MODULE 1 : Coulomb's Law, Electric Field Intensity and Flux Density

1.1 Experimental law of Coulomb

Coulomb's law states that the electrostatic force F between two point charges q1 and q2 is directly proportional to the product of the magnitude of the charges, and inversely proportional to the square of the distance between them., and it acts along the line joining the two charges.
Then, as per the Coulomb's Law,

$$F \propto kq1q2$$

Or $$F = (kq1q2)/(r^2) \text{ N}$$

Where k is the constant of proportionality whose value varies with the system of units. R^ is the unit vector along the line joining the two charges.

In SI unit, k= $\frac{1}{4\pi\epsilon_0}$.

Where ϵ_0 is called the permittivity of the free space.

It has an assigned value given as ϵ_0 =8.834 $\times 10^{-12}$ F/m.

Force on a point charge:

The forces of attraction/repulsion between two point charges Q_1 and Q_2 (charges whose size is much smaller than the distance between them) are given by Coulomb's law:

$$\mathbf{F}_1 = k \cdot \frac{Q_1 Q_2}{R^2} \mathbf{a}_{21}$$

$$\mathbf{F}_2 = k \cdot \frac{Q_1 Q_2}{R^2} \mathbf{a}_{12}$$

where $k \approx 9 \times 10^9$ m/F in SI units, and R is the distance between the two charges. Here, $\mathbf{F}_1$ is the force exerted on Q_1, and $\mathbf{F}_2$ is the force acting on Q_2. The unit vector $\mathbf{a}_{21}$ points from charge 2 toward charge 1. Accordingly, $\mathbf{a}_{12} = -\mathbf{a}_{21}$.

Force on Q1 is given by

$$F1 = \frac{1}{4\pi\epsilon_0} \frac{Q_1 Q_2}{R^2} \mathbf{a}_{21} \text{ Newtons}$$

Force due to several charges

Let there be many point charges q1,q2,q3.........qn at distances r1,r2,r3.....rn from charge q. The force F1, F2, F3..........Fn at the charges q1,q2,q3...........qn respectively are:

$$q\{\frac{q_1}{4\pi\varepsilon r_1}\hat{r} + \frac{q_2}{4\pi\varepsilon r_2^2}\hat{r}_2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\}$$

F=Fq1+Fq2+Fq3...............

Hence, $F = q\{\frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{n}\frac{qi}{ri^2}\ \hat{r}i\}$ N

1.1 Objectives

After going through this section, the students are able
to 1. State Coulombs law
Application of Coulombs Law to point charge as well as several charges

1.2 Electric field intensity

Electric field intensity at any point in an electric field is the force experienced by positive unit charge placed at that point.

Consider a charge Q located at a point A. At the point B in the electric fields set up by Q, it is required to find the electric field intensity E.

Let the charge at B be Δq and let the charge Q be fixed at A. Let r be the distance between A and B. As per the Coulomb's Law, the force between Q and *q* is given by:

F= $Q\Delta q/(4\pi \in^0 r^2)$ $\hat{r}$ N

If it is a unit positive charge, then by definition, F in the above equation gives the magnitude of the electric field intensity E.

i.e. E=F when $\Delta q = 1.$

Therefore, the magnitude of the electric field strength is: E=Q/(4r $\pi \in_0{}^2)$

Let r be the unit vector along the line joining A and B. Thus, the vector relation between E is written as:

E=Q/(4 $\pi \in$or²) V/m

1.2.1 Electric Field intensity due to several charges

Let there be many point charges q1,q2,q3......... qn at distances r1,r2,r3 .....rn be the corresponding unit vectors. The field E1, E2, E3.......... En at the charges q1,q2,q3........... qn respectively are:

$\hat{r}$ +

E=Eq1+Eq2+Eq3 ....

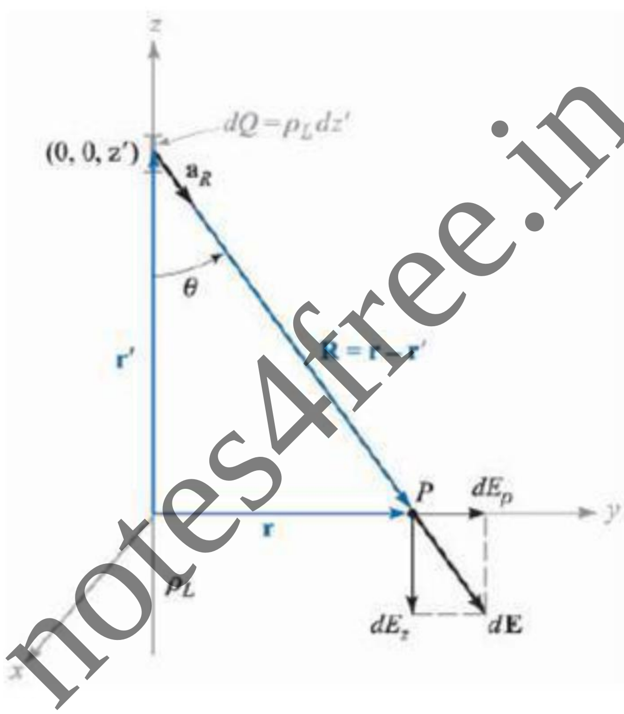
Hence, $E=\frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{n}\frac{qi}{ri^2}\,\hat{r}i$

1.2 .2 Electric field intensity at a point due to a infinite sheet of charge

Let us assume a straight line charge extending along Z axis in a cylindrical coordinate system from -∞ to +∞ as shown in the figure 1.1. Consider an incremental length dl at a point on the conductor. The incremental length has an incremental charge of dQ= ρl dl= ρldz' Coulombs.

Considering the charge dQ, the incremental field intensity at point p is given by,

$$d\mathbf{E} = \frac{\rho_L dz'(\mathbf{r} - \mathbf{r}')}{4\pi\epsilon_0 |\mathbf{r} - \mathbf{r}'|^3}$$



Where

$$\mathbf{r} = y\mathbf{a}_y = \rho\mathbf{a}_\rho$$
$$\mathbf{r}' = z'\mathbf{a}_z$$

,
and

$$\mathbf{r} - \mathbf{r}' = \rho\mathbf{a}_\rho - z'\mathbf{a}_z$$

Therefore,

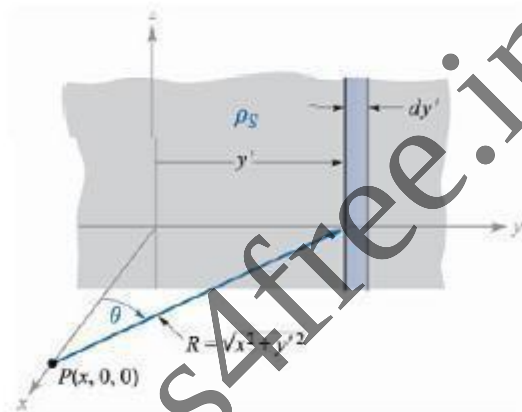
$$E_\rho = \int_{-\infty}^{\infty} \frac{\rho_L \rho dz'}{4\pi\epsilon_0(\rho^2 + z'^2)^{3/2}}$$

Integrating the above and substituting z'=ρ cot θ, we get and

$$E_\rho = \frac{\rho_L}{2\pi\epsilon_0 \rho}$$

1.2.3 Electric field intensity at a point due to a infinite sheet of charge:

Let us assume a infinite sheet of charge with surface charge density ρs as shown in the figure 1.2. Divide the sheet of charge into differential width strips. number of str Consider an incremental length dl at a point on the conductor. The line charge density ρl= ρs dy'.



The differential Electric field intensity at point P,

$$dE_x = \frac{\rho_S\, dy'}{2\pi\epsilon_0\sqrt{x^2 + y'^2}}\cos\theta = \frac{\rho_S}{2\pi\epsilon_0}\frac{x dy'}{x^2 + y'^2}$$

adding the effects of all the strips,

$$E_x = \frac{\rho_S}{2\pi\epsilon_0}\int_{-\infty}^{\infty}\frac{x\, dy'}{x^2 + y'^2} = \frac{\rho_S}{2\pi\epsilon_0}\tan^{-1}\frac{y'}{x}\Big]_{-\infty}^{\infty} = \frac{\rho_S}{2\epsilon_0}$$

Therefore,

$$\mathbf{E} = \frac{\rho_S}{2\epsilon_0}\mathbf{a}_N$$

1.2.4 Electric field at a point on the axis of charged circular ring:

Let ρ be the charge density of the ring.

So, $\rho = dq/dl$

$dq = \rho dl$

Electric field due to an infinitely small element = $dE = dq/4\pi\varepsilon_o r^2\ \hat{r}$

where $\hat{r}$ is the unit vector along AP.

dE can resolved into two rectangular components, dEx and dEy. Now, $dEx = dE\cos\theta$.

Taking the magnitude of dE from above, the equation becomes,

$$dEx = \frac{dq\cos\theta}{4\pi\varepsilon^0 r^2}$$

$$\cos\theta = \frac{x}{r}$$

substituting for dq from above, we have;

$$dEx = \frac{\rho h dl}{4\pi\varepsilon r^3}$$

The component dEy is directed downwards. If we consider an element of the ring at a point diametrically opposite to A, then its dEy component points upwards and hence, cancels with that due to element A. The dEx components add up.

$$\int dEy = 0.$$

The total field at P is the sum of the fields due to all the elements of the ring.

Therefore, $E = \int dE = \int dEx + \int dEy = \int dEx$

$$E = \int dEx = \frac{\rho x}{4\pi\varepsilon r^3}.\int_0^{2\pi R} dl$$

$$= \frac{\rho x(2\pi R)}{4\pi\varepsilon r^3}$$

But, $r = (R^2 + x^2)^{1/2}$

Therefore, $$E = \frac{2\pi R\rho x}{4\pi\varepsilon (R^2+x^2)^{3/2}}\, a_x$$

Where, a_x is the unit vector along the x axis.

1.2 Objectives

At the end of this section the students are able to

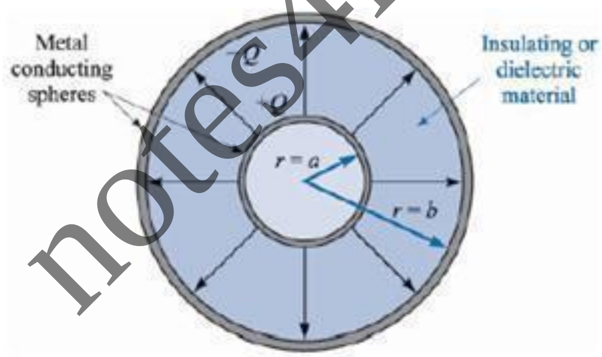
1. Define Electric field Intensity
2. Derive Electric field intensity at a due to several charges
3. Derive Electric field Intensity at a point due to sheet of charge
4. Derive Electric field intensity at a point on the axis of charged circular ring

1.3 Electric flux:

The concept of electric flux is useful in association with Gauss' law. The electric flux through a planar area is defined as the electric field times the component of the area perpendicular to the field. If the area is not planar, then the evaluation of the flux generally requires an area integral since the angle will be continually changing.

When the area A is used in a vector operation like this, it is understood that the magnitude of the vector is equal to the area and the direction of the vector is perpendicular to the area.

Consider a concentric sphere having radius of 'a'm charged up to +Q C. This sphere is then placed in another sphere having a radius of 'b' m as shown in the figure 1.4.



There is no electrical connection between them. The outer sphere is momentarily charged, then it found that the charge on the outer sphere is equal to the charge on the inner sphere. This is depicted by the radial lines. This is referred as displacement flux. Therefore,

$$\Psi = Q.$$

1.3.1 Electric flux density:

If +Q C of charge on the inner sphere produces the electric flux of ψ, tthen electric flux ψ uniformly distributed over the surface area $4\Pi a^2\ m^2$, where a is the radius of the inner sphere.

The electric flux density si given by

$$\mathbf{D}\Big|_{r=a} = \frac{Q}{4\pi a^2}\mathbf{a}_r \qquad \text{(inner sphere)}$$

Similarly for the outer sphere,

$$\mathbf{D}\Big|_{r=b} = \frac{Q}{4\pi b^2}\mathbf{a}_r \qquad \text{(outer sphere)}$$

If the inner sphere becomes smaller and smaller retaining a charge of Q C, it becomes a point charge. The flux density at appoint ‘r’ from the point charge is given by,

$$\boxed{\mathbf{D} = \frac{Q}{4\pi r^2}\mathbf{a}_r}$$

The electric field intensity due to point charge in free space is given by,

$$\mathbf{E} = \frac{Q}{4\pi\epsilon_0 r^2}\mathbf{a}_r$$

Therefore in free space,

$$\boxed{\mathbf{D} = \epsilon_0 \mathbf{E}}$$

1.3 Objective

After going through this section the students should be able to
1. Define Electric flux
2. Explain Electric flux density

1.4 Gauss law:

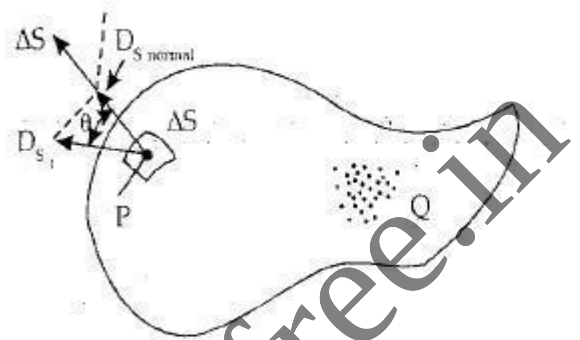
The Gauss's law states that. "The electric flux passing through any closed surface is equal to the total charge enclosed by the surface"

For the Gaussian-surface shown in the following figure, the Gauss' law can be expressed mathematically, .

$$\psi = \oint \vec{D}_S \cdot d\vec{s} = Q$$

Where

Ψ = flux passing through the closed surface
§s =1 surface integral
Ds =, flux density (vector quantity) normal to the surface Q = Total charge enclosed in the surface

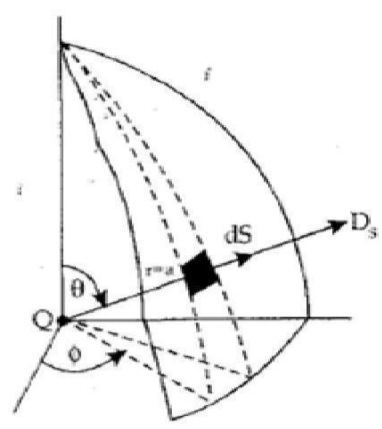


Gauss law for charge Q enclosed in a closed surface:

Let Q be the point charge placed at the origin of imaginary sphere in spherical co-ordinate system with a radius of "a" as illustrated in the figure
The electrical field intensity cf the point charge is found to be equal to

$$\vec{E} = \frac{Q}{4\pi \epsilon_0 r^2} \vec{a}_r \qquad \text{-----(1)}$$

Where r = Cl
and we al so know that the relation between E and D as,



$$\vec{D} = \epsilon_0 E \qquad \text{-----(2)}$$

Therefore from (1) and (2) we get.

$$\vec{D} = \epsilon_0 \frac{Q}{4\pi \epsilon_0 r^2} \vec{a}_r$$

$$= \frac{Q}{4\pi a^2} \vec{a}_r$$

at the surface of the sphere,

$$\vec{D} = \frac{Q}{4\pi a^2} \vec{a}_r$$

The differential element of area on a spherical surface is, in spherical coordinate form is given by,

$$ds = r^2 \sin\theta \, d\theta \, d\phi = a^2 \sin\theta \, d\theta \, d\phi$$

$$\text{Or} \quad d\vec{s} = a^2 \sin\theta \, d\theta \, d\phi \, \vec{a}_r$$

Then the required integrand

$$= \vec{D}_s \cdot d\vec{s}$$

$$= \frac{Q}{4\pi a^2} \cdot a^2 \sin\theta \, d\theta \, d\phi \; \vec{a}_r \cdot \vec{a}_r$$

$$= \frac{Q}{4\pi} \sin\theta \, d\theta \, d\phi \qquad (\because \vec{a}_r \cdot \vec{a}_r = 1 \text{ from vector basics})$$

Then the integration over the surface as required for Gauss' law.

$$\oint_S \vec{D}_S \cdot d\vec{s} = \int_{\phi=0}^{\phi=2\pi} \int_{\theta=0}^{\theta=\pi} \frac{Q}{4\pi} \sin\theta d\theta d\phi$$

The limits placed for integral indicate that the integration over the entire sphere in spherical co-ordinate system on integration we get

$$= \int_0^{2\pi} \frac{Q}{4\pi}(-\cos\theta)_0^{\pi}\, d\phi$$

$$= \int_0^{2\pi} \frac{Q}{2\pi} d\phi$$

$$= Q$$

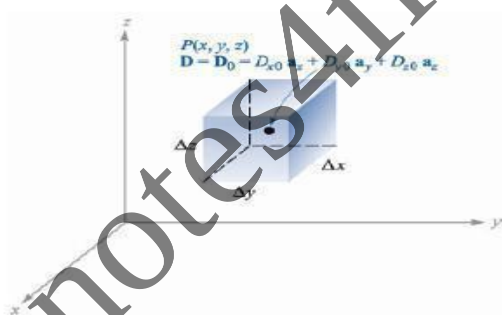
Thus we get, comparing LHS of Gauss' law as

$$\boxed{\psi = Q}$$

This indicates that, Q coulombs of electric flux are crossing the surface as the enclosed charge is Q coulombs.

1.4 .1 Application of Gauss law:

In case of asymmetry, we need to choose a very closed surface such that D is almost constant over the surface. Consider any point P shown in the figure 1.6 located in the rectangular co-ordinate system.



The value of D at point P, may be expressed in rectangular components as,

$D = D_{x0}a_x + D_{y0}a_y + D_{z0}a_z$. . From Gauss law, we have

$$\boxed{\oint_S \mathbf{D} \cdot d\mathbf{S} = Q}$$

In order to evaluate the integral over the closed surface, the integral must be broken into six integrals, one over each surface,

$$\oint_S \mathbf{D} \cdot d\mathbf{S} = \int_{\text{front}} + \int_{\text{back}} + \int_{\text{right}} + \int_{\text{top}} + \int_{\text{bottom}}.$$

The surface element is very small & hence D is essentially constant ,

$$\int_{\text{front}} + \int_{\text{back}} \doteq \frac{\partial D_x}{\partial x} \Delta x \, \Delta y \, \Delta z$$,

Similarly,

$$\int_{\text{right}} + \int_{\text{left}} \doteq \frac{\partial D_y}{\partial y} \Delta x \, \Delta y \, \Delta z$$

and,

$$\int_{\text{top}} + \int_{\text{bottom}} \doteq \frac{\partial D_z}{\partial z} \Delta x \, \Delta y \, \Delta z$$.

Therefore collectively,

$$\oint_S \mathbf{D} \cdot d\mathbf{S} \doteq \left(\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} \right) \Delta x \, \Delta y \, \Delta z$$

or

$$\oint_S \mathbf{D} \cdot d\mathbf{S} = Q \doteq \left(\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} \right) \Delta v$$

Charge enclosed in volume Δv,

$$\text{Charge enclosed in volume } \Delta v \doteq \left(\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} \right) \times \text{volume } \Delta v$$

1.4 Objectives

At the end of this section the students are able to
1. State and prove Gauss Law
2. Apply Gauss law to find the charge enclosed in differential volume

1.5 Divergence:

From Gauss law, we know that,

$$\left(\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} \right) \doteq \frac{\oint_S \mathbf{D} \cdot d\mathbf{S}}{\Delta v} = \frac{Q}{\Delta v}$$

And applying limits,

$$\left(\frac{\partial D_x}{\partial x}+\frac{\partial D_y}{\partial y}+\frac{\partial D_z}{\partial z}\right)=\lim_{\Delta v\to 0}\frac{\oint_S \mathbf{D}\cdot d\mathbf{S}}{\Delta v}=\lim_{\Delta v\to 0}\frac{Q}{\Delta v}$$

The last term in the equation is the volume charge density, ρv.

$$\left(\frac{\partial D_x}{\partial x}+\frac{\partial D_y}{\partial y}+\frac{\partial D_z}{\partial z}\right)=\lim_{\Delta v\to 0}\frac{\oint_S \mathbf{D}\cdot d\mathbf{S}}{\Delta v}=\rho_v$$

We shall write it as two separate equations,

$$\left(\frac{\partial D_x}{\partial x}+\frac{\partial D_y}{\partial y}+\frac{\partial D_z}{\partial z}\right)=\lim_{\Delta v\to 0}\frac{\oint_S \mathbf{D}\cdot d\mathbf{S}}{\Delta v}$$

And

$$\left(\frac{\partial D_x}{\partial x}+\frac{\partial D_y}{\partial y}+\frac{\partial D_z}{\partial z}\right)=\rho_v$$

Divergence is defined as,

$$\text{div }\mathbf{D}=\left(\frac{\partial D_x}{\partial x}+\frac{\partial D_y}{\partial y}+\frac{\partial D_z}{\partial z}\right)$$

$$\text{Divergence of }\mathbf{A}=\text{div }\mathbf{A}=\lim_{\Delta v\to 0}\frac{\oint_S \mathbf{A}\cdot d\mathbf{S}}{\Delta v}$$

Statement: The flux crossing the closed surface is equal to the integral of the divergence of the flux density throughout the enclosed volume, as the volume shrinks to zero.

Divergence in Cartesian system,

$$\text{div }\mathbf{D}=\frac{\partial D_x}{\partial x}+\frac{\partial D_y}{\partial y}+\frac{\partial D_z}{\partial z}\qquad\text{(cartesian)}$$

Divergence in Cylindrical system,

$$\text{div }\mathbf{D}=\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho D_\rho)+\frac{1}{\rho}\frac{\partial D_\phi}{\partial\phi}+\frac{\partial D_z}{\partial z}\qquad\text{(cylindrical)}$$

Divergence in Spherical system,

$$\text{div }\mathbf{D}=\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 D_r)+\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\, D_\theta)+\frac{1}{r\sin\theta}\frac{\partial D_\phi}{\partial\phi}\qquad\text{(spherical)}$$

1.6.1 Maxwell's First equation:

From divergence theorem, we have

$$\text{div } \mathbf{D} = \lim_{\Delta v \to 0} \frac{\oint_S \mathbf{D} \cdot d\mathbf{S}}{\Delta v}$$

$$\text{div } \mathbf{D} = \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}$$

$$\text{div } \mathbf{D} = \rho_v$$

From Gauss law,

$$\oint_S \mathbf{A} \cdot d\mathbf{S} = Q$$

Per unit volume,

$$\frac{\oint_S \mathbf{A} \cdot d\mathbf{S}}{\Delta v} = \frac{Q}{\Delta v}$$

As the volume shrinks to zero,

$$\lim_{\Delta v \to 0} \frac{\oint_S \mathbf{A} \cdot d\mathbf{S}}{\Delta v} = \lim_{\Delta v \to 0} \frac{Q}{\Delta v}$$

Therefore, div D = ρv.

1.6.2 Divergence theorem:

The del operator is defined as a vector operator.

$$\nabla = \frac{\partial}{\partial x}\mathbf{a}_x + \frac{\partial}{\partial y}\mathbf{a}_y + \frac{\partial}{\partial z}\mathbf{a}_z$$

.

In Cartesian coordinate system,

$$\nabla \cdot \mathbf{D} = \frac{\partial}{\partial x}(D_x) + \frac{\partial}{\partial y}(D_y) + \frac{\partial}{\partial z}(D_z)$$

Which is equal to,

$$\nabla \cdot \mathbf{D} = \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}$$

.

Therefore,

$$\text{div } \mathbf{D} = \nabla \cdot \mathbf{D} = \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}$$.

From Gauss law, we have

$$\oint_S \mathbf{D} \cdot d\mathbf{S} = Q$$

And by letting,

$$Q = \int_{\text{vol}} \rho_v dv \quad \& \quad \nabla \cdot \mathbf{D} = \rho_v$$.

Hence we have,

$$\oint_S \mathbf{D} \cdot d\mathbf{S} = \int_{\text{vol}} \nabla \cdot \mathbf{D}\, dv$$

1.6 Objectives

At the end of this section the students are able to

1. Explain the concept of divergence
2. Derive Maxwell's First Equation
3. State and prove Divergence theorem
4.
5.

1.7 Recommended Questions

1. State Coulomb's law of force between any 2 point charges & indicate the units of the quantities involved.

2. Derive the general expression for electric field vector due to infinite line charge using Gauss law.

3. State and prove Gauss law.

4. Derive the general expression for E at a height h(h<a) , along the axis of the ring charge & normal to its plane.

5. From gauss law show that .D=σv

6. State and prove divergence theorem for symmetric condition.

7. State and prove divergence theorem for asymmetric condition

1.8 Further Readings

1. Energy Electromagnetics, William H Hayt Jr. and John A Buck, Tata McGraw-Hill, 7th edition,2006.
2. Electromagnetics with Applications, John Krauss and Daniel A Fleisch McGraw-Hill, 5th edition, 1999
3. Electromagnetic Waves And Radiating Systems, Edward C. Jordan and Keith G Balmain, Prentice – Hall of India / Pearson Education, 2nd edition, 1968.Reprint 2002
4. Field and Wave Electromagnetics, David K Cheng, Pearson Education Asia, 2nd edition, 1989, Indian Reprint – 2001

MODULE 2: Gauss's law and Divergence, Energy and Potential, Conductors Dielectrics and Capacitance

2.1 Energy expended in moving a point charge in an electric field
2.2 Line integral
2.3 Definition of potential difference and potential
2.4 Potential field of a point charge & system of charges
2.5 Potential gradient,

2.6 Energy density in an electrostatic field.

2.7 Current and current density
2.8 Continuity of current
2.9 metallic conductors
2.11 Dielectric properties and boundary conditions for dielectrics, Conductor properties and boundary conditions for perfect
2.12 dielectrics,

2.0 Objectives

1. To Understand the concept of Potential and Potential Difference
2. To Learn the concepts of Energy density, current density
3. To derive current continuity equation
4. To understand the boundary Conditions

2.1 Energy expended in moving a point charge in an electric field

Electric field intensity is defined as the force experienced by unit test charge at a point p. If the test charge is moved against the electric field, then we have to exert a force equal and opposite to that exerted by the field and this requires work to be done.

Suppose we need to move a charge fo Q C a distance dl in an electric field E. The force on Q arising from the electric field is,

$$\mathbf{F}_E = Q\mathbf{E}$$

The differential amount of work done in moving charge Q over a distance dl is given by, $dW = -Q\mathbf{E} \cdot d\mathbf{L}$, as F =QE

Thus the work done to move the charge for the finite distance is given by,

$$W = -Q\int_{\text{init}}^{\text{final}} \mathbf{E} \cdot d\mathbf{L}$$

2.3 Definition of Potential Difference and potential

Potential difference(V) is defined as the work done in moving unit positive charge from one point to another point in an electric field.

