int_{0}^{c}$$

$$= -\frac{3F}{4c}\left[2c - \frac{1}{3c^2}(c^3+c^3)\right]$$

$$= -\frac{3F}{4c}\left[2c-\frac{2c}{3}\right]$$

$$= -\frac{3F}{2}\left[1-\frac{1}{3}\right]$$

$$= \frac{-3F}{2}\left[\frac{2}{3}\right]$$

$$= -F$$

To find moment

Moment = Force × dist

= Stress × Area × distance

$$= \sigma_x \times (dy \times 1)\, y$$

$$= \left(-\frac{3F}{2c^3}xy + P\right)dy.y$$

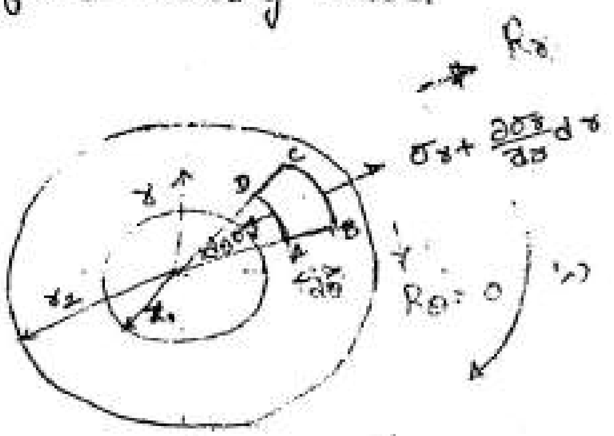
$$= \int_{-c}^{c}\left(P-\frac{3F}{2c^3}xy\right)y\,dy = 2\int_{0}^{c}\left(Py-\frac{3F}{2c^3}xy^2\right)dy$$

$$= 2\left[\frac{Py^2}{2} - \frac{3F}{2c^3}\frac{xy^3}{3}\right]^{c}$$

Axisymmetric Problems with body forces (or) Rotational Symmetry

Stresses in rotating disc of Uniform thickness :-

The stresses produced in a disc rotating at high speed is important in many practical purposes among which the design of disc wheels in steam & gas turbines. The stresses due to tangential forces being transmitted are usually small. In this cases, & the large stresses are due to the centrifugal forces of the rotating disc.



Rotational Symmetry :- The stress distribution is symmetrical about axis of rotation is called rotational symmetry.

Consider an element of the circular disc are shown in fig. having unit thickness (plane stress problem $A > t$).

Let ω = angular velocity of disc rad/s

ρ = mass density of disc kg/m³

R_r = Body force per unit volume in radial dirⁿ

R_θ = Body force per unit volume along tangential dirⁿ

Let the body force is the centrifugal force in radial dirⁿ

$$\therefore F_c = R_r = m\omega^2 r$$
$$= \rho\omega^2 r$$

$\rho = \frac{m}{V}$

$\boxed{\rho = m}$ $\because V = 1$

It is evident that the stress distribution in the disc symmetrical about axis of rotation.

we know that the equilibrium eqn along radial direction

$\Sigma F_r = 0$

$$\frac{d\sigma_r}{dr} + \frac{1}{r}\frac{d\tau_{r\theta}}{d\theta}^{\to 0} + \frac{(\sigma_r - \sigma_\theta)}{r} + R_r = 0$$

$$\frac{d\sigma_r}{dr} + \frac{1}{r}(\sigma_r - \sigma_\theta) + R_r = 0$$

x^{ly} by 'r'

$$r\frac{d\sigma_r}{dr} + (\sigma_r - \sigma_\theta) + rR_r = 0$$

$$\underbrace{\left(r\frac{d\sigma_r}{dr} + \sigma_r\right)}_{\downarrow} - \sigma_\theta + (r.\rho\omega^2 r) = 0$$

$$\frac{d}{dr}(r\sigma_r) - \sigma_\theta + r^2\rho\omega^2 = 0 \quad \text{——— (1)}$$

Assume the stress function $\left.\begin{array}{l}\phi = r\sigma_r \\ \sigma_r = \frac{\phi}{r}\end{array}\right\}$ ——— (2)

$$\sigma_\theta = \frac{d\phi}{dr} + \rho\omega^2 r^2 \quad \text{——— (2a)}$$

~~The~~

we also know that the strain

{ The displacement component 'u' is function of 'r' & 'v' is zero }

$$\epsilon_r = \frac{du}{dr}$$

$$\epsilon_\theta = \frac{u}{r} \implies u = \epsilon_\theta r$$

$$\frac{d\epsilon_\theta}{dr} = \frac{1}{r}\frac{du}{dr} - \frac{1}{r^2}u$$

$$= \frac{1}{r}\frac{du}{dr} - \frac{1}{r^2}r\epsilon_\theta$$

$$= \frac{1}{r}\epsilon_r - \frac{1}{r}\epsilon_\theta$$

$$= \frac{1}{r}(\epsilon_r - \epsilon_\theta)$$

$$\boxed{r\frac{d\epsilon_\theta}{dr} + \epsilon_\theta - \epsilon_r = 0} \quad \text{——— (3)}$$

of strain

To determine Airy's stress function

Using Hooke's law (or) stress-strain relation

$$\left.\begin{aligned} e_r &= \frac{1}{E}\left[\sigma_r - \nu\sigma_\theta\right] \\ e_\theta &= \frac{1}{E}\left[\sigma_\theta - \nu\sigma_r\right] \end{aligned}\right] \quad \text{—— (4)}$$

Substituting the values of σ_θ & σ_r in (4)

$$e_r = \frac{1}{E}\left[\frac{\phi}{r} - \nu\left(\frac{d\phi}{dr} + \rho\omega^2 r^2\right)\right]$$

$$= \frac{1}{E}\left(\frac{\phi}{r}\right) - \frac{\nu}{E}\frac{d\phi}{dr} - \frac{\nu}{E}\rho\omega^2 r^2$$

Ily $\quad e_\theta = \frac{1}{E}\left[\frac{d\phi}{dr} + \rho\omega^2 r^2 - \frac{\nu\phi}{r}\right]$

$$= \frac{1}{E}\frac{d\phi}{dr} + \frac{\rho\omega^2 r^2}{E} - \frac{\nu\phi}{Er}$$

Sub value of e_θ & e_r in (3)

$$r\frac{d}{dr}\left[\frac{1}{\cancel{E}}\frac{d\phi}{dr} + \frac{\rho\omega^2 r^2}{\cancel{E}} - \frac{\nu\phi}{\cancel{E}r}\right] + \frac{1}{\cancel{E}}\left[\frac{d\phi}{dr} + \rho\omega^2 r^2 - \frac{\nu\phi}{r}\right]$$

$$- \frac{1}{\cancel{E}}\left[\frac{\phi}{r} - \frac{\nu}{E}\frac{d\phi}{dr} - \nu\rho\omega^2 r^2\right] = 0$$

$$r\frac{d}{dr}\left[\frac{d\phi}{dr} + \rho\omega^2 r^2 - \nu\frac{\phi}{r}\right] + \frac{d\phi}{dr} + \rho\omega^2 r^2 - \frac{\nu\phi}{r} - \frac{\phi}{r}$$

$$+ \nu\frac{d\phi}{dr} + \nu\rho\omega^2 r^2 = 0$$

$$r\left[\frac{d^2\phi}{dr^2} + \rho\omega^2 2r - \nu\left\{\frac{1}{r}\frac{d\phi}{dr} - \frac{1}{r^2}\phi\right\}\right] + \frac{d\phi}{dr} + \rho\omega^2 r^2 - \frac{\nu\phi}{r} - \frac{\phi}{r}$$

$$+ \nu\frac{d\phi}{dr} + \nu\rho\omega^2 r^2 = 0$$

$$r\frac{d^2\phi}{dr^2} + 2\rho\omega^2 r^2 - \cancel{\nu\frac{d\phi}{dr}} + \cancel{\frac{\nu}{r}\phi} + \frac{d\phi}{dr} + \rho\omega^2 r^2 - \cancel{\frac{\nu\phi}{r}} - \frac{\phi}{r} + \cancel{\nu\frac{d\phi}{dr}} + \nu\rho\omega^2 r^2 = 0$$

$$r\frac{d^2\phi}{dr^2} + (3+\nu)\rho\omega^2 r^2 + \frac{d\phi}{dr} - \frac{\phi}{r} = 0$$

