Module 1:
Introduction

1.1.1 Signal definition

A **signal** is a function representing a physical quantity or variable, and typically it contains information about the behaviour or nature of the phenomenon.

For instance, in a RC circuit the signal may represent the voltage across the capacitor or the current flowing in the resistor. Mathematically, a signal is represented as a function of an independent variable **‘t’**. Usually **‘t’** represents time. Thus, a signal is denoted by **x(t)**.

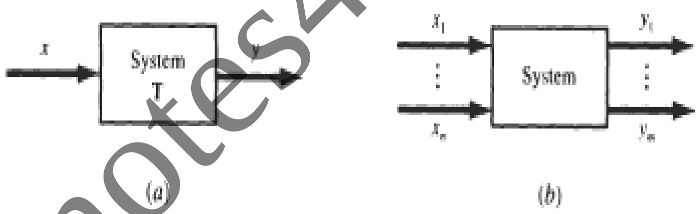
1.1.2 System definition

A system is a mathematical model of a physical process that relates the input (or excitation) signal to the output (or response) signal.

Let x and y be the input and output signals, respectively, of a system. Then the system is viewed as a transformation (or mapping) of **x** into y. This transformation is represented by the mathematical notation

y= Tx --(1.1)

where **T** is the operator representing some well-defined rule by which **x** is transformed into y. Relationship (1.1) is depicted as shown in Fig. 1-1(a). Multiple input and/or output signals are possible as shown in Fig. 1-1(b). We will restrict our attention for the most part in this text to the single-input, single-output case.



1.1 System with single or multiple input and output signals

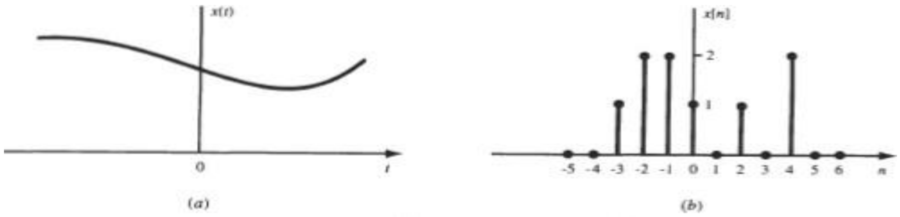
1.2 Classification of signals

Basically seven different classifications are there:

- Continuous-Time and Discrete-Time Signals
- Analog and Digital Signals
- Real and Complex Signals
- Deterministic and Random Signals
- Even and Odd Signals
- Periodic and Nonperiodic Signals
- Energy and Power Signals

Continuous-Time and Discrete-Time Signals

A signal x(t) is a continuous-time signal if t is a continuous variable. If t is a discrete variable, that is, x(t) is defined at discrete times, then x(t) is a discrete-time signal. Since a discrete-time signal is defined at discrete times, a discrete-time signal is often identified as a sequence of numbers, denoted by {x,) or x[n], where n = integer. Illustrations of a continuous-time signal x(t) and of a discrete-time signal x[n] are shown in Fig. 1-2.



1.2 Graphical representation of (a) continuous-time and (b) discrete-time signals

Analog and Digital Signals

If a continuous-time signal x(t) can take on any value in the continuous interval (a, b), where a may be $-\infty$ and b may be $+\infty$ then the continuous-time signal x(t) is called an analog signal. If a discrete-time signal x[n] can take on only a finite number of distinct values, then we call this signal a digital signal.

Real and Complex Signals

A signal x(t) is a real signal if its value is a real number, and a signal x(t) is a complex signal if its value is a complex number. A general complex signal x(t) is a function of the form

$$x(t) = x_1(t) + jx_2(t) \text{ ------------------------------ } 1.2$$

where $x_1(t)$ and $x_2(t)$ are real signals and $j = \sqrt{-1}$

Note that in Eq. (1.2) **'t'** represents either a continuous or a discrete variable.

Deterministic and Random Signals:

Deterministic signals are those signals whose values are completely specified for any given time. Thus, a deterministic signal can be modelled by a known function of time **'t'.**

Random signals are those signals that take random values at any given time and must be characterized statistically.

Even and Odd Signals

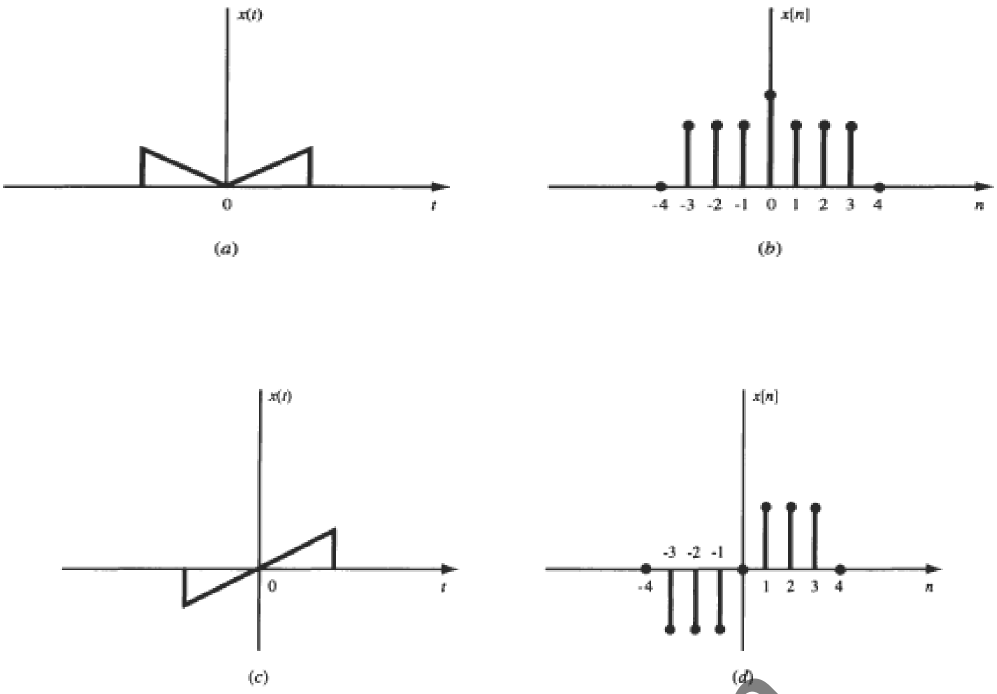
A signal ***x (t)*** or ***x[n]*** is referred to as an ***even*** signal if

$$x(-t) = x(t)$$
$$x[-n] = x[n] \text{ -------------(1.3)}$$

A signal ***x (t)*** or ***x[n]*** is referred to as an ***odd*** signal if

$$x(-t) = -x(t)$$
$$x[-n] = -x[n] \text{--------------(1.4)}$$

Examples of even and odd signals are shown in Fig. 1.3.



1.3 Examples of even signals (a and b) and odd signals (c and d).

Any signal x(t) or x[n] can be expressed as a sum of two signals, one of which is even and one of which is odd. That is,

$$x(t) = x_o(t) + x_e(t) \quad \text{-------(1.5)}$$

Where,

$$x_e(t) = \frac{1}{2}\left(x(t) + x(-t)\right)$$

$$x_o(t) = \frac{1}{2}\left(x(t) - x(-t)\right) \quad \text{-----(1.6)}$$

Similarly for x[n],

$$x[n] = x_o[n] + x_e[n] \quad \text{-------(1.7)}$$

Where,

$$x_e[n] = \frac{1}{2}\left(x[n] + x[-n]\right)$$

$$x_o[n] = \frac{1}{2}\left(x[n] - x[-n]\right) \quad \text{--------(1.8)}$$

Note that the product of two even signals or of two odd signals is an even signal and that the product of an even signal and an odd signal is an odd signal.