We know that,

$$W = -Q\int_{\text{init}}^{\text{final}} \mathbf{E}\cdot d\mathbf{L}$$

Therefore V=W/Q= $\int_{\text{init}}^{\text{final}} \mathbf{E}\cdot d\mathbf{L}$

V_{AB} signifies potential difference between points A & B and the work done in moving the unit charge from B to A. Thus B is the initial point & A is the final point.

$$V_{AB} = -\int_{B}^{A} \mathbf{E}\cdot d\mathbf{L} \quad \text{V}$$

From the previous example, the work done in moving charge Q from ρ= b to ρ= a was,

$$W = \frac{Q\rho_L}{2\pi\epsilon_0}\ln\frac{b}{a}$$

Thus the potential difference between the points a & b is given by,

$$V_{ab} = \frac{W}{Q} = \frac{\rho_L}{2\pi\epsilon_0}\ln\frac{b}{a}$$

Absolute electric potential is defined as the work done in moving a unit positive charge from infinity to that point against the field.

Electric field is defined as force on unit charge.

E= F/Q.

By moving the charge Q aganist an electric field between the two points a & b work is done. Thus ,

Edl= Fxdl/Q =work/ charge.

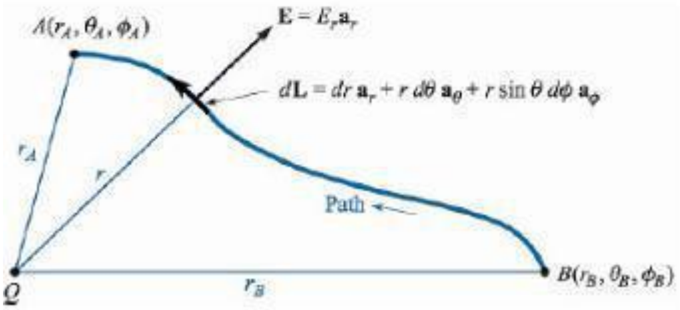
This work done per charge is the electric potential difference. Potential difference between points a and b at a radial distance of r_a and r_b from a point charge Q is given by, If the potential at point a is V_A and at point B is V_B, then

$$V_{AB} = V_A - V_B$$

Equipotential Surface is defined as "It is a surfacehaving the same value of potential" on composed of all- points such surfaces no work is charge, hence no potential difference involved in moving a unit between any two points on this surface.

2.4 Potential field of a point charge & system of charges

Consider a point charge Q to be placed in the origin of a spherical coordinate system. Consider 2 points A & B as shown in the figure.



Electric Potential difference between A & B, V_{AB} is given by,

$$V_{AB} = -\int_B^A \mathbf{E} \cdot d\mathbf{L}$$

dl in spherical co ordinate system is given the figure above and $E = Q/4\Pi\epsilon r^2$.
Therefore,

$$V_{AB} = -\int_B^A \mathbf{E} \cdot d\mathbf{L} = -\int_{r_B}^{r_A} \frac{Q}{4\pi\epsilon_0 r^2} dr = \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{r_A} - \frac{1}{r_B}\right)$$

And

$$V_{AB} = V_A - V_B$$

Potential at a point has been defined as the work done in moving unit positive charge from zero reference to the point. Potential is independent of the path taken from one point to the other. Potential due to a single charge is given by

$V(r) = Q1/4\Pi\epsilon R$. If Q1 is at r1 & point p at r, then

$$V(\mathbf{r}) = \frac{Q_1}{4\pi\epsilon_0 |\mathbf{r} - \mathbf{r}_1|}$$

Potential arising from 2 charges, Q1 at r1 and Q2 at r2, is given by

$$V(\mathbf{r}) = \frac{Q_1}{4\pi\epsilon_0 |\mathbf{r} - \mathbf{r}_1|} + \frac{Q_2}{4\pi\epsilon_0 |\mathbf{r} - \mathbf{r}_2|}$$

Potential due to n number of charges, is given by

$$V(\mathbf{r}) = \frac{Q_1}{4\pi\epsilon_0 |\mathbf{r} - \mathbf{r}_1|} + \frac{Q_2}{4\pi\epsilon_0 |\mathbf{r} - \mathbf{r}_2|} + \ldots + \frac{Q_n}{4\pi\epsilon_0 |\mathbf{r} - \mathbf{r}_n|}$$

Or

$$V(\mathbf{r}) = \sum_{m=1}^{n} \frac{Q_m}{4\pi\epsilon_0 |\mathbf{r} - \mathbf{r}_m|}$$

If point charge is a small element in the continuous volume charge distribution then,

$$V(\mathbf{r}) = \frac{\rho_v(\mathbf{r}_1)\Delta v_1}{4\pi\epsilon_0|\mathbf{r}-\mathbf{r}_1|} + \frac{\rho_v(\mathbf{r}_2)\Delta v_2}{4\pi\epsilon_0|\mathbf{r}-\mathbf{r}_2|} + \ldots + \frac{\rho_v(\mathbf{r}_n)\Delta v_n}{4\pi\epsilon_0|\mathbf{r}-\mathbf{r}_n|}$$

As number of point charges in the volume charge distribution tends to infinity,

$$V(\mathbf{r}) = \int_{vol} \frac{\rho_v(\mathbf{r}')dv'}{4\pi\epsilon_0|\mathbf{r}-\mathbf{r}'|}$$

Similarly if the point charges takes the form of a straight line then,

$$V(\mathbf{r}) = \int \frac{\rho_L(\mathbf{r}')dL'}{4\pi\epsilon_0|\mathbf{r}-\mathbf{r}'|}$$

Similarly if the point charges takes the form of a surface charge then,

$$V(\mathbf{r}) = \int_S \frac{\rho_S(\mathbf{r}')dS'}{4\pi\epsilon_0|\mathbf{r}-\mathbf{r}'|}$$

Potential is a function of inverse distance. Hence we can conclude that for a zero reference at infinity, then:

I Potential due to a single point charge is the work done in moving unit positive charge from zero reference to the point. Potential is independent of the path taken from one point to the other

II Potential field due to number of charges is the sum of the individual potential fields arising from each charge.

III. Potential due to continuous charge distribution is found by carrying a unit charge from infinity to the point under consideration.

$$V_{AB} = V_A - V_B = -\int_B^A \mathbf{E} \cdot d\mathbf{L}$$

is independent on the path chosen for the line integral, regardless of the source of the E field.

Hence we can conclude that no work is done in carrying a unit positive charge around any closed path, or

$$\oint \mathbf{E} \cdot d\mathbf{L} = 0$$

Any field that satisfies an equation of the form above is said to be conservative field

---

2.5 Potential Gradient

Potential at any point is given by $V = -\int \mathbf{E} \cdot d\mathbf{L}$

Potential difference between 2 points separated by a very short length ΔL along which E is essentially constant, is given by

$$\Delta V \doteq -\mathbf{E} \cdot \Delta\mathbf{L}$$

In rectangular co ordinate system,

$$dV = \frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy + \frac{\partial V}{\partial z}dz$$

, As V is a unique function of x,y,z. Then,

$$dV = -\mathbf{E}\cdot d\mathbf{L} = -E_x\,dx - E_y\,dy - E_z\,dz$$

Since both the expressions are true with respect dx,dy & dz, we can write

$$E_x = -\frac{\partial V}{\partial x}$$

$$E_y = -\frac{\partial V}{\partial y}$$

$$E_z = -\frac{\partial V}{\partial z}$$

Therefore,

$$\mathbf{E} = -\left(\frac{\partial V}{\partial x}\mathbf{a}_x + \frac{\partial V}{\partial y}\mathbf{a}_y + \frac{\partial V}{\partial z}\mathbf{a}_z\right)$$

In rectangular co ordinate system,

$$\text{grad } V = \frac{\partial V}{\partial x}\mathbf{a}_x + \frac{\partial V}{\partial y}\mathbf{a}_y + \frac{\partial V}{\partial z}\mathbf{a}_z$$

Combining all the above equations allows us to use a compact expression that relates E & V, $\mathbf{E} = -\nabla V$

Gradient in other coordinate system is as given below,

$$\nabla V = \frac{\partial V}{\partial \rho}\mathbf{a}_\rho + \frac{1}{\rho}\frac{\partial V}{\partial \phi}\mathbf{a}_\phi + \frac{\partial V}{\partial z}\mathbf{a}_z \qquad \text{(cylindrical)}$$

$$\nabla V = \frac{\partial V}{\partial r}\mathbf{a}_r + \frac{1}{r}\frac{\partial V}{\partial \theta}\mathbf{a}_\theta + \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}\mathbf{a}_\phi \qquad \text{(spherical)}$$

Given the potential field, $V = 2x^2y - 5z$, and a point $P(-4, 3, 6)$, we wish to find several numerical values at point P: the potential V, the electric field intensity $\mathbf{E}$, the direction of $\mathbf{E}$, the electric flux density $\mathbf{D}$, and the volume charge density ρ_v.

Solution. The potential at $P(-4, 5, 6)$ is

$$V_P = 2(-4)^2(3) - 5(6) = 66 \quad \text{V}$$

Next, we may use the gradient operation to obtain the electric field intensity,

$$\mathbf{E} = -\nabla V = -4xy\mathbf{a}_x - 2x^2\mathbf{a}_y + 5\mathbf{a}_z \quad \text{V/m}$$

The value of $\mathbf{E}$ at point P is

$$\mathbf{E}_P = 48\mathbf{a}_x - 32\mathbf{a}_y + 5\mathbf{a}_z \quad \text{V/m}$$

and

$$|\mathbf{E}_P| = \sqrt{48^2 + (-32)^2 + 5^2} = 57.9 \quad \text{V/m}$$

The direction of $\mathbf{E}$ at P is given by the unit vector

$$\mathbf{a}_{E,P} = (48\mathbf{a}_x - 32\mathbf{a}_y + 5\mathbf{a}_z)/57.9$$
$$= 0.829\mathbf{a}_x - 0.553\mathbf{a}_y + 0.086\mathbf{a}_z$$

If we assume these fields exist in free space, then

$$\mathbf{D} = \epsilon_0\mathbf{E} = -35.4xy\mathbf{a}_x - 17.71x^2\mathbf{a}_y + 44.3\mathbf{a}_z \quad \text{pC/m}^3$$

6 Energy Density in an Electric Field

Consider a surface without charge. Bringing a charge Q1 from infinity to any point on the surface requires no work as there is no field present. The positioning of Q2 at appoint in the field of Q1 requires an amount of work to be done which is given by

$$\text{Work to position } Q_2 = Q_2 V_{2,1}$$

Similarly work required to position each additional charge in the field is given by,

$$\text{Work to position } Q_3 = Q_3 V_{3,1} + Q_3 V_{3,2}$$
$$\text{Work to position } Q_4 = Q_4 V_{4,1} + Q_4 V_{4,2} + Q_4 V_{4,3}$$

Total positioning work = Potential energy of the field

$$= W_E = Q_2 V_{2,1} + Q_3 V_{3,1} + Q_3 V_{3,2} + Q_4 V_{4,1} + Q_4 V_{4,2} + Q_4 V_{4,3} + \dots$$

Bringing the charges in the reverse order, the work done is given by,

$$W_E = Q_1 V_{1,2} + Q_1 V_{1,3} + Q_2 V_{2,3} + Q_1 V_{1,4} + Q_2 V_{2,4} + Q_3 V_{3,4} + \dots$$

Adding the 2 energy expressions, we get

$$2W_E = Q_1(V_{1,2} + V_{1,3} + V_{1,4} + \dots) + Q_2(V_{2,1} + V_{2,3} + V_{2,4} + \dots) + Q_3(V_{3,1} + V_{3,2} + V_{3,4} + \dots)$$

For n number of charges,

$$W_E = \tfrac{1}{2}(Q_1 V_1 + Q_2 V_2 + Q_3 V_3 + \dots) = \tfrac{1}{2}\sum_{m=1}^{m=N} Q_m V_m$$

2.7 Potential energy in a continuous charge distribution:

For the region with continuous charge distribution, the equation for W_E=
By vector identity which is true for any scalar function V & vector D,

$$\nabla \cdot (V\mathbf{D}) \equiv V(\nabla \cdot \mathbf{D}) + \mathbf{D} \cdot (\nabla V)$$

Then,

$$W_E = \tfrac{1}{2}\int_{\text{vol}} \rho_v V dv = \tfrac{1}{2}\int_{\text{vol}} (\nabla \cdot \mathbf{D}) V\, dv$$
$$= \tfrac{1}{2}\int_{\text{vol}} [\nabla \cdot (V\mathbf{D}) - \mathbf{D} \cdot (\nabla V)]\, dv$$

From Guass law, We can write

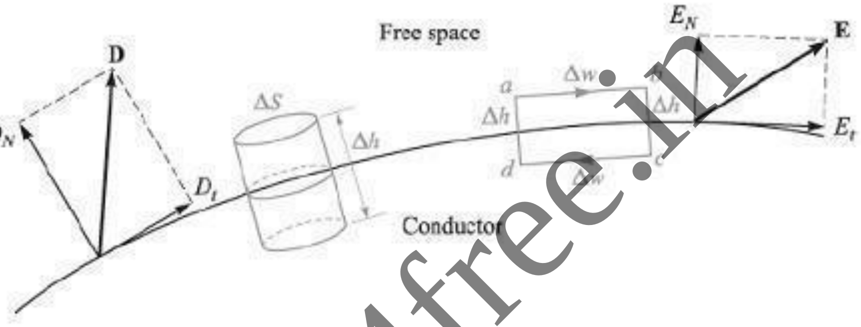
$$W_E = \frac{1}{2}\oint_S (V\mathbf{D}) \cdot d\mathbf{S} - \frac{1}{2}\int_{vol} \mathbf{D} \cdot (\nabla V)\, dv$$

and from gradient

$$W_E = \frac{1}{2}\int_{vol} \mathbf{D} \cdot \mathbf{E}\, dv = \frac{1}{2}\int_{vol} \epsilon_0 E^2\, dv$$

2.8 Boundary condition for conductor free space interface:

Consider a closed path at the boundary between conductor and a dielectric, such that Δh→0.



We know that work done in moving a charge over a closed path is zero i.e.,

$$\oint \mathbf{E} \cdot d\mathbf{L} = 0$$.

Therefore the integral can be broken up as,

$$\int_a^b + \int_b^c + \int_c^d + \int_d^a = 0$$.

Let the length from a to b or c to d be ΔW and from a to d or b to c be Δh , hence we obtain,

$$E_t \Delta w - E_{N,\text{at } b}\frac{1}{2}\Delta h + E_{N,\text{at } a}\frac{1}{2}\Delta h = 0$$

. Hence we obtain EΔW=0 & therefore E_t=0

Hence at the conductor dielectric interface tangential component of the electric field intensity is zero.

Consider a gaussian cylinder of radius ρ and height Δh at the boundary, Applying Gauss law,

$$\oint_S \mathbf{D} \cdot d\mathbf{S} = Q$$

& then integrating over the distinct surfaces we get

$$\int_{top} + \int_{bottom} + \int_{sides} = Q$$.

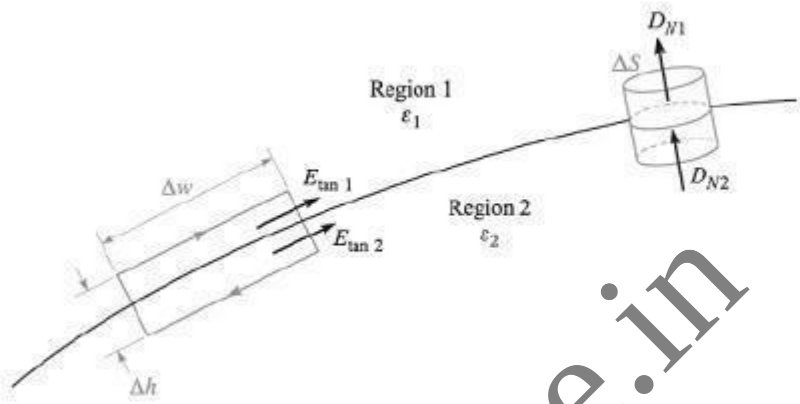
Flux experienced by the lateral surface is zero & Flux experienced by the bottom surface is zero as charge inside the conductor is zero. Therefore

$D_N \Delta S = Q = \rho_S \Delta S$ or $D_N = \rho_S$.

At the conductor dielectric interface normal component of the electric flux density is equal to the surface charge density.

2.8 Boundary condition for perfect dielectric:

Consider a closed path abcda at the dielectric dielectric interface & Δh→0. The work done in moving a unit charge over a closed path is zero. Therefore,



We know that the work done in moving a unit charge over a closed path is zero. Therefore,

$$\oint \mathbf{E} \cdot d\mathbf{L} = 0$$

, and hence

$$E_{\tan 1}\,\Delta w - E_{\tan 2}\,\Delta w = 0$$

.

The small contribution of the normal component of E due to Δh becomes negligible. Therefore,

$$E_{\tan 1} = E_{\tan 2}$$

. & as D = € E we get,

$$\frac{D_{\tan 1}}{\epsilon_1} = E_{\tan 1} = E_{\tan 2} = \frac{D_{\tan 2}}{\epsilon_2} \quad \text{or} \quad \frac{D_{\tan 1}}{D_{\tan 2}} = \frac{\epsilon_1}{\epsilon_2}$$

.

At the dielectric – dielectric boundary tangential component of the E is continuous where as tangential component of electric flux density is discontinuous.

Consider a gaussian cylinder of radius ρ and height Δh at the boundary, Applying Gauss law, & then integrating over the distinct surfaces we get

$$\int_{top} + \int_{bottom} + \int_{sides} = Q$$

. Flux experienced by the lateral surface is zero. Therefore

$$D_{N1}\Delta S - D_{N2}\Delta S = \Delta Q = \rho_S \Delta S$$

.

From which,

$$\boxed{D_{N1} - D_{N2} = \rho_S}$$

.

For perfect dielectric, $D_{N1} = D_{N2}$, then $€_2 E_2 = €_1 E_1$.

At the dielectric dielectric boundary normal component of the flux density is continuous. Normal components of D are continuous,

$$D_{N1} = D_1 \cos\theta_1 = D_2 \cos\theta_2 = D_{N2}$$

. The ratio of the tangential components,

$$\frac{D_{\tan 1}}{D_{\tan 2}} = \frac{D_1 \sin\theta_1}{D_2 \sin\theta_2} = \frac{\epsilon_1}{\epsilon_2}$$

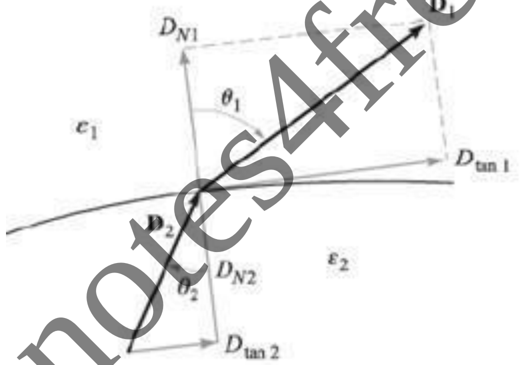
Or $\epsilon_2 D_1 \sin\theta_1 = \epsilon_1 D_2 \sin\theta_2$.

And

$$\frac{\tan\theta_1}{\tan\theta_2} = \frac{\epsilon_1}{\epsilon_2}.$$

The magnitude of D is given by,

$$D_2 = D_1 \sqrt{\cos^2\theta_1 + \left(\frac{\epsilon_2}{\epsilon_1}\right)^2 \sin^2\theta_1}.$$



Out comes

At the end of the unit the students are able to understand the concepts of Potential and Potential difference, energy and current densities, current continuity equation, and different boundary conditions.

Recommended questions

1. Define electric scalar potential. Establish the relationship between intensity and potential.
2. Discuss the boundary conditions between 2 perfect dielectrics.
3. State & explain the principle of charge conservation.
4. Derive for energy stored in an electrostatic field.

5. Derive for energy expended in moving a point charge in an electric field.

6. Define Potential & potential difference.

7. Prove that E is Grad of V

8. Write a short note on dipole

9. Three point charges, 0.4 μC each, are located at (0,0,-1), (0,0,0) and (0,0,1) in free space.
(a). Find an expression for the absolute potential as a function of Z along the lne x=0, y=1.
(b) Sketch V(Z).

Further Reading

1. Energy Electromagnetics, William H Hayt Jr. and John A Buck, Tata McGraw-Hill, 7th edition,2006.

2. Electromagnetics with Applications, John Krauss and Daniel A Fleisch McGraw-Hill, 5th edition, 1999

3. Electromagnetic Waves And Radiating Systems, Edward C. Jordan and Keith G Balmain, Prentice – Hall of India / Pearson Education, 2nd edition, 1968.Reprint 2002

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MODULE 3: POISSONS AND LAPLACES EQUATION, STEADY MAGNETIC FIELD

STRUCTURE

1.1 Derivation of Poisson’s equation and Laplace’s equation
1.2 Uniqueness theorem,
1.3 Examples of the solutions Laplace Equations and Poisson’s Equations

Objectives

1. To derive the Poissons and Laplaces equation
2. To derive the Uniqueness theorem
3. Application of Laplaces equation to parallel plate capacitor…

<u>Laplace’s & Poisson’s equation:</u>

Laplace’s & Poisson’s equation enable us to find potential fields within regions bounded by known potentials or charge densities.

Derivation of Laplace’s & Poisson’s equation:

From Gauss law in point form, we have

$$\nabla \cdot \mathbf{D} = \rho_v$$ --------------------------------(1).

By definition, D = €E. & from gradient relationship,

.

By substituting the above in equation 1, we get

$$\nabla \cdot \mathbf{D} = \nabla \cdot (\epsilon \mathbf{E}) = -\nabla \cdot (\epsilon \nabla V) = \rho_v$$

Or --------------------2

$$\nabla \cdot \nabla V = -\frac{\rho_v}{\epsilon}$$

For a homogeneous region in which € is constant. Equation 2 is poisson's equation.
In rectangular co-ordinates,

$$\nabla \cdot \nabla V = \frac{\partial}{\partial x}\left(\frac{\partial V}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial V}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial V}{\partial z}\right)$$
$$= \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}$$

Therefore,

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = -\frac{\rho_v}{\epsilon}$$

If $\rho_V = 0$, indicating zero volume charge density, but allowing point charges, line charges & surface charge density to exist at singular locations as sources of the field, then

$$\nabla^2 V = 0$$

which is Laplace's equation. The ∇^2 operorator is called the Laplacian of V.

In rectangular coordinates Laplace equation is,

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0 \qquad \text{(cartesian)}$$

, In cylindrical coordinates,

$$\nabla^2 V = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial V}{\partial \rho}\right) + \frac{1}{\rho^2}\left(\frac{\partial^2 V}{\partial \phi^2}\right) + \frac{\partial^2 V}{\partial z^2} \qquad \text{(cylindrical)}$$

& in spherical coordinates,

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial V}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 V}{\partial \phi^2} \qquad \text{(spherical)}$$

very conductor produces a field for which ∇^2

If $\rho_V = 0$, indicating zero volume charge density, but allowing point charges, line charges & surface charge density to exist at singular locations as sources of the field, then

$$\nabla^2 V = 0$$

which is Laplace's equation. The ∇^2 operator is called the Laplacian of V.

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$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0 \qquad \text{(cartesian)}$$

, In cylindrical coordinates,

$$\nabla^2 V = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial V}{\partial \rho}\right) + \frac{1}{\rho^2}\left(\frac{\partial^2 V}{\partial \phi^2}\right) + \frac{\partial^2 V}{\partial z^2} \qquad \text{(cylindrical)}$$

& in spherical coordinates,

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial V}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 V}{\partial \phi^2} \qquad \text{(spherical)}$$

Every conductor produces a field for which $\nabla^2 V=0$. In examples if it satisfies the boundary conditions and Laplace equation, then it is the only possible answer.

V=0. In examples if it satisfies the boundary conditions and Laplace equation, then it is the only possible answer.

Uniqueness theorem:

Let us assume we have two solutions of Laplace equation, V_1 and V_2, both general functions of the coordinates used. Therefore

$\nabla^2 V_1 = 0$ and $\nabla^2 V_2 = 0$. From which $\nabla^2(V_1 - V_2) = 0$.

On the boundary, $V_{b1}=V_{b2}$. Let the difference between V_1 & V_2 be V_d. Therefore $V_d= V_1-V_2$. From Laplace equation,

$\nabla^2 V_d= \nabla^2 V_2-\nabla^2 V_1$. On the boundary $V_d=0$.

From Divergence theorem,

$$\oint_S \mathbf{D}\cdot d\mathbf{S} = \int_{vol} \nabla\cdot\mathbf{D}\, dv$$

Using vector identity,

$$\nabla\cdot(V\mathbf{D}) \equiv V(\nabla\cdot\mathbf{D}) + \mathbf{D}\cdot(\nabla V)$$

We get,

$$\int_{vol} \nabla\cdot[(V_1 - V_2)\nabla(V_1 - V_2)]\, dv \equiv \int_{vol}(V_1 - V_2)[\nabla\cdot\nabla(V_1 - V_2)]dv + \int_{vol}[\nabla(V_1 - V_2)]^2\, dv$$

As $V_1= V_2$,

$$\int_{vol} \nabla\cdot[(V_1 - V_2)\nabla(V_1 - V_2)]\, dv = \oint_S [(V_{1b} - V_{2b})\nabla(V_{1b} - V_{2b})]\cdot d\mathbf{S} = 0$$

Surface consists of boundaries and hence

$$\int_{vol} [\nabla(V_1 - V_2)]^2 \, dv = 0$$

Therefore

$$[\nabla(V_1 - V_2)]^2 = 0$$

And

$$\nabla(V_1 - V_2) = 0$$

As

$$V_1 - V_2 = V_{1b} - V_{2b} = 0$$

We obtain,

$$V_1 = V_2$$

2.8 Example of solution of Laplace's equation:

Example 1: For a Parallel plate capacitor:

Let us assume V is a function of x. Laplace's equation reduces to,

$$\frac{\partial^2 V}{\partial x^2} = 0$$

Since V is not a function of y & z.

Integrating the above equation twice we obtain,

$$V = Ax + B$$

Where A & B are integration constants.

If V=0 at x=0 and V= V_0 at x = d, then,

A= V_0/d and B = 0.