$\div$ by r

$$\frac{d^2\phi}{dr^2} + (3+\nu)\rho\omega^2 r + \frac{1}{r}\frac{d\phi}{dr} - \frac{\phi}{r^2} = 0$$

$$\frac{d^2\phi}{dr^2} + \frac{1}{r}\frac{d\phi}{dr} - \frac{\phi}{r^2} = -(3+\nu)\rho\omega^2 r$$

$$\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}(\phi r)\right] = -(3+\nu)\rho\omega^2 r \quad \text{——— (5)}$$

$$\frac{1}{r}\frac{d}{dr}(r\phi)$$

$$\frac{1}{r}\left[\phi + r\frac{d\phi}{dr}\right]$$

$$= \frac{d\phi}{dr} + \frac{\phi}{r}$$

diff w r to 'r'

$$\frac{d^2\phi}{dr^2} + \frac{1}{r}\frac{d\phi}{dr} - \frac{1}{r^2}\phi$$

Integrating Eqn (5)

$$\frac{1}{r}\frac{d}{dr}(\phi r) = -(3+\nu)\rho\omega^2\frac{r^2}{2} + C_1$$

$$\frac{d}{dr}(r\phi) = -(3+\nu)\rho\omega^2\frac{r^3}{2} + C_1 r$$

Integrate once again w.r.to 'r'.

$$r\phi = -\frac{(3+\nu)\rho\omega^2}{2}\frac{r^4}{4} + \frac{C_1 r^2}{2} + C_2$$

$$\boxed{\phi = -(3+\nu)\frac{\rho\omega^2 r^3}{8} + C_1\frac{r}{2} + \frac{C_2}{r}} \quad \text{——— (6)}$$

ϕ is Airy's stress function

where C_1 & C_2 are const of integration

To determine stress components

$\frac{d}{dx}\{1/x\} = -1/x^2$

wkt $\sigma_r = \frac{\phi}{r}$

$\sigma_\theta = \frac{d\phi}{dr} + \rho\omega^2 r^2$

$$\therefore \cancel{\phi}\,\sigma_r = -(3+\nu)\rho\omega^2\frac{r^2}{8} + \frac{c_1}{2} + \frac{c_2}{r^2} \quad \text{—— (7)}$$

$$\sigma_\theta = \frac{d}{dr}\left[-(3+\nu)\rho\omega^2\frac{r^3}{8} + \frac{c_1 r}{2} + \frac{c_2}{r}\right] + \rho\omega^2 r^2$$

$$= -(3+\nu)\rho\omega^2\frac{3r^2}{8} + \frac{c_1}{2} - \frac{c_2}{r^2} + \rho\omega^2 r^2$$

$$= \rho\omega^2 r^2\left[1 - (3+\nu)\frac{3}{8}\right] + \frac{c_1}{2} - \frac{c_2}{r^2} \qquad \left\{1 - 9/8 + 3\nu/8\right\}$$

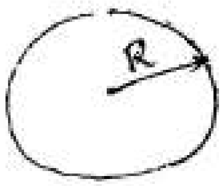
$$= \rho\omega^2 r^2\left[-\frac{1}{8} + \frac{3\nu}{8}\right] + \frac{c_1}{2} - \frac{c_2}{r^2} \qquad \left\{\frac{8-9}{8} + 3\nu/8\right\}$$

$$\left\{-1/8 - 3\nu/8\right\}$$

$$\sigma_\theta = -\frac{\rho\omega^2 r^2}{8}(1+3\nu) + \frac{c_1}{2} - \frac{c_2}{r^2} \quad \text{—— (8)}$$

Case 1 :- Solid disc of radius $r_2 = R$

$r_1 = 0 \quad r_2 = R,$



Consider solid disc of radius 'R', No external forces applied at the boundary. $\sigma_r = 0, \ \tau_{r\theta} = 0$

when $r = 0$ at the centre of disc the stresses becomes infinite. To avoid infinite stresses we must take $\boxed{c_2 = 0}$

Eqn (7) becomes

$$\sigma_r = -(3+\nu)\frac{\rho\omega^2 r^2}{8} + \frac{c_1}{2} + 0$$

$$0 = -\frac{(3+\nu)\rho\omega^2R^2}{8} + \frac{c_1}{2}$$

$$\frac{c_1}{2} = \frac{(3+\nu)\rho\omega^2R^2}{8\,4}$$

$$\boxed{c_1 = \frac{(3+\nu)}{4}\rho\omega^2R^2} \quad \text{—— (9)}$$

Sub value of c_1 in (7)

$$\sigma_r = -\frac{(3+\nu)\rho\omega^2r^2}{8} + \frac{(3+\nu)}{8}\rho\omega^2R^2$$

$$\sigma_r = \frac{(3+\nu)}{8}\rho\omega^2(R^2-r^2) \quad \text{—— (10)}$$

$\Rightarrow$ (8)
$$\sigma_\theta = \frac{-\rho\omega^2r^2}{8}(1+3\nu) + \frac{(3+\nu)}{8}\rho\omega^2R^2 \quad \text{—— (11)}$$

$$= \cancel{\frac{\rho\omega^2R^2}{8}}$$

(i) At $r=0$, ie at the centre.

$\Rightarrow$ (10)
$$\boxed{\sigma_{r(max)} = \frac{(3+\nu)}{8}\rho\omega^2R^2}$$

$$\boxed{\sigma_{\theta_{max}} = \frac{3+\nu}{8}\rho\omega^2R^2}$$

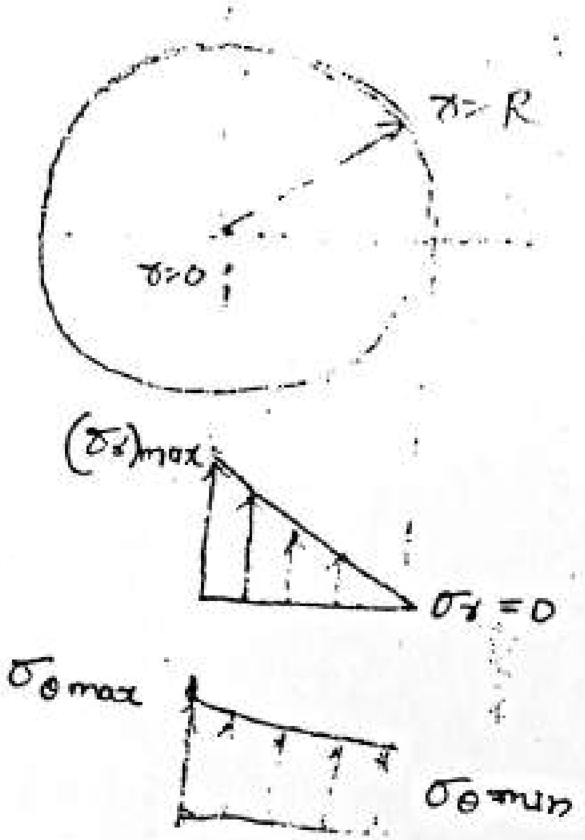
(ii) At $r=R$ ie at boundary

(10) $\Rightarrow$ $\boxed{\sigma_r = 0}$

(11) $\Rightarrow$ $$\sigma_\theta = +\frac{\rho\omega^2R^2}{8}\left[-1-3\nu+3+\nu\right]$$

$$= \frac{\rho\omega^2R^2}{8}\left[2-2\nu\right]$$

$$\boxed{(\sigma_\theta) = \frac{\rho\omega^2R^2}{\ldots}\left[1-\nu\right]}$$

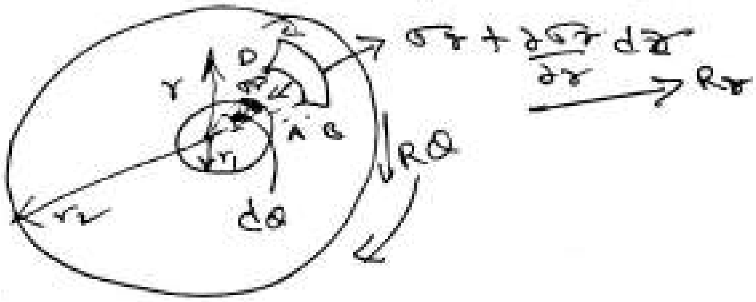


Stress distribution

Axisymmetric & Torsional Prob.

→ Stresses in rotating disc: (Uniform thk)

The stresses produced in a disc rotating at high speed is important in many practical purposes among which the design of disc wheels in steam & gas turbine. The stresses due to tangential forces being transmitted are usually small. In this case, & the large stresses are due to the centrifugal forces of the rotating disc.



Rotational Symmetry: The stress distribution is symmetrical about axis of rotation is called rotational symmetry.