Periodic and Nonperiodic Signals

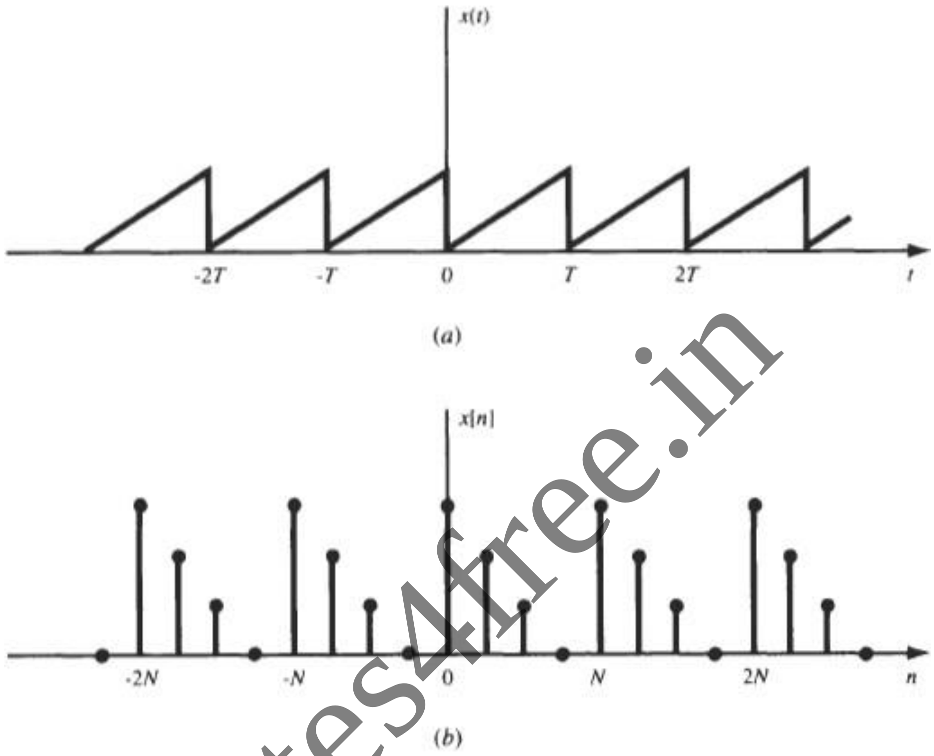
A continuous-time signal x (t) is said to be periodic with period T if there is a positive nonzero value of T for which

$$x(t + T) = x(t) \qquad \text{all } t \qquad \text{...........(1.9)}$$

An example of such a signal is given in Fig. 1-4(a). From Eq. (1.9) or Fig. 1-4(a) it follows that

$$x(t + mT) = x(t)$$ ---------------------------(1.10)

for all t and any integer m. The fundamental period T, of x(t) is the smallest positive value of T for which Eq. (1.9) holds. Note that this definition does not work for a constant



1.4 Examples of periodic signals.

signal x(t) (known as a dc signal). For a constant signal x(t) the fundamental period is undefined since x(t) is periodic for any choice of T (and so there is no smallest positive value). Any continuous-time signal which is not periodic is called a nonperiodic (or aperiodic) signal.

Periodic discrete-time signals are defined analogously. A sequence (discrete-time signal) x[n] is periodic with period N if there is a positive integer N for which

$$x[n + N] = x[n] \qquad \text{all } n$$ ……….(1.11)

An example of such a sequence is given in Fig. 1-4(b). From Eq. (1.11) and Fig. 1-4(b) it follows that

$$x[n + mN] = x[n]$$ ……………………..(1.12)

for all n and any integer m. The fundamental period N_o of x[n] is the smallest positive integer N for which Eq.(1.11) holds. Any sequence which is not periodic is called a nonperiodic (or aperiodic sequence.

Note that a sequence obtained by uniform sampling of a periodic continuous-time signal may not be periodic. Note also that the sum of two continuous-time periodic signals may not be periodic but that the sum of two periodic sequences is always periodic.

Energy and Power Signals

Consider v(t) to be the voltage across a resistor R producing a current i(t). The instantaneous power p(t) per ohm is defined as

$$p(t) = \frac{v(t)i(t)}{R} = i^2(t)$$
…………(1.13)

Total energy E and average power P on a per-ohm basis are

$$E = \int_{-\infty}^{\infty} i^2(t)\,dt \quad \text{joules}$$

$$P = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} i^2(t)\,dt \quad \text{watts}$$
……(1.14)

For an arbitrary continuous-time signal x(t), the normalized energy content E of x(t) is defined as

$$E = \int_{-\infty}^{\infty} |x(t)|^2\,dt$$
…………………(1.15)

The normalized average power P of x(t) is defined as

$$P = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2\,dt$$
(1.16)

Similarly, for a discrete-time signal x[n], the normalized energy content ***E*** of x[n] is defined as

$$E = \sum_{n=-\infty}^{\infty} |x[n]|^2$$
(1.17)

The normalized average power P of x[n] is defined as

$$P = \lim_{N \to \infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x[n]|^2$$
(1.18)

Based on definitions (1.15) to (1.18), the following classes of signals are defined:

1. x(t) (or x[n]) is said to be an energy signal (or sequence) if and only if $0 < E < m$, and so $P = 0$.
2. x(t) (or x[n]) is said to be a power signal (or sequence) if and only if $0 < P < m$, thus implying that $E = m$.
3. Signals that satisfy neither property are referred to as neither energy signals nor power signals.

Note that a periodic signal is a power signal if its energy content per period is finite, and then the average power of this signal need only be calculated over a period

1.3 Basic Operations on signals

The operations performed on signals can be broadly classified into two kinds

- Operations on dependent variables
- Operations on independent variables

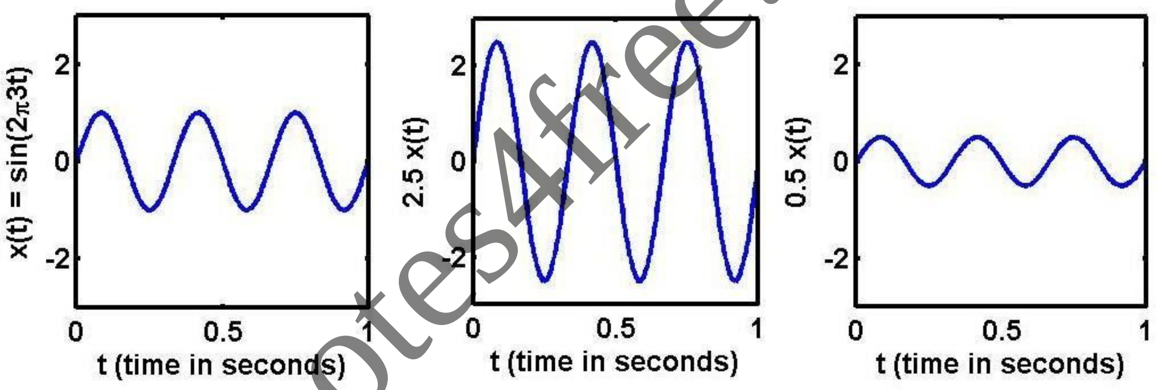
Operations on dependent variables

The operations of the dependent variable can be classified into five types: amplitude scaling, addition, multiplication, integration and differentiation.