Therefore,

$$V = \frac{V_0 x}{d}$$

Hence we have,

$$V = V_0 \frac{x}{d}$$

$$\mathbf{E} = -\frac{V_0}{d}\mathbf{a}_x$$

$$\mathbf{D} = -\epsilon \frac{V_0}{d}\mathbf{a}_x$$

$$\mathbf{D}_S = \mathbf{D}\Big|_{x=0} = -\epsilon \frac{V_0}{d}\mathbf{a}_x$$

$$\mathbf{a}_N = \mathbf{a}_x$$

$$D_N = -\epsilon \frac{V_0}{d} = \rho_S$$

$$Q = \int_S \frac{-\epsilon V_0}{d} dS = -\epsilon \frac{V_0 S}{d}$$

And the capacitance is

$$C = \frac{|Q|}{V_0} = \frac{\epsilon S}{d}$$

Example 2: Capacitance of a co-axial cylindrical conductor:

Assuming variation with respect to ρ Laplace equation becomes,

$$\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial V}{\partial \rho}\right) = 0$$

Integrating twice on both sides we obtain,

$$\rho \frac{dV}{d\rho} = A,$$

$$V = A \ln \rho + B.$$

Assuming V = V0 at ρ = A and V= 0 at ρ = B, We get

$$V = V_0 \frac{\ln(b/\rho)}{\ln(b/a)}$$

$$\mathbf{E} = \frac{V_0}{\rho}\frac{1}{\ln(b/a)}\mathbf{a}_\rho$$

$$D_{N(\rho=a)} = \frac{\epsilon V_0}{a\ln(b/a)}$$

$$Q = \frac{\epsilon V_0 2\pi a L}{a\ln(b/a)}$$

$$C = \frac{2\pi\epsilon L}{\ln(b/a)}$$

Example 3: Spherical capacitor:

Assuming variation with respect to r Laplace equation becomes,

$$\frac{1}{r^2\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dV}{d\theta}\right) = 0$$

Integrating twice on both sides we obtain,

$$\sin\theta\frac{dV}{d\theta} = A$$

$$V = \int\frac{A\,d\theta}{\sin\theta} + B$$

Assuming V = V0 at θ = Π/2 and V= 0 at θ = α, We get

$$V = V_0\frac{\ln\left(\tan\frac{\theta}{2}\right)}{\ln\left(\tan\frac{\alpha}{2}\right)}$$

$$\mathbf{E} = -\nabla V = \frac{1}{r}\frac{\partial V}{\partial\theta}\mathbf{a}_\theta = -\frac{V_0}{r\sin\theta\ln\left(\tan\frac{\alpha}{2}\right)}\mathbf{a}_\theta$$

$$\rho_S = \frac{-\epsilon V_0}{r\sin\alpha\ln\left(\tan\frac{\alpha}{2}\right)}$$

$$Q = \frac{-\epsilon V_0}{\sin\alpha\ln\left(\tan\frac{\alpha}{2}\right)}\int_0^\infty\int_0^{2\pi}\frac{r\sin\alpha\,d\phi\,dr}{r}$$

$$= \frac{-2\pi\epsilon_0 V_0}{\ln\left(\tan\frac{\alpha}{2}\right)}\int_0^\infty dr$$

and

$$C \doteq \frac{2\pi\epsilon r_1}{\ln\left(\cot\frac{\alpha}{2}\right)}$$.

Outcomes

The students are able to state and derive the poisons and laplace's equation and apply it to derive the capacitance of parallel plate capacitor, cylindrical conductor and spherical ring & show that Laplaces equation has only one solution

Recommended Questions

1. Derive Poisson's & Laplace's equation.

2. Using Laplace's equation , Prove that the potential distribution at any point in the region between two concentric cylinders of radii A & B as
V=Voln ρ̉/B /ln A/B

3. State and prove uniqueness theorem

4. Derive for Capacitance of Parallel plate capacitor

5. Derive for Capacitance of Concentric spherical capacitor.

6. Let $V = 2xy^2z^3$ and $\varepsilon = \varepsilon_0$. Given point P(1,2,-1), Find (a) V at P; (b) E at P; (c) ρ_v at P; (d) the equation of the equipotential surface passing through P; (e) the equation of the streamline passing through P; (f) Does V satisfy the Laplaces Equation

Further Reading

TEXT BOOK:

1. Energy Electromagnetics, William H Hayt Jr . and John A Buck, Tata McGraw-Hill, 7th edition,2006.

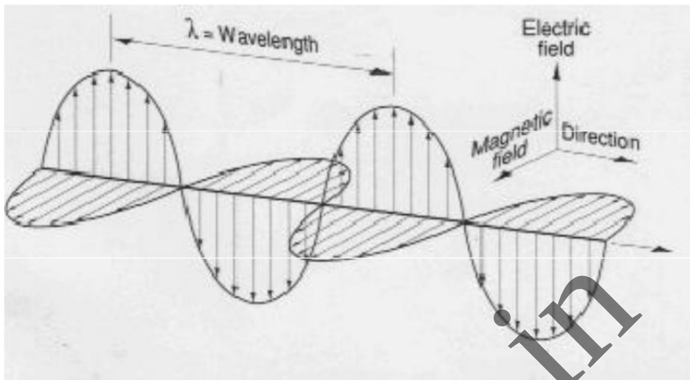
REFERENCE BOOKS:

2. Electromagnetics with Applications, John Krauss and Daniel A Fleisch McGraw-Hill, 5th edition, 1999
3. Electromagnetic Waves And Radiating Systems, Edward C. Jordan and Keith G Balmain, Prentice – Hall of India / Pearson Education, 2nd edition, 1968.Reprint 2002
4. Field and Wave Electromagnetics, David K Cheng, Pearson Education Asia, 2nd edition, -1989, Indian Reprint – 2001.

MODULE-IV

Plane Wave:

A uniform plane wave is the wave that the electric field, E or magnetic field, H in same direction, same magnitude and same phase in infinite planes perpendicular to the direction of propagation. A plane wave has no electric field, and magnetic field, components along its direction of propagation.



Wave Equations:

If the wave is in simple (linear, isotropic and homogeneous) nonconducting medium (=0), Maxwell's equation reduce to,

$$\vec{\nabla} \times \vec{E} = -\mu \frac{\partial \vec{H}}{\partial t} \qquad \vec{\nabla} \cdot \vec{E} = 0$$

$$\vec{\nabla} \times \vec{H} = \varepsilon \frac{\partial \vec{E}}{\partial t} \qquad \vec{\nabla} \cdot \vec{H} = 0$$

The first-order differential equations in the two variables E and H. They can combine to give E or H alone using second-order equation.

Using Maxwell's equation,

$$\vec{\nabla} \times \vec{E} = -\mu \frac{\partial \vec{H}}{\partial t} \quad (1) \qquad \vec{\nabla} \times \vec{H} = \varepsilon \frac{\partial \vec{E}}{\partial t} \quad (2) \qquad \vec{\nabla} \cdot \vec{E} = 0 \quad (3)$$

The curl of equation of (1)

$$\vec{\nabla} \times \vec{\nabla} \times \vec{E} = -\mu \frac{\partial}{\partial t}\left(\vec{\nabla} \times \vec{H}\right)$$

Replacing in equation (2)

$$\vec{\nabla} \times \vec{\nabla} \times \vec{E} = -\mu\varepsilon \frac{\partial^2 \vec{E}}{\partial t^2}$$

We know that $\vec{\nabla} \times \vec{\nabla} \times \vec{E} = \vec{\nabla}(\vec{\nabla} \cdot \vec{E}) - \vec{\nabla}^2 \vec{E}$ because of equation (3), thus the wave equation is

$$\vec{\nabla}^2 \vec{E} - \mu\varepsilon \frac{\partial^2 \vec{E}}{\partial t^2} = 0$$

The wave equation also can written as

$$\vec{\nabla}^2 \vec{E} - k^2 \vec{E} = 0$$ --------------------------(a)

Assuming an implicit time dependence $e^{j\omega t}$ in the field vector. Equation (a) also called Helmholtz equation. The k is called the wave number or propagation constant.

$$k - k_o \sqrt{\varepsilon_r} = \frac{2\pi f}{c}\sqrt{\varepsilon_r}$$ **and** $$c - \frac{1}{\sqrt{\varepsilon\mu}}$$

where c is the velocity of light in free space.

For magnetic intensity domain, H, we have,

$$\vec{\nabla}^2 \vec{H} - \mu\varepsilon \frac{\partial^2 \vec{H}}{\partial t^2} = 0$$ **or** $$\vec{\nabla}^2 \vec{H} - \mu_r \varepsilon_r k_o^2 \vec{H} = 0$$

For a uniform plane wave with an electric field $E\, \hat{x}\, E_x$ traveling in the z-direction, the wave equation can be reduced as

$$\frac{\partial^2 \vec{E}_x(z)}{\partial z^2} - k^2 \vec{E}_x(z) = 0$$

The solution of this wave equation,

$$\vec{E}(z) = \hat{x}E_x$$
$$= \hat{x}E_o e^{-kz}$$
$$= \hat{x}E_o e^{-\alpha z} e^{-j\beta z}$$

Where is the attenuation constant of the medium and is its phase constant.

The associated magnetic field, H,

$$\vec{H}(z) = \hat{y}H_y$$
$$= \hat{y}\frac{\vec{E}_x}{\eta}$$
$$= \hat{y}\frac{E_o}{\eta} e^{-\alpha z} e^{-j\beta z}$$

where is the intrinsic impedance of the medium.

The k is called the wave number or propagation constant.

$$k^2 = k_o^2 \varepsilon_r \mu_r$$
$$k^2 = k_o^2 \mu_r (\varepsilon_r' - j\varepsilon_r'')$$

The wave number can also be written in terms of and .

$$k^2 = (\alpha + j\beta)^2$$
$$= (\alpha^2 - \beta^2) + j2\alpha\beta$$

Thus,

$$\alpha^2 - \beta^2 = k_o^2 \mu_r \varepsilon_r' \qquad \textbf{(1)}$$

$$2\alpha\beta = -k_o^2 \mu_r \varepsilon_r'' \qquad \textbf{(2)}$$

By solving (1) & (2),

$$\alpha = \sqrt{\frac{k_o^2 \mu_r \varepsilon_r'}{2}\left(\sqrt{1+\left(\frac{\varepsilon_r''}{\varepsilon_r'}\right)^2} - 1\right)}$$

$$\beta = \sqrt{\frac{k_o^2 \mu_r \varepsilon_r'}{2}\left(\sqrt{1+\left(\frac{\varepsilon_r''}{\varepsilon_r'}\right)^2} + 1\right)}$$

So for different medium,

Lossless Medium $(\sigma = 0)$	Low-loss Medium $(\varepsilon''/\varepsilon' \neq 0)$	Conductor $(\varepsilon''/\varepsilon' \to \infty)$
$\alpha = 0$	$\alpha = \frac{\sigma}{2}\sqrt{\frac{\mu}{\varepsilon}}$	$\alpha = \sqrt{\pi f \mu \sigma}$
$\beta = \omega\sqrt{\mu\varepsilon}$	$\beta = \omega\sqrt{\mu\varepsilon}$	$\beta = \sqrt{\pi f \mu \sigma}$

Electromagnetic Phenomena are described by using four Maxwell's equations:

	Maxwell's equation		
Gauss's Law (Electric fields)	**Integral form:** $\underbrace{\varepsilon_o \oint \vec{E} \cdot d\vec{S}}_{Left} = \underbrace{q}_{Right}$	**Description** **Left side:** The number of electric field lines – perpendicularly passing through to a closed surface, $\vec{S}$ **Right side:** Total amount of charge, q contained within that surface, .	**Information** Electric charge produces an electric field, $\vec{E}$ and the flux of that field passing through any closed surface is proportional to the total charge, q contained within that surface. Charge on an insulated conductor moves outward surface.
	Differential form: $\underbrace{\varepsilon_o \vec{\nabla} \cdot \vec{E}}_{Left} = \underbrace{\rho}_{Right}$	**Left side:** Divergence of the electric field, $\vec{E}$ – the tendency of the field to "flow" away from a specified location. **Right side:** Electric charge density, ρ	The electric field, $\vec{E}$ produced by electric charge diverges from positive charge and converges upon negative charge. The electric field, $\vec{E}$ is tendency to propagate perpendicularly away from a surface charge.

Gauss's Law (Magnetic fields)	**Integral form:** $\underbrace{\mu_o \oint \vec{H} \cdot d\vec{S}}_{Left} = \underbrace{0}_{Right}$	**Left side:** The number of magnetic field lines – perpendicularly passing through a closed surface. **Right side:** Identically zero.	The total magnetic flux passing through any closed surface is zero. Flux enter the closed surface is same with the flux come out from the surface. The divergence of the magnetic field at any point is zero.
	Differential form: $\underbrace{\mu_o \vec{\nabla} \cdot \vec{H}}_{Left} = \underbrace{0}_{Right}$	**Left side:** Divergence of the magnetic field – the tendency of the field to "flow" away from a point than toward it. **Right side:** Identically zero.	

Faraday's Law	**Integral form:** $\underbrace{\oint_C \vec{E} \cdot d\vec{l}}_{Left} = \underbrace{-\mu_o \int_S \frac{\partial \vec{H}}{\partial t} \cdot d\vec{S}}_{Right}$	**Left side:** The circulation of the vector electric field, $\vec{E}$ around a closed path, C. **Right side:** The rate of change with time (d/dt) of magnetic field, through any surface, $\vec{S}$.	Changing magnetic flux through a surface induces an emf in any boundary path, C of that surface, and a changing magnetic field, $\vec{H}$ induces a circulating electric field.
	Differential form: $\underbrace{\vec{\nabla} \times \vec{E}}_{Left} = \underbrace{-\mu_o \frac{\partial \vec{H}}{\partial t}}_{Right}$	**Left side:** Curl of the electric field, – the tendency of the field lines to circulate around a point. **Right side:** The rate of change of the magnetic field, $\vec{H}$ over time (d/dt)	A circulating electric field, is produced by a magnetic field, $\vec{H}$ that changes with time.

Ampere's Law	**Integral form:** $$\underbrace{\oint_C \vec{H}\cdot d\vec{l}}_{Left} = \underbrace{\int_S \left(\vec{J}_c + \varepsilon_o \frac{\partial \vec{E}}{\partial t}\right)\cdot d\vec{S}}_{Right}$$	**Left side:** The circulation of the magnetic field, $\vec{H}$ around a closed path, C. **Right side:** Two sources for the magnetic field, $\vec{H}$; a steady conduction current, $\vec{J}_c$ and a changing electric field, $\vec{E}$ through any surface, bounded by closed path, C.	An electric current or a changing electric flux through a surface produces a circulating magnetic field around any path, C that bounds that surface.
	Differential form: $$\underbrace{\bar{\nabla}\times\vec{H}}_{Left} = \underbrace{\vec{J}_c + \varepsilon_o \frac{\partial \vec{E}}{\partial t}}_{Right}$$	**Left side:** Curl of the magnetic field, – the tendency of the field lines to circulate around a point. **Right side:** Two terms represent the electric current density, $\vec{J}_c$ and the time rate of change of the electric field, $\vec{E}$.	A circulating electric field, is produced by a magnetic field, $\vec{H}$ that changes with time. An electric current, or a changing electric field, through a surface produces a circulating magnetic field, $\vec{H}$ around any path that bounds that surface.

Poynting Vector and Power Flow in Electromagnetic Fields:

Electromagnetic waves can transport energy from one point to another point. The electric and magnetic field intensities associated with a travelling electromagnetic wave can be related to the rate of such energy transfer.

Let us consider Maxwell's Curl Equations:

$$\nabla\times\vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\nabla\times\vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$$

Using vector identity

$$\nabla.\left(\vec{E}\times\vec{H}\right) = \vec{H}.\nabla\times\vec{E} - \vec{E}.\nabla\times\vec{H}$$

The above curl equations we can write

$$\vec{H}.\frac{\partial \vec{B}}{\partial t} = \frac{\partial}{\partial t}\left(\frac{1}{2}\mu H^2\right) \quad , \quad \vec{E}.\frac{\partial \vec{D}}{\partial t} = \frac{\partial}{\partial t}\left(\frac{1}{2}\mu E^2\right)$$

And $\vec{E}.\vec{J} = \sigma E^2$.

In simple medium where $\in, \mu$ and σ are constant, we can write

$$\therefore \nabla.\left(\vec{E}\times\vec{H}\right) = -\frac{\partial}{\partial t}\left(\frac{1}{2}\in E^2 + \frac{1}{2}\mu H^2\right) - \sigma E^2$$

Applying Divergence theorem we can write,

$$\oint_S \left(\vec{E}\times\vec{H}\right).d\vec{S} = -\frac{\partial}{\partial t}\int_V\left(\frac{1}{2}\in E^2 + \frac{1}{2}\mu H^2\right)dV - \int_V \sigma E^2 dV \quad \text{……………(a)}$$

The term $\frac{\partial}{\partial t}\int_V\left(\frac{1}{2}\in E^2 + \frac{1}{2}\mu H^2\right)dV$ represents the rate of change of energy stored in the electric and magnetic fields and the term $\int_V \sigma E^2 dV$ represents the power dissipation within the volume. Hence right hand side of the equation (a) represents the total decrease in power within the volume under consideration.

The left hand side of equation (6.36) can be written as $\oint_S \left(\vec{E}\times\vec{H}\right).d\vec{S} = \oint_S \vec{P}.d\vec{S}$ where $\vec{P} = \vec{E}\times\vec{H}$ (W/mt^2) is called the Poynting vector and it represents the power density vector associated with the electromagnetic field. The integration of the Poynting vector over any closed surface gives the net power flowing out of the surface. Equation (6.36) is referred to as Poynting theorem and it states that the net power flowing out of a given volume is equal to the time rate of decrease in the energy stored within the volume minus the conduction losses.

Poynting vector for the time harmonic case:

For time harmonic case, the time variation is of the form $e^{j\omega t}$, and we have seen that instantaneous value of a quantity is the real part of the product of a phasor quantity and $e^{j\omega t}$ when $\cos\omega t$ is used as reference. For example, if we consider the phasor

$$\vec{E}(z) = \hat{a}_x E_x(z) = \hat{a}_x E_0 e^{-j\beta z}$$

then we can write the instanteneous field as

$$\vec{E}(z,t) = \text{Re}\left[\vec{E}(z)e^{j\omega t}\right] = E_0\cos(\omega t - \beta z)\hat{a}_x$$

when E_0 is real.

Let us consider two instanteneous quantities A and B such that

$$A = \text{Re}\left(Ae^{j\omega t}\right) = |A|\cos(\omega t + \alpha), \quad B = \text{Re}\left(Be^{j\omega t}\right) = |B|\cos(\omega t + \beta)$$

where A and B are the phasor quantities. i.e, $A = |A|e^{j\alpha}$

$$B = |B|e^{j\beta}$$

Therefore,

$$AB = |A|\cos(\omega t + \alpha)|B|\cos(\omega t + \beta)$$

$$= \frac{1}{2}|A||B|\left[\cos(\alpha - \beta) + \cos(2\omega t + \alpha + \beta)\right]$$

Since A and B are periodic with period $T = \frac{2\pi}{\omega}$, the time average value of the product form AB, denoted by $\overline{AB}$ can be written as

$$\overline{AB} = \frac{1}{T}\int_0^T AB\,dt$$

$$\overline{AB} = \frac{1}{2}|A||B|\cos(\alpha - \beta)$$

Further, considering the phasor quantities *A* and *B*, we find that

$$AB^* = |A|e^{j\alpha}|B|e^{-j\beta} = |A||B|e^{j(\alpha-\beta)}$$

and $\mathrm{Re}(AB^*) = |A||B|\cos(\alpha - \beta)$, where * denotes complex conjugate.

$$\therefore \overline{AB} = \frac{1}{2}\mathrm{Re}(AB^*)$$

The poynting vector $\vec{P} = \vec{E}\times\vec{H}$ can be expressed as

$$\vec{P} = \hat{a}_x(E_yH_z - E_zH_y) + \hat{a}_y(E_zH_x - E_xH_z) + \hat{a}_z(E_xH_y - E_yH_x) \quad \text{.............................(b)}$$

If we consider a plane electromagnetic wave propagating in +z direction and has only E_x component, from (b) we can write:

$$\vec{P}_z = E_x(z,t)H_y(z,t)\hat{a}_3$$

Using (6.41)

$$\vec{P}_{zav} = \frac{1}{2}\mathrm{Re}\left(E_x(z)H_y^*(z)\hat{a}_z\right)$$

$$\vec{P}_{zav} = \frac{1}{2}\mathrm{Re}\left(E_x(z)\times H_y(z)\right)$$

where $\vec{E}(z) = E_x(z)\hat{a}_x$ and $\vec{H}(z) = H_y(z)\hat{a}_y$, for the plane wave under consideration.
For a general case, we can write

$$\vec{P}_{av} = \frac{1}{2}\mathrm{Re}\left(\vec{E}\times\vec{H}^*\right)$$

We can define a complex Poynting vector

$$\vec{S} = \frac{1}{2}\vec{E}\times\vec{H}^*$$

and time average of the instantaneous Poynting vector is given by $\vec{P}_{av} = \mathrm{Re}(\vec{S})$.

Polarisation of plane wave:

The polarization of a plane wave can be defined as the orientation of the electric field vector as a function of time at a fixed point in space. For an electromagnetic wave, the specification of the orientation of the electric field is sufficient as the magnetic field components are related to electric field vector by the Maxwell's equations.
Let us consider a plane wave travelling in the +z direction. The wave has both E_x and E_y components.

$$\vec{E} = \left(\hat{a}_x E_{ox} + \hat{a}_y E_{oy}\right) e^{-j\beta z}$$

The corresponding magnetic fields are given by,

$$\vec{H} = \frac{1}{\eta}\hat{a}_z \times \vec{E}$$
$$= \frac{1}{\eta}\hat{a}_z \times \left(\hat{a}_x E_{ox} + \hat{a}_y E_{oy}\right) e^{-j\beta z}$$
$$= \frac{1}{\eta}\left(-E_{oy}\hat{a}_x + E_{ox}\hat{a}_x\right) e^{-j\beta z}$$

Depending upon the values of E_{ox} and E_{oy} we can have several possibilities:

1. If $E_{oy} = 0$, then the wave is linearly polarised in the *x*-direction.
2. If $E_{oy} = 0$, then the wave is linearly polarised in the *y*-direction.
3. If E_{ox} and E_{oy} are both real (or complex with equal phase), once again we get a linearly polarised wave with the axis of polarisation inclined at an angle $\tan^{-1}\frac{E_{oy}}{E_{ox}}$, with respect to the x-axis. This is shown in fig 6.4.

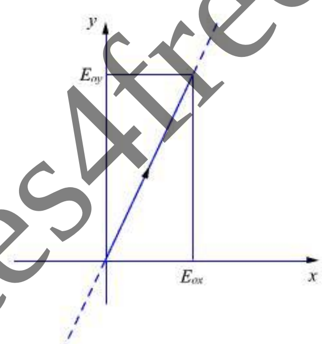


Fig 6.4 : Linear Polarisation

If *Eox* and *Eoy* are complex with different phase angles, $\vec{E}$ will not point to a single spatial direction. This is explained as follows:

Let $E_{ox} = |E_{ox}|e^{ja}$ $E_{oy} = |E_{oy}|e^{jb}$

Then,

and ……………………………(c)

To keep the things simple, let us consider a =0 and $b = \frac{\pi}{2}$. Further, let us study the nature of the electric field on the z =0 plain.

From equation (c) we find that,

$$E_x(o,t) = |E_{ox}|\cos\omega t$$
$$E_y(o,t) = |E_{oy}|\cos\left(\omega t + \frac{\pi}{2}\right) = |E_{oy}|(-\sin\omega t)$$

$$\therefore \left(\frac{E_x(o,t)}{|E_{ox}|}\right)^2 + \left(\frac{E_y(o,t)}{|E_{oy}|}\right)^2 = \cos^2 \omega t + \sin^2 \omega t = 1$$

and the electric field vector at z = 0 can be written as

$$\vec{E}(o,t) = |E_{ox}|\cos(\omega t)\hat{a}_x - |E_{oy}|\sin(\omega t)\hat{a}_y \quad \text{.........(d)}$$

Assuming $|E_{ox}| > |E_{oy}|$, the plot of $\vec{E}(o,t)$ for various values of t is hown in figure 6.5.

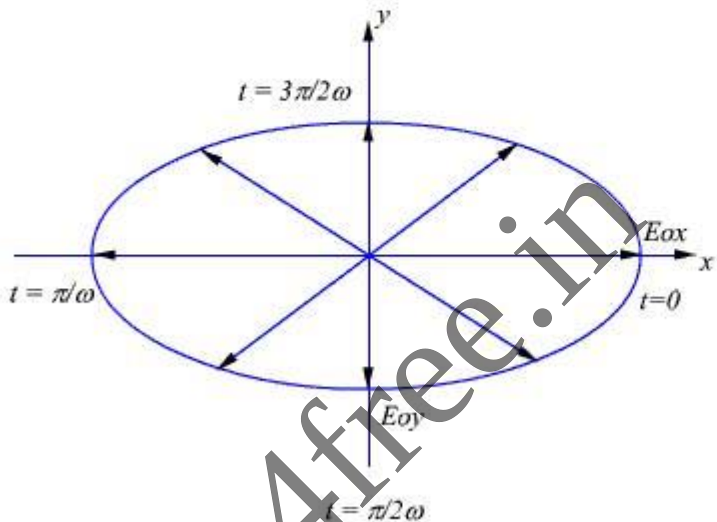


Figure 6.5 : Plot of *E*(*o*,*t*)

From equation (d) and figure (6.5) we observe that the tip of the arrow representing electric field vector traces an ellipse and the field is said to be elliptically polarized.

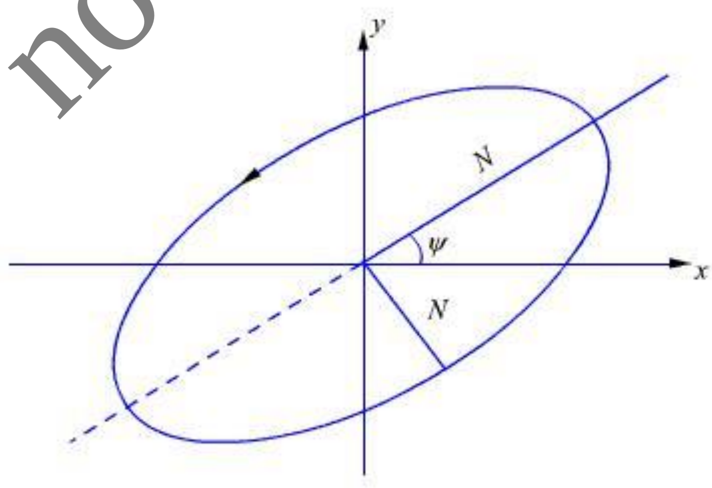


Figure 6.6: Polarisation ellipse

The polarisation ellipse shown in figure 6.6 is defined by its axial ratio(M/N, the ratio of semimajor to semiminor axis), tilt angle ψ (orientation with respect to xaxis) and sense of rotation(i.e., CW or CCW). Linear polarisation can be treated as a special case of elliptical polarisation, for which the axial ratio is infinite.

In our example, if $|E_{ox}| = |E_{oy}|$, from equation (6.47), the tip of the arrow representing electric field vector traces out a circle. Such a case is referred to as Circular Polarisation. For circular polarisation the axial ratio is unity.

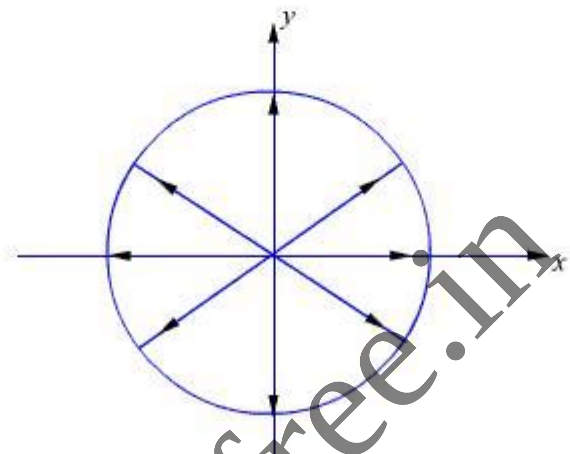


Figure 6.7: Circular Polarisation (RHCP)

Further, the circular polarisation is aside to be right handed circular polarisation (RHCP) if the electric field vector rotates in the direction of the fingers of the right hand when the thumb points in the direction of propagation-(same and CCW). If the electric field vector rotates in the opposite direction, the polarisation is asid to be left hand circular polarisation (LHCP) (same as CW).