Consider an circular disc element having unit thk.

let ω = ang. vel. of disc rad/s

ρ = mass density of disc kg/m³

R_r = Body force per unit vol. in radial dirn

R_θ = Body force per unit vol. along tangential dir.

→ Body force — centrifugal force in radial dirn.

$\therefore F_c = R_r = m\omega^2 r$

$= \rho\omega^2 r$

$\rho = \frac{m}{V}$

$\rho = m \quad (\text{if } V = 1)$

ρ → density (m³)

* The stress distribution in sheet is symmetrical about axis of rotation

$\therefore$ Equilibrium along radial direction

$\Sigma F_r = 0,$

$$\frac{d\sigma_r}{dr} + \frac{1}{r}\cancel{\frac{d\tau_{\theta r}}{d\theta}}^{0} + \frac{(\sigma_r - \sigma_\theta)}{r} + R_r = 0$$

[Explanation: We add the inertia force term ~~(…)~~ and equate it to inertia force (R_r) in z-dir.]

Now, $\times$ly by 'r'

$$r\frac{d\sigma_r}{dr} + (\sigma_r - \sigma_\theta) + r \cdot R_r = 0$$

$$\left(r\frac{\partial\sigma_r}{\partial r} + \sigma_r\right) - \sigma_\theta + (r \cdot \rho\omega^2 r) = 0$$

$$\frac{d}{dr}(r \cdot \sigma_r) - \sigma_\theta + r^2\rho\omega^2 = 0 \quad \text{—— (1)}$$

Assume the stress fxn $\phi = r\sigma_r$, $\sigma_r = \frac{\phi}{r}$ —— (2)

(or) $\sigma_\theta = \frac{d\phi}{dr} + \rho\omega^2 r^2$ —— (2a)

$\rightarrow$ W.K.T, Strain displacement component is a fxn of 'r' & 'γ' is zero

$$\epsilon_r = \frac{du}{dr}, \quad \epsilon_\theta = \frac{u}{r} \Rightarrow \underline{\underline{u = \epsilon_\theta r}}$$

$$\frac{d\epsilon_\theta}{dr} = \frac{1}{r}\frac{du}{dr} - \frac{1}{r^2}u = \frac{1}{r}\frac{du}{dr} - \frac{1}{r^{\cancel{2}}}\cancel{r} \cdot \epsilon_\theta$$

$$= \frac{1}{r}\epsilon_r - \frac{1}{r}\epsilon_\theta$$

$$= \frac{1}{r}(\epsilon_r - \epsilon_\theta) \text{ (or) } \boxed{r\frac{d\epsilon_\theta}{dr} + \epsilon_\theta - \epsilon_r = 0} \quad (3)$$

To det. Airy's stress fxn

Using Hooke's law

$$\left.\begin{aligned}\epsilon_r &= \frac{1}{E}[\sigma_r - \nu\sigma_\theta] \\ \epsilon_\theta &= \frac{1}{E}[\sigma_\theta - \nu\sigma_r]\end{aligned}\right\} \text{— } (4)$$

Subs. the values of σ_θ & σ_r in (4)

$$\epsilon_r = \frac{1}{E}\left[\frac{\phi}{r} - \nu\left(\frac{d\phi}{dr} + \rho\omega^2 r^2\right)\right]$$

$$= \frac{1}{E}\left(\frac{\phi}{r}\right) - \frac{\nu}{E}\frac{d\phi}{dr} - \frac{\nu}{E}\rho\omega^2 r^2$$

IIIly $\epsilon_\theta = \frac{1}{E}\left[\frac{d\phi}{dr} + \rho\omega^2 r^2 - \frac{\nu\phi}{r}\right]$

$$= \frac{1}{E}\frac{d\phi}{dr} + \frac{\rho\omega^2 r^2}{E} - \frac{\nu\phi}{Er}$$

Sub value of ϵ_θ & ϵ_r in (3)

$$\Rightarrow r\frac{d}{dr}\left[\frac{1}{\cancel{E}}\frac{d\phi}{dr} + \frac{\rho\omega^2 r^2}{\cancel{E}} - \frac{\nu\phi}{\cancel{E}r}\right] + \frac{1}{E}\left[\frac{d\phi}{dr} + \rho\omega^2 r^2 - \frac{\nu\phi}{r}\right]$$

$$- \frac{1}{E}\left[\frac{\phi}{r} - \frac{\nu}{1}\cdot\frac{d\phi}{dr} - \nu\rho\omega^2 r^2\right] = 0$$

$$\Rightarrow r\frac{d}{dr}\left[\frac{d\phi}{dr} + \rho\omega^2 r^2 - \nu\frac{\phi}{r}\right] + \frac{d\phi}{dr} + \rho\omega^2 r^2 - \frac{\nu\phi}{r} - \frac{\phi}{r}$$

$$+ \nu\frac{d\phi}{dr} + \nu\rho\omega^2 r^2 = 0$$

$$\Rightarrow r\left[\frac{d^2\phi}{dr^2} + \rho\omega^2 \cdot 2r - \nu\left\{\frac{1}{r}\frac{d\phi}{dr} - \frac{1}{r^2}\phi\right\}\right] + \frac{d\phi}{dr} + \rho\omega^2 r^2$$

$$- \frac{\nu\phi}{r} - \frac{\phi}{r} + \nu\frac{d\phi}{dr} + \nu\rho\omega^2 r^2 = 0$$

$$\Rightarrow r\frac{d^2\phi}{dr^2} + 2\rho\omega^2 r^2 - \nu\frac{d\phi}{dr} + \frac{\nu}{r}\phi + \frac{d\phi}{dr} + \rho\omega^2 r^2$$

$$- \frac{\nu\phi}{r} - \frac{\phi}{r} + \nu\frac{d\phi}{dr} + \nu\rho\omega^2 r^2 = 0$$

$$\Rightarrow r\frac{d^2\phi}{dr^2} + (3+\nu)\rho\omega^2 r^2 + \frac{d\phi}{dr} - \frac{\phi}{r} = 0$$

÷ by 'r'

$$\Rightarrow \frac{d^2\phi}{dr^2} + (3+\nu)\rho\omega^2 r + \frac{1}{r}\frac{d\phi}{dr} - \frac{\phi}{r^2} = 0$$

$$\Rightarrow \frac{d^2\phi}{dr^2} + \frac{1}{r}\frac{d\phi}{dr} - \frac{\phi}{r^2} = -(3+\nu)\rho\omega^2 r$$

$$\Rightarrow \frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}(\phi r)\right] = -(3+\nu)\rho\omega^2 r \quad \text{—— (5)}$$

Integrating Eqn (5)

$$\frac{1}{r}\frac{d}{dr}(\phi r) = -(3+\nu)\rho\,\omega^2\frac{r^2}{2} + C_1$$

$$\frac{d}{dr}(r,\phi) = -(3+\nu)\rho\omega^2\frac{r^3}{2} + C_1 r$$

Integrate once again w.r.t 'r'

$$r\phi = -\frac{(3+\nu)\rho\omega^2}{2}\frac{r^4}{4} + \frac{C_1 r^2}{2} + C_2$$

$$\boxed{\phi = -(3+\nu)\frac{\rho\omega^2 r^3}{8} + C_1\frac{r}{2} + \frac{C_2}{r}} \quad \text{—— (6)}$$

ϕ – Airy's stress fan

Part 1

→ Rotating Shafts & Cylinders (Stresses)

$$\frac{d}{dr}(r\sigma_r) - \sigma_\theta + \rho\omega^2 r^2 = 0$$

The strain component are

$$\varepsilon_r = \frac{du_r}{dr}, \quad \varepsilon_\theta = \frac{u_r}{r}, \quad \varepsilon_z = \frac{\partial u_z}{\partial z} = 0$$

⇒ From Hooke's law.

$$\varepsilon_r = \frac{1}{E}[\sigma_r - \nu(\sigma_\theta + \sigma_z)]$$

$$\varepsilon_\theta = \frac{1}{E}[\sigma_\theta - \nu(\sigma_r + \sigma_z)]$$

$$\varepsilon_z = \frac{1}{E}[\sigma_z - \nu(\sigma_r + \sigma_\theta)]$$

⇒ Since $\varepsilon_z = 0$ (plane strain)

$$\sigma_z = \nu(\sigma_r + \sigma_\theta)$$

⇒ & hence, substituting in eqn for ε_r & ε_θ

$$\varepsilon_r = \frac{1+\nu}{E}\left[(1-\nu)\sigma_r - \nu\sigma_\theta\right]$$

$$\varepsilon_\theta = \frac{1+\nu}{E}\left[(1-\nu)\sigma_\theta - \nu\sigma_r\right]$$