Amplitude scaling

Amplitude scaling of a signal $x(t)$ given by equation 1.19, results in amplification of $x(t)$ if $a > 1$, and attenuation if $a < 1$.

$$y(t) = ax(t) \ldots\ldots\ldots (1.20)$$

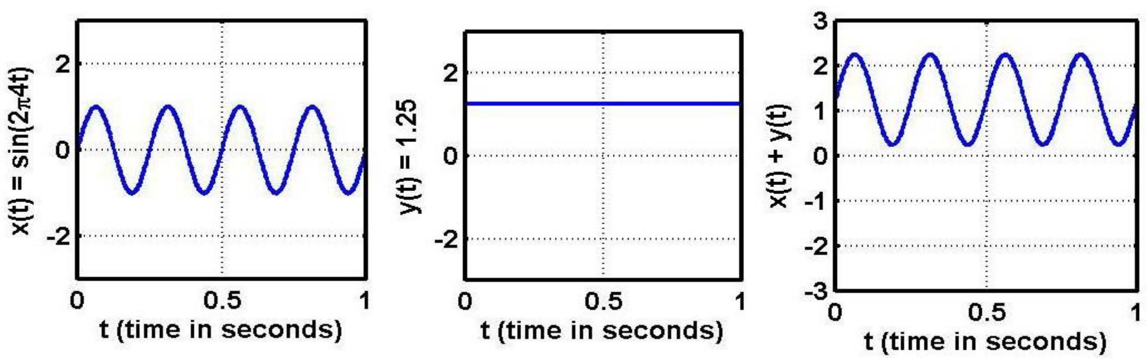


1.5 Amplitude scaling of sinusoidal signal

Addition

The addition of signals is given by equation of 1.21.

$$y(t) = x1(t) + x2\,(t) \ldots\ldots (1.21)$$



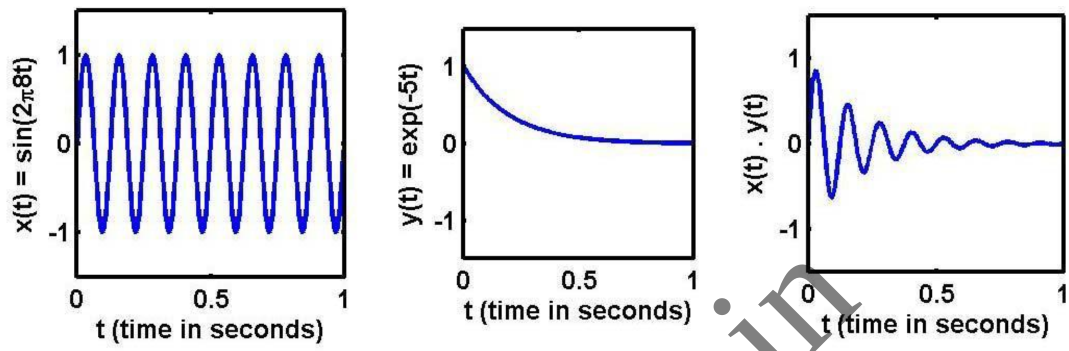
1.6 Example of the addition of a sinusoidal signal with a signal of constant amplitude (positive constant)

Physical significance of this operation is to add two signals like in the addition of the background music along with the human audio. Another example is the undesired addition of noise along with the desired audio signals.

Multiplication

The multiplication of signals is given by the simple equation of 1.22.

$y(t) = x1(t).x2\ (t)$……..(1.22)



1.7 Example of multiplication of two signals

Differentiation

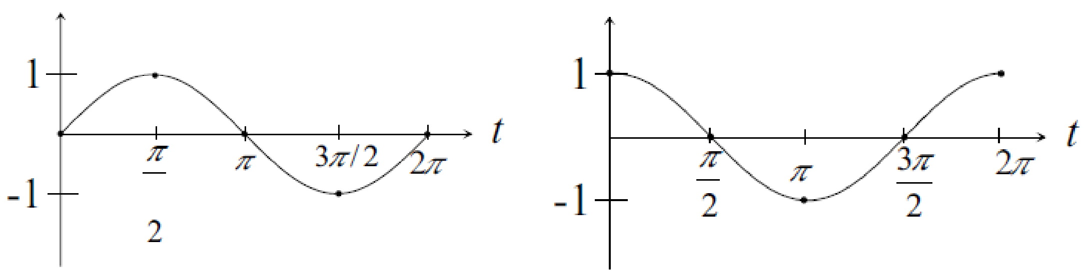
The differentiation of signals is given by the equation of 1.23 for the continuous.

$$y(t) = \frac{d}{dt} x(t) \quad …..1.23$$

The operation of differentiation gives the rate at which the signal changes with respect to time, and can be computed using the following equation, with Δt being a small interval of time.

$$\frac{d}{dt} x(t) = \lim_{\Delta t \to 0} \frac{x(t+\Delta t) - x(t)}{\Delta t} \quad ….1.24$$

If a signal doesn"t change with time, its derivative is zero, and if it changes at a fixed rate with time, its derivative is constant. This is evident by the example given in figure 1.8.