In AM radio broadcast, the radiated electromagnetic wave is linearly polarised with the $\vec{E}$ field vertical to the ground(vertical polarisation) where as TV signals are horizontally polarised waves. FM broadcast is usually carried out using circularly polarised waves.
In radio communication, different information signals can be transmitted at the same frequency at orthogonal polarisation (one signal as vertically polarised other horizontally polarised or one as RHCP while the other as LHCP) to increase capacity. Otherwise, same signal can be transmitted at orthogonal polarisation to obtain diversity gain to improve reliability of transmission.

Behaviour of Plane waves at the inteface of two media:
We have considered the propagation of uniform plane waves in an unbounded homogeneous medium. In practice, the wave will propagate in bounded regions where several values of ε, μ, σ will be present. When plane wave travelling in one medium meets a different medium, it is partly reflected and partly transmitted. In this section, we consider wave reflection and transmission at planar boundary between two media.

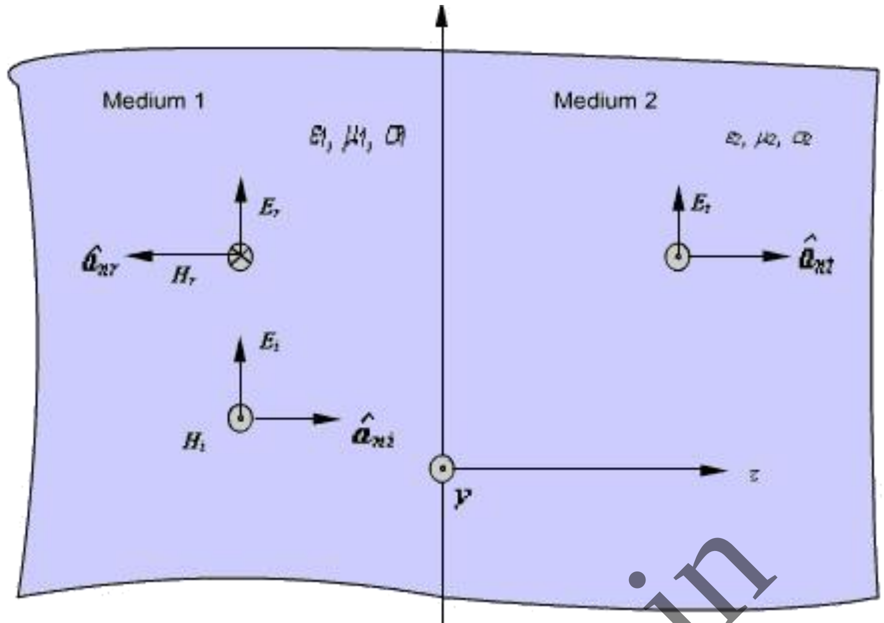


Fig 6.8 : Normal Incidence at a plane boundary

Case1: Let z = 0 plane represent the interface between two media. Medium 1 is characterised by $(\varepsilon_1, \mu_1, \sigma_1)$ and medium 2 is characterized by $(\varepsilon_2, \mu_2, \sigma_2)$.
Let the subscripts '*i*' denotes incident, '*r*' denotes reflected and '*t*' denotes transmitted field components respectively.

The incident wave is assumed to be a plane wave polarized along *x* and travelling in medium 1 along $\hat{a}_z$ direction. From equation (6.24) we can write

$$\vec{E}_i(z) = E_{io}e^{-\gamma_1 z}\hat{a}_x \quad \text{..................(e)}$$

$$\vec{H}_i(z) = \frac{1}{\eta_i}\hat{a}_z \times E_{io}(z) = \frac{E_o}{\eta_i}e^{-\gamma_1 z}\hat{a}_y \quad \text{.....................(f)}$$

where $\gamma_1 = \sqrt{j\omega\mu_1(\sigma_1 + j\omega\varepsilon_1)}$ and $\eta_1 = \sqrt{\frac{j\omega\mu_1}{\sigma_1 + j\omega\varepsilon_2}}$.

Because of the presence of the second medium at *z* =0, the incident wave will undergo partial reflection and partial transmission. The reflected wave will travel along $\hat{a}_z$ in medium 1.
The reflected field components are:

$$\vec{E}_r = E_{ro}e^{\gamma_1 z}\hat{a}_x \quad \text{...(g)}$$

$$\vec{H}_r = \frac{1}{\eta_1}\left(-\hat{a}_z\right)\times E_{ro}e^{\gamma_1 z}\hat{a}_x = -\frac{E_{ro}}{\eta_1}e^{\gamma_1 z}\hat{a}_y \quad \text{.........(h)}$$

The transmitted wave will travel in medium 2 along $\hat{a}_z$ for which the field components are

$$\vec{E}_t = E_{t0}e^{-\gamma_2 z}\hat{a}_x \quad \text{.......................................(i)}$$

$$\vec{H}_t = \frac{E_{t0}}{\eta_2} e^{-\gamma_2 z} \hat{a}_y \qquad \text{...(j)}$$

where $\gamma_2 = \sqrt{j\omega\mu_2\left(\sigma_2 + j\omega\varepsilon_2\right)}$ and $\eta_2 = \sqrt{\dfrac{j\omega\mu_2}{\sigma_2 + j\omega\varepsilon_2}}$

In medium 1,

$$\vec{E}_1 = \vec{E}_i + \vec{E}_r \text{ and } \vec{H}_1 = \vec{H}_i + \vec{H}_r$$

and in medium 2,

$$\vec{E}_2 = \vec{E}_t \text{ and } \vec{H}_2 = \vec{H}_t$$

Applying boundary conditions at the interface z = 0, i.e., continuity of tangential field components and noting that incident, reflected and transmitted field components are tangential at the boundary, we can write

$$\vec{E}_i(0) + \vec{E}_r(0) = \vec{E}_i(0)$$

$$\& \quad \vec{H}_i(0) + \vec{H}_r(0) = \vec{H}_t(0)$$

From equation (e) to (j) we get,

$$E_{io} + E_{ro} = E_{to} \qquad \text{..(k)}$$

$$\frac{E_{io}}{\eta_1} - \frac{E_{ro}}{\eta_1} = \frac{E_{to}}{\eta_2} \qquad \text{..(l)}$$

Eliminating E_{to} ,

$$\frac{E_{io}}{\eta_1} - \frac{E_{ro}}{\eta_1} = \frac{1}{\eta_2}\left(E_{io} + E_{ro}\right)$$

$$\text{or,} \quad E_{io}\left(\frac{1}{\eta_1} - \frac{1}{\eta_2}\right) = E_{ro}\left(\frac{1}{\eta_1} + \frac{1}{\eta_2}\right)$$

$$\text{or,} \quad E_{ro} = \tau E_{io}$$

$$\tau = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} \qquad \text{...............(m)}$$

is called the reflection coefficient.

From equation (k) & (l), we can write

$$2E_{io} = E_{to}\left[1 + \frac{\eta_1}{\eta_2}\right]$$

$$\text{or,} \quad E_{to} = \frac{2\eta_2}{\eta_1 + \eta_2} E_{io} = TE_{io}$$

$$T = \frac{2\eta_2}{\eta_1 + \eta_2}$$

is called the transmission coefficient.

We observe that,

$$T = \frac{2\eta_2}{\eta_1 + \eta_2} = \frac{\eta_2 - \eta_1 + \eta_1 + \eta_2}{\eta_1 + \eta_2} = 1 + \tau$$

The following may be noted

(i) both τ and T are dimensionless and may be complex

(ii) $0 \leq |\tau| \leq 1$

Let us now consider specific cases:

Case I: Normal incidence on a plane conducting boundary

The medium 1 is perfect dielectric $(\sigma_1 = 0)$ and medium 2 is perfectly conducting $(\sigma_2 = \infty)$.

$$\therefore \eta_1 = \sqrt{\frac{\mu_1}{\epsilon_1}}$$

$$\eta_2 = 0$$

$$\gamma_1 = \sqrt{(j\omega\mu_1)(j\omega\epsilon_1)}$$

$$= j\omega\sqrt{\mu_1 \epsilon_1} = j\beta_1$$

From (k) and (l)

τ = -1

and T =0

Hence the wave is not transmitted to medium 2, it gets reflected entirely from the interface to the medium 1.

$$\therefore \vec{E}_1(z) = E_{io}e^{-j\beta_1 z}\hat{a}_x - E_{io}e^{j\beta_1 z}\hat{a}_x = -2jE_{io}\sin\beta_1 z\,\hat{a}_x$$

& $$\therefore \vec{E}_1(z,t) = \mathrm{Re}\left[-2jE_{io}\sin\beta_1 z e^{j\omega t}\right]\hat{a}_x = 2E_{io}\sin\beta_1 z\sin\omega t\,\hat{a}_x$$

Proceeding in the same manner for the magnetic field in region 1, we can show that,

$$\vec{H}_1(z,t) = \hat{a}_y\frac{2E_{io}}{\eta_1}\cos\beta_1 z\cos\omega t$$

The wave in medium 1 thus becomes a **standing wave** due to the super position of a forward travelling wave and a backward travelling wave. For a given $'t'$, both $\vec{E}_1$ and $\vec{H}_1$ vary sinusoidally with distance measured from z = 0. This is shown in figure 6.9.

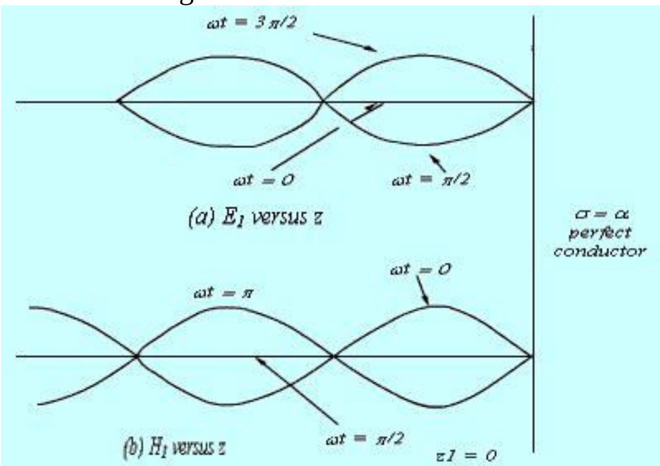


Figure 6.9: Generation of standing wave

Zeroes of $E_1(z,t)$ and

Maxima of $H_1(z,t)$.

$$\left.\right\} \text{occur at } \beta_1 z = -n\pi \quad \text{or } z = -n\frac{\lambda}{2}$$

Maxima of $E_1(z,t)$ and

zeroes of $H_1(z,t)$.

$$\left.\right\} \text{occur at } \beta_1 z = -(2n+1)\frac{\pi}{2} \quad \text{or } z = -(2n+1)\frac{\lambda}{4},\ n = 0,1,2,....$$

Case2: Normal incidence on a plane dielectric boundary

If the medium 2 is not a perfect conductor (i.e. $\sigma_2 \neq \infty$) partial reflection will result. There will be a reflected wave in the medium 1 and a transmitted wave in the medium 2.Because of the reflected wave, standing wave is formed in medium 1.
From above equations we can write

$$\vec{E}_1 = E_{io}\left(e^{-\gamma_1 z} + \Gamma e^{\gamma_1 z}\right)\hat{a}_x$$

Let us consider the scenario when both the media are dissipation less i.e. perfect dielectrics ($\sigma_1 = 0, \sigma_2 = 0$)

$$\gamma_1 = j\omega\sqrt{\mu_1\varepsilon_1} = j\beta_1 \qquad \eta_1 = \sqrt{\frac{\mu_1}{\varepsilon_1}}$$

$$\gamma_2 = j\omega\sqrt{\mu_2\varepsilon_2} = j\beta_2 \qquad \eta_2 = \sqrt{\frac{\mu_2}{\varepsilon_2}}$$

In this case both η_1 and η_2 become real numbers.

$$\begin{aligned}\vec{E}_1 &= \hat{a}_x E_{io}\left(e^{-j\beta_1 z} + \Gamma e^{j\beta_1 z}\right) \\ &= \hat{a}_x E_{io}\left((1+T)e^{-j\beta_1 z} + \Gamma\left(e^{j\beta_1 z} - e^{-j\beta_1 z}\right)\right) \\ &= \hat{a}_x E_{io}\left(Te^{-j\beta_1 z} + \Gamma(2j\sin\beta_1 z)\right)\end{aligned}$$ ....(n)

From (n), we can see that, in medium 1 we have a traveling wave component with amplitude TE_{io} and a standing wave component with amplitude $2JE_{io}$.
The location of the maximum and the minimum of the electric and magnetic field components in the medium 1from the interface can be found as follows. The electric field in medium 1 can be written as

$$\vec{E}_1 = \hat{a}_x E_{io} e^{-j\beta_1 z}\left(1 + \Gamma e^{j2\beta_1 z}\right)$$

If $\eta_2 > \eta_1$ i.e. $\Gamma > 0$
The maximum value of the electric field is

$$\left|\vec{E}_1\right|_{max} = E_{io}(1+T)$$

and this occurs when

$$2\beta_1 z_{max} = -2n\pi$$

or $z_{max} = -\frac{n\pi}{\beta_1} = -\frac{n\pi}{2\pi/\lambda_1} = -\frac{n}{2}\lambda_1$, n = 0, 1, 2, 3......................(o)

The minimum value of $|\vec{E}_1|$ is

$$|\vec{E}_1|_{min} = E_{io}(1-\Gamma) \quad \text{.................(p)}$$

And this occurs when

$$2\beta_1 z_{min} = -(2n+1)\pi$$

or $z_{min} = -(2n+1)\frac{\lambda_1}{4}$, n = 0, 1, 2, 3...........................(q)

For $\eta_2 < \eta_1$ i.e. $\Gamma < 0$

The maximum value of $|\vec{E}_1|$ is $E_{io}(1-\Gamma)$ which occurs at the z_{min} locations and the minimum value of $|\vec{E}_1|$ is $E_{io}(1+\Gamma)$ which occurs at z_{max} locations as given by the equations (o) and (q).

From our discussions so far we observe that $\frac{|E|_{max}}{|E|_{min}}$ can be written as

$$S = \frac{|E|_{max}}{|E|_{min}} = \frac{1+|\Gamma|}{1-|\Gamma|}$$

The quantity *S* is called as the standing wave ratio.

As $0 \le |\Gamma| \le 1$ the range of *S* is given by $1 \le S \le \infty$

We can write the expression for the magnetic field in medium 1 as

$$\vec{H}_1 = \hat{a}_y \frac{E_{io}}{\eta_1} e^{-j\beta_1 z}\left(1-\Gamma e^{j2\beta_1 z}\right)$$

From above equation we can see that $|\vec{H}_1|$ will be maximum at locations where $|\vec{E}_1|$ is minimum and vice versa.

In medium 2, the transmitted wave propagates in the + z direction.

Oblique Incidence of EM wave at an interface

So far we have discuss the case of normal incidence where electromagnetic wave traveling in a lossless medium impinges normally at the interface of a second medium. In this section we shall consider the case of oblique incidence. As before, we consider two cases

i. When the second medium is a perfect conductor.
ii. When the second medium is a perfect dielectric.

A plane incidence is defined as the plane containing the vector indicating the direction of propagation of the incident wave and normal to the interface. We study two specific cases when the incident electric field $\vec{E}_i$ is perpendicular to the plane of incidence (perpendicular polarization) and $\vec{E}_i$ is parallel to the plane of incidence (parallel polarization). For a general case, the incident wave may have arbitrary polarization but the same can be expressed as a linear combination of these two individual cases.

Oblique Incidence at a plane conducting boundary

i. Perpendicular Polarization

The situation is depicted in figure 6.10.

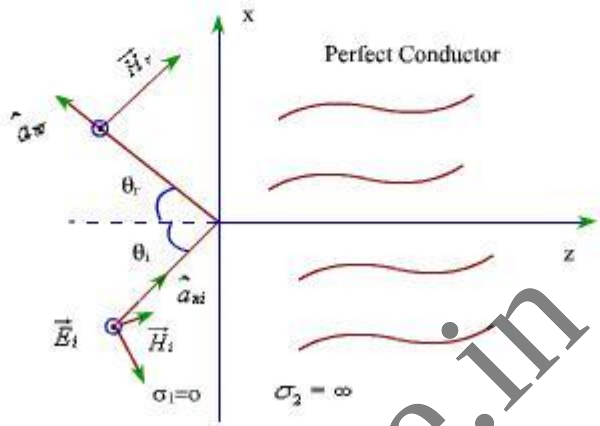


Figure 6.10: Perpendicular Polarization

As the EM field inside the perfect conductor is zero, the interface reflects the incident plane wave. $\hat{a}_{ni}$ and $\hat{a}_{nr}$ respectively represent the unit vector in the direction of propagation of the incident and reflected waves, θ_i is the angle of incidence and θ_r is the angle of reflection.

We find that

$$\hat{a}_{ni} = \hat{a}_z \cos\theta_i + \hat{a}_x \sin\theta_i$$
$$\hat{a}_{nr} = -\hat{a}_z \cos\theta_r + \hat{a}_x \sin\theta_r$$

Since the incident wave is considered to be perpendicular to the plane of incidence, which for the present case happens to be xz plane, the electric field has only y-component. Therefore,

$$\vec{E}_i(x,z) = \hat{a}_y E_{io} e^{-j\beta_1 \bar{a}_{ni}\cdot\bar{r}}$$
$$= \hat{a}_y E_{io} e^{-j\beta_1(x\sin\theta_i + z\cos\theta_i)}$$

The corresponding magnetic field is given by

$$\vec{H}_i(x,z) = \frac{1}{\eta_1}\left[\hat{a}_n \times \vec{E}_i(x,z)\right]$$
$$= \frac{1}{\eta_1}\left[-\cos\theta_i \hat{a}_x + \sin\theta_i \hat{a}_z\right] E_{io} e^{-j\beta_1(x\sin\theta_i + z\cos\theta_i)}$$

Similarly, we can write the reflected waves as

$$\vec{E}_r(x,z) = \hat{a}_y E_{ro} e^{-j\beta_1 \vec{a}_{nr}\cdot\vec{r}}$$
$$= \hat{a}_y E_{ro} e^{-j\beta_1(x\sin\theta_r - z\cos\theta_r)}$$

Since at the interface z=o, the tangential electric field is zero.

$$E_{io} e^{-j\beta_1 x\sin\theta_i} + E_{ro} e^{-j\beta_1 x\sin\theta_r} = 0$$

The above equation is satisfied if we have

$$E_{ro} = -E_{io}$$
$$\text{and } \theta_i = \theta_r$$

The condition $\theta_i = \theta_r$ is Snell's law of reflection.

$$\therefore \vec{E}_r(x,z) = -\hat{a}_y E_{io} e^{-j\beta_1(x\sin\theta_i - z\cos\theta_i)}$$

$$\text{and } \vec{H}_r(x,z) = \frac{1}{n_1}\left[\hat{a}_{nr} \times \vec{E}_r(x,z)\right]$$
$$= \frac{E_{io}}{n_1}\left[-\hat{a}_x \cos\theta_i - \hat{a}_z \sin\theta_i\right] e^{-j\beta_1(x\sin\theta_i - z\cos\theta_i)}$$

The total electric field is given by

$$\vec{E}_1(x,z) = \vec{E}_i(x,z) + \vec{E}_r(x,z)$$
$$= -\hat{a}_y 2jE_{io}\sin(\beta_1 z\cos\theta_i)e^{-j\beta_1 x\sin\theta_i}$$

Similarly, total magnetic field is given by

$$\vec{H}_1(x,z) = -2\frac{E_{io}}{n_1}\left[\hat{a}_x \cos\theta_i \cos(\beta_1 z\cos\theta_i) e^{-j\beta_1 x\sin\theta_i} + \hat{a}_z j\sin\theta_i \sin(\beta_1 z\cos\theta_i) e^{-j\beta_1 x\sin\theta_i}\right]$$

From above two equations we observe that

1. Along z direction i.e. normal to the boundary

   y component of $\vec{E}$ and x component of $\vec{H}$ maintain standing wave patterns according to $\sin\beta_{1z}z$ and $\cos\beta_{1z}z$ where $\beta_{1z} = \beta_1\cos\theta_i$. No average power propagates along z as y component of $\vec{E}$ and x component of $\vec{H}$ are out of phase.

2. Along x i.e. parallel to the interface

   y component of $\vec{E}$ and z component of $\vec{H}$ are in phase (both time and space) and propagate with phase velocity

$$v_{p1x} = \frac{\omega}{\beta_{1x}} = \frac{\omega}{\beta_1 \sin\theta_i}$$
$$\text{and } \lambda_{1x} = \frac{2\pi}{\beta_{1x}} = \frac{\lambda_1}{\sin\theta_i}$$

The wave propagating along the x direction has its amplitude varying with z and hence constitutes a **non uniform**plane wave. Further, only electric field is perpendicular to the direction of propagation (i.e. x), the magnetic field has component along the direction of propagation. Such waves are called transverse electric or TE waves.

ii. **Parallel Polarization**:

In this case also $\hat{a}_{ni}$ and $\hat{a}_{nr}$ are given by the derived equations. Here $\vec{H}_i$ and $\vec{H}_r$ have only y component.

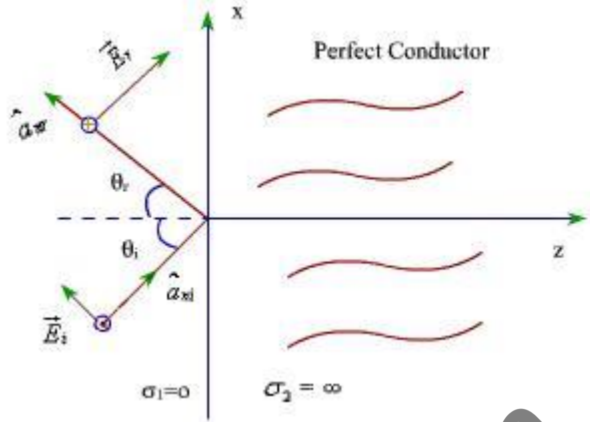


Figure 6.11: Parallel Polarization

With reference to fig (6.11), the field components can be written as:

Incident field components:

$$\vec{E}_i(x,z) = E_{io}\left[\cos\theta_i \hat{a}_x - \sin\theta_i \hat{a}_z\right] e^{-j\beta_1(x\sin\theta_i + z\cos\theta_i)}$$

$$\vec{H}_i(x,z) = \hat{a}_y \frac{E_{io}}{n_1} e^{-j\beta_1(x\sin\theta_i + z\cos\theta_i)}$$

..........................(r)

Reflected field components:

$$\vec{E}_r(x,z) = E_{ro}\left[\hat{a}_x \cos\theta_r + \hat{a}_z \sin\theta_r\right] e^{-j\beta_1(x\sin\theta_r - z\cos\theta_r)}$$

$$\vec{H}_r(x,z) = -\hat{a}_y \frac{E_{ro}}{n_1} e^{-j\beta_1(x\sin\theta_r - z\cos\theta_r)}$$

Since the total tangential electric field component at the interface is zero.

$$E_i(x,0) + E(x,0) = 0$$

Which leads to $E_{io} = -E_{ro}$ and $\theta_i = \theta_r$ as before.

Substituting these quantities in (r) and adding the incident and reflected electric and magnetic field components the total electric and magnetic fields can be written as

$$\vec{E}_i(x,z) = -2E_{io}\left[\hat{a}_x j\cos\theta_i \sin(\beta_1 z\cos\theta_i) + \hat{a}_z \sin\theta_i \cos(\beta_1 z\cos\theta_i)\right] e^{-j\beta_1 x\sin\theta_i}$$

and $$\vec{H}_i(x,z) = \hat{a}_y \frac{2E_{io}}{n_1}\cos(\beta_1 z\cos\theta_i) e^{-j\beta_1 x\sin\theta_i}$$

Once again, we find a standing wave pattern along z for the x and y components of $\vec{E}$ and $\vec{H}$, while a non uniform plane wave propagates along x with a phase velocity given by $v_{p1x} = \dfrac{v_{p1}}{\sin\theta_i}$

where $v_{p1} = \frac{\omega}{\beta_1}$. Since, for this propagating wave, magnetic field is in transverse direction, such waves are called transverse magnetic or TM waves.

Oblique incidence at a plane dielectric interface

We continue our discussion on the behavior of plane waves at an interface; this time we consider a plane dielectric interface. As earlier, we consider the two specific cases, namely parallel and perpendicular polarization.

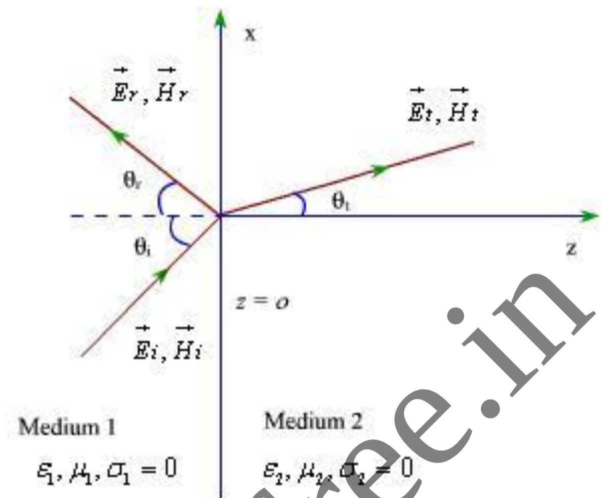


Fig 6.12: Oblique incidence at a plane dielectric interface

For the case of a plane dielectric interface, an incident wave will be reflected partially and transmitted partially.

In Fig(6.12), θ_i, θ_0 and θ_t corresponds respectively to the angle of incidence, reflection and transmission.