⇒ From strain-disp. relations, given

$$\varepsilon_r = \frac{d}{dr}(r\varepsilon_\theta) \qquad \left[\varepsilon_r = \frac{du_r}{dr} \quad \because u_r = r\cdot\varepsilon_\theta\right]$$

& using the above expressions for ε_r & ε_θ

$$(1-\nu)\sigma_r - \nu\sigma_\theta = \frac{d}{dr}\left[(1-\nu)r\sigma_\theta - \nu r\sigma_r\right]$$

With $r\sigma_r = y$.

$$\sigma_\theta = \frac{dy}{dr} + \rho\omega^2 r^2$$

Substituting for σ_r & σ_θ

$$(1-\nu)\frac{y}{r} - \nu\frac{dy}{dr} - \nu\rho\omega^2 r^2 = \frac{d}{dr}\left[(1-\nu)\left(r\frac{dy}{dr} + \rho\omega^2 r^3\right) - \nu \ldots\right]$$

$$r^2\left(\frac{d^2y}{dr^2}\right) + r\left(\frac{dy}{dr}\right) - y + \left(\frac{3-2\nu}{1-\nu}\rho\omega^2 r^3\right) = 0$$

The solution for this differential eq'n is

$$y = Cr + C_1\frac{1}{r} - \frac{(3-2\nu)}{8(1-\nu)}\rho\omega^2 r^3$$

Hence,

$$\boxed{\sigma_r = C + C_1\frac{1}{r^2} - \frac{(3-2\nu)}{8(1-\nu)}\rho\omega^2 r^2}$$

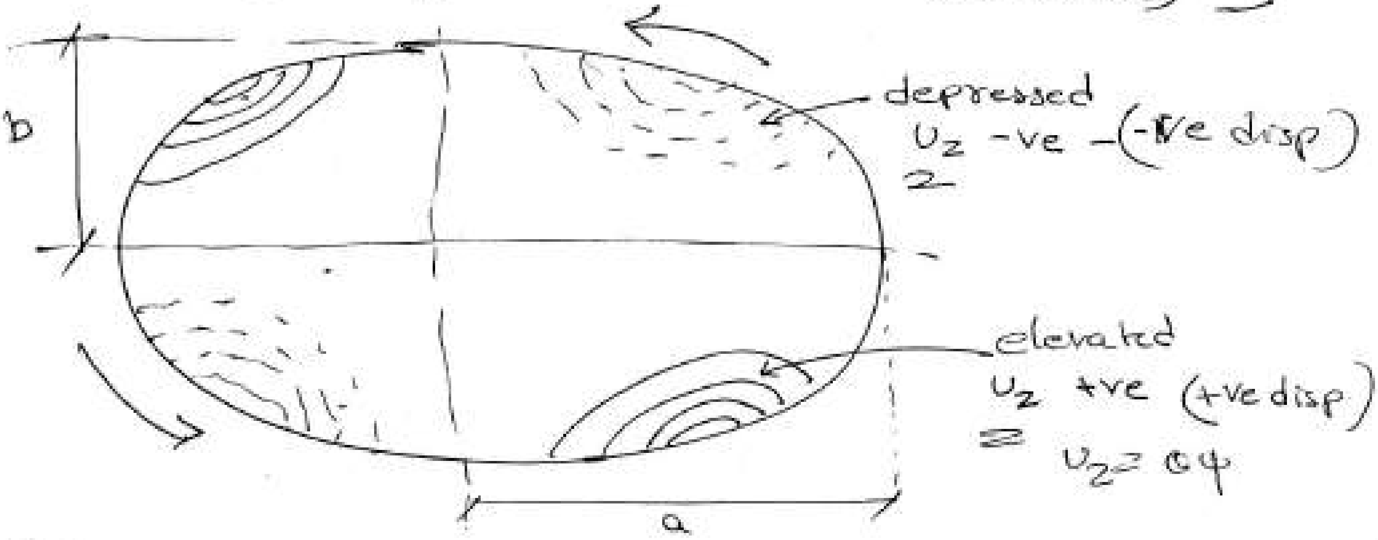
$$\boxed{\sigma_\theta = C - C_1\frac{1}{r^2} - \frac{(1+2\nu)}{8(1-\nu)}\rho\omega^2 r^2}$$

$$\boxed{\sigma_z = \nu\left[2C - \frac{1}{2(1-\nu)}\rho\omega^2 r^2\right]}$$

→ Torsion of Circular & Elliptical Bars

[Torsion in general prismatic bar (solid section)

we have, $\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = \nabla^2 \psi = 0$, Laplace eqn, ψ – wrapping fn (Harmonic)]



(Cross-Section of an elliptical bar + Contour lines of U_z)

- Torque 'T' is applied
- U_z is –ve for dotted line } Grain displacement
- U_z is +ve for Solid line }
- If the ends are free – no Normal Stresses
- If one end is fixed then wrapping at that end is prevented
- Due to this normal stresses are induced.

(i) The simplest sol. to the laplace eqn

ψ = const. = c

with $\psi = c$, the bc's given

$$-\frac{dx}{ds} = \cos(n,y) = \frac{dy}{dn} \qquad \left(\frac{\partial \psi}{\partial x} - y\right)\frac{dy}{ds} -$$

$$-\frac{dy}{ds} = \cos(n,x) = \frac{dx}{dn} \qquad \left(\frac{\partial \psi}{\partial y} + x\right)\frac{dx}{ds} = 0 \quad \text{—— (1)}$$

$$\Rightarrow \; -y\frac{dy}{ds} - x\frac{dx}{ds} = 0$$

$$\Rightarrow \; \frac{d}{ds}\frac{x^2+y^2}{2} = 0$$

$$\Rightarrow \; x^2 + y^2 = \text{Const.}$$

where (x, y) are the co-ordinates of any pt. on the boundary. Hence the boundary is a circle.

$$J = \iint_R (x^2+y^2)\,dx\,dy = I_p$$

the polar moment of inertia for the section

$$T = G I_p \theta$$

$$\theta = \frac{T}{G I_p}$$

$$\therefore \; u_z = \theta c = \frac{Tc}{G I_p}$$, which is a Const.

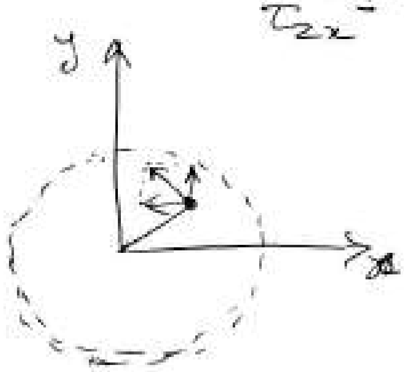
Since the fixed end has zero u_z at least at one point u_z is zero at every c/s. Thus, the c/s does not wrap.

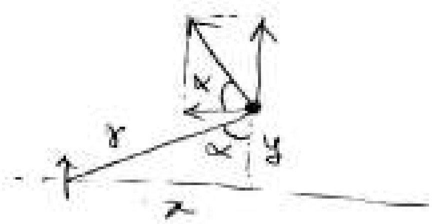
$\therefore$ The shear stresses are given by

$$\tau_{yz} = G\theta x = \frac{Tx}{I_p} \qquad \left(\tau_{yz} = G\theta\left(\frac{\partial\psi}{\partial y} + x\right)\right)$$

$$\tau_{zx} = -G\theta y = -\frac{Ty}{I_p} \qquad \left(\tau_{zx} = G\theta\left(\frac{\partial\psi}{\partial x} - y\right)\right)$$

$$\tan\alpha = \frac{\tau_{zy}}{\tau_{zx}} = -\frac{G\theta x}{G\theta y} = -\frac{x}{y}$$, where α – direction of resultant shear τ





Hence, the resultant shear is perpendicular to the radius.

$$\therefore \tau^2 = \tau_{yz}^2 + \tau_{zx}^2 = \frac{\tau^2 (x^2+y^2)}{I_p^2}$$

(or) $\tau = \frac{Tr}{I_p}$

where r is the radial dist. of pt. (x,y)

(ii) Next case, $\psi = Axy$

$A \to$ Const., This also satisfies Laplace eqn

w.k.T

$$\left(\frac{\partial\psi}{\partial x} - y\right)\frac{dy}{ds} - \left(\frac{\partial\psi}{\partial y} + x\right)\frac{dx}{ds} = 0$$

$$\Rightarrow (Ay - y)\frac{dy}{ds} - (Ax + x)\frac{dx}{ds} = 0$$

$$\Rightarrow y(A-1)\frac{dy}{ds} - x(A+1)\frac{dx}{ds} = 0$$

$$\Rightarrow (A+1)2x\frac{dx}{ds} - (A-1)2y\frac{dy}{ds} = 0$$

$$\frac{d}{ds}\left[(A+1)x^2 - (A-1)y^2\right] = 0$$

Integrating we get $\rightarrow$

$$(1+A)x^2 + (1-A)y^2 = c$$

Now, this is of the form $x^2/a^2 + y^2/b^2 = 1$

These two are identical if

$$\frac{a^2}{b^2} = \frac{1-A}{1+A}$$

(or) $A = \frac{b^2 - a^2}{b^2 + a^2}$

$$\psi = \frac{b^2 - a^2}{b^2 + a^2} xy$$

This fn represents the warping fxn for an elliptic cylinder with semi-axes a & b under torsion.