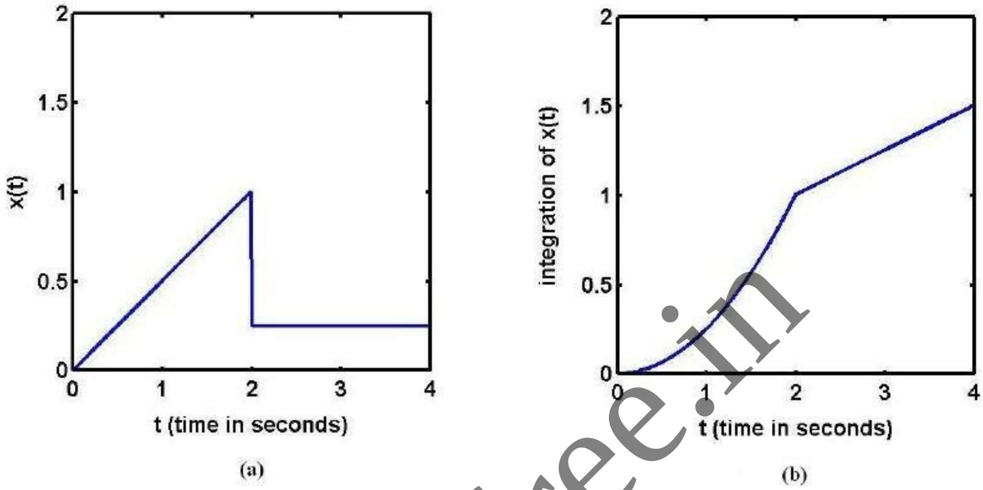


1.8 Differentiation of Sine - Cosine

Integration

The integration of a signal $x(t)$, is given by equation 1.25

$$y(t) = \int_{-\infty}^{t} x(\tau)d\tau \qquad \ldots\ldots 1.25$$



1.9 Integration of x(t)

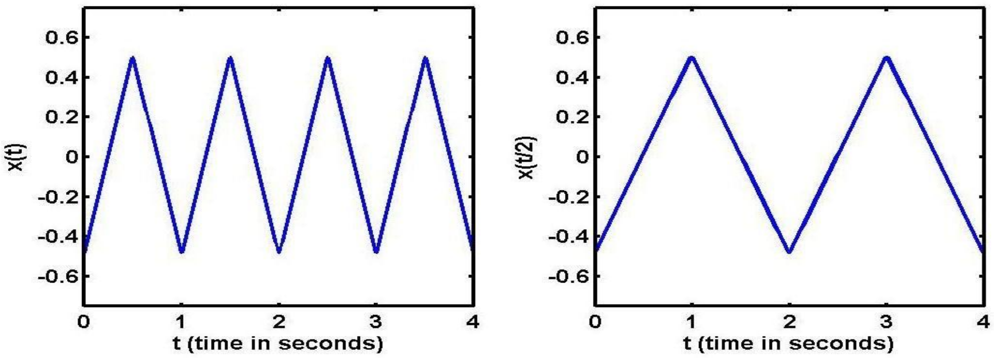
Operations on independent variables

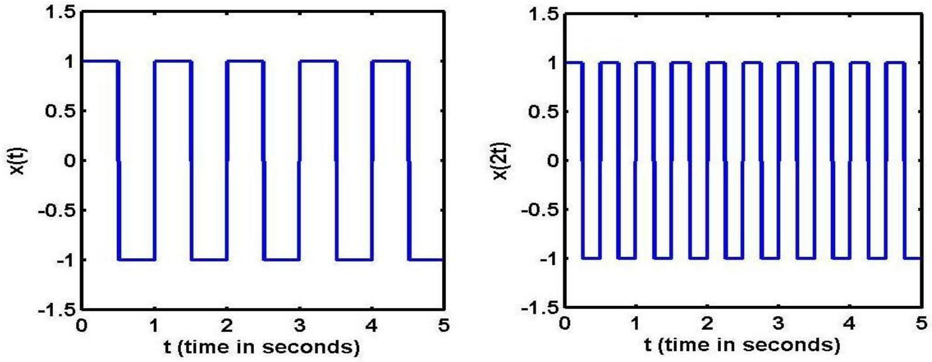
Time scaling

Time scaling operation is given by equation 1.26

$y(t) = x(at)$ 1.26

This operation results in expansion in time for a<1 and compression in time for a>1, as evident from the examples of figure 1.10.





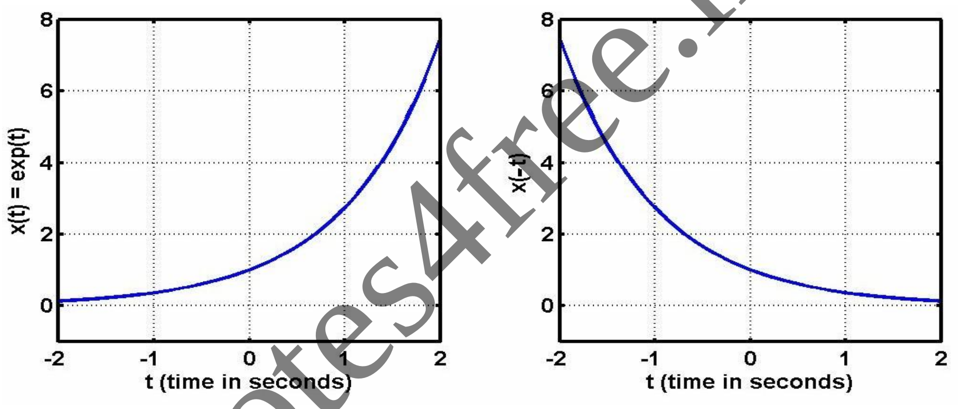
1.10 Examples of time scaling of a continuous time signal

An example of this operation is the compression or expansion of the time scale that results in the „*fast-forward'* or the „*slow motion'* in a video, provided we have the entire video in some stored form.

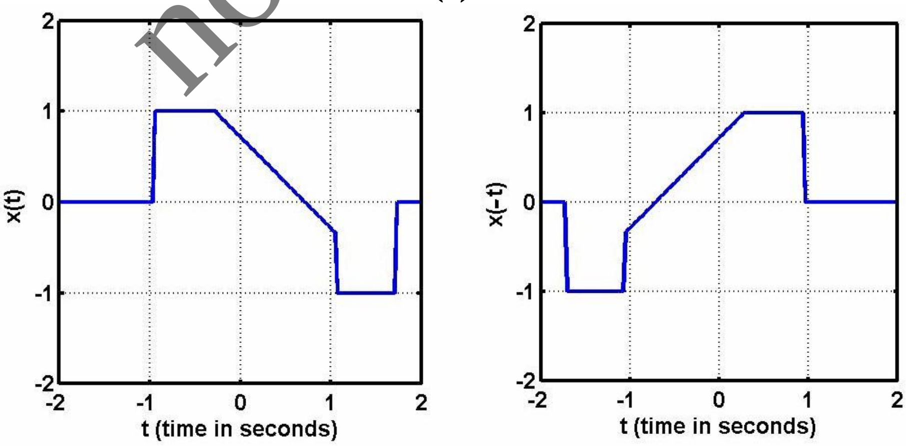
Time reflection

Time reflection is given by equation (1.27), and some examples are contained in fig1.11.

$$y(t) = x(-t) \text{.............} 1.27$$



(a)



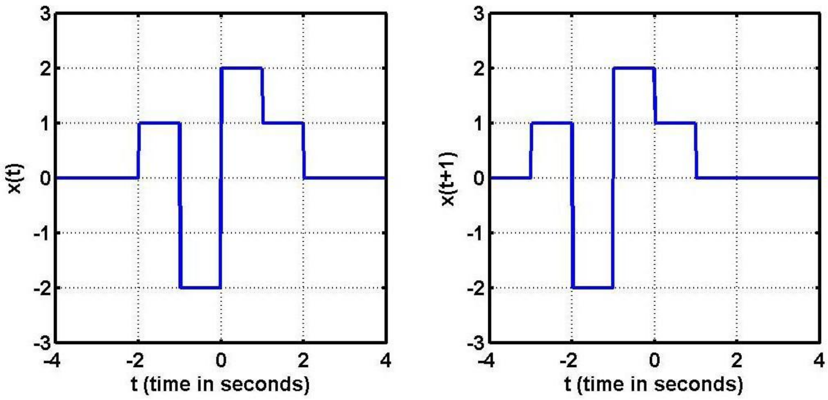
(b)

1.11 Examples of time reflection of a continuous time signal

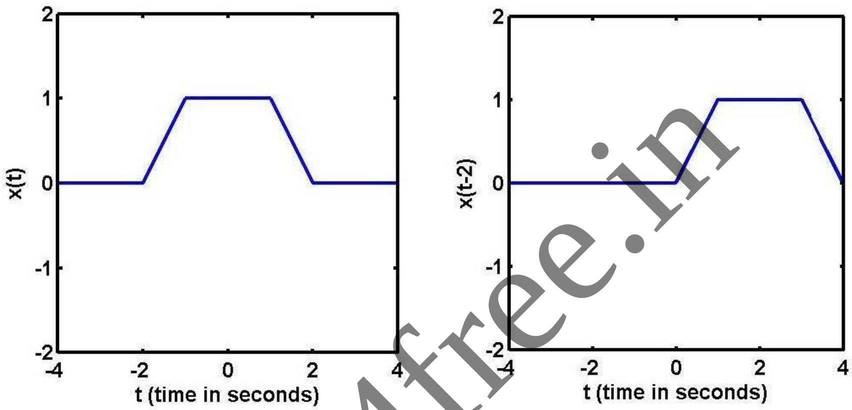
Time shifting

The equation representing time shifting is given by equation (1.28), and examples of this operation are given in figure 1.12.

$$y(t) = x(t - t0) \ldots\ldots\ldots\ldots\ldots 1.28$$



(a)