1. Parallel Polarization

As discussed previously, the incident and reflected field components can be written as

$$\vec{E}_i(x,z) = E_{io}\left[\cos\theta_i \hat{a}_x - \sin\theta_i \hat{a}_z\right] e^{-j\beta_1(x\sin\theta_i + z\cos\theta_i)}$$

$$\vec{H}_i(x,z) = \hat{a}_y \frac{E_{io}}{\eta_1} e^{-j\beta_1(x\sin\theta_i + z\cos\theta_i)}$$

$$\vec{E}_r(x,z) = E_{ro}\left[\hat{a}_x \cos\theta_r + \hat{a}_z \sin\theta_r\right] e^{-j\beta_1(x\sin\theta_r - z\cos\theta_r)}$$

$$\vec{H}_r(x,z) = -\hat{a}_y \frac{E_{ro}}{\eta_1} e^{-j\beta_1(x\sin\theta_r - z\cos\theta_r)}$$

In terms of the reflection coefficient Γ

$$\vec{E}_r(x,z) = \Gamma E_{io}\left[\hat{a}_x \cos\theta_r + \hat{a}_z \sin\theta_r\right] e^{-j\beta_1(x\sin\theta_r - z\cos\theta_r)}$$

$$\vec{H}_r(x,z) = -\hat{a}_y \frac{\Gamma E_{io}}{\eta_1} e^{-j\beta_1(x\sin\theta_r - z\cos\theta_r)}$$

The transmitted filed can be written in terms of the transmission coefficient T

$$\vec{E}_t(x,z) = TE_{io}\left[\hat{a}_x \cos\theta_t - \hat{a}_z \sin\theta_t\right] e^{-j\beta_2(x\sin\theta_t + z\cos\theta_t)}$$

$$\vec{H}_t(x,z) = \hat{a}_y \frac{TE_{io}}{\eta_2} e^{-j\beta_2(x\sin\theta_t + z\cos\theta_t)}$$

We can now enforce the continuity of tangential field components at the boundary i.e. *z=0*

$$\cos\theta_i e^{-j\beta_1 x\sin\theta_i} + \Gamma\cos\theta_r e^{-j\beta_1 x\sin\theta_r} = T\cos\theta_t e^{-j\beta_2 x\sin\theta_t}$$

$$\text{and } \frac{1}{\eta_1} e^{-j\beta_1 x\sin\theta_i} - \frac{\Gamma}{\eta_1} e^{-j\beta_1 x\sin\theta_r} = \frac{T}{\eta_2} e^{-j\beta_2 x\sin\theta_t} \quad \text{..........................(s)}$$

If both E_x and H_y are to be continuous at *z=0* for all *x* , then form the phase matching we have

$$\beta_1\sin\theta_i = \beta_1\sin\theta_r = \beta_2\sin\theta_t$$

∴ We find that

$$\theta_i = \theta_r$$

$$\text{and } \beta_1\sin\theta_i = \beta_2\sin\theta_t \quad \text{..........................(t)}$$

Further, from equations (s) and (t) we have

$$\cos\theta_i + \Gamma\cos\theta_i = T\cos\theta_t$$

$$\text{and } \frac{1}{\eta_1} - \frac{\Gamma}{\eta_1} = \frac{T}{\eta_2}$$

$$\therefore \quad \cos\theta_i(1+\Gamma) = T\cos\theta_t$$

$$\text{and } \frac{1}{\eta_1}(1-\Gamma) = \frac{T}{\eta_2}$$

$$\therefore T = \frac{\eta_2}{\eta_1}(1-\Gamma)$$

$$\cos\theta_i(1+\Gamma) = \frac{\eta_2}{\eta_1}(1-\Gamma)\cos\theta_t$$

$$\therefore (\eta_1\cos\theta_i + \eta_2\cos\theta_t)\Gamma = \eta_2\cos\theta_t - \eta_1\cos\theta_i$$

or

$$\Gamma = \frac{\eta_2\cos\theta_t - \eta_1\cos\theta_i}{\eta_2\cos\theta_t + \eta_1\cos\theta_i}$$

$$\text{and } T = \frac{\eta_2}{\eta_1}(1-\Gamma)$$

$$= \frac{2\eta_2\cos\theta_i}{\eta_2\cos\theta_t + \eta_1\cos\theta_i} \quad \text{..........................(u)}$$

From equation (u) we find that there exists specific angle $\theta_i = \theta_b$ for which $\Gamma = 0$ such that

$$\eta_2\cos\theta_t = \eta_1\cos\theta_b$$

or

$$\sqrt{1-\sin^2\theta_t} = \frac{\eta_1}{\eta_2}\sqrt{1-\sin^2\theta_b}$$

Further,

$$\sin\theta_t = \frac{\beta_1}{\beta_2}\sin\theta_b$$

For non magnetic material $\mu_1 = \mu_2 = \mu_0$
Using this condition

$$1-\sin^2\theta_t = \frac{\varepsilon_1}{\varepsilon_2}\left(1-\sin^2\theta_b\right)$$

$$\text{and } \sin^2\theta_t = \frac{\varepsilon_1}{\varepsilon_2}\sin^2\theta_b \qquad \text{.......................(v)}$$

rom equation (v), solving for $\sin\theta_b$ we get

$$\sin\theta_b = \frac{1}{\sqrt{1+\frac{\varepsilon_1}{\varepsilon_2}}}$$

This angle of incidence for which Γ = *0* is called Brewster angle. Since we are dealing with parallel polarization we represent this angle by $\theta_{b\|}$ so that

$$\sin\theta_{b\|} = \frac{1}{\sqrt{1+\frac{\varepsilon_1}{\varepsilon_2}}}$$

2. **Perpendicular**

Polarization For this case

$$\vec{E}_i(x,z) = \hat{a}_y E_{io} e^{-j\beta_1(x\sin\theta_i + z\cos\theta_i)}$$

$$\vec{H}_i(x,z) = \frac{E_{io}}{\eta_1}\left[-\hat{a}_x\cos\theta_i + \hat{a}_z\sin\theta_i\right]e^{-j\beta_1(x\sin\theta_i + z\cos\theta_i)}$$

$$\vec{E}_r(x,z) = \hat{a}_y \Gamma E_{io} e^{-j\beta_1(x\sin\theta_r - z\cos\theta_r)}$$

$$\vec{H}_r(x,z) = \frac{\Gamma E_{io}}{\eta_1}\left[\hat{a}_x\cos\theta_r + \hat{a}_z\sin\theta_r\right]e^{-j\beta_1(x\sin\theta_r - z\cos\theta_r)}$$

$$\vec{E}_t(x,z) = \hat{a}_y T E_{io} e^{-j\beta_1(x\sin\theta_t + z\cos\theta_t)}$$

$$\vec{H}_t(x,z) = \frac{TE_{io}}{\eta_2}\left[-\hat{a}_x\cos\theta_t + \hat{a}_z\sin\theta_t\right]e^{-j\beta_2(x\sin\theta_t + z\cos\theta_t)}$$

Using continuity of field components at *z=0*

$$e^{-j\beta_1 x\sin\theta_i} + \Gamma e^{-j\beta_1 x\sin\theta_r} = TE_{io}e^{-j\beta_2 x\sin\theta_t}$$

$$\text{and } -\frac{1}{\eta_1}\cos\theta_i e^{-j\beta_1 x\sin\theta_i} + \frac{\Gamma}{\eta_1}\cos\theta_r e^{-j\beta_1 x\sin\theta_r} = -\frac{T}{\eta_2}\cos\theta_t e^{-j\beta_2 x\sin\theta_t}$$

As in the previous case

$$\beta_1\sin\theta_i = \beta_1\sin\theta_r = \beta_2\sin\theta_t$$

$$\therefore \quad \theta_i = \theta_r$$

$$\text{and } \sin\theta_t = \frac{\beta_1}{\beta_2}\sin\theta_i$$

Using these conditions we can write

$$1 + \Gamma = T$$

$$-\frac{\cos\theta_i}{n_1} + \frac{\Gamma\cos\theta_i}{n_1} = -\frac{T\cos\theta_t}{n_2} \qquad \text{........................(w)}$$

From equation (w) the reflection and transmission coefficients for the perpendicular polarization can be computed as

$$\Gamma = \frac{n_2\cos\theta_i - n_1\cos\theta_t}{n_2\cos\theta_i + n_1\cos\theta_t}$$

$$\text{and} \quad T = \frac{2n_2\cos\theta_i}{n_2\cos\theta_i + n_1\cos\theta_t}$$

We observe that if $\Gamma = 0$ for an angle of incidence $\theta_i = \theta_b$

$$n_2\cos\theta_b = n_1\cos\theta_t$$

$$\therefore \cos^2\theta_t = \frac{n_2}{n_1}\cos^2\theta_b$$

$$= \frac{\mu_2\varepsilon_1}{\mu_1\varepsilon_2}\cos^2\theta_b$$

$$\therefore 1 - \sin^2\theta_t = \frac{\mu_2\varepsilon_1}{\mu_1\varepsilon_2}\left(1 - \sin^2\theta_b\right)$$

Again

$$\sin\theta_t = \frac{\beta_1}{\beta_2}\sin\theta_b$$

$$\therefore \sin^2\theta_t = \frac{\mu_1\varepsilon_1}{\mu_2\varepsilon_2}\sin^2\theta_b$$

$$\therefore \left(1 - \frac{\mu_1\varepsilon_1}{\mu_2\varepsilon_2}\sin^2\theta_b\right) = \frac{\mu_2\varepsilon_1}{\mu_1\varepsilon_2} - \frac{\mu_2\varepsilon_1}{\mu_1\varepsilon_2}\sin^2\theta_b$$

or

$$\sin^2\theta_b\left(\frac{\mu_1\varepsilon_1}{\mu_2\varepsilon_2} - \frac{\mu_2\varepsilon_1}{\mu_1\varepsilon_2}\right) = \left(1 - \frac{\mu_2\varepsilon_1}{\mu_1\varepsilon_2}\right)$$

or

$$\sin^2\theta_b\left(\frac{\mu_1^2 - \mu_2^2}{\mu_1\mu_2\varepsilon_2}\right)\varepsilon_1 = \left(\frac{\mu_1\varepsilon_2 - \mu_2\varepsilon_1}{\mu_1\varepsilon_2}\right)$$

or

$$\sin^2\theta_b = \frac{\mu_2\left(\mu_1\varepsilon_2 - \mu_2\varepsilon_1\right)}{\varepsilon_1\left(\mu_1^2 - \mu_2^2\right)} \qquad \text{........................(x)}$$

We observe if $\mu_1 = \mu_2 = \mu_0$ i.e. in this case of non magnetic material Brewster angle does not exist as the denominator or equation (x) becomes zero. Thus for perpendicular polarization in dielectric media, there is Brewster angle so that Γ can be made equal to zero.

From our previous discussion we observe that for both polarizations

$$\sin\theta_t = \frac{\beta_1}{\beta_2}\sin\theta_i$$

If $\mu_1 = \mu_2 = \mu_0$

$$\sin\theta_t = \sqrt{\frac{\varepsilon_1}{\varepsilon_2}}\sin\theta_i$$

For $\varepsilon_1 > \varepsilon_2$; $\theta_t > \theta_i$

The incidence angle $\theta_i = \theta_c$ for which $\theta_t = \frac{\pi}{2}$ i.e. $\theta_c = \sin^{-1}\sqrt{\frac{\varepsilon_2}{\varepsilon_1}}$ is called the critical angle of incidence. If the angle of incidence is larger than θ_c total internal reflection occurs. For such case an evanescent wave exists along the interface in the x direction (w.r.t. fig (6.12)) that attenuates exponentially in the normal i.e. z direction. Such waves are tightly bound to the interface and are called surface waves.

QUESTIONS:

1.Write down Maxwell's field equations in the differential and integral form for time harmonic fields
2.Derive the expressions for energy stored in electric and magnetic field. Which field is efficient.

3.In a uniform plane wave, E and H are at right angles to each other. Prove.
4.A lossy dielectric is characterized by $_R$=1.5, $_R$=1 and / =2.5x10^{-4}. At a frequency of 200MHz, how far can a uniform plane wave propagate in the material before

(i)it undergoes an attenuation 1Np

(ii)its amplitude is halved

5. Deduce the integral form of the theoram of Poynting and state the significance of the three terms appearing in the equation.

6.What are the properties of uniform plane wave?

7.Write Maxwell's equation in integral form and interpret

8.Show that characteristic impedance of free space is 377ohm

9.State and explain Poynting Vector(P) and Poynting theorem.

10.A brass(conductivity=10^{7}mho/m) pipe with inner and outer diameter of 3.4 and 4 cm carries a total current of 100A dc. Find Electric field (E), Magnetic field(H) and Poynting Vector(P) within the brass

TIME VARYING MAGNETIC FIELDS AND MAXWELL'S EQUATIONS

Introduction

Electrostatic fields are usually produced by static electric charges whereas magnetostatic fields are due to motion of electric charges with uniform velocity (direct current) or static magnetic charges (magnetic poles); time-varying fields or waves are usually due to accelerated charges or time-varying current.

- Stationary charges → Electrostatic fields
- Steady current → Magnetostatic fields
- Time-varying current → Electromagnetic fields (or waves)

Faraday discovered that the induced emf, V_{emf} (in volts), in any closed circuit is equal to the time rate of change of the magnetic flux linkage by the circuit

This is called Faraday's Law, and it can be expressed as

$$V_{emf} = -\frac{d\lambda}{dt} = -N\frac{d\Psi}{dt} \qquad 1.1$$

where N is the number of turns in the circuit and ψ is the flux through each turn. The negative sign shows that the induced voltage acts in such a way as to oppose the flux producing it. This is known as Lenz's Law, and it emphasizes the fact that the direction of current flow in the circuit is such that the induced magnetic filed produced by the induced current will oppose the original magnetic field.

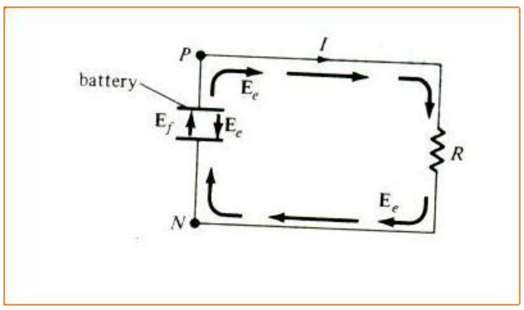


Fig. 1 A circuit showing emf-producing field E_f and electrostatic field E_e

Having considered the connection between emf and electric field, we may examine how Faraday's law links electric and magnetic fields. For a circuit with a single (N = 1), eq. (1.1) becomes

$$V_{emf} = -N \frac{d\Psi}{dt} \qquad 1.2$$

In terms of **E** and **B,** eq. (1.2) can be written as

$$V_{emf} = \oint_L E \cdot dl = -\frac{d}{dt}\int_S B \cdot dS \qquad 1.3$$

where, ψ has been replaced by $\int_S B \cdot dS$ and S is the surface area of the circuit bounded by the closed path L. It is clear from eq. (1.3) that in a time-varying situation, both electric and magnetic fields are present and are interrelated. Note that d**l** and d**S** in eq. (1.3) are in accordance with the right-hand rule as well as Stokes's theorem. This should be observed in Figure 2. The variation of flux with time as in eq. (1.1) or eq. (1.3) may be caused in three ways:

1. By having a stationary loop in a time-varying **B** field
2. By having a time-varying loop area in a static **B** field
3. By having a time-varying loop area in a time-varying **B** field.

A. STATIONARY LOOP IN TIME-VARYING B FIELD (TRANSFORMER EMF)

This is the case portrayed in Figure 2 where a stationary conducting loop is in a time varying magnetic **B** field. Equation (1.3) becomes

$$V_{emf} = \oint_L E \cdot dl = -\int_S \frac{\partial B}{\partial t} \cdot dS \qquad 1.4$$

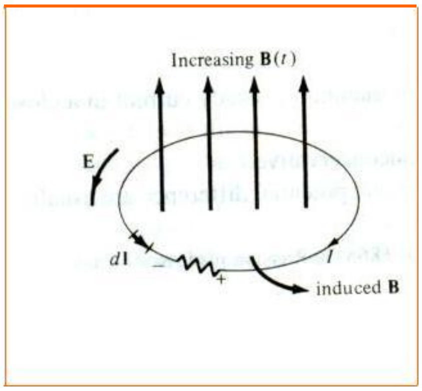


Fig. 2: Induced emf due to a stationary loop in a time varying **B** field.

This emf induced by the time-varying current (producing the time-varying **B** field) in a stationary loop is often referred to as *transformer emf* in power analysis since it is due to transformer action. By applying Stokes's theorem to the middle term in eq. (1.4), we obtain

$$\int_S (\nabla \times E) \cdot dS = -\int_S \frac{\partial B}{\partial t} \cdot dS \qquad 1.5$$

For the two integrals to be equal, their integrands must be equal; that is,

$$\nabla \times E = -\frac{\partial B}{\partial t} \qquad 1.6$$

This is one of the Maxwell's equations for time-varying fields. It shows that the time varying E field is not conservative (∇ x E ≠ 0). This does not imply that the principles of energy conservation are violated. The work done in taking a charge about a closed path in a time-varying electric field, for example, is due to the energy from the time-varying magnetic field.

B. MOVING LOOP IN STATIC B FIELD (MOTIONAL EMF)

When a conducting loop is moving in a static **B** field, an emf is induced in the loop. We recall from eq. (1.7) that the force on a charge moving with uniform velocity **u** in a magnetic field **B** is

$$\mathbf{F_m} = Qu \times \mathbf{B} \qquad 1.7$$

We define the *motional electric field* E_m as

$$E_m = \frac{F_m}{Q} = u \times B \qquad 1.8$$

If we consider a conducting loop, moving with uniform velocity **u** as consisting of a large number of free electrons, the emf induced in the loop is

$$V_{emf} = \oint_L E_m \cdot dl = \oint_L (u \times B) \cdot dl \qquad 1.9$$

This type of emf is called *motional emf or flux-cutting emf* because it is due to motional action. It is the kind of emf found in electrical machines such as motors, generators, and alternators.

C. MOVING LOOP IN TIME-VARYING FIELD

This is the general case in which a moving conducting loop is in a time-varying magnetic field. Both transformer emf and motional emf are present. Combining equation 1.4 and 1.9 gives the total emf as

$$V_{emf} = \oint_L E \cdot dl = -\int_S \frac{\partial B}{\partial t} \cdot dS + \oint_L (u \times B) \cdot dl \qquad 1.10$$

$$\nabla \times E_m = \nabla \times (u \times B) \qquad 1.11$$

or from equations 1.6 and 1.11.

$$\nabla \times E = -\frac{\partial B}{\partial t} + \nabla \times (u \times B) \qquad 1.12$$

DISPLACEMENT CURRENT

For static EM fields, we recall that

$$\nabla \times \mathbf{H} = \mathbf{J} \qquad 1.13$$

But the divergence of the curl of any vector field is identically zero.

Hence,

$$\nabla \cdot (\nabla \times H) = 0 = \nabla \cdot J \qquad 1.14$$

The continuity of current requires that

$$\nabla \cdot J = -\frac{\partial \rho_v}{\partial t} \neq 0 \qquad 1.15$$

Thus eqs. 1.14 and 1.15 are obviously incompatible for time-varying conditions. We must modify eq. 1.13 to agree with eq. 1.15. To do this, we add a term to eq. 1.13, so that it becomes

$$\nabla \times H = J + J_d \qquad 1.16$$

where J_d is to be determined and defined. Again, the divergence of the curl of any vector is zero. Hence:

$$\nabla \cdot (\nabla \times H) = 0 = \nabla \cdot J + \nabla \cdot J_d \qquad 1.17$$

In order for eq. 1.17 to agree with eq. 1.15,

$$\nabla \cdot J_d = -\nabla \cdot J = \frac{\partial \rho_v}{\partial t} = \frac{\partial}{\partial t}(\nabla \cdot D) = \nabla \cdot \frac{\partial D}{\partial t} \qquad 1.18$$

or

$$J_d = \frac{\partial D}{\partial t} \qquad 1.19$$

Substituting eq. 1.19 into eq. 1.15 results in

$$\nabla \times H = J + \frac{\partial D}{\partial t} \qquad 1.20$$

This is Maxwell's equation (based on Ampere's circuit law) for a time-varying field. The term $J_d = \partial D/\partial t$ is known as *displacement current density and* J is the conduction current density ($J = \sigma E$)[3].

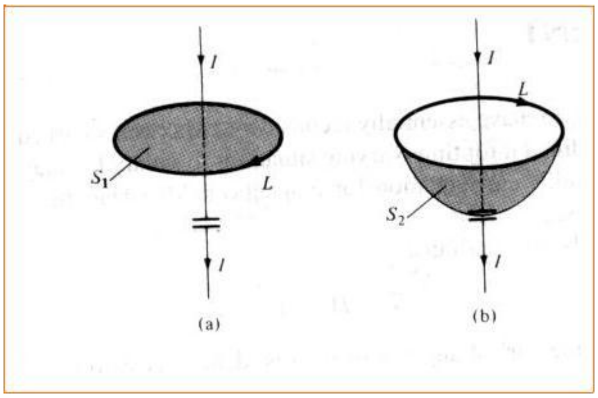


Fig. 3 Two surfaces of integration showing the need for J_d in Ampere's circuit law

The insertion of J_d into eq. 1.13 was one of the major contribution of Maxwell. Without the term J_d, electromagnetic wave propagation (radio or TV waves, for example) would be impossible. At low frequencies, J_d is usually neglected compared with J. however, at radio frequencies, the two terms are comparable. At the time of Maxwell, high-frequency sources were not available and eq. 1.20 could not be verified experimentally.

Based on displacement current density, we define the displacement current as

$$I_d = \int J_d \cdot dS = \int \frac{\partial D}{\partial t} \cdot dS \qquad 1.21$$

We must bear in mind that displacement current is a result of time-varying electric field. A typical example of such current is that through a capacitor when an alternating voltage source is applied to its plates.

PROBLEM: A parallel-plate capacitor with plate area of 5 cm^2 and plate separation of 3 mm has a voltage 50 sin 10^3 t V applied to its plates. Calculate the displacement current assuming $\varepsilon = 2\,\varepsilon_0$.

Solution:

$$D = \varepsilon E = \varepsilon \frac{V}{d}$$

$$J_d = \frac{\partial D}{\partial t} = \frac{\varepsilon}{d} \frac{dV}{dt}$$

Hence,

$$I_d = J_d \cdot S = \frac{\varepsilon S}{d} \frac{dV}{dt} = C \frac{dV}{dt}$$

which is the same as the conduction current, given by

$$I_c = \frac{dQ}{dt} = S \frac{d\rho_s}{dt} = S \frac{dD}{dt} = \varepsilon S \frac{dE}{dt} = \frac{\varepsilon S}{d} \frac{dV}{dt} = C \frac{dV}{dt}$$

$$I_d = 2 \cdot \frac{10^{-9}}{36\pi} \cdot \frac{5 \times 10^{-4}}{3 \times 10^{-3}} \cdot 10^3 \times 50 \cos 10^3 t$$

$= 147.4 \cos 10^3 \, t$ nA

EQUATION OF CONTINUITY FOR TIME VARYING FIELDS

Equation of continuity in point form is

$$\nabla \cdot J = -\rho_v$$

where,

J = conduction current density (A/M^2)

Pυ = volume charge density (C/M^3), $\rho_v = \frac{\partial \rho_v}{\partial t}$

∇ = vector differential operator (1/m)

$$\nabla = a_x \frac{\partial}{\partial x} + a_y \frac{\partial}{\partial y} + a_z \frac{\partial}{\partial z}$$

Proof: Consider a closed surface enclosing a charge *Q*. There exists an outward flow of current given by

$$I = \oint_S J \cdot dS$$

This is equation of continuity in **integral form.**

From the principle of conservation of charge, we have

$$I = \oint_S J \cdot dS = \frac{-dQ}{dt}$$

From the divergence theorem, we have

$$I = \oint_S J \cdot dS = \int_v (\nabla \cdot J) d\upsilon$$

Thus, $$\int_\upsilon (\nabla \cdot J) d\upsilon = \frac{-dQ}{dt}$$

By definition, $Q = \int_{\upsilon} \rho_{\upsilon}\, d\upsilon$

where, ρ_{υ} = volume charge density (C/m^3)

So, $\int_{\upsilon} (\nabla \cdot J) d\upsilon = \int \frac{\partial \rho_{\upsilon}}{\partial t} d\upsilon = \int - \dot{\rho}_{\upsilon}\, d\upsilon$

where $\dot{\rho}_{\upsilon} = \frac{\partial \rho_{\upsilon}}{\partial t}$

The volume integrals are equal only if their integrands are equal.

Thus, $\nabla \cdot \mathbf{J} = - \dot{\rho}_{\upsilon}$

MAXWELL'S EQUATIONS FOR STATIC EM FIELDS

Differential (or Point) Form	Integral Form	Remarks
$\nabla \cdot \mathbf{D} = \rho_v$	$\oint_S D \cdot dS = \int_v \rho_v\, dv$	Gauss's law
$\nabla \cdot \mathbf{B} = 0$	$\oint_S B \cdot dS = 0$	Nonexistence of magnetic monopole
$\nabla \times \mathbf{E} = -\frac{\partial B}{\partial t}$	$\oint_L E \cdot dl = -\frac{\partial}{\partial t}\int_s B \cdot dS$	Faraday's Law
$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial D}{\partial t}$	$\oint_L H \cdot dl = \int_s J \cdot dS$	Ampere's circuit law

MAXWELL'S EQUATIONS FOR TIME VARYING FIELDS

These are basically four in number.

Maxwell's equations in **differential form** are given by

$$\nabla \times H = \frac{\partial D}{\partial t} + J$$

$$\nabla \times E = -\frac{\partial B}{\partial t}$$

∂t

$\nabla . D = \rho_\upsilon$

$\nabla . B = 0$

Here,

H = magnetic field strength (A/m)

D = electric flux density, (C/m^2)

($\partial D/\partial t$) = displacement electric current density

(A/m^2) J = conduction current density (A/m^2)

E = electric field (V/m)

B = magnetic flux density wb/m^2 or Tesla

($\partial B/\partial t$) = time-derivative of magnetic flux density (wb/m^2 -

sec) B is called magnetic current density (V/m^2) or Tesla/sec

P_υ = volume charge density (C/m^3)

Maxwell's equations for time varying fields in **integral form** are given by

$$\oint_L H \cdot dL = \int_S \left(\dot{D} + J \right) \cdot dS$$

$$\oint_L E \cdot dL = -\int_S \dot{B} \cdot dS$$

$$\oint_S D \cdot dS = \int_\upsilon \rho_\upsilon \, d\upsilon$$

$$\oint_S B \cdot dS = 0$$

MEANING OF MAXWELL'S EQUATIONS

1. The first Maxwell's equation states that the magnetomotive force around a closed path is equal to the sum of electric displacement and, conduction currents through any surface bounded by the path.

2. The second law states that the electromotive force around a closed path is equal to the inflow of magnetic current through any surface bounded by the path.

3. The third law states that the total electric displacement flux passing through a closed surface (Gaussian surface) is equal to the total charge inside the surface.

4. The fourth law states that the total magnetic flux passing through any closed surface is zero.

MAXWELL'S EQUATIONS FOR STATIC FIELDS

Maxwell's Equations for static fields are:

$$\nabla\times H = J \leftrightarrow \oint_L H \cdot dL = \int_S J \cdot dS$$

$$\nabla\times E = 0 \leftrightarrow \oint_L E \cdot dL = 0$$

$$\nabla\cdot D = \rho_\upsilon \leftrightarrow \oint_S D \cdot dS = \oint_\upsilon \rho_\upsilon \, d\upsilon$$

$$\nabla\cdot B = 0 \leftrightarrow \oint_S B \cdot dS = 0$$

As the fields are static, all the field terms which have time derivatives are zero, that is, $\frac{\partial D}{\partial t} = 0$, $\frac{\partial B}{\partial t} = 0$.

PROOF OF MAXWELLS EQUATIONS

1. From Ampere's circuital law, we have

$$\nabla \times H = J$$

Take dot product on both sides

$$\nabla \cdot \nabla \times H = \nabla \cdot J$$

As the divergence of curl of a vector is zero,

$$RHS = \nabla \cdot J = 0$$

But the equation of continuity in point form is

$$\nabla\cdot J = \frac{-\partial\rho_\upsilon}{\partial t} = -\rho_\upsilon$$

This means that if $\nabla \times H = J$ is true, it is resulting in $\nabla \cdot J = 0$.