The value of $J = \iint_R (x^2 + y^2 + Ax^2 - Ay^2)\, dx\, dy$

$$= (A+1)\iint x^2\, dx\, dy + (1-A)\iint y^2\, dx\, dy$$

$$= (A+1) I_y + (1-A) I_x$$

Substituting $I_x = \frac{\pi a b^3}{4}$ & $I_y = \frac{\pi a^3 b}{4}$, we get

$$J = \frac{\pi a^3 b^3}{a^2 + b^2}$$

W.K.T, $T = GJ\theta = G\theta \frac{\pi a^3 b^3}{a^2 + b^2}$

(or) $\theta = \frac{T}{G} \frac{a^2 + b^2}{\pi a^3 b^3}$

$$\tau_{yz} = G\theta\left(\frac{\partial \psi}{\partial y} + x\right)$$

$$= T \frac{a^2 + b^2}{\pi a^3 b^3}\left(\frac{b^2 - a^2}{b^2 + a^2} + 1\right)x$$

$$= \frac{2Tx}{\pi a^3 b}$$

The resultant shearing stress at any pt. (x, y)

$$\tau = \left[\tau_{yz}^2 + \tau_{zx}^2\right]^{1/2} = \frac{2T}{\pi a^3 b^3}\left[b^4 x^2 + a^4 y^2\right]^{1/2}$$

→ To det. where the max. shear stress occurs, we substitute for x^2 from

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad (or) \quad x^2 = a^2\left(1 - \frac{y^2}{b^2}\right)$$

$$\therefore \tau = \frac{2T}{\pi a^3 b^3}\left[a^2 b^2 + a^2(a^2 - b^2) y^2\right]^{1/2}$$

$$\tau_{max} = \frac{2T}{\pi a^3 b^3}\left(a^4 b^2\right)^{1/2} = \frac{2T}{\pi a b^2}$$

* When $y = b$ → τ_{max} occurs

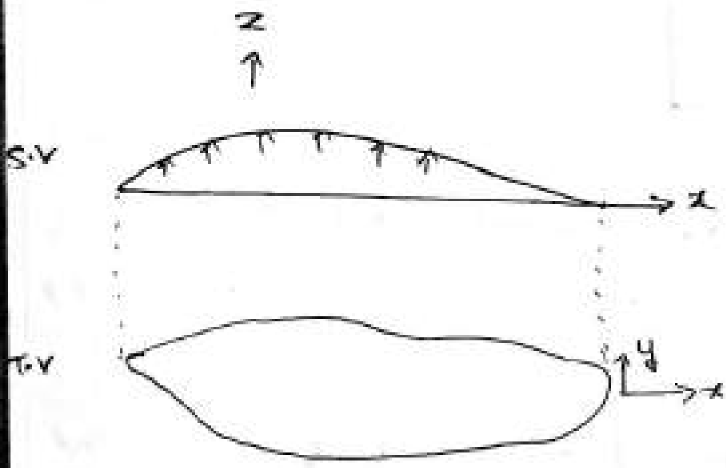
$$\therefore \tau_{max} = \frac{2T}{\pi a^3 b^3}\left(a^4 b^2\right)^{1/2} = \frac{2T}{\pi a b^2}$$

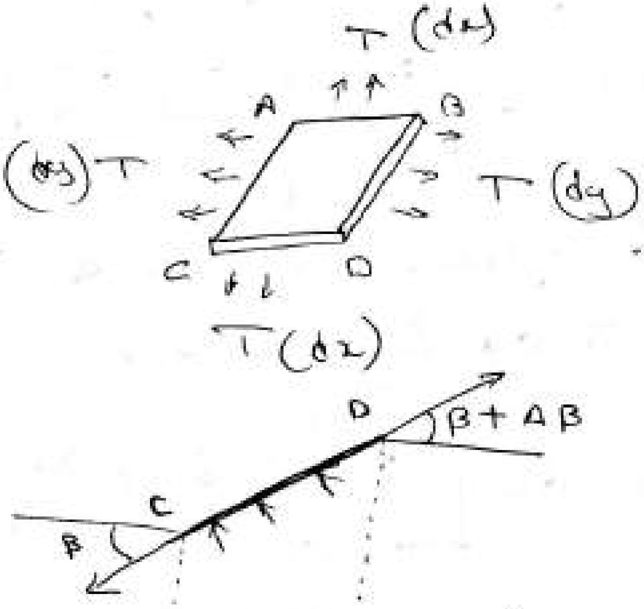
we have $u_z = \theta \psi$

$$u_z = \frac{T(b^2 - a^2)}{\pi a^3 b^3 G} xy$$

Prandtl's Membrane Analogy

* Consider a thin homogeneous membrane, like a thin rubber sheet, stretched with uniform tension fixed at it's edge





1. Membrane is subjected to constant lateral pressure 'p'
2. Therefore undergoes small deformation/disp. z
   $z = f(x, y)$
3. Now consider equillibrium element ABCD
4. 'T' be the uniform tension on the membrane
5. On face AC, the force acting is T.dy
6. AC is inclined at an angle β to x-axis
7. AB is inclined at an angle β ∴ slope $= \tan\beta = \frac{\partial z}{\partial x}$
8. Component of T dy in z-dir. is $\left(-T\,dy\,\frac{dz}{dx}\right)$
9. The force on face BD is also T dy but is inclined at an angle $(\beta + \Delta\beta)$ to x-axis

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∴ Slope of BD is

$$\frac{\partial z}{\partial x} + \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)dx$$

And force component in z-dir

$$T\,dy\left[\frac{\partial z}{\partial x} + \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)dx\right]$$

(slope)

IIIly force component T.dx on AB & CD are

$$-T\,dx\,\frac{\partial z}{\partial y} \quad \text{— AB}$$

$$T\,dx\left[\frac{\partial z}{\partial y} + \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)dy\right] \quad \text{— CD}$$

∴ Resultant force in z-direction due to tension F

(AB + BC + CD + DC = 0)

$$-T\,dy\,\frac{\partial z}{\partial x} + T\,dy\left[\frac{\partial z}{\partial x} + \frac{\partial^2 z}{\partial x^2}dx\right] - T\,dx\,\frac{\partial z}{\partial y} + T\,dx\left[\frac{\partial z}{\partial y} + \frac{\partial^2 z}{\partial y^2}dy\right] = 0$$

* let force 'p' acting upwards on membrane ABCD is p dx dy

$$T\left(\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}\right)dx\,\cancel{dy} = -p\,dx\,\cancel{dy}$$

$$\boxed{\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = -\frac{p}{T}}$$

Now, if the membrane tension 'T' or the air pressure 'p' is adjusted in such a way that p/T becomes numerically equal to $2G\theta$, then eq'n $\left(\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}=\frac{-p}{T}\right)$ of the membrane becomes identical to the equation $\left(\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}=2G\theta\right)$ or (Stress fn method) of the torsion stress fn ϕ.

Also, if the membrane height z remains zero at the boundary contour of the section, then the membrane height 'z' ~~remains zero at~~ becomes numerically equal to the torsion stress fn $\phi=0$.