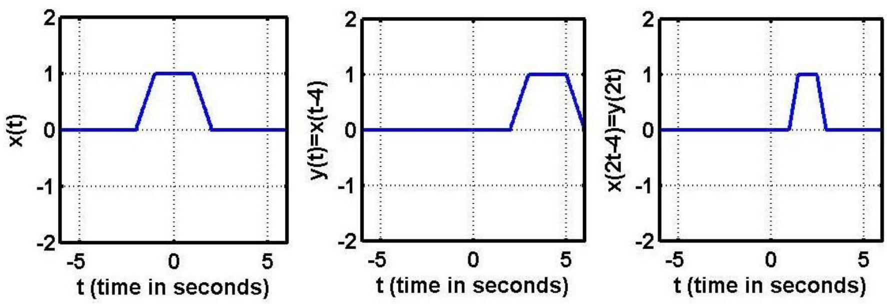
(b)

1.12 Examples of time shift of a continuous time signal

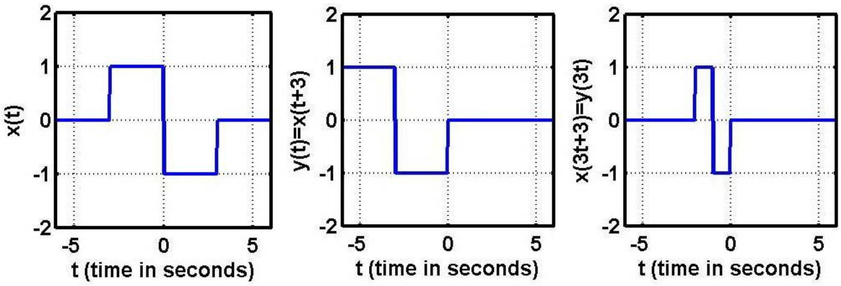
Time shifting and scaling

The combined transformation of shifting and scaling is contained in equation (1.29), along with examples in figure 1.13. Here, time shift has a higher precedence than time scale.

$$y(t) = x(at - t0) \ldots\ldots\ldots\ldots\ldots 1.29$$



(a)



(b)

1.13 Examples of simultaneous time shifting and scaling. The signal has to be shifted first and then time scaled.

1.4 Elementary signals

Exponential signals:

The exponential signal given by equation (1.29), is a monotonically increasing function if $a > 0$, and is a decreasing function if $a < 0$.

$$x(t) = e^{at} \qquad \text{……………………(1.29)}$$

It can be seen that, for an exponential signal,

$$x(t + a^{-1}) = e.x(t)$$

$$x(t - a^{-1}) = e^{-1}.x(t) \qquad \text{…………………..(1.30)}$$

Hence, equation (1.30), shows that change in time by $\pm 1/a$ seconds, results in change in magnitude by $e\pm1$. The term $1/a$ having units of time, is known as the time-constant. Let us consider a decaying exponential signal

$$x(t) = e^{-at} \quad \text{for } t \geq 0. \qquad \text{……………(1.31)}$$

This signal has an initial value $x(0) = 1$, and a final value $x(\infty) = 0$. The magnitude of this signal at five times the time constant is,

$$x(5/a) = 6.7x10^{-3} \qquad \text{…………………(1.32)}$$

while at ten times the time constant, it is as low as,

$$x(10/a) = 4.5x10^{-5} \qquad \text{……………(1.33)}$$

-

It can be seen that the value at ten times the time constant is almost zero, the final value of the signal. Hence, in most engineering applications, the exponential signal can be said to have reached its final value in about ten times the time constant. If the time constant is 1 second, then final value is achieved in 10 seconds!! We have some examples of the exponential signal in figure 1.14.

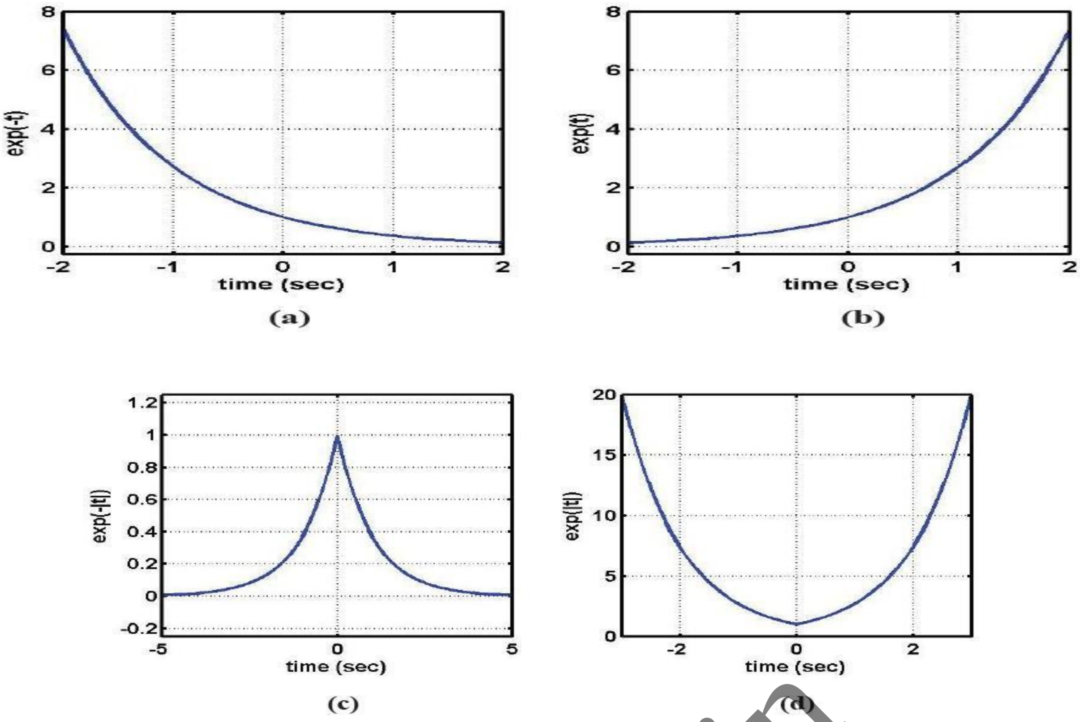


Fig 1.14 The continuous time exponential signal (a) e^{-t} , (b) e^{t} , (c) $e^{-|t|}$, and (d) $e^{|t|}$

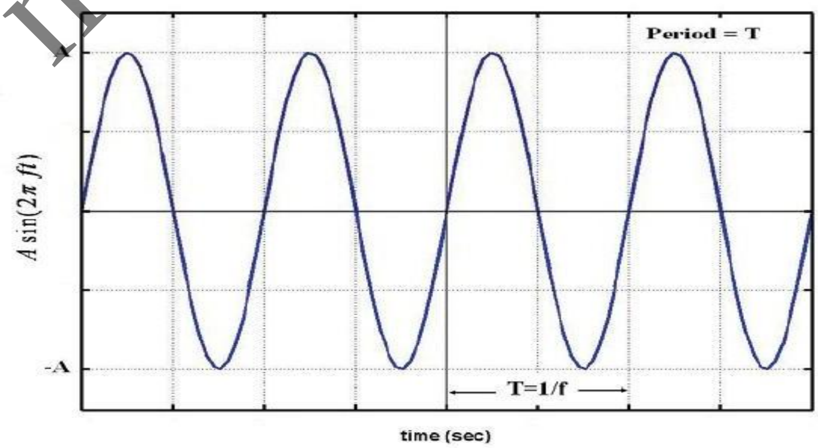
The sinusoidal signal:

The sinusoidal continuous time periodic signal is given by equation 1.34, and examples are given in figure 1.15

$$x(t) = A\sin(2\pi ft) \quad \text{..........................(1.34)}$$

The different parameters are:

Angular frequency $\omega = 2\pi f$ in radians,
Frequency f in Hertz, (cycles per second)
Amplitude A in Volts (or Amperes)
Period T in seconds



The complex exponential:

We now represent the complex exponential using the Euler"s identity (equation (1.35)),

$$e^{j\theta} = \left(\cos\theta + j\sin\theta\right)$$

...............(1.35)

to represent sinusoidal signals. We have the complex exponential signal given by equation (1.36)

$$e^{j\omega t} = \left(\cos(\omega t) + j\sin(\omega t)\right)$$
$$e^{-j\omega t} = \left(\cos(\omega t) - j\sin(\omega t)\right)$$

………(1.36)

Since sine and cosine signals are periodic, the complex exponential is also periodic with the same period as sine or cosine. From equation (1.36), we can see that the real periodic sinusoidal signals can be expressed as:

$$\cos(\omega t) = \left(\frac{e^{j\omega t} + e^{-j\omega t}}{2}\right)$$

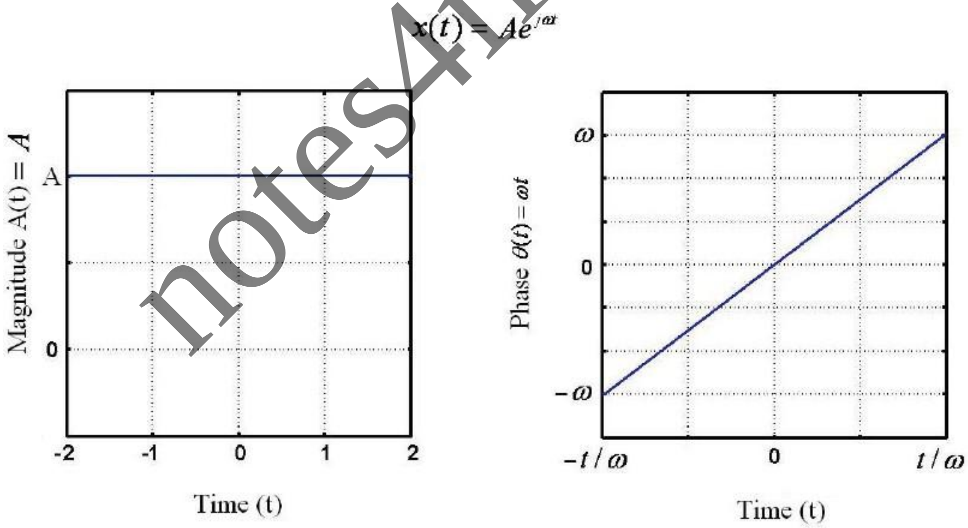
$$\sin(\omega t) = \left(\frac{e^{j\omega t} - e^{-j\omega t}}{2j}\right)$$

………………..(1.37)

Let us consider the signal $x(t)$ given by equation (1.38). The sketch of this is given in fig 1.15

$$x(t) = A(t)e^{j\theta(t)}$$

……………………..(1.38)



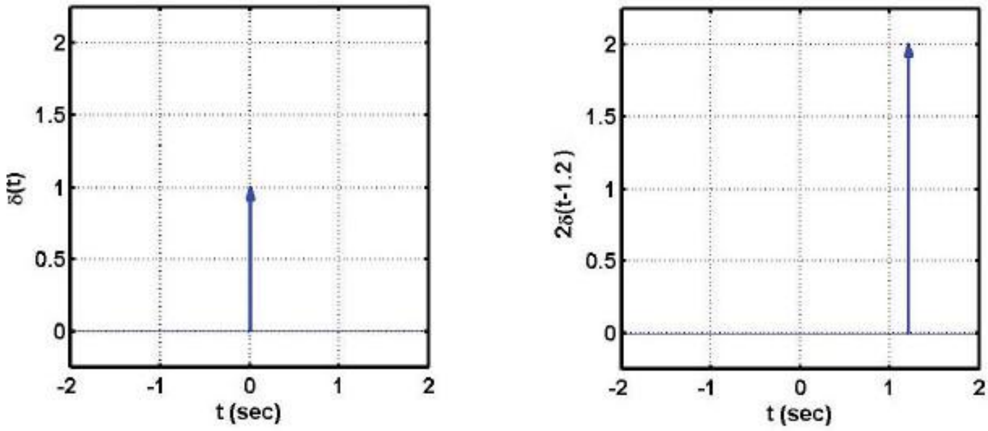
The unit impulse:

The unit impulse usually represented as δ (t) , also known as the dirac delta function, is given by,

$$\delta(t) = 0 \quad for \quad t \neq 0; \quad and \quad \int_{-\infty} \delta(t)dt = 1$$

…….(1.38)

From equation (1.38), it can be seen that the impulse exists only at $t = 0$, such that its area is 1. This is a function which cannot be practically generated. Figure 1.16, has the plot of the impulse function



The unit step:

The unit step function, usually represented as $u(t)$, is given by,

$$u(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases}$$

……………….(1.39)

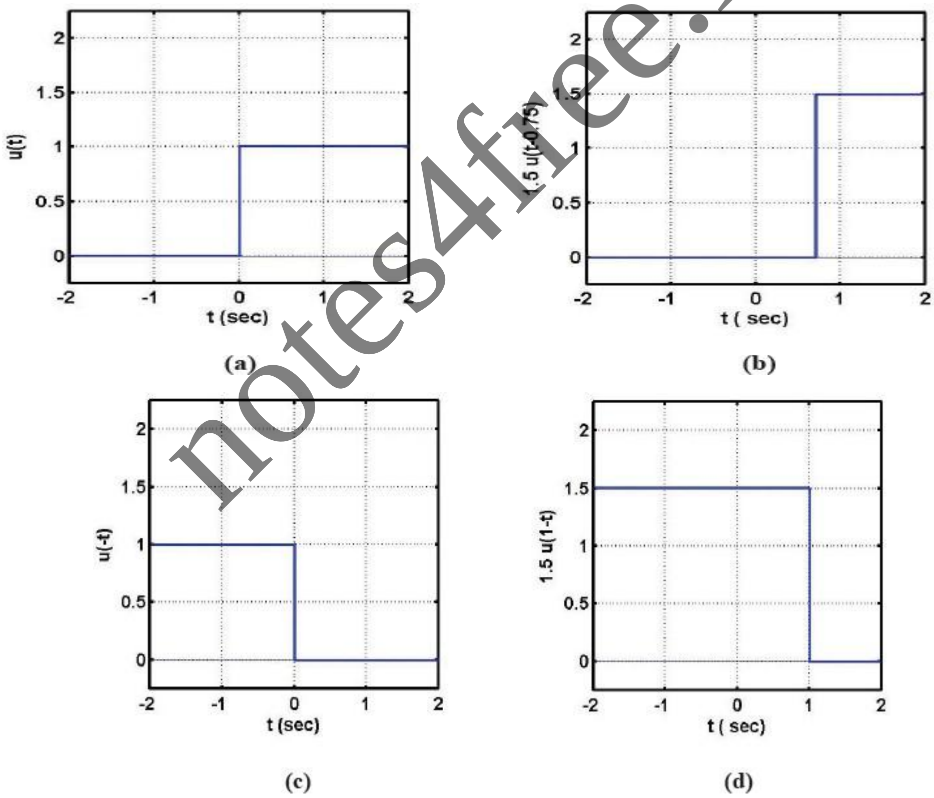


Fig 1.17 Plot of the unit step function along with a few of its transformations

The unit ramp:

The unit ramp function, usually represented as $r(t)$, is given by,

$$r(t) = \begin{cases} t & t \geq 0 \\ 0 & t < 0 \end{cases}$$

…………….(1.40)