As the equation of continuity is more fundamental, Ampere's circuital law should be modified. Hence we can write

$$\nabla \times H = J + F$$ Take

dot product on both sides

$$\nabla \cdot \nabla \times H = \nabla \cdot J + \nabla \cdot F$$

that is, $\nabla \cdot \nabla \times H = 0 = \nabla \cdot J + \nabla \cdot F$

Substituting the value of $\nabla \cdot J$ from the equation of continuity in the above expression, we get

$$\nabla \cdot F + (-\rho_\upsilon) = 0$$

or, $\nabla \cdot F = -\rho_\upsilon$

The point form of Gauss's law is

$$\nabla \cdot D = \rho_\upsilon$$

or, $\nabla \cdot D = \rho_\upsilon$

From the above expressions, we get

$$\nabla \cdot F = \nabla \cdot D$$

The divergence of two vectors are equal only if the vectors are identical,

that is, $F = D$

So, $\nabla \times H = D + J$

Hence proved.

2. According to Faraday's law,

$$emf = \frac{-d\phi}{dt}$$

ϕ = magnetic flux, (wb)

and by definition,

$$emf = \oint_L E \cdot dL$$

$$\oint_L E \cdot dL = \frac{-d\phi}{dt}$$

But $$\phi = \int_S B \cdot dS$$

$$\oint_L E \cdot dL = -\int_S \frac{\partial B}{\partial t} \cdot dS$$

$$= -\int_S B \cdot dS, \quad B = \frac{\partial B}{\partial t}$$

Applying Stoke's theorem to LHS, we get

$$\oint_L E \cdot dL = -\int_S (\nabla \times E) \cdot dS$$

$$\int_S (\nabla \times E) \cdot dS = \int_S -B \cdot dS$$

Two surface integrals are equal only if their integrands are equal,

that is, $\nabla \times E = -B$

Hence proved.

3. From Gauss's law in electric field, we have

$$\oint_S D \cdot dS = Q = \int_\upsilon \rho_\upsilon \, d\upsilon$$

Applying divergence theorem to LHS, we get

$$\oint_S D \cdot dS = \int_\upsilon (\nabla \cdot D) d\upsilon = \int_\upsilon \rho_\upsilon \, d\upsilon$$

Two volume integrals are equal if their integrands are equal,

that is, $\nabla \cdot D = \rho_\upsilon$

Hence proved.

4. We have Gauss's law for magnetic fields as

$$\oint_S B \cdot dS = 0$$

RHS is zero as there are no isolated magnetic charges and the magnetic flux lines are closed loops.

Applying divergence theorem to LHS, we get

$$\oint_\upsilon \nabla \cdot B \, d\upsilon = 0$$

or,

$\nabla \cdot B = 0$ Hence proved.

PROBLEM 1:

Given $E = 10 \sin(\omega t - \beta y)\, a_y$ V/m, in free space, determine D, B and H.

Solution:

$E = 10 \sin(\omega t - \beta y)\, a_y$, V/m

$D = \epsilon_0 E$, $\epsilon_0 = 8.854 \times 10^{-12}$ F/m

$D = 10\epsilon_0 \sin(\omega t - \beta y)\, a_y$, C/m^2

Second Maxwell's equation is

$\nabla \times E = -B$

That is,
$$\nabla \times E = \begin{vmatrix} a_x & a_y & a_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & E_y & 0 \end{vmatrix}$$

or,
$$\nabla \times E = a_x\left[-\frac{\partial}{\partial z} E_y\right] + 0 + a_z\left[\frac{\partial}{\partial x} E_y\right]$$

As $E_y = 10 \sin(\omega t - \beta z)$ V/m

$$\frac{\partial E_y}{\partial x} = 0$$

Now, $\nabla \times E$ becomes

$$\nabla \times E = -\frac{\partial E_y}{\partial z} a_x$$

$= 10\, \beta \cos(\omega t - \beta z)\, a_x$

$$= -\frac{\partial B}{\partial t}$$

$$B = -\int 10\beta \cos(\omega t - \beta z)\, dt\, a_x$$

or
$$B = \frac{10\beta}{\omega} \sin(\omega t - \beta z) a_z, \; wb/m^2$$

and
$$H = \frac{B}{\mu_0} = \frac{10\beta}{\mu_0 \omega} \sin(\omega t - \beta z) a_z, \; A/m$$

PROBLEM 2: If the electric field strength, E of an electromagnetic wave in free space is given by $E = 2\cos\omega\left(t-\frac{z}{\upsilon_0}\right)$ V/m, find the magnetic field, H.

Solution: We have

$$\partial B/\partial t = -\nabla \times E$$

$$= \begin{vmatrix} a_x & a_y & a_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & E_y & 0 \end{vmatrix}$$

$$= -\left[a_x\left[-\frac{\partial E_y}{\partial z}\right] + a_y(0) + a_z\left[\frac{\partial E_y}{\partial x}\right]\right]$$

$$= \frac{\partial E_y}{\partial z} a_x$$

$$= \frac{2\omega}{\upsilon_0}\sin\omega\left(t-\frac{z}{\upsilon_0}\right)a_x$$

$$B = \frac{2\omega}{\upsilon_0}\int \sin\omega\left(t-\frac{z}{\upsilon_0}\right)dt\, a_x$$

or, $$B = \frac{-2\omega}{\upsilon_0\omega}\cos\omega\left(t-\frac{z}{\upsilon_0}\right)$$

or, $$H = \frac{B}{\mu_0} = \frac{-2}{\upsilon_0\mu_0}\cos\omega\left(t-\frac{z}{\upsilon_0}\right) \qquad \eta_0 = \sqrt{\frac{\mu_0}{\epsilon_0}} = 120\pi\Omega$$

Thus, $$H = \frac{-2}{\eta_0}\cos\omega\left(t-\frac{z}{\upsilon_0}\right) \qquad \left[\upsilon_0 = \frac{1}{\sqrt{\mu_0\epsilon_0}}\right]$$

$$H = \frac{-1}{60\pi}\cos\omega\left(t-\frac{z}{\upsilon_0}\right)a_x \quad A/m$$

PROBLEM 3: If the electric field strength of a radio broadcast signal at a TV receiver is given by

$$E = 5.0 \cos(\omega t - \beta y)\, a_z, \ V/m,$$

determine the displacement current density. If the same field exists in a medium whose conductivity is given by 2.0 x 10^3 (mho)/cm, find the conduction current density.

Solution:

E at a TV receiver in free space

$$= 5.0 \cos(\omega t - \beta y)\, a_z, \ V/m$$

Electric flux density

$$D = \epsilon_0 E = 5\, \epsilon_0 \cos(\omega t - \beta y)\, a_z, \ V/m$$

The displacement current density

$$J_d = D = \frac{\partial D}{\partial t}$$

$$= \frac{\partial}{\partial t}\left[-5 \epsilon_0 \cos(\omega t - \beta y) a_z\right]$$

$$J_d = -5\epsilon_0\, \omega \sin(\omega t - \beta y)\, a_z, \ V/m^2$$

The conduction current density,

$$J_c = \sigma E$$

$$\sigma = 2.0 \times 10^3 \text{ (mho) /cm}$$

$$= 2 \times 10^5 \text{ mho /m}$$

$$J_c = 2 \times 10^5 \times 5 \cos(\omega t - \beta y)\, a_z$$

$$\mathbf{J_c = 10^6 \cos(\omega t - \beta y)\, a_z \ V/m^2}$$

UNIFORM PLANE WAVES

In free space (source-less regions where $\rho = \vec{J} = \sigma = 0$), The wave equation for electric field, in free-space is,

$$\nabla^2 \vec{E} = \mu \in \frac{\partial^2 \vec{E}}{\partial t^2} \_\_\_\_\_\_\_\_\_ (2)$$

The wave equation (2) is a composition of these equations, one each component wise,

ie,

$$\frac{\partial^2 Ex}{\partial x^2} = \mu \in \frac{\partial^2 Ey}{\partial t^2} \_\_\_\_\_\_\_\_\_ (2)\ a$$

$$\frac{\partial^2 Ey}{\partial y^2} = \mu \in \frac{\partial^2 Ey}{\partial t^2} \_\_\_\_\_\_\_\_\_ (2)\ b$$

$$\frac{\partial^2 Ez}{\partial z^2} = \mu \in \frac{\partial^2 Ez}{\partial t^2} \_\_\_\_\_\_\_\_\_ (2)\ c$$

Further, eqn. (1) may be written as

$$\frac{\partial Ex}{\partial x} + \frac{\partial Ey}{\partial y} + \frac{\partial Ez}{\partial z} = 0 \_\_\_\_\_\_\_\_\_ (1)\ a$$

For the UPW, $\vec{E}$ is independent of two coordinate axes; x and y axes, as we have assumed.

$$\therefore \quad \frac{\partial}{\partial x} = \frac{\partial}{\partial y} = 0$$

Therefore eqn. (1) reduces to

$$\frac{\partial E_z}{\partial z} = 0 \_\_\_\_\_\_\_\_ (3)$$

ie., there is no variation of E_z in the z direction.

Also we find from 2 (a) that $\frac{\partial^2 Ez}{\partial t^2} = 0$ ____(4)

These two conditions (3) and (4) require that E_z can be

(1) Zero

(2) Constant in time or

(3) Increasing uniformly with time.

A field satisfying the last two of the above three conditions cannot be a part of wave motion. Therefore E_z can be put equal to zero, (the first condition).

$$\boxed{E_z = 0}$$

The uniform plane wave (traveling in z direction) does not have any field components of $\vec{E}$ & $\vec{H}$ in its direction of travel.

Therefore the UPWs are transverse., having field components (of $\vec{E}$ & $\vec{H}$) only in directions perpendicular to the direction of propagation does not have any field component only the direction of travel.

RELATION BETWEEN $\vec{E}$ & $\vec{H}$ in a uniform plane wave.

We have, from our previous discussions that, for a UPW traveling in z direction, both $\vec{E}$ & $\vec{H}$ are independent of x and y; and $\vec{E}$ & $\vec{H}$ have no z component. For such a UPW, we have,

$$\nabla \times \vec{E} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x}(=0) & \frac{\partial}{\partial y}(=0) & \frac{\partial}{\partial z} \\ Ex & Ey & Ez(=0) \end{vmatrix} = \hat{i}\left(-\frac{\partial E_y}{\partial_z}\right) + \hat{j}\left(\frac{\partial E_x}{\partial_z}\right) \_\_\_\_\_\_ (5)$$

$$\nabla \times \vec{H} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x}(=0) & \frac{\partial}{\partial y}(=0) & \frac{\partial}{\partial z} \\ Hx & Hy & Hz(=0) \end{vmatrix} = \hat{i}\left(-\frac{\partial H_y}{\partial_z}\right) + \hat{j}\left(\frac{\partial H_x}{\partial_z}\right) \_\_\_\_\_\_ (6)$$

Then Maxwell's curl equations (1) and (2), using (5) and (6), (2) becomes,

$$\nabla \times \vec{H} = \in \frac{\partial \vec{E}}{\partial t} = \quad \in \frac{\partial Ex}{\partial t}\,\hat{i} + \in \frac{\partial Ey}{\partial t}\,\hat{j} = \hat{i}\left(-\frac{\partial Hy}{\partial z}\right) + \hat{j}\left(\frac{\partial Hx}{\partial z}\right) \_\_\_\_\_\_ (7)$$

and

$$\nabla \times \vec{E} = -\mu \frac{\partial \vec{H}}{\partial t} = \quad \mu \frac{\partial Hx}{\partial t}\,\hat{i} - \mu \frac{\partial Hy}{\partial t}\,\hat{j} = i\left(-\frac{\partial Ey}{\partial z}\right) + \hat{j}\left(\frac{\partial Ex}{\partial z}\right) \_\_\_\_\_\_ (8)$$

Thus, rewriting (7) and (8) we get

$$-\frac{\partial Hy}{\partial z}\hat{i}+\frac{\partial Hx}{\partial z}\hat{j} \quad = \quad \in\left(\frac{\partial Ex}{\partial t}\hat{i}+\frac{\partial Ey}{\partial t}\hat{j}\right) \_\_\_\_\_\_\_\_ (7)$$

$$-\frac{\partial Ey}{\partial z}\hat{i}+\frac{\partial Ex}{\partial z}\hat{j} \quad = -\mu\left(\frac{\partial Hx}{\partial t}\hat{i}-\frac{\partial Hy}{\partial t}\hat{j}\right) \_\_\_\_\_\_\_\_ (8)$$

Equating $\hat{i}$ th and j th terms, we get

$$-\frac{\partial Hy}{\partial z} = \in\frac{\partial Ex}{\partial t} \qquad \_\_\_\_\_\_\_\_ 9(a)$$

$$\frac{\partial Hx}{\partial z} = \in\frac{\partial Ey}{\partial t} \qquad \_\_\_\_\_\_\_\_ 9(b)$$

$$-\frac{\partial Ey}{\partial z}\hat{i}-\mu\frac{\partial Hx}{\partial t} \qquad \_\_\_\_\_\_\_\_ 9(c)$$

and

$$\frac{\partial Ex}{\partial z} = \mu\frac{\partial Hy}{\partial t} \qquad \_\_\_\_\_\_\_\_ 9(d)$$

Let

$$Ey = f_1(z-\nu_0 t); \qquad \nu_0=\frac{1}{\sqrt{\mu E}}. \quad Then,$$

$$\frac{\partial Ey}{\partial t} = f_1(z-\nu_0 t)(-\nu_0). = -\nu_0 f_1.$$

$$\therefore From\ eqn.\ 9(c),\ we\ get,$$

$$\frac{\partial Hx}{\partial t} = -\frac{\nu_0}{\mu}f' \quad = \quad -\sqrt{\frac{\in_0}{\mu_0}}\ f'_1$$

$$\therefore Hx = -\sqrt{\frac{\in}{\mu_0}}\int f_1'\,dz + c.$$

$$Now$$

$$\frac{\partial f_1^{'}}{\partial z} = f_1^{'} \frac{\partial (z - v_0 t)}{\partial z} = f_1^{'}$$

$$\therefore H_z = -\sqrt{\frac{\in}{\mu}} \int \frac{\partial f_1}{\partial z} + C$$

$$Now$$

$$\frac{\partial f_1^{'}}{\partial z} = f_1^{'} \frac{\partial (z - v_0 t)}{\partial z} = f_1^{'}$$

$$\therefore = -\sqrt{\frac{\in}{\mu}} \int \frac{\partial f_1}{\partial z} dz + c = -\sqrt{\frac{\in}{\mu}} f_1 + c$$

$$Hx = -\sqrt{\frac{\in}{\mu}} Ey + c$$

The constan C indicates that a field independent of Z could be present. Evidently this is not a part of the wave motion and hence is rejected.

Thus the relation between H_X and E_Y becomes,

$$H_x = -\sqrt{\frac{\in}{\mu}} E_y$$

$$\therefore \frac{E_y}{H_x} = -\sqrt{\frac{\mu}{\in}} \quad \text{\_\_\_\_\_\_\_\_\_\_\_\_} \quad (10)$$

Similarly it can be shown that

$$\frac{E_x}{H_y} = \sqrt{\frac{\mu}{\in}} \quad \text{\_\_\_\_\_\_\_\_\_\_\_\_} \quad (11)$$

In our UPW, $\vec{E} = E_x\hat{i} + E_y\hat{j}$

$$\nabla^2 \vec{E} = -\mu \frac{\partial}{\partial t}\left(\sigma \vec{E} + \in \frac{\partial \vec{E}}{\partial t} \right)$$

$$\nabla^2 \vec{E} = -\mu\sigma \frac{\partial \vec{E}}{\partial t} - \mu \in \frac{\partial^2 \vec{E}}{\partial t^2} \quad \_\_\_\_\_\_\_\_ \quad (xi)$$

$$But \ \nabla \square \vec{E} = \frac{\rho}{\in_0}$$

$$\vec{E}$$

DERIVATION OF WAVE EQUATION FOR A CONDUCTING MEDIUM:

In a conducting medium, $\square = \square_0$, $\square = \square_0$. Surface charges and hence surface currents exist, static fields or charges do not exist.

For the case of conduction media, the point form of maxwells equations are:

$$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} = \sigma \vec{E} + \in \frac{\partial \vec{E}}{\partial t} \text{ \_\_\_\_\_\_\_\_\_\_ } (i)$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} = -\mu \frac{\partial \vec{H}}{\partial t} \text{ \_\_\_\_\_\_\_\_\_\_ } (ii)$$

$$\nabla \cdot \vec{D} = \nabla \cdot \in \vec{E} = \in \nabla \cdot \vec{E} = 0 \text{ \_\_\_\_\_\_\_\_\_\_ } (iii)$$

$$\nabla \cdot \vec{B} = \nabla \cdot \mu \vec{H} = \mu \nabla \cdot \vec{H} = 0 \text{ \_\_\_\_\_\_\_\_\_\_ } (iv)$$

Taking curl on both sides of equation (i), *we get*

$$\nabla \times \nabla \times \vec{H} = \nabla \times \left(\sigma \vec{E} + \in \frac{\partial \vec{E}}{\partial t} \right)$$

$$= \sigma \nabla \times \vec{E} + \in \frac{\partial}{\partial t} \nabla \times \vec{E} \text{ \_\_\_\_\_\_\_\_\_\_ } (v)$$

substituting eqn. (ii) *in eqn.* (v), *we get*

$$\nabla \times \nabla \times \vec{H} = \sigma \left(-\mu \frac{\partial \vec{H}}{\partial t} \right) + \in \left(-\mu \frac{\partial^2 \vec{H}}{\partial t^2} \right) \text{ \_\_\_\_\_\_\_\_\_\_ } (vi)$$

But $\quad \nabla \times \nabla \times \vec{H} = \nabla \left(\nabla \cdot \vec{H} \right) - \nabla^2 \vec{H} \text{ \_\_\_\_\_\_\_\_\_\_ } (vii)$

$\therefore$ *eqn.* (vi) *becomes*

$$\therefore \nabla \left(\nabla \cdot \vec{H} \right) - \nabla^2 \vec{H} = -\mu\sigma \frac{\partial \vec{H}}{\partial t} - \mu \in \frac{\partial^2 \vec{H}}{\partial t^2} \text{ \_\_\_\_\_\_\_\_\_\_ } (viii)$$

But $\nabla \cdot \vec{H} = \nabla \cdot \frac{\vec{B}}{\mu} = \frac{1}{\mu} \nabla \cdot \vec{B} = \frac{1}{\mu} \cdot 0 = 0$

$\therefore$ *eqn.* $(viii)$ *becomes,*

$$\nabla^2 \vec{H} - \mu\sigma \frac{\partial \vec{H}}{\partial t} - \mu \in \frac{\partial^2 \vec{H}}{\partial t^2} = 0 \text{ \_\_\_\_\_\_\_\_\_\_ } (ix)$$

This is the wave equation for the magnetic field $\vec{H}$ in a conducting medium.

Next we consider the second Maxwell's curl equation (ii)

$$\nabla \times \vec{E} = -\mu \frac{\partial \vec{H}}{\partial t} \quad \_\_\_\_\_\_\_\_\_\_ \quad (ii)$$

Taking curl on both sides of equation (ii) we get

$$\nabla \times \nabla \times \vec{E} = -\mu \nabla \times \frac{\partial \vec{H}}{\partial t} = -\mu \frac{\partial \left(\nabla \times \vec{H} \right)}{\partial t} \quad \_\_\_\_\_\_\_\_\_\_ \quad (x)$$

$$But\ \nabla \times \nabla \times \vec{E} = \nabla \left(\nabla \cdot \vec{E} \right) - \nabla^2 \vec{E}\ ;$$

Vector identity and substituting eqn. (1) in eqn (2), we get

$$\nabla \left(\nabla \cdot \vec{E} \right) - \nabla^2 \vec{E} = -\mu \frac{\partial}{\partial t} \left(\sigma \vec{E} + \in \frac{\partial \vec{E}}{\partial t} \right)$$

$$= -\mu\sigma \frac{\partial \vec{E}}{\partial t} - \mu \in \frac{\partial^2 \vec{E}}{\partial t^2} \quad \_\_\_\_\_\_\_\_\_\_ \quad (xi)$$

$$But\ \nabla \cdot \vec{E} = \frac{\rho}{\in_0}$$

(Point form of Gauss law) However, in a conductor, ρ = 0, since there is no net charge within a conductor,

Therefore we get $\nabla \cdot \vec{E} = 0$

Therefore eqn. (xi) becomes,

$$\nabla^2 \vec{E} = -\mu\sigma \frac{\partial \vec{E}}{\partial t} - \mu \in \frac{\partial^2 \vec{E}}{\partial t^2} \quad \_\_\_\_\_\_\_\_\_\_ \quad (xii)$$

This is the wave equation for electric field $\vec{E}$ in a conducting medium.

Wave equations for a conducting medium:

1. Regions where conductivity is non-zero.
2. Conduction currents may exist.

For such regions, for time varying fields

The Maxwell's eqn. Are:

$$\nabla \times \vec{H} = \vec{J} + \in \frac{\partial \vec{E}}{\partial t} \_\_\_\_\_\_\_\_\_\_\_ (1)$$

$$\nabla \times \vec{E} = -\mu \frac{\partial \vec{H}}{\partial t} \_\_\_\_\_\_\_\_\_\_\_ (2)$$

$$\vec{J} = \sigma \vec{E} \qquad \sigma : Conductivity\ (\Omega / m)$$

= conduction current density.

Therefore eqn. (1) becomes,

$$\nabla \times \vec{H} = \sigma \vec{E} + \in \frac{\partial \vec{E}}{\partial t} \_\_\_\_\_\_\_\_\_\_\_ (3)$$

Taking curl of both sides of eqn. (2), we get

$$\nabla \times \nabla \times \vec{E} = -\mu \frac{\partial}{\partial t}\left(\nabla \times \vec{H}\right)$$

$$= -\mu \frac{\partial^2 \vec{E}}{\partial t^2} - \mu\sigma \frac{\partial \vec{E}}{\partial t} \_\_\_\_\_\_\_\_\_\_\_ (4)$$

But

$$\nabla \times \nabla \times \vec{E} = \nabla\left(\nabla \cdot \vec{E}\right) - \nabla^2 \vec{E}\ (vector\ identity)$$

$u \sin g\ this\ eqn.\ (4)\ becomes\ vector\ identity,$

$$\nabla\left(\nabla \cdot \vec{E}\right) - \nabla^2 \vec{E} = -\mu\sigma \frac{\partial \vec{E}}{\partial t} - \mu \frac{\partial^2 \vec{E}}{\partial t^2} \_\_\_\_\_\_\_\_\_\_ (5)$$

$$\therefore\ But\ \nabla \cdot \vec{D} = \rho$$

$$\because \in is\ cons\tan t, \quad \nabla \cdot \vec{E} = \frac{1}{\in} \nabla \cdot \vec{D}$$

Since there is no net charge within a conductor the charge density is zero (there can be charge on the surface), we get.

$$\nabla \cdot \vec{E} = \frac{1}{\in} \nabla \cdot \vec{D} = 0$$

Therefore using this result in eqn. (5)

we get

$$\nabla^2 \vec{E} - \mu\sigma \frac{\partial \vec{E}}{\partial t} - \mu \in \frac{\partial^2 \vec{E}}{\partial t^2} = 0 \_\_\_\_\_\_\_\_\_\_ (6)$$

This is the wave eqn. For the electric field $\vec{E}$ in a conducting medium.

This is the wave eqn. for $\vec{E}$. The wave eqn. for $\vec{H}$ is obtained in a similar manner.

Taking curl of both sides of (1), we get

$$\nabla \times \nabla \times \vec{H} = \in \nabla \times \frac{\partial \vec{E}}{\partial t} + \sigma \, \nabla \times \vec{E} \quad \_\_\_\_\_\_\_\_\_\_ \ (7)$$

$$But \ \nabla \times \vec{E} = -\mu \frac{\partial \vec{H}}{\partial t} \quad \_\_\_\_\_\_\_\_\_\_ \ (2)$$

$$\therefore (1) \ becomes,$$

$$\nabla \times \nabla \times \vec{H} = -\mu \in \frac{\partial^2 \vec{H}}{\partial t^2} - \mu\sigma \frac{\partial \vec{H}}{\partial t} \quad \_\_\_\_\_\_\_\_\_\_ \ (8)$$

As before, we make use of the vector identity.

$$\nabla \times \nabla \times \vec{H} = \nabla \left(\nabla \cdot \vec{H} \right) - \nabla^2 \vec{H}$$

in eqn. (8) and get

$$\nabla \left(\nabla \cdot \vec{H} \right) - \nabla^2 \vec{H} = -\mu\sigma \frac{\partial \vec{H}}{\partial t} - \mu \in \frac{\partial^2 \vec{H}}{\partial t^2} \quad \_\_\_\_\_\_\_\_ \ (9)$$

$$But$$

$$\nabla \cdot \vec{H} = \nabla \cdot \frac{\vec{B}}{\mu} = \frac{1}{\mu} \nabla \cdot \vec{B} = \frac{1}{\mu} \cdot 0 = 0$$

$$\therefore eqn.(9) becomes$$

$$\nabla^2 \vec{H} = \mu\sigma \frac{\partial \vec{H}}{\partial t} - \mu \in \frac{\partial^2 \vec{H}}{\partial t^2} \quad \_\_\_\_\_\_\_\_ \ (10)$$

This is the wave eqn. for $\vec{H}$ in a conducting medium.

<u>Sinusoidal Time Variations:</u>

In practice, most generators produce voltage and currents and hence electric and magnetic fields which vary sinusoidally with time. Further, any periodic variation can be represented as a weight sum of fundamental and harmonic frequencies.

Therefore we consider fields having sinusoidal time variations, for example,

$E = E_m \cos \omega t$

$E = E_m \sin \omega t$

Here, w = $2\pi f$, f = frequency of the variation.

Therefore every field or field component varies sinusoidally, mathematically by an additional term. Representing sinusoidal variation. For example, the electric field $\vec{E}$ can be represented as

$$\vec{E}(x, y, z, t)\, as$$

$$ie., \tilde{\vec{E}}(\vec{r}, t); \vec{r}(x, y, z)$$

Where $\tilde{\vec{E}}$ is the time varying field.

The time varying electric field can be equivalently represented, in terms of corresponding phasor quantity $\tilde{\vec{E}}$ (r) as

$$\tilde{\vec{E}}(\vec{r}, t) = R_e\left[\vec{E}(r)e^{j\omega t}\right] \_\_\_\_\_\_\_\_\_\_ (11)$$

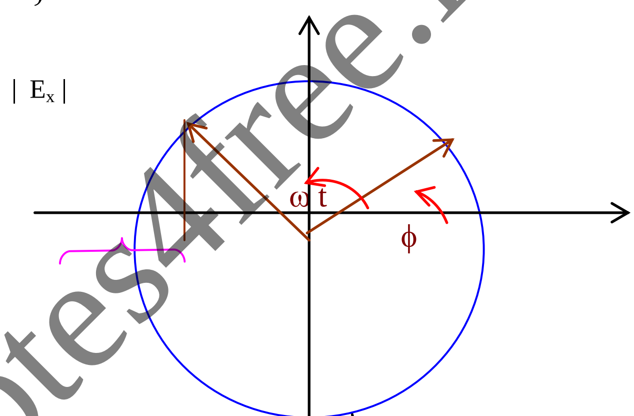
The symbol ‘tilda’ placed above the E vector represents that $\vec{E}$ is time – varying quantity.

<u>The phasor notation:</u>

We consider only one component at a time, say E_x.