→ Torsion of thin-walled sections

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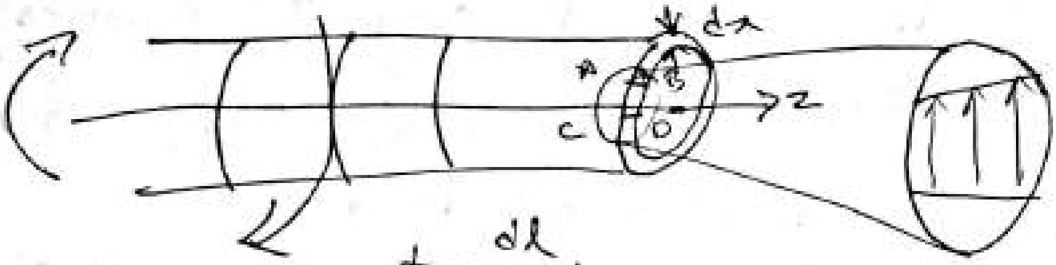
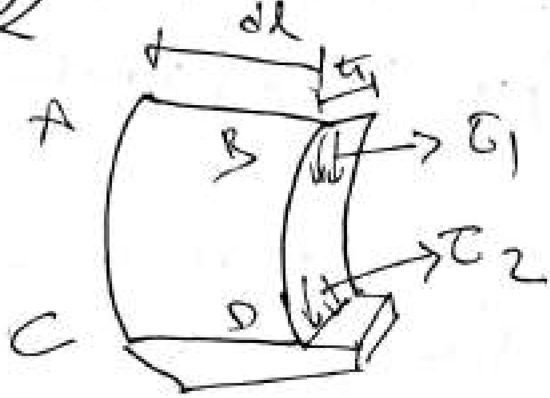




→ Area of AB cut face $t_1 dl$

→ Area of CD cut face $t_2 dl$

The shear stresses are τ_1 & τ_2

for eqbl. in z dir.,

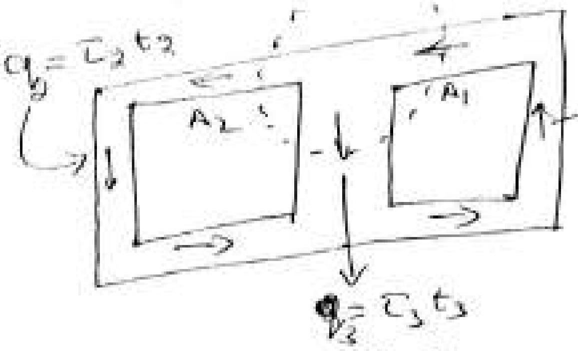
$$\tau_1 t_1 dl + \tau_2 t_2 dl = 0$$

$$\therefore \tau_1 t_1 = \tau_2 t_2 = q = \text{constant.}$$

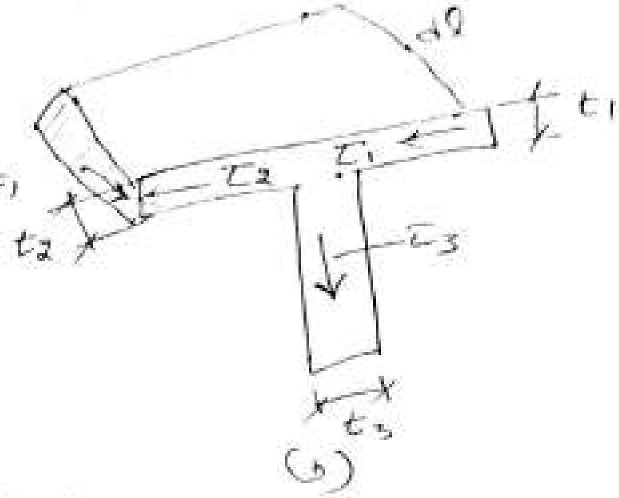
$$\boxed{\therefore \tau t \text{ is a const}}$$

This is called the shear flow q.

→ Torsion of thin-walled multiple-cell closed sections



(a)



(b)

→ Consider the jn in fig(b),

→ Let the element be in a equillibrium

→ ∴ In the direction of axis of the tube

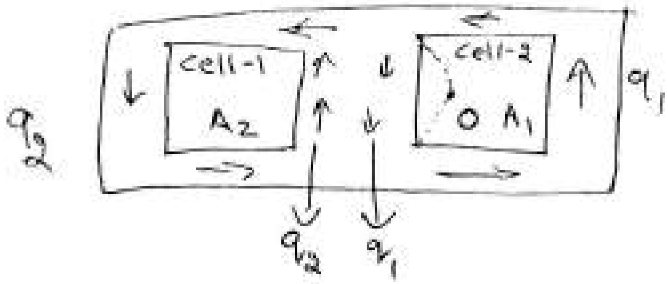
$$\Rightarrow -\tau_1 t_1\, dl + \tau_2 t_2\, dl + \tau_3 t_3\, dl = 0$$

$$\Rightarrow \tau_1 t_1 = \tau_2 t_1 + \tau_3 t_3$$

$$\Rightarrow q_1 = q_2 + q_3$$

→ Now this can be compared to a fluid flow, deviding itself into 2 streams.

∴ Choose moment axis by point 'O'.



Here $q_3 = q_1 - q_2$

[Note: The shear flow is considered to be made of q_1 and q_2 only]

(1) Cell-1 $\quad M_{t_1} = 2 q_1 A_1$

(2) Cell-2 $\quad M_{t_2} = 2 q_2 [A_1 + A_2] - 2 q_2 A_1$

} about O.

Total torque, $M_t = M_{t1} + M_{t2}$

$$\boxed{M_t = 2q_1 A_1 + 2q_2 A_2} \quad \text{——— (1)}$$

→ To find Twist (θ)

* For continuity, the twist of each cell should be same

S - Surface area

$\therefore \theta = \frac{q}{2AG} \oint \frac{ds}{t}$

(or) $2G\theta = \frac{1}{A} \int \frac{q\,ds}{t}$

let $a_1 = \oint \frac{ds}{t}$ for cell 1 including web

$a_2 = \oint \frac{ds}{t}$ for cell 2 including web

$a_{12} = \oint \frac{ds}{t}$ for the web only

Then for cell 1

$$\boxed{2G\theta = \frac{1}{A_1}(a_1 q_1 - a_2 q_2)} \quad \text{——— (2)}$$

For cell 2,

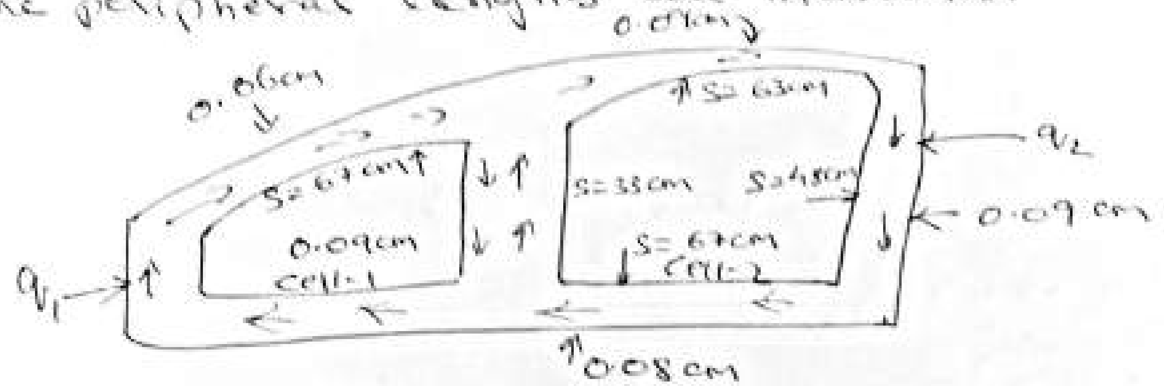
$$\boxed{2G\theta = \frac{1}{A_2}(a_2 q_2 - a_{12} q_1)} \quad \text{——— (3)}$$

Eqn. (1), (2) & (3) are sufficient for solving problems.