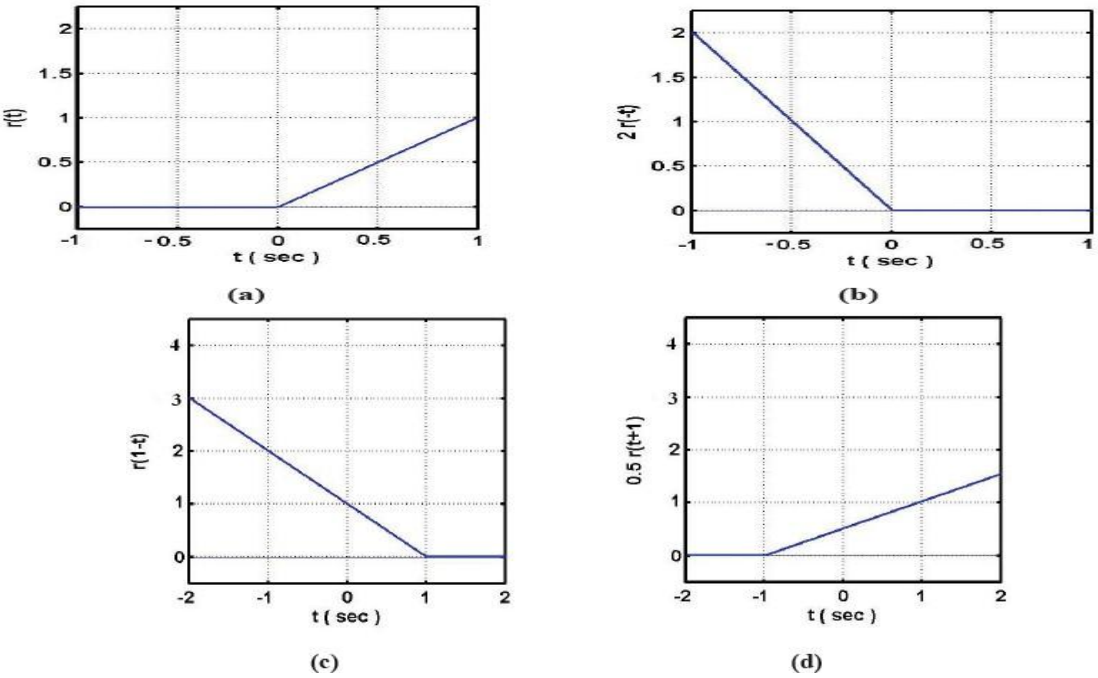


Fig 1.18 Plot of the unit ramp function along with a few of its transformations

The signum function:

The signum function, usually represented as $\text{sgn}(t)$, is given by

$$\text{sgn}(t) = \begin{cases} 1 & t > 0 \\ 0 & t = 0 \\ -1 & t < 0 \end{cases}$$

…………………………….(1.41)

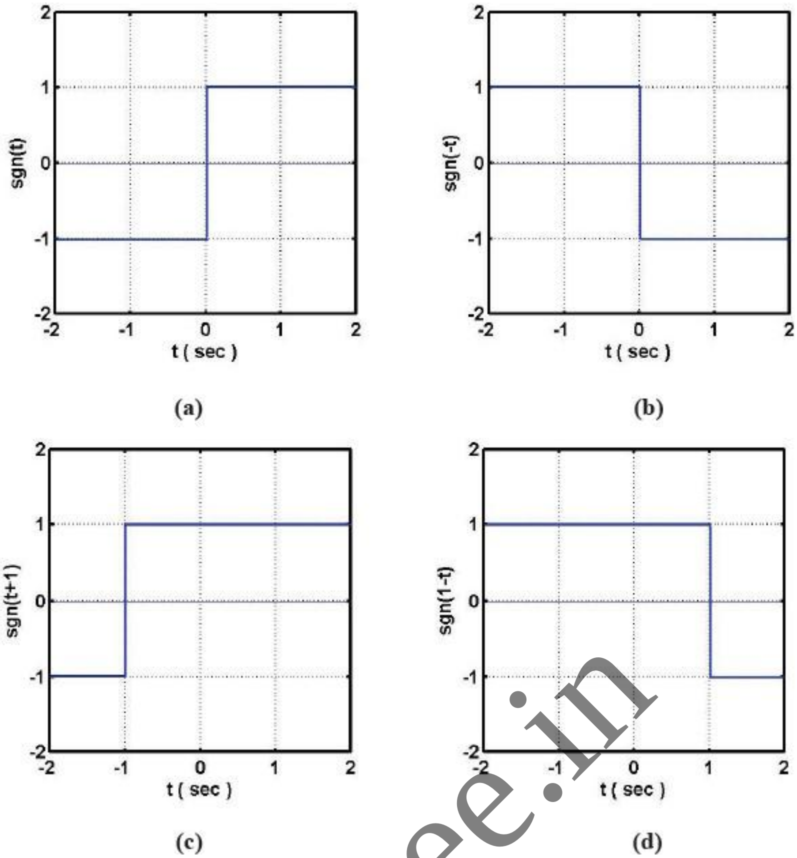


Fig 1.19 Plot of the unit signum function along with a few of its transformations

1.5 System viewed as interconnection of operation:

This article is dealt in detail again in chapter 2/3. This article basically deals with system connected in series or parallel. Further these systems are connected with adders/subtractor, multipliers etc.

1.6 Properties of system:

In this article discrete systems are taken into account. The same explanation stands for continuous time systems also.

The discrete time system:
The discrete time system is a device which accepts a discrete time signal as its input, transforms it to another desirable discrete time signal at its output as shown in figure 1.20

Fig 1.20 DT system

Stability

A system is stable if „bounded input results in a bounded output". This condition, denoted by BIBO, can be represented by:

$$\sum_{n=-\infty}^{\infty} |x[n]| < \infty \quad implies \quad \sum_{n=-\infty}^{\infty} |y[n]| < \infty \quad for \quad all \quad n$$

…….(1.42)

Hence, a finite input should produce a finite output, if the system is stable. Some examples of stable and unstable systems are given in figure 1.21