The phasor E_x is defined by

$$\tilde{E}_x(\vec{r}, t) = R_e\left\{E_x(r)\ \ e^{j\omega t}\right\} \_\_\_\_\_\_\_\_\_\_ (12)$$



$E_x(\vec{r})$ denotes E_x as a function of space (x,y,z). In general $E_x(\vec{r})$ is complex and hence can be represented as a point in a complex and hence can be represented as a point in a complex plane. (see fig) Multiplication by e^{jwt} results in a rotation through an angle wt measured from the angle □. At t increases, the point E_x e^{jwt} traces out a circle with center at the origin. Its projection on the real axis varies sinusoidally with time & we get the time-harmonically varying electric field $\tilde{E}x$ (varying sinusoidally with time). We note that the phase of the sinusoid is determined by □, the argument of the complex number E_x.

Therefore the time varying quantity may be expressed as

$$\tilde{E}_x = R_e\left\{\left|E_x\right| e^{j\phi}\ e^{j\omega t}\right\} \_\_\_\_\_\_\_\_\_ (13)$$

$$= \left|E_x\right| \cos(\omega t + \phi) \_\_\_\_\_\_\_\_\_ (14)$$

<u>Maxwell’s eqn. in phasor notation:</u>

In time – harmonic form, the Maxwell’s first curl eqn. is:

$$\nabla \times \tilde{\tilde{H}} = \tilde{\tilde{J}} + \frac{\partial \tilde{\tilde{D}}}{\partial t} \_\_\_\_\_\_\_\_ (15)$$

using phasor notation, this eqn. becomes,

$$\nabla \times R_e\left(\vec{H}e^{j\omega t}\right) = \frac{\partial}{\partial t} R_e\left[\vec{D}e^{j\omega t}\right] + R_e\left[\vec{J}e^{j\omega t}\right] \_\_\_\_\_\_\_\_ (16)$$

The diff. Operator ∇ & R_e part operator may be interchanged to get,

$$R_e\left(\nabla \times \vec{H}e^{j\omega t}\right) = R_e\left[\frac{\partial}{\partial t}\left(\vec{D}e^{j\omega t}\right) + R_e\left[\vec{J}e^{j\omega t}\right]\right.$$

$$= R_e\left[j\omega\vec{D} \quad e^{j\omega t}\right] + R_e\left[\vec{J}e^{j\omega t}\right]$$

$\therefore$

$$R_e\left[\left(\nabla \times \vec{H} - j\omega\vec{D} - \vec{J}\right)e^{j\omega t}\right] = 0$$

This relation is valid for all t. Thus we get

$$\nabla \times \vec{H} = \vec{J} + j\omega\vec{D} \_\_\_\_\_\_\_\_\_\_\_ (17)$$

This phasor form can be obtained from time-varying form by replacing each time derivative by

$$jw\left(ie., \quad \frac{\partial}{\partial t} is\ to\ be\ replaced\ by \int \omega\right)$$

For the sinusoidal time variations, the Maxwell's equation may be expressed in phasor form as:

(17) $\nabla \times \vec{H} = \vec{J} + j\omega\vec{D}$ $\quad \oint_L \vec{H} \cdot d\vec{L} = \int_S \left(J + j\omega\vec{D}\right) \cdot d\vec{s}$

(18) $\nabla \times \vec{E} = -j\omega\vec{B}$ $\quad \oint_L \vec{E} \cdot d\vec{l} = \int_S -j\omega\vec{B} \cdot d\vec{s}$

(19) $\nabla \cdot \vec{D} = \rho$ $\quad \oint_S \vec{D} \cdot d\vec{s} = \int_V \rho_V d_V$

(20) $\nabla \cdot \vec{B} = 0$ $\quad \oint_S \vec{B} \cdot d\vec{s} = 0$

The continuity eqn., contained within these is,

$$\nabla \cdot \vec{J} = -j\omega\rho \qquad \oint_S \vec{J} \cdot d\vec{s} = -\int_{vol} j\omega\rho\, dv \_\_\_\_\_\_\_\_ (21)$$

The constitutive eqn. retain their forms:

$$\vec{D} = \in \vec{E}$$
$$\vec{B} = \mu\vec{H}$$
$$\vec{J} = \sigma\vec{E} \qquad \_\_\_\_ (22)$$

For sinusoidal time variations, the wave equations become

$$\left\{\nabla^2 \vec{E} = -\omega^2 \mu \in \vec{E} \qquad (\,for\ electric\ \ field\,)\right\}$$

$$\left\{\nabla^2 \vec{H} = -\omega^2 \mu \in \vec{H} \qquad (\,for\ electric\ \ field\,)\right\} \quad \_\_\_\_\_\_\_\_ (23)$$

Vector Helmholtz eqn.

In a conducting medium, these become

$$\nabla^2 \vec{E} + \left(\omega^2 \mu \in - j\omega\mu\sigma\right)\vec{E} = 0$$

$$\nabla^2 \vec{H} + \left(\omega^2 \mu \in - j\omega\mu\sigma\right)\vec{H} = 0 \quad \_\_\_\_\_\_\_ (24)$$

Wave propagation in a loss less medium:

In phasor form, the wave eqn. for VPW is

$$\left.\begin{aligned}\frac{\partial^2 \vec{E}}{\partial x^2} &= -\omega^2 \mu \in \vec{E} \\ &= -\beta^2 \vec{E}\end{aligned}\right\};\ \frac{\partial^2 E_y}{\partial x^2} = -\beta^2 E_y \quad \_\_\_\_\_\_\_\_ (25)$$

$$\therefore E_y = C_1 \quad e^{-j\beta x} + C_2 e^{j\beta x} \quad \_\_\_\_\_\_\_\_ (26)$$

C_1 & C_2 are arbitrary constants.

The corresponding time varying field is

$$\begin{aligned}\tilde{E}_y\left(x,t\right) &= R_e\left[E_y\left(x\right)e^{j\omega t}\right] \\ &= R_e\left[C_1\ e^{j(\omega t - \beta z)} + C_2\ e^{j(\omega t + \beta z)}\ \right] \_\_\_\_\_\_\_ (27) \\ &= C_1 \cos\left(\omega t - \beta z\right) + C_2\ \cos\left(\omega t + \beta z\right) \_\_\_\_\_\_\_ (28)\end{aligned}$$

When C_1 and C_2 are real.

Therefore we note that, in a homogeneous, lossless medium, the assumption of sinusoidal time variations results in a space variation which is also sinusoidal.

Eqn. (27) and (28) represent sum of two waves traveling in opposite directions.

If $C_1 = C_2$, the two traveling waves combine to form a simple standing wave which does not progress.

If we rewrite eqn. (28) with E_y as a f_n of (x-□t),

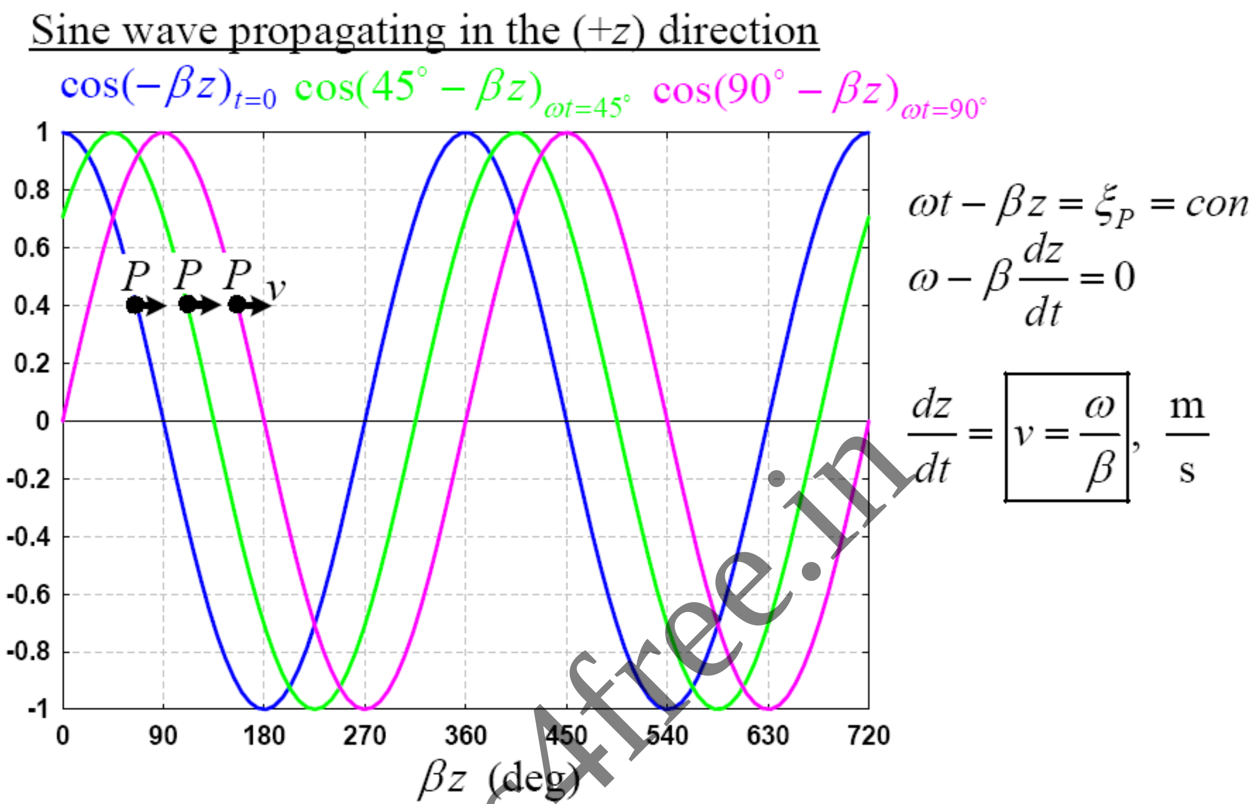
we get □ = $\dfrac{\omega}{\rho}$

Let us identify some point in the waveform and observe its velocity; this point is $\left(\omega t - \beta x\right) = a$ constant

Then $$\upsilon = \frac{dx}{dt} = \frac{\omega}{\beta} \qquad \because \frac{\partial x}{\partial t} = \frac{\partial^{\left(\frac{'a' - \omega t}{}\right)}}{\beta} = \frac{\omega}{\beta}$$

This velocity is called phase velocity, the velocity of a phase point in the wave.

β is called the phase shift constant of the wave.



Wavelength: These distance over which the sinusoidal waveform passes through a full cycle of 2π radians

ie.,

$$\beta\lambda = 2\pi$$

$$\beta = \frac{2\pi}{\lambda} \quad or \quad \lambda = \frac{2\pi}{\beta}$$

But

$$\beta = \frac{\omega}{\upsilon} \qquad \therefore \lambda = \frac{2\pi\upsilon}{\omega} = \frac{\upsilon}{f}$$

or

$$\upsilon = f\lambda; \qquad f \text{ in } H_Z$$

$$\upsilon : \frac{\omega}{\beta} = \frac{1}{\sqrt{\mu \in}} = \upsilon_o$$

Wave propagation in a conducting medium

We have,

$$\nabla^2 \vec{E} - \gamma^2 \vec{E} = 0$$
$$\gamma^2 = -\omega^2 \mu \in + j\omega\mu\sigma$$
$$= j\omega\mu(\sigma + j\omega \in)$$

Where

□ is called the propagation constant is, in general, complex.

Therefore, □ = □ + j□

□ = Attenuation constant

□ = phase shift constant.

The eqn. for UPW of electric field strength is

$$\frac{\partial^2 \vec{E}}{\partial x^2} = \gamma^2 \vec{E}$$

One possible solution is

$$\vec{E}(x) = E_0 e^{-\gamma x}$$

Therefore in time varying form, we get

$$\tilde{\vec{E}}(x,t) = R_e\left[E\, e^{-\gamma x}\, e^{j\omega t}\right]$$
$$= e^{-\alpha x}\, R_e\left[E_0 e^{jwt}\right]$$

This eqn. shown that a up wave traveling in the +x direction and attenuated by a factor $e^{-\alpha x}$.

The phase shift factor

$$\beta = \frac{2\pi}{\lambda}$$

$$and\ velocity = f\lambda = \frac{\omega}{\beta}$$

□ = Real part of □ = RP $\sqrt{j\omega\mu(\sigma + j\omega t)}$

$$\omega\sqrt{\frac{\mu \in}{2}\left(\sqrt{1+\frac{\sigma^2}{\omega^2 \in^2}} - 1\right)}$$

$$= \quad \beta = \omega\sqrt{\frac{\mu \in}{2}\left(\sqrt{1+\frac{\sigma^2}{\omega^2 \in^2}} + 1\right)}$$

Conductors and dielectrics:

We have the phasor form of the 1st Maxwell's curl eqn.

$$\nabla \times \vec{H} = \sigma\vec{E} + j\omega \in \vec{E} = J_c + J_{disp}$$

where $J_c = \sigma \vec{E} =$ conduction current density (A/m^2)

$J_{disp} = j\omega \in \vec{E} =$ displacement current density (A/m^2)

$$\therefore \left| \frac{J_{cond}}{J_{disp}} \right| = \frac{\sigma}{\omega \in}$$

We can choose a demarcation between dielectrics and conductors;

$$\frac{\sigma}{\omega \in} = 1$$

* $\frac{\sigma}{\omega \in} > 1$ is conductor. Cu: $3.5*10^8$ @ 30 GHz

* $\frac{\sigma}{\omega \in} < 1$ is dielectric. Mica: 0.0002 @ audio and RF

* For good conductors, □ & □ are independent of freq.

* For most dialectics, □ & □ are function of freq.

* $\frac{\sigma}{\omega \in}$ is relatively constant over frequency range of interest

Therefore dielectric " constant "

* $\frac{\sigma}{\omega \in}$ dissipation factor D

if D is small, dissipation factor is practically as the power factor of the dielectric.

PF = sin□

□ = $\tan^{-1}$D

PF & D difference by <1% when their values are less than 0.15.

Example 11.1

1. Express

$$E_y = 100 \cos\left(2\pi 10^8 t - 0.5z + 30^0\right) v/m \; as \; a \; phasor$$

$$E_y = R_e \left[100\, e^{j 2\pi \times 10^6 t - 0.5z + 30^0} \right]$$

Drop R_e and suppress e^{jwt} term to get phasor

Therefore phasor form of E_{ys} = $100e^{-0.5z+30^0}$

Whereas E_y is real, E_{ys} is in general complex.

Note: 0.5z is in radians; 30^0 in degrees.

Example 11.2

Given

$$\vec{E}_s = 100\ \angle 30^0\, \hat{a}x + 20\ \angle -50^0\, \hat{a}y + 40\ \angle 210^0\, \hat{a}z, V/m$$

find its time varying form representation

Let us rewrite $\vec{E}_s$ as

$$\vec{E}_s = 100e^{j30^0}\,\hat{a}x + 20e^{-j50^0}\,\hat{a}y + 40e^{j210^0}\,\hat{a}z. V/m$$

$$\therefore \tilde{\vec{E}} = R_e\left[E_s \quad e^{j\omega t}\right]$$

$$= R_e\left[100e^{j\left(\omega t+30^0\right)} + 20e^{j\left(\omega t-50^0\right)} + 40e^{j\left(\omega t+210^0\right)}\right] V/m$$

$$\tilde{\vec{E}} = 100\cos\left(\omega t + 30^0\right) 20\cos\left(\omega t - 50^0\right) + 40\cos\left(\omega t + 210^0\right) V/m$$

None of the amplitudes or phase angles in this are expressed as a function of x,y or z.

Even if so, the procedure is still effective.

2. Consider

$$H_s = 20e^{-(0.1+j20)z}\,\hat{a}x\ A/m$$

$$\tilde{\vec{H}}(t) = R_e\left[20e^{-(0.1+j20)z}\ \hat{a}x \quad e^{j\omega t}\right]$$

$$= 20e^{-0.1z}\cos\left(\omega t - 20z\right)\hat{a}x\ A/m$$

$$E_x = E_x\left(x, y, z\right)$$

$$Note: \qquad consider\ \frac{\partial E_x}{\partial t} = \frac{\partial}{\partial t} R_e\left[E_x\left(x,y,z\right) \quad e^{j\omega t}\right]$$

$$= R_e\left[j\omega E_x e^{j\omega t}\right]$$

Therefore taking the partial derivative of any field quantity wrt time is equivalent to multiplying the corresponding phasor by j □.

Example

Given

$$\vec{E}_{0s} = \left(500\angle -40^0\,\hat{a}y + \left(200 - j600\right)\hat{a}z\right)e^{-j0.4x}\ V/m$$

$$Find\ (a)\ \omega$$

$$(b)\ \vec{E}\ at\ (2,3,1)\ at\ t = 0$$

$$(c)\ \vec{E}\ at\ (2,3,1)\ at\ t = 10\ ns.$$

$$(d)\ \vec{E}\ at\ (3,4,2)\ at\ t = 20\ ns.$$

Q. From given data,

$$\beta = 0.4 = \omega\sqrt{\mu_0 \in_0}$$

$$\therefore \omega = \frac{0.4\times3\times10^8}{\sqrt{4\pi\times10^{-7}\times\frac{10^{-9}}{36\pi^{-9}}}} = 120\times10^6$$

$$f = 19.1\times10^6\,Hz$$

R. Given,

$$\vec{E}_s = \left(500\angle-40^0\ \hat{a}y + (200 - j600)\,\hat{a}z\right)e^{-j0.4x}$$

$$= 500e^{-j40}\ e^{-j0.4x}\ \hat{a}y + 632.456e^{-j71.565^0}\ e^{-j0.4x}\ \hat{a}z$$

$$= 500e^{-j\left(0.4x+40^0\right)}\hat{a}y + 632.456e^{-j\left(0.4x+71.565^0\right)}\hat{a}z$$

$$\vec{E}(t) = 500\ R_e\left[e^{+j\omega t}\ e^{-j\left(0.4x+40^0\right)}\ \hat{a}y + 632.456\,e^{j\omega t}\ e^{-j\left(0.4x+71.565^0\right)}\ \hat{a}z\right]$$

$$= 500\cos\left(\omega t - 0.4x - 40^0\right)\hat{a}y + 632.456\cos\left(\omega t - 0.4x - 71.565\right)\hat{a}z$$

$$\vec{E}\ at\ (2,3,1)\,t = 0 = 500\cos\left(-0.4x - 40^0\right)\hat{a}y + 632.456\left(-0.4x - 71.565\right)\hat{a}z$$

$$= 36.297\,\hat{a}y - 291.076\,\hat{a}z\ V/m$$

c)

$$\vec{E}\ at\ (t = 10\ ns)\ at\ (2,3,1)$$

$$= 500\cos\left(120\times10^6\times10\times10^{-9} - 0.4\times2 - 40^0\right)\hat{a}y$$

$$+ 632.456\cos\left(120\times10^6\times10\times10^{-9} - 0.4\times2 - 71.565^0\right)\hat{a}z$$

$$= 477.823\,\hat{a}y + 417.473\,\hat{a}z\ V/m$$

d)

at t = 20 ns,

$$\vec{E}\ at\ (2,3,1)$$

$$= 438.736\,\hat{a}y + 631.644\,\hat{a}z\ V/m$$

D 11.2:

Given $\vec{H}_s = \left(2\angle -40^0\ \hat{a}x - 3\angle 20\ \hat{a}y\right) e^{-j0.07z} \quad A/m$ for a UPW traveling in free space. Find

(a) □ (b) H_x at p(1,2,3) at t = 31 ns. (c) $\left|\vec{H}\right|$ at t=0 at the origin.

(a) we have p = 0.07 $(e^{-j\beta z}\ term)$

$$\therefore \omega\sqrt{\mu \in} = 0.07$$

$$\omega = \frac{0.07}{\sqrt{\mu \in}} = 0.07 \times 3 \times 10^8 = 21.0 \times 10^6\ rad/\sec$$

$$= 21.0 \times 10^6\ rad/\sec$$

(b)

$$\vec{H}(t) = R_e\left\{\left[2\,e^{-j40^0}\ e^{-j0.07z}\ \hat{a}x - 3\,e^{j20^0}\ e^{-j0.07z}\ \hat{a}y\right]e^{j\omega t}\right\}$$

$$= 2\cos\left(\omega t - 0.07z - 40^0\right)\hat{a}x - 3\cos\left(\omega t - 0.07z + 20^0\right)\hat{a}y$$

$$H_x(t) = 2\cos\left(\omega t - 0.07z - 40^0\right)$$

$$H_x(t)\ at\ p(1,2,3)$$

$$= 2\cos\left(2.1\times 10^6 t - 0.21 - 40^0\right)$$

$$At\ t = 31n\sec; = 2\cos\left(2.1\times 10^6 \times 31\times 10^{-9} - 0.21 - 40^0\right)$$

$$= 2\cos\left(651\times 10^{-3} - 0.21 - 40^0\right)$$

$$= 1.9333 \quad A/m$$

(c)

$$\vec{H}(t)\ at\ t = 0 = 2\cos\left(-0.07z - 0.7\right)\hat{a}x - 3\cos\left(-0.7z + 0.35\right)\hat{a}y$$

$$\vec{H}(t) = 2\cos(0.7)\hat{a}x - 3\cos(0.3)\hat{a}y$$

$$= 1.53\hat{a}x - 2.82\hat{a}y$$

$$= 3.20666 \quad A/m$$

In free space,

$$E(z,t) = 120\sin(\omega t - \beta z)\,\hat{a}y \quad V/m$$

$$find \quad H(z,t)$$

$$we\ have \quad \frac{E_y}{H_x} = -\eta = -120\pi$$

$$\therefore H_x = -\frac{E_y}{120\pi} = -\frac{120}{120\pi}\sin(\omega t - \beta z)\,\hat{a}y$$

$$= -\frac{1}{\pi}\sin(\omega t - \beta z)$$

$$\therefore \vec{H}(z,t) = -\frac{1}{\pi}\sin(\omega t - \beta z)\,\hat{a}x$$

Problem 3. J&B

Non uniform plans waves also can exist under special conditions. Show that the function

$$F = e^{-\alpha z}\sin\frac{\omega}{\upsilon}(x - \upsilon t)$$

satisfies the wave equation $\nabla^2 F = \frac{1}{c^2}\frac{\partial^2 F}{\partial t^2}$

provided the wave velocity is given by

$$\upsilon = e\sqrt{\left(1 + \frac{\alpha^2 c^2}{\omega^2}\right)}$$

Ans:

From the given eqn. for F, we note that F is a function of x and z,

$$\therefore \nabla^2 F = \frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2}$$

$$\frac{\partial F}{\partial x} = e^{-\alpha z}\,\frac{\omega}{\upsilon}\cos\frac{\omega}{\upsilon}(x - \upsilon t)$$

$$\frac{\partial^2 F}{\partial x^2} = -e^{-\alpha z}\left(\frac{\omega}{\upsilon}\right)\left(\frac{\omega}{\upsilon}\right)\sin\frac{\omega}{\upsilon}(x - \upsilon t) = -\frac{\omega^2 e^{-\alpha z}}{\upsilon^2}F$$

$$\frac{\partial F}{\partial z} = -e^{-\alpha z}\sin\frac{\omega}{\upsilon}(x - \upsilon t)$$

$$\frac{\partial^2 F}{\partial z^2} = +\alpha^2 e^{-\alpha z}\sin\frac{\omega}{\upsilon}(x - \upsilon t) = \alpha^2 F$$

$$\therefore \nabla^2 F = \left(-\frac{\omega^2}{\upsilon^2} + \alpha^2 \right) F$$

$$\frac{dF}{dt} = e^{-\alpha z} \left(\frac{\omega}{\upsilon} \right) (-\upsilon) \cos \frac{\omega}{\upsilon} (x - \upsilon t)$$

$$\frac{d^2 F}{dt^2} = -e^{-\alpha z} \left(\frac{\omega}{\upsilon} \right) \frac{\omega}{\upsilon} (-\upsilon)(-\upsilon) \sin (x - \upsilon t)$$

$$= -\omega^2 F$$

The given wave equation is

$$\nabla^2 F = \frac{1}{c^2} \frac{\partial^2 F}{\partial t^2}$$

$$\therefore \left(\alpha^2 - \frac{\omega^2}{\upsilon^2} \right) F = \frac{1}{c^2} \left(-\omega^2 \right) F$$

$$\therefore \alpha^2 - \frac{\omega^2}{\upsilon^2} = -\frac{\omega^2}{c^2}$$

$$\alpha^2 + \frac{\omega^2}{c^2} = \frac{\omega^2}{\upsilon^2}$$

$$\upsilon^2 = \frac{\omega^2}{\alpha^2 + \frac{\omega^2}{c^2}}$$

$$\therefore \upsilon^2 = \frac{\omega^2 c^2}{\alpha^2 c^2 + \omega^2} = \frac{c^2}{1 + \frac{\alpha^2 c^2}{\omega^2}}$$

$$or\ \upsilon = \frac{c}{\sqrt{1 + \frac{\alpha^2 c^2}{\omega^2}}}$$

Example

The electric field intensity of a uniform plane wave in air has a magnitude of 754 V/m and is in the z direction. If the wave has a wave length □ = 2m and propagating in the y direction.

Find

(i) Frequency and □ when the field has the form $A\cos(\omega t - \beta z)$.

(ii) Find an expression for $\vec{H}$.

In air or free space,

$$\upsilon = c = 3\times10^{8}\ m/\sec$$

(i)

$$f = \frac{e}{\lambda} = \frac{3\times10^{8}}{2m}\, m/\sec = 1.5\times10^{8}\, Hz = 150MHz$$

$$\beta = \frac{2\pi}{\lambda} = \frac{2\pi}{2m} = 3.14\ rad/m$$

$$\therefore E_z = 754\cos\left(2\pi\times150\times10^{6}t - \pi y\right)$$

(ii)

For a wave propagating in the +y direction,

$$\frac{E_z}{H_z} = \eta = -\frac{E_x}{H_z}$$

For the given wave,

$$E_z = 754\,V/m; \quad E_x = 0$$

$$\therefore H_x = 754\times\eta = \frac{754}{120\pi} = \frac{754}{377}\,A/m$$

$$\therefore \vec{H} = 2\cos\left(2\pi\times150\times10^{6}t - \pi y\right)\hat{a}x \quad A/m$$

Example

find □ for copper having □ = $5.8*10^{7}$ (□/m) at 50Hz, 3MHz, 30GHz.

$$\delta = \sqrt{\frac{2}{\omega\mu\sigma}} = \sqrt{\frac{1}{\pi f \mu\sigma}}$$

$$= \sqrt{\frac{1}{\pi} \times \frac{1}{4\pi \times 10^{-7}} \times \frac{1}{5.8 \times 10^{7}} \times \frac{1}{f}}$$

$$= \sqrt{\frac{1}{4\pi^2 \times 5.8} \times \frac{1}{f}} = \sqrt{\frac{1}{23.2\pi^2 f}} = \frac{66 \times 10^{-3}}{\sqrt{f}}$$

$$(i) = \frac{66 \times 10^{-3}}{\sqrt{50}} = 9.3459 \times 10^{-3} m$$

$$(ii) = \frac{66 \times 10^{-3}}{\sqrt{3 \times 10^{6}}} = 3.8105 \times 10^{-5} m$$

$$(iii) = \frac{66 \times 10^{-3}}{\sqrt{3 \times 10^{6}}} = 3.8105 \times 10^{-7} m$$

<u>Wave Propagation in a loss less medium:</u>

Definition of uniform plane wave in Phasor form:

In phasor form, the uniform plane wave is defined as one for which the equiphase surface is also an equiamplitude surface, it is a uniform plane wave.