① The fig. gives 2-cell tubular section as formed by a conventional airfoil shape & having one interior web. An axial torque of 10000 Nm is acting in a clockwise direction. Det. the internal shear flow distribution. The cell areas are as follows:

$A_1 = 680\ cm^2$, $A_2 = 2000\ cm^2$

The peripheral lengths are indicated



Sol: → For cell 1, $a_1 = \oint \frac{ds}{t}$ [include the web]

$$= \frac{67}{0.06\,cm} + \frac{33}{0.09} \Rightarrow \underline{\underline{a_1 = 148.3}}$$

→ For cell 2, $a_2 = \frac{33}{0.09} + \frac{63}{0.07} + \frac{48}{0.09} + \frac{67}{0.08}$

$$\underline{\underline{a_2 = 2409}}$$

→ Web, $a_{12} = \frac{33}{0.09} = 366$

→ Now, for cell 1, $2G\theta = \frac{1}{A_1}(a_1 q_1 - a_{12} q_2)$

$$= \frac{1}{680}(1483 q_1 - 366 q_2)$$

$$\therefore 2G\theta = 2.181 q_1 - 0.514 q_2 \quad \text{—— (1)}$$

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→ for Cell 2

$$290 = \frac{1}{A_2}\left(a_2 q_2 - a_{12} q_1\right)$$

$$= \frac{1}{2000}\left(2409\, q_2 - 366\, q_1\right)$$

$$\therefore\ 290 = 1.20 q_2 - 0.18 q_1 \quad \text{——— (2)}$$

Equating (1) & (2)

$$2.18 q_1 - 0.54 q_2 = 1.20 q_2 - 0.18 q_1$$

$$(\text{or})\quad 2.36 q_1 - 1.74 q_2 = 0$$

$$\boxed{q_2 = 1.36 q_1}$$

$$\rightarrow\ M_t = 2q_1 A_1 + 2q_2 A_2$$

$$\cancel{10000 \times 100 = 1360 q_1}$$

$$10000 \times 100 = 2q_1 \times 680 + 2q_2 \times 2000$$

$$= 1360 q_1 + 4000 q_2$$

Subs. q_2,

$$10000 \times 100 = 1360 q_1 + 4000 \times 1.36 q_1$$

$$\therefore \boxed{q_1 = 147\,N} \quad \& \quad \boxed{q_2 = 200\,N}$$

Module - 5

→ Thermoelastic Stress - Strain Relation

Consider a body to be made of large no. of. small cubical elements. If the element temp. is uniformly raised & if the boundary is not in any constrained, then the cubical elements will expand uniformly.

But, if these cubical elements is not uniformly heated, it will expand un-uniformly and distortion will take place which will lead to stresses.

Thus total strain on each part of a body is made of 2 parts

1. Uniform expansion due to temperature $\varepsilon = \alpha T$
   This is equal in all direction and only normal strain act, no shearing strain acts.
2. Strain at each cube due to the stress components

∴ The total strain at each points.

$$\varepsilon_x = \frac{1}{E}\left[\sigma_x - \nu(\sigma_y + \sigma_z)\right] + \alpha T$$

$$\varepsilon_y = \frac{1}{E}\left[\sigma_y - \nu(\sigma_x + \sigma_z)\right] + \alpha T$$

$$\varepsilon_z = \frac{1}{E}\left[\sigma_z - \nu(\sigma_x + \sigma_y)\right] + \alpha T$$

$$\gamma_{xy} = \frac{1}{G}\tau_{xy}, \quad \gamma_{yz} = \frac{1}{G}\tau_{yz}, \quad \gamma_{zx} = \frac{1}{G}\tau_{zx}$$

→ Equations of Equillibrium

The equation of equillibrium are the same as those of isothermal elasticity since they are based on purely mechanical consideration. In rectangular co-ordinates are given by the same

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} + Y_x = 0$$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} + Y_y = 0$$

$$\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} + Y_z = 0$$

where Y_x, Y_y & Y_z are body force comp.

→ Thermal Stresses in thin circular discs

Consider a thin disk subjected to a temp. distribution which varies with r only & is independent of θ.

The stresses σ_r & σ_θ ∴ satisfy equillibrium eq^n

$$\frac{d\sigma_r}{dr} + \frac{\sigma_r - \sigma_\theta}{r} = 0$$

- Body forces are ignored
- Also, because of symmetry $\tau_{r\theta} = 0$
- with $\sigma_z = 0$ (thin discs)

$$\therefore \; \varepsilon_r = \frac{1}{E}(\sigma_r - \nu \sigma_\theta) + \alpha T$$

$$\varepsilon_\theta = \frac{1}{E}(\sigma_\theta - \nu \sigma_r) + \alpha T$$

Solving these equations for σ_r & σ_θ,

$$\sigma_r = \frac{E}{1-\nu^2}\left[\varepsilon_r + \nu\,\varepsilon_\theta - (1+\nu)\alpha T\right]$$

$$\sigma_\theta = \frac{E}{1-\nu^2}\left[\varepsilon_\theta + \nu\,\varepsilon_r - (1+\nu)\alpha T\right]$$

Substitute these in Equillibrium Eq'n $\boxed{\frac{d\sigma_r}{dr} + \frac{\sigma_r - \sigma_\theta}{r} = 0}$ and solving the same, we get

$$\Rightarrow r\frac{d}{dr} + (1-\nu)(\varepsilon_r - \varepsilon_\theta) = (1+\nu)\alpha r\frac{dT}{dr} \quad \text{--- } (*)$$

The strain - displacement relation for a symmetrically strained body, $\varepsilon_r = \frac{dU_r}{dr}$, $e_\theta = \frac{U_r}{r}$

Now, Substituting in eqn (*) $\quad \frac{d^2U_r}{dr^2} + \frac{1}{r}\frac{dU_r}{dr} - \frac{U_r}{r^2} = (1+\nu)\alpha\frac{dT}{dr}$

$$\text{(or)} \quad \frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}(rU_r)\right] = (1+\nu)\alpha\frac{dT}{dr}$$

Integration of this, we get

$$U_r = (1+\nu)\alpha\frac{1}{r}\int_a^r Tr.\,dr + C_1 r + \frac{C_2}{r}$$

Now, this eq'n becomes similar to

$U_r = C_1 r + \frac{C_2}{r}$ (Thick-walled cylinder subjected to internal & External pressure - Lame's Prob.)

where U_r - stresses in

[. It is similar, only if $T = 0$]*

→ Here ① Disk with a hole a = inner radius

② Solid disc a = inner radius = $\underline{\underline{0}}$

Thus stress components are determined by substituting the value of U_r in σ_r & σ_θ

$$\varepsilon_r = \frac{dU_r}{dr}, \quad \varepsilon_\theta = \frac{U_r}{r} \quad \& \quad \sigma_r = \frac{E}{1-\nu^2}\left[\varepsilon_r + \nu\varepsilon_\theta - (1+\nu)\alpha T\right]$$

$$\sigma_\theta = \frac{E}{1-\nu^2}\left[\varepsilon_\theta + \nu\varepsilon_r - (1+\nu)\alpha T\right]$$

$$\therefore \boxed{\sigma_r = -\alpha E \frac{1}{r^2}\int_a^r Tr\,dr + \frac{E}{1-\nu^2}\left[C_1(1+\nu) - C_2(1-\nu)\frac{1}{r^2}\right]}$$

$$\boxed{\sigma_\theta = \alpha E \frac{1}{r^2}\int_a^r Tr\,dr - \alpha E T + \frac{E}{1-\nu^2}\left[C_1(1+\nu) + C_2(1-\nu)\frac{1}{r^2}\right]}$$

Thermal Stresses in Long Circular Cylinder

Here, the temperature is symmetrical along the axis. Let z-axis be the axis of the cylinder & r the radius.

Let $T = f(r)$ and is independent of z.

Now, analyze the prob. with U_z, the axial disp. assumed to be zero.

And because of symmetry, shear stress = 0

Only normal stresses will be there

Normal stresses are σ_r, σ_θ and σ_z

∴ The stress-strain relations are

$$\varepsilon_r = \frac{1}{E}\left[\sigma_r - \nu(\sigma_\theta + \sigma_z)\right] + \alpha T$$

$$\varepsilon_\theta = \frac{1}{E}\left[\sigma_\theta - \nu(\sigma_r + \sigma_z)\right] + \alpha T$$

$$\varepsilon_z = \frac{1}{E}\left[\sigma_z - \nu(\sigma_r + \sigma_\theta)\right] + \alpha T$$

But $U_z = 0$ ∴ $\varepsilon_z = 0$, ∴ $\sigma_z = \nu(\sigma_r + \sigma_\theta) - ET\alpha$