For a uniform plane wave having no variations in x and y directions, the wave equation in phasor form may be expressed as

$$\frac{\partial^2 \vec{E}}{\partial Z^2} = -\omega^2 \mu \in \vec{E} \quad Or \quad \frac{\partial^2 \vec{E}}{\partial Z^2} = -\beta^2 \vec{E} \text{ \_\_\_\_\_\_\_\_\_\_ } (i)$$

where $\beta = \omega\sqrt{\mu \in}$. Let us consider eqn.(i) for, the E_y component, we get

$$\frac{\partial^2 E_y}{\partial Z^2} = -\beta^2 E_y$$

E_y has a solution of the form,

$$E_y = C_1 e^{-j\beta z} + C_2 e^{+j\beta z} \text{ \_\_\_\_\_\_\_\_\_\_ } (2)$$

Where C_1 and C_2 are arbitrary complex constants. The corresponding time varying form of E_y is

$$\tilde{E}_y(z,t) = R_e\left\{E_y(z) \quad e^{j\omega t}\right\}$$

$$= R_e\left[\left(C_1 \quad e^{-j\beta z} + C_2 \quad e^{j\beta z}\right) e^{j\omega t}\right] \text{ \_\_\_\_\_\_\_\_ } (3)$$

If C_1 and C_2 are real, the result of real part extraction operation is,

$$\therefore E_y(z,t) = C_1 \cos(\omega t - \beta z) + C_2 \cos(\omega t + \beta z) \_\_\_\_\_\_\_\_ (4)$$

From (3) we note that, in a homogeneous lossless medium, sinusoidal time variation results in space variations which is also sinusoidal.

Equations (3) and (4) represent sum of two waves traveling in opposite directions.

If $C_1 = C_2$, the two wave combine to form a standing wave which does not progress.

Phase velocity and wavelength:

The wave velocity can easily obtained when we rewrite E_y as a function and $(z \pm \upsilon t)$, as in eqn. (4). This shows that

$$\upsilon = \frac{\omega}{\beta} \_\_\_\_\_\_\_\_\_\_ (5)$$

In phasor form, identifying a some reference point on the waveform and observing its velocity may obtain the same result. For a wave traveling in the +Z direction, this point is given by $\omega t - \beta z = a$ constant.

$$\therefore \upsilon = \frac{dz}{dt} = \frac{\omega}{\beta}$$, as in eqn. (5)

This velocity of some point on the sinusoidal waveform is called the phase velocity. □ is called the phase-shift constant and is a measure of phase shift in radians per unit length.

Wavelength: Wavelength is defined as that distance over which the sinusoidal waveform passes through a full cycle of 2□ radius.

ie.,

$$\beta \lambda = 2\pi$$

$$\therefore \lambda = \frac{2\pi}{\beta} = \frac{2\pi}{\omega\sqrt{\mu \in}} = \frac{2\pi}{2\pi f\sqrt{\mu \in}} = \frac{\upsilon}{f}; \upsilon = \frac{1}{\sqrt{\mu \in}} \_\_\_\_\_\_\_\_\_ (7)$$

$$\therefore \upsilon = f\lambda, \quad f \text{ in } Hz \quad \_\_\_\_\_\_\_\_\_ (8)$$

For the value of □ given in eqn. (1), the phase velocity is,

$$\upsilon = \frac{\omega}{\beta} = \frac{\omega}{\omega\sqrt{\mu \in}} = \frac{1}{\sqrt{\mu \in}} = \upsilon_0 \_\_\_\_\_\_\_\_\_ (9)$$

$$\upsilon_0 = C \quad ; \quad C = 3\times 10^8 \ m/\sec$$

Wave propagation in conducting medium:

The wave eqn. written in the form of Helmholtz eqn. is

$$\nabla^2 \vec{E} - \gamma^2 \vec{E} = 0 \_\_\_\_\_\_\_\_ (10)$$

$$where \quad \gamma^2 = \left(-\omega^2 \mu \in - j\omega\mu\sigma\right) = j\omega\mu(\sigma + j\omega) \_\_\_\_\_\_\_\_ (11)$$

□, the propagation constant is complex = □ + j□ _________(12)

We have, for the uniform plane wave traveling in the z direction, the electric field $\vec{E}$ must satisfy

$$\frac{\partial^2 \vec{E}}{\partial Z^2} = \gamma^2 \vec{E} \quad \_\_\_\_\_\_\_\_\_(13)$$

This equation has a possible solution

$$\vec{E}(Z) = E_0 e^{-\gamma Z} \quad \_\_\_\_\_\_\_\_(14)$$

In time varying form this is becomes

$$\tilde{\vec{E}}(z,t) = R_e\left\{E_0 \quad e^{-\gamma Z} \quad e^{j\omega t}\right\} \quad \_\_\_\_\_\_\_\_(15)$$

$$= e^{-\alpha z}\ R_e\left\{E_0 \quad e^{j(\omega t - \beta z)}\right\} \quad \_\_\_\_\_\_\_\_\_(16)$$

This is the equation of a wave traveling in the +Z direction and attenuated by a factor $e^{-\alpha Z}$. The phase shift factor and the wavelength phase, velocity, as in the lossless case, are given by

$$\beta = \frac{2\pi}{\lambda} \qquad \upsilon = f\lambda = \frac{\omega}{\beta}$$

The propagation constant

We have, $\gamma = \sqrt{j\omega\mu(\sigma + j\omega \in)}$ _________(11)

$$\therefore \gamma^2 = (\alpha + j\beta)^2 = \alpha^2 + 2j\alpha\beta - \beta^2 = j\omega\mu\sigma - \omega^2\mu \in \quad \_\_\_\_\_\_\_\_\_(17)$$

$$\therefore \alpha^2 - \beta^2 = -\omega^2\mu \in; \quad \beta^2 = \alpha^2 + \omega^2\mu \in \quad \_\_\_\_\_\_\_\_\_(18)$$

$$\alpha\beta = \omega\mu\sigma$$

$$\therefore \alpha = \frac{\omega\mu\sigma}{2\beta} \quad \_\_\_\_\_\_\_\_\_(19)$$

Therefore (19) in (18) gives:

$$\beta^2 = \left(\frac{\omega\mu\sigma}{4\beta}\right)^2 + \omega^2\mu \in$$

$$4\beta^4 - 4\beta^2\omega^2\mu \in -\omega^2\mu^2\sigma^2 = 0$$

$$\beta^4 - \beta^2\omega^2\mu \in -\frac{\omega^2\mu^2\sigma^2}{4} = 0$$

$$\beta^2 = \frac{\omega^2\mu \in \pm\sqrt{\omega^4\mu^2\sigma^2 + \omega^2\mu^2\sigma^2}}{2}$$

$$= \frac{\omega^2\mu \in \pm\omega^2\mu \in \sqrt{\left(1+\frac{\omega^2\sigma^2}{\in^2}\right)}}{2}$$

$$= \frac{\omega^2\mu \in}{2}\left(1 \pm \sqrt{1+\frac{\omega^2\sigma^2}{\omega^2 \in^2}}\right)$$

$$\therefore \beta = \omega\sqrt{\frac{\mu \in}{2}\sqrt{\left(1+\frac{\sigma^2}{\omega^2 \in^2}\right)}+1} \quad \_\_\_\_\_\_\_\_(20)$$

and

$$\alpha = \omega\sqrt{\frac{\mu \in}{2}\sqrt{\left(1+\frac{\sigma^2}{\omega^2 \in^2}\right)}-1} \quad \_\_\_\_\_\_\_\_(21)$$

We choose some reference point on the wave, the cosine function,(say a rest). The value of the wave ie., the cosine is an integer multiple of 2□ at erest.

$\therefore k_0 z = 2m\pi$ at m[th] erest.

Now let us fix our position on the wave as this m[th] erest and observe time variation at this position, nothing that the entire cosine argument is the same multiple of 2□ for all time in order to keep track of the point.

ie., $\omega t - k_0\beta_0 z = 2m\pi = \omega\left(t - z/c\right)$

Thus at t increases, position z must also increase to satisfy eqn. ( ). Thus the wave erest (and the entire wave moves in a +ve direction) with a speed given by the above eqn. Similarly, eqn. ( ) having a cosine argument $\left(\omega t + \beta_0 z\right)$ describes a wave that moves in the negative direction (as + increases z must decrease to keep the argument constant). These two waves are called the traveling waves.

Let us further consider only +ve z traveling wave:

We have

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} = 0 & \frac{\partial}{\partial y} = 0 & \frac{\partial}{\partial z} \\ E_x & E_y & 0 \end{vmatrix}$$

$$\nabla \times \vec{E}_s = -j\omega\mu\vec{H}_s$$

$$i\left(-\frac{\partial E_y}{\partial z}\right) + j\frac{\partial E_x}{\partial z} + \hat{k}_0 = -j\omega\mu\left(iH_0 x + j + b_y\right)$$

$$\therefore \frac{\partial E_{xs}}{\partial z} = -j\omega\mu H_{0y}$$

$$\therefore H_{oy} = -\frac{1}{j\omega\mu}\left(E_{z0} \quad e^{-jk_0 z}\right) = E_{x0}\sqrt{\frac{\in_0}{\mu_0}} \quad e^{-j\beta_0 z}$$

$$\therefore H_y(z,t) = E_{x0}\sqrt{\frac{\in_0}{\mu_0}} \cos\left(\omega t - \beta_0 z\right)$$

$$\frac{E_x}{H_y} = \sqrt{\frac{\mu}{\in}} = \eta \quad ; \quad \eta = \eta_0 = 377\Omega = 120\pi\Omega$$

E_y and H_x are in phase in time and space. The UPW is called so because is uniform thought any plane Z = constant.

Energy flow is in +Z direction.

E and H are perpendicular to the direction of propagation; both lie in a plane that is transverse to the direction of propagation. Therefore also called a TEM wave.

11.1. The electric field amplitude of a UPW in the $\hat{a}z$ direction is 250 V/m. If $\vec{E}$ = $E_x\hat{a}x$ and = 1m rad/sec, find (i) f (ii) (iii) period (iv) amplitude of $\vec{H}$.

$$f = \frac{\omega}{2\pi} = \frac{2\pi f}{2\pi} = \frac{10^6}{2\pi} = 159.155\ KHz$$

$$\lambda = \frac{C}{f} = 1.88495\ km$$

$$period = \frac{1}{f} = 6.283\ \mu s$$

$$amplitude\ of\ \ H_y = \frac{E_x}{H_y} = \eta = 120\pi$$

$$\therefore H_y = \frac{E_x}{120\pi} = \frac{250}{120\pi} = 0.6631\ A/m$$

1. Given $\vec{H}_s = \left(2\angle -40^0\,\hat{a}x - 3\angle 20^0\,\hat{a}y\right)e^{-j0.07z}\,A/m$ for a certain UPW traveling in free space.

Find (i)□, (ii)H_x at p(1,2,3) at t = 31ns and (iii) $|\vec{H}|$ at t = 0 at the orign.

<u>Wave propagation in dielectrics:</u>

For an isotopic and homogeneous medium, the wave equation becomes

$$\nabla^2 \vec{E}_s = -k^2 \in_s$$

$$k = \omega\sqrt{\mu \in} = k_0\sqrt{\mu_r \in_r} = \beta_0\sqrt{\mu_r \in_r}$$

For E_x component

We have

$$\frac{d^2 E_{xs}}{dz^2} = -k^2 E_{xs}$$

for E_x comp. Of electric field wave traveling in Z – direction.

k can be complex one of the solutions of this eqn. is,

$$jk = \alpha + j\beta$$

$$E_{xs} = E_{x0}e^{-\alpha z}\ e^{-j\beta z}$$

Therefore its time varying part becomes,

$$E_{xs} = E_{x0}e^{-\alpha z}\ \cos\left(\omega t - \beta z\right)$$

This is UPW that propagates in the +Z direction with phase constant □ but losing its amplitude with increasing $Z\left(e^{-\alpha z}\right)$. Thus the general effect of a complex valued k is to yield a traveling wave that changes its amplitude with distance.

If □ is +ve $\longrightarrow$ □ = attenuation coefficient if □ is +ve wave decays

If □ is -ve $\longrightarrow$ □ = gain coefficient $\longrightarrow$ wave grows

In passive media, □ is +ve □ is measured in repers per meter

In amplifiers (lasers) □ is –ve.

<u>Wave propagation in a conducting medium for medium for time-harmonic fields:</u>

<u>(Fields with sinusoidal time variations)</u>

For sinusoidal time variations, the electric field for lossless medium (□ = 0) becomes

$$\nabla^2 \vec{E} = -\omega^2 \mu \in \vec{E}$$

In a conducting medium, the wave eqn. becomes for sinusoidal time variations:

$$\nabla^2 \vec{E} + \left(\omega^2 \mu \in - j\omega\sigma\right) \vec{E} = 0$$

<u>Problem:</u>

Using Maxwell's eqn. (1) show that

$\nabla.\vec{D} = 0$ in a conductor

if ohm's law and sinusoidal time variations are assumed. When ohm's law and sinusoidal time variations are assumed, the first Maxwell's curl equation is

$$\nabla \times \vec{H} = \sigma \vec{E} + j\omega \in \vec{E}$$

Taking divergence on both sides, we get,

$$\nabla\left(\nabla \times \vec{H}\right) = \sigma\ \nabla \square \vec{E} + j\omega \in \nabla \square \vec{E} = 0$$

$$\therefore \nabla \square \vec{E}\left(\sigma + j\omega \in\right) = 0$$

$$or\ \nabla \square \vec{D}\left(\frac{\sigma}{\in} + j\omega\right) = 0$$

$$\sigma, \in \&\ \omega\ are$$

constants and of finite values and $\therefore \neq 0$

$$\nabla \square \vec{D} = 0$$

<u>Wave propagation in free space:</u>

The Maxwell's equation in free space, ie., source free medium are,

$$\nabla \times \vec{H} = \in_0 \frac{\partial \vec{E}}{\partial t} \vec{H} \text{ \_\_\_\_\_\_\_\_\_\_(1)}$$

$$\nabla \times \vec{E} = -\mu \frac{\partial \vec{H}}{\partial t} \text{ \_\_\_\_\_\_\_\_\_\_(2)}$$

$$\nabla \square \vec{D} = 0 \text{ \_\_\_\_\_\_\_\_\_\_(3)}$$

$$\nabla \square \vec{B} = 0 \text{ \_\_\_\_\_\_\_\_\_\_(4)}$$

Note that wave motion can be inferred from the above equation.

How? Let us see,

Eqn. (1) states that if electric field $\vec{E}$ is changing with time at some, point then magnetic field $\vec{H}$ has a curl at that point; thus $\vec{H}$ varies spatially in a direction normal to its orientation direction. Further, if $\vec{E}$ varies with time, then $\vec{H}$ will, in general, also change with time; although not necessarily in the same way.

Next

From (2) we note that a time varying $\vec{H}$ generates $\vec{E}$; this electric field, having a curl, therefore varies spatially in a direction normal to its orientation direction.

We thus have once more a time changing electric field, our original hypothesis, but this field is present a small distance away from the point of the original disturbance.

The velocity with which the effect has moved away from the original disturbance is the velocity of light as we are going to prove later.

UNIFORM PLANE WAVE:

Uniform plane wave is defined as a wave in which (1) both fields $\vec{E}$ and $\vec{H}$ lie in the transverse plane. Ie., the plane whose normal is the direction of propagation; and (2) both $\vec{E}$ and $\vec{H}$ are of constant magnitude in the transverse plane.

Therefore we call such a wave as transverse electro magnetic wave or TEM wave.

The spatial variation of both $\vec{E}$ and $\vec{H}$ fields in the direction normal to their orientation (travel) ie., in the direction normal to the transverse plane.

Differentiating eqn. (7) with respect to Z_1 we get

$$\frac{\partial^2 E_x}{\partial Z^2} = -\mu_0 \frac{\partial}{\partial Z}\left(\frac{\partial Hy}{\partial t}\right) = -\mu_0 \frac{\partial^2 \vec{H}}{\partial t \partial Z} \text{ \_\_\_\_\_\_\_\_\_\_(9)}$$

Differentiating (8) with respect to t_1 we get

$$\frac{\partial^2 H}{\partial t \partial Z} = - \in_0 \frac{\partial^2 E_x}{\partial t^2} \_\_\_\_\_\_\_\_\_\_(10)$$

Therefore substituting (10) into (9) gives,

$$\frac{\partial^2 E_x}{\partial t^2} = +\mu_0 \in_0 \frac{\partial^2 E_x}{\partial t^2} \_\_\_\_\_\_\_\_\_\_(11)$$

This eqn.(11) is the wave equation for the x-polarized TEM electric field in free space.

The constant $\frac{1}{\sqrt{\mu_0 \in_0}}$ is the velocity of the wave in free space, denoted c and has a value $3\times10^8 m/\sec$, on substituting the values, $\mu_0 = 4\pi \times 10^{-7} H/m \ and \in_0 = \frac{10^{-9}}{36\pi}$ Differentiating (10) with respect to Z and differentiating (9) with respect to 't' and following the similar procedure as above, we get

$$\frac{\partial^2 H_y}{\partial Z^2} = \mu_0 \in_0 \frac{\partial^2 H_y}{\partial t^2} \_\_\_\_\_\_\_\_\_\_(13)$$

eqn. (11 and (13) are the second order partial differential eqn. and have solution of the form, for instance,

$$E_x(Z,t) = f_1(t - Z/\upsilon) + f_2(t - Z/\upsilon) \_\_\_\_\_\_\_\_(14)$$

Let $\vec{E} = E_x \hat{a}x$ (ie., the electric field is polarized (!) in the x- direction !) traveling along Z direction. Therefore variations of $\vec{E}$ occurs only in Z direction.

Form (2) in this case, we get

$$\nabla \times \vec{E} = \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ \frac{\partial}{\partial x}(=0) & \frac{\partial}{\partial y}(=0) & \frac{\partial}{\partial z} \\ E_x & 0 & 0 \end{vmatrix} = -\frac{\partial E_x}{\partial z}\hat{j} = -\mu_0 \frac{\partial \vec{H}}{\partial t} = -\mu_0 \frac{\partial \vec{H}}{\partial t}\hat{j} \_\_\_\_\_\_\_\_\_(5)$$

Note that the direction of the electric field $\vec{E}$ determines the direction of $\vec{H}$, we is now along the y direction.

Therefore in a UPW, $\vec{E}$ and $\vec{H}$ are mutually orthogonal. (ie., perpendicular to each other). This in a UPW .

(i) $\vec{E}$ and $\vec{H}$ are perpendicular to each other (mutually orthogonal and

(ii) $\vec{E}$ and $\vec{H}$ are also perpendicular to the direction of travel.

Form eqn. (1), for the UPW, we get

$$\nabla \times \vec{H} = -\frac{\partial H_y}{\partial Z}\hat{a}x = \in_0 \frac{\partial \vec{E}}{\partial t} = t_0 \frac{\partial E_x}{\partial t}\hat{a}x$$

(using the mutually orthogonal property) _______________(6)

Therefore we have obtained so far,

$$\frac{\partial E_x}{\partial Z} = -\mu_0 \frac{\partial H_y}{\partial t} \text{ \_\_\_\_\_\_\_\_\_\_\_\_ } (7)$$

$$\frac{\partial H_y}{\partial Z} = -\in_0 \frac{\partial E_x}{\partial t} \text{ \_\_\_\_\_\_\_\_\_\_\_\_ } (8)$$

f_1 and f_2 can be any functions who se argument is of the form $t \pm Z/\upsilon$.

The first term on RHS represents a forward propagating wave ie., a wave traveling along positive Z direction.

The second term on RHS represents a reverse propagating wave ie., a wave traveling along negative Z direction.

(Real instantaneous form and phaser forms).

The expression for E_x (z,t) can be of the form

$$\begin{aligned} E_x(z,t) &= E_x(z,t) + E^1{}_x(z,t) \\ &= E_{x0}\cos\left[\omega\left(t - Z/\upsilon_p\right) + \phi_1\right] + E^1{}_{x0}\cos\left[\omega\left(t - Z/\upsilon_p\right) + \phi_2\right] \\ &= E_{x0}\cos\left(\omega t - k_0 z + \phi_1\right) + E^1{}_{x0}\cos\left(\omega t + k_0 z + \phi_2\right) \text{ \_\_\_\_\_\_\_\_ } (15) \end{aligned}$$

υ_p is called the phase velocity = c in free space k_0 is called the wave number in free space = $\frac{\omega}{c}$ rad/m _________(16)

eqn. (15) is the real instantaneous forms of the electric (field) wave. (experimentally measurable)

□$_0$t and k_0z have the units of angle usually in radians.

□ : radian time frequency, phase shift per unit time in rad/sec.

k_0 : spatial frequency, phase shift per unit distance in rad/m.

k_0 is the phase constant for lossless propagation.

Wavelength in free space is the distance over which the spatial phase shifts by 2□ radians, (time fixed)

ie.,

$$k_0 z = k_0 \lambda = 2\pi$$

$$or \quad \lambda = \frac{2\pi}{k_0}$$

(in free space) _________(17)

Let us consider some point, for instance, the crest or trough or zero crossing (either –ve to +ve or +ve to –ve). Having chosen such a reference, say the crest, on the forward-propagating cosine function, ie., the function $\cos\left(\omega t - k_0 z + \phi_1\right)$. For a erest to occur, the argument of the cosine must be an integer multiple of 2□. Consider the mth erest of the wave from our reference point, the condition becomes,

K_0z = 2m□, m an integer.

This point on the cosine wave we have chosen, let us see what happens as time increases.

The entire cosine argument must have the same multiple of 2□ for all times, in order to keep track of the chosen point.

Therefore we get, $\omega t - k_0 z = \omega\left(t - Z/\upsilon\right) = 2m\pi$ _______(18)

As time increases, the position Z must also increase to satisfy (18). The wave erest, and the entire wave, moves in the positive Z-direction with a phase velocity C (in free space).

Using the same reasoning, the second term on the RHS of eqn. (15) having the cosine argument $\left[\omega t + k_0 z\right]$ represents a wave propagating in the Z direction, with a phase velocity C, since as time t increases, Z must decrease to keep the argument constant.

POLARISATION:

It shows the time varying behavior of the electric field strength vector at some point in space.

Consider of a UPW traveling along Z direction with $\vec{\tilde{E}}$ and $\vec{\tilde{H}}$ vectors lying in the x-y plane.

1. If $\tilde{E}y = 0$ and only $\tilde{E}x$ is present, the wave is said to be polarized in the x-direction.

2. If $\tilde{E}x$ = 0 and only $\tilde{E}y$ is present, the wave is said to be polarized in the y-direction.

Therefore the direction of $\vec{E}$ is the direction of polarization

3. If both $\tilde{E}x$ and $\tilde{E}y$ are present and are in phase, then the resultant electric field $\vec{E}$ has a direction that depends on the relative magnitudes of $\tilde{E}x$ and $\tilde{E}y$.

The angle which this resultant direction makes with the x axis is $\tan^{-1}\dfrac{\tilde{E}y}{\tilde{E}x}$; and this angle will be constant with time.

1. Linear polarization:

In all the above three cases, the direction of the resultant vector is constant with time and the wave is said to be linearly polarized.

If $\tilde{E}x$ and $\tilde{E}y$ are not in phase ie., they reach their maxima at different instances of time, then the direction of the resultant electric vector will vary with time. In this case it can be shown that the locus of the end point of the resultant $\vec{\tilde{E}}$ will be an ellipse and the wave is said to be elliptically polarized.

In the particular case where $\tilde{E}x$ and $\tilde{E}y$ have equal magnitudes and a 90^0 phase difference, the locus of the resultant $\vec{\tilde{E}}$ is a circle and the wave is circularly polarized.

Linear Polarisation:

Consider the phasor form of the electric field of a UPW traveling in the Z-direction:

$$\vec{E} \leq (Z) = E_0 e^{-j\beta z}$$

Its time varying or instanious time form is

$$\tilde{\vec{E}}(Z,t) = R_e \left\{ E_0 e^{-j\beta z} e^{j\omega t} \right\}$$

The wave is traveling in Z-direction.

Therefore $\vec{E}_z$ lies in the x-y plane. In general, $\vec{E}_0$ is a complex vector ie., a vector whose components are complex numbers.

Therefore we can write $\vec{E}_0$ as,

$$\vec{E}_0 = \vec{E}_r + j\vec{E}_{0i}$$

Where $\vec{E}_0$ and $\vec{E}_{0i}$ are real vectors having, in general, different directions.

At some point in space, (say z = 0) the resultant time varying electric field is

$$\tilde{\vec{E}}(0,t) = R_e \left\{ \left(\vec{E}_{0r} + j\,\vec{E}_{0i} \right) e^{j\omega t} \right\}$$
$$= \vec{E}_{0r} \cos \omega t - \vec{E}_{0i} \sin \omega t$$

Therefore $\vec{E}$ not only changes its magnitude but also changes its direction as time varies.

Circular Polarisation:

Here the x and y components of the electric field vector are equal in magnitude.

If E_y leads E_x by 90^0 and E_x and E_y have the same amplitudes,

Ie., $|E_x| = |E_y|$, we have, $\tilde{\vec{E}} = (\hat{a}x + j\,\hat{a}y) E_0$

The corresponding time varying version is,

$$\tilde{\vec{E}}(0,t) = [\hat{a}x \cos \omega t - \hat{a}y \sin \omega t] \vec{E}_0$$
$$\therefore E_x = E_0 \cos \omega t$$
$$and\ E_y = E_0 \sin \omega t$$
$$\therefore E_x^2 + E_y^2 = E_0^2$$

Which shows that the end point of $\tilde{\vec{E}}_0(0,t)$ traces a circle of radius E_0 as time progresses.

Therefore the wave is said to the circularly polarized. Further we see that the sense or direction of rotation is that of a left handed screw advancing in the Z-direction (ie., in the direction of propagation). Then this wave is said to be left circularly polarized.

Similar remarks hold for a right-circularly polarized wave represented by the complex vector,

$$\tilde{\vec{E}} = (\hat{a}x + j\,\hat{a}y) E_0$$

It is apparent that a reversal of the sense of rotation may be obtained by a 180^0 phase shift applied either to the x component of the electric field.

Elliptical Polarisation:

Here x and y components of the electric field differ in amplitudes $\left(\tilde{E}_x \neq \tilde{E}_y\right)$.

Assume that E_y leads E_x by 90^0.

Then,

$$E_0 \hat{a}x\ A + j\ \hat{a}y\ B$$

Where A and B are +ve real constants.

Its time varying form is

$$\tilde{\vec{E}}\left(0,t\right) = \hat{a}xA\cos\omega t - \hat{a}yB\sin\omega t$$

$$\therefore \tilde{E}_x = A\cos\omega t$$

$$\tilde{E}_y = -B\sin\omega t$$

$$\therefore \frac{\tilde{E}_{x^2}}{A^2} + \frac{\tilde{E}_{y^2}}{B^2} = 1$$

Thus the end point of the $\tilde{\vec{E}}\left(0,t\right)$ vector traces out an ellipse and the wave is elliptically polarized; the sense of polarization is left-handed.

Elliptical polarization is a more general form of polarization. The polarization is completely specified by the orientation and axial ratio of the polarization ellipse and by the sense in which the end point of the electric field moves around the ellipse.