→ Substituting σ_z in ε_r & ε_θ

$$\varepsilon_r = \frac{1-\nu^2}{E}\left(\sigma_r - \frac{\nu}{1-\nu}\sigma_\theta\right) + (1+\nu)\alpha T$$

$$\varepsilon_\theta = \frac{1-\nu^2}{E}\left(\sigma_\theta - \frac{\nu}{1-\nu}\sigma_r\right) + (1+\nu)\alpha T$$

$$\left[\text{Let } \frac{E}{1-\nu^2} = E_1 \;;\; \frac{\nu}{1-\nu} = \nu_1 \;;\; (1+\nu)\alpha = \alpha_1\right]$$

$$\varepsilon_r = \frac{1}{E_1}(\sigma_r - \nu_1 \sigma_\theta) + \alpha_1 T$$

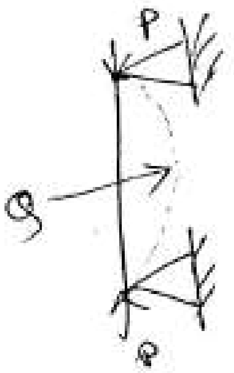
$$\varepsilon_\theta = \frac{1}{E_1}(\sigma_\theta - \nu_1 \sigma_r) + \alpha_1 T$$

$$\therefore U_r = \frac{1+\nu}{1-\nu}\alpha\frac{1}{r}\int_a^r Tr\cdot dr + C_1 r + \frac{C_2}{r}$$

$$\sigma_r = -\frac{\alpha E}{1-\nu}\frac{1}{r^2}\int_a^r Tr\cdot dr + \frac{E}{1+\nu}\left(\frac{C_1}{1-2\nu} + \frac{C_2}{r^2}\right)$$

$$\sigma_\theta = \frac{E}{1-\nu}\frac{1}{r^2}\int_a^r Tr\cdot dr - \frac{\alpha ET}{1-\nu} + \frac{E}{1+\nu}\left(\frac{C_1}{1-2\nu} + \frac{C_2}{r^2}\right)$$

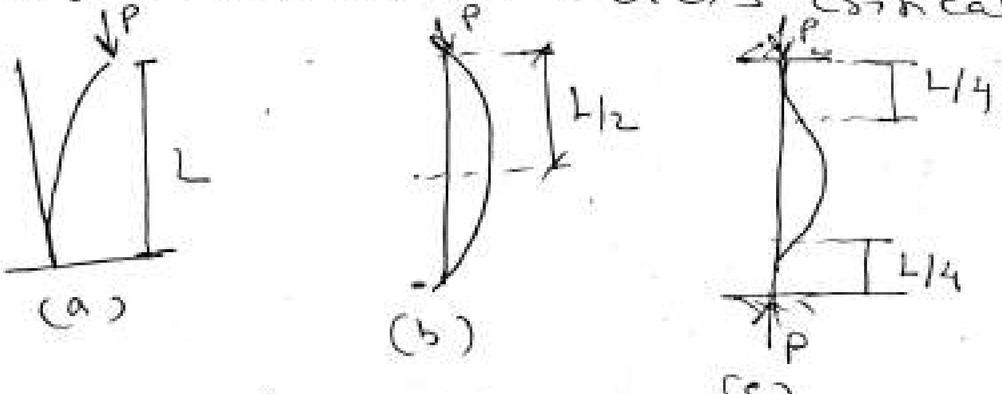
→ Euler's Buckling Load



- Consider a slender column subjected to an axial force P.

Now,
- If an small lateral force Q is applied the member will act like a beam

- A small deflection will be remain until 'Q' is applied, when it is removed it will come back to its original shape
- But, there will be an critical axial load P_{cr} where the column will remain slightly buckled for load 'Q'
- P_{cr} is known as Euler's critical load



∴ Critical load for

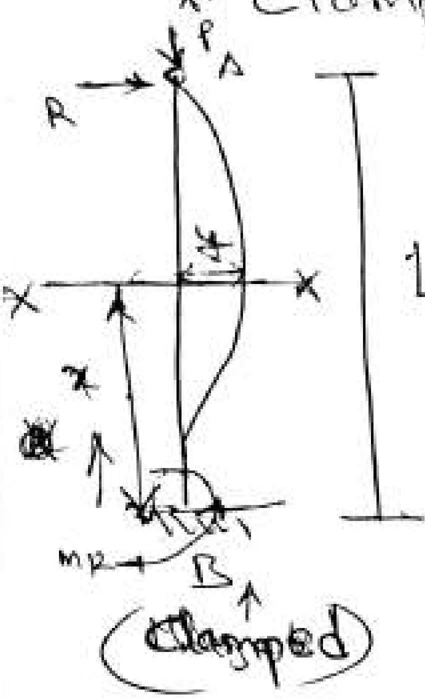
(a) One end fixed – $\frac{\pi^2 EI}{4L^2} = P_{cr}$

(b) both end hinged – $\frac{\pi^2 EI}{L^2} = P_{cr}$

(c) Column with both end fixed – $\frac{4\pi^2 EI}{L^2} = P_{cr}$

→ Euler's Column Buckling load

* Clamped - Hinged



• Consider a centrally loaded column with the lower end built in & upper end hinged

→ The bending moment at any section 'x' is

$$M = -Py - R(L-x)$$

$$\rightarrow M = EI\frac{d^2y}{dx^2}$$

(or) $EI\frac{d^2y}{dx^2} = -Py + R(L-x)$

$[\text{Let } K^2 = P/EI]$

The differential eqn then becomes

$$\frac{d^2y}{dx^2} = -K^2y + \frac{R}{EI}(L-x)$$

The gen. sol. $y = C_1\cos Kx + C_2\sin Kx + \frac{R}{P}(L-x)$

The constants C_1 & C_2 and the reaction R will have to be determined from the bc.

These are, $y = 0$ at $x = 0$ & $x = L$

$$\frac{dy}{dx} = 0 \text{ at } x = 0$$

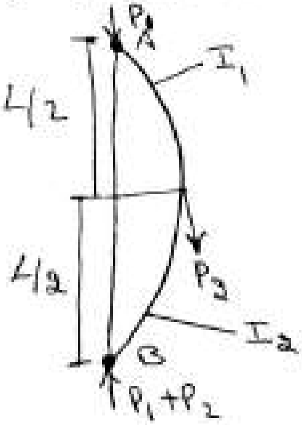
substituting these, we obtain the foll.

$$C_1 + \frac{R}{P} L = 0$$

$$C_1 \cos kL + C_2 \sin kL = 0$$

$$kC_2 - \frac{R}{P} = 0$$

→ Euler's Buckling Load – (Hinged – Hinged)



- A col. AB with hinged ends is compressed by two forces P_1 + P_2
- Moment of Inertia for the Length L_1 of the column is I_1 + L_2 is I_2

* To determine the critical value of the force $P_1 + P_2$.

- Let y_1 be the deflection at any section of the L_1 portion & y_2 the deflection at any section of the L_2 portion.
- If the beam is in equillibrium, then it is neccessary to have a Reaction $RL = P_2\delta$

→ @ L_1 – Moment, $M = P_1 y_1 + R(L-x)$

$$-EI_1 \frac{d^2y_1}{dx^2} = P_1 y_1 + \frac{\delta P_2}{L}(L-x)$$

→ @ L_2 – Moment, $M = P_1 y_2 + R(L-x) - P_2(\delta - y_2)$

$$-EI_2 \frac{d^2y_2}{dx^2} = P_1 y_2 + \frac{\delta P_2}{L}(L-x) - P_2(\delta - y_2)$$

taking

$$\frac{P_1}{EI_1} = K_1^2 \; ; \; \frac{P_2}{EI_2} = K_2^2 \; ; \; \frac{P_1+P_2}{EI_2} = K_3^2$$

$$\frac{P_2}{EI_1} = K_4^2$$

the differential eqn becomes

$$\frac{d^2y_1}{dx^2} = -K_1^2 y_1 - \frac{\delta}{L} K_4^2 (L-x)$$

$$\frac{d^2y_2}{dx^2} = -K_3^2 y_2 - \frac{\delta}{L} K_2^2 x$$

Solving, we get

$$y_1 = C_1 \sin K_1 x + C_2 \cos K_2 x - \frac{\delta}{L} \frac{K_4^2}{K_1^2} (L-x)$$

$$y_2 = C_3 \sin K_3 x + C_4 \cos K_4 x + \frac{\delta}{L} \frac{K_2^2}{K_3^2} x$$

The boundary conditions are

$y_1 = 0$ at $x = L$; $y_1 = \delta$ at $x = L_2$; $y_2 = \delta$ at $x = L_2$

$y_2 = 0$ at $x = 0$; $\left(\frac{dy_1}{dx}\right) = \left(\frac{dy_2}{dx}\right)$ at $x = L_2$

The first four conditions yield

$$C_1 = \frac{\delta(K_1^2 L + K_4^2 L_1)}{K_1^2 L(\sin K_1 L_2 - \tan K_1 L \cos K_1 L_2)}$$

$$C_2 = -C_1 \tan K_1 L \; ; \; C_3 = \frac{\delta(K_3^2 L - K_2^2 L_2)}{K_3^2 L \sin K_3 L_2}$$

$C_4 = 0$