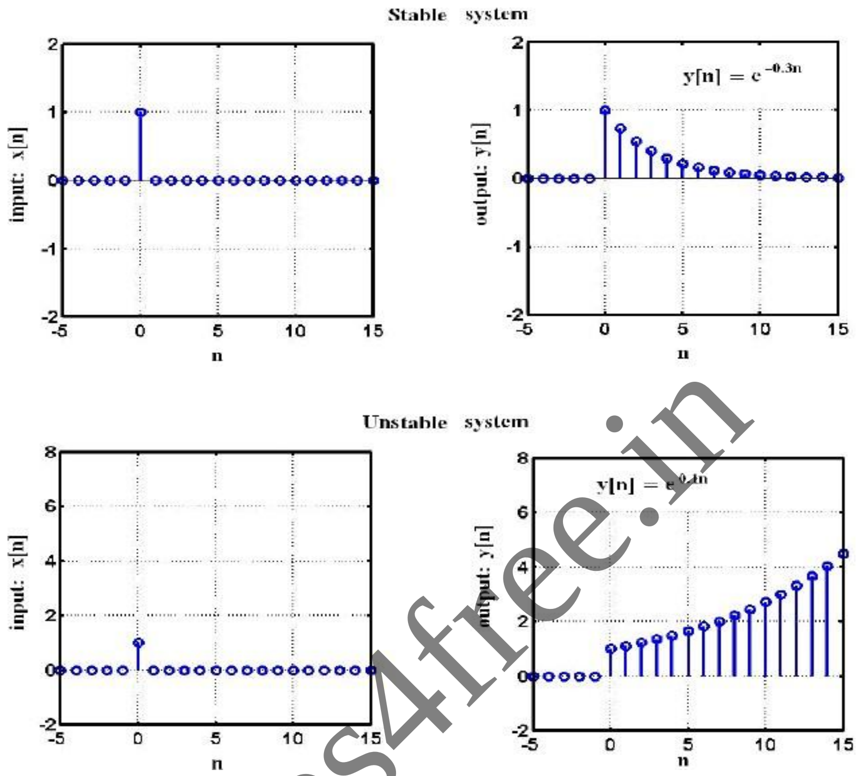


Fig 1.21 Examples for system stability

Memory

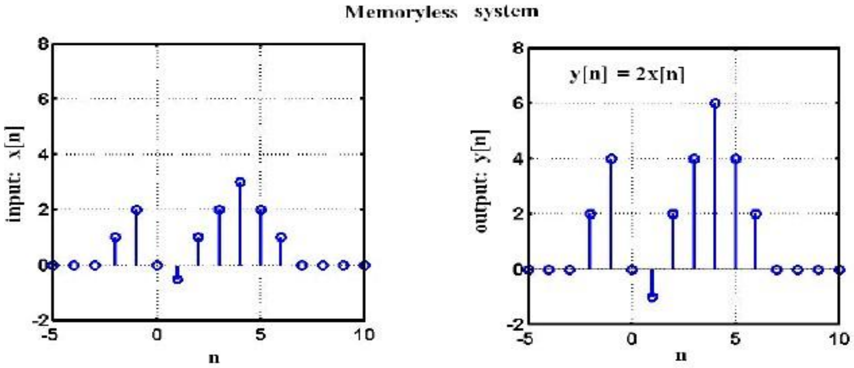
The system is memory-less if its instantaneous output depends only on the current input. In memory-less systems, the output does not depend on the previous or the future input.

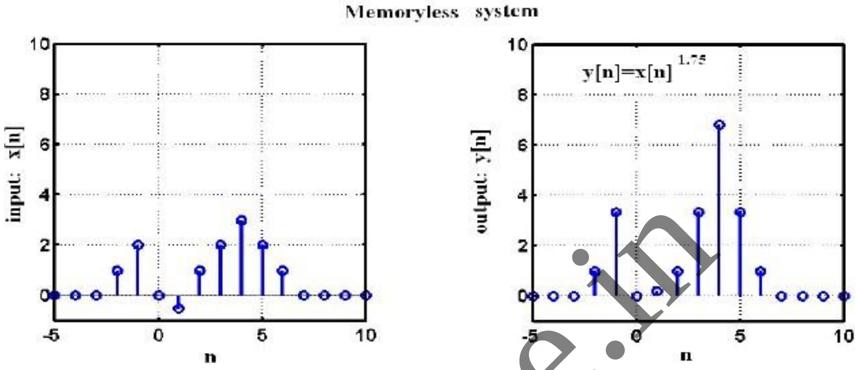
Examples of memory less systems:

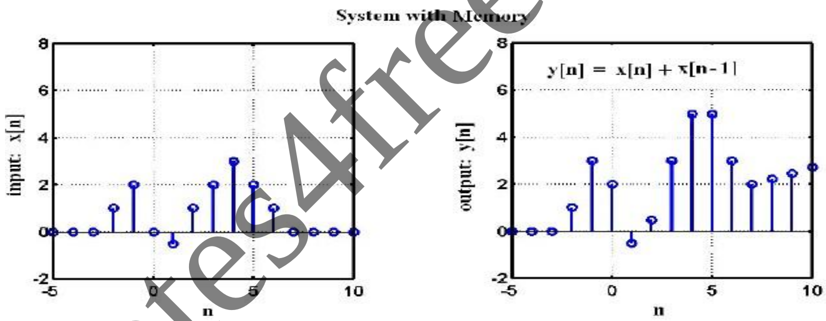
$$y[n] = ax[n]$$

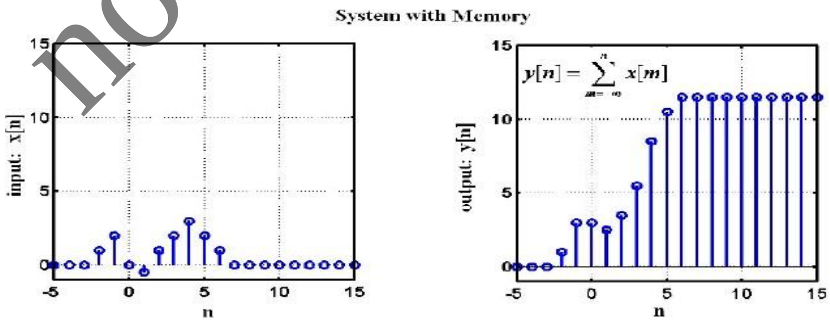
$$y[n] = ax^2[n]$$

$$i[n] = a_0 + a_1 v[n] + a_2 v^2[n] + a_3 v^3[n] + ...$$









Causality:

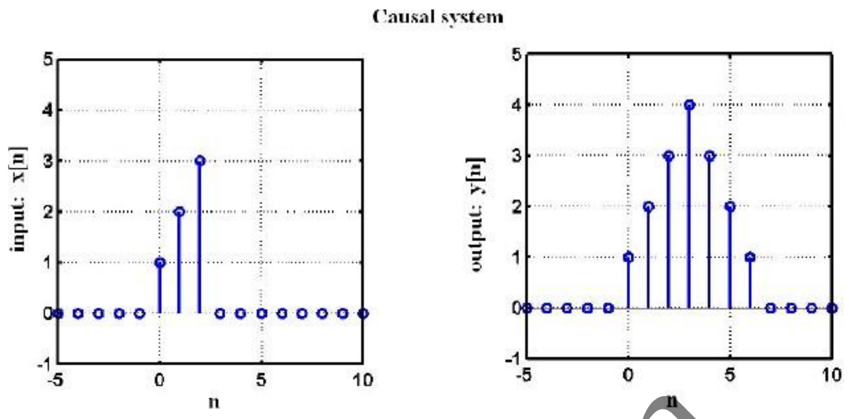
A system is causal, if its output at any instant depends on the current and past values of input. The output of a causal system does not depend on the future values of input. This can be represented as:

$$y[n] = F\{x[m]\} \quad \text{for } m \le n$$

For a causal system, the output should occur only after the input is applied, hence,

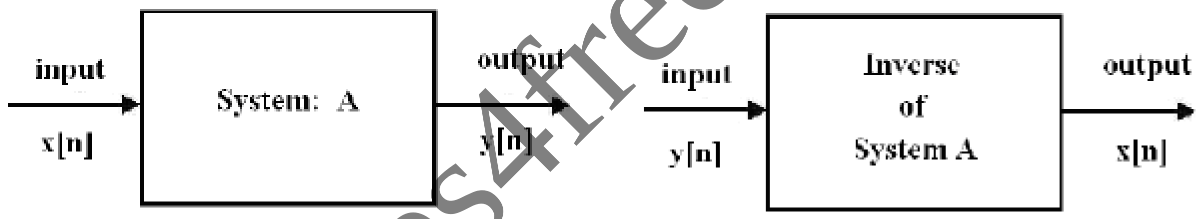
$$x[n] = 0 \text{ for } n < 0 \text{ implies } \quad y[n] = 0 \text{ for } n < 0$$

All physical systems are causal (examples in figure 7.5). Non-causal systems do not exist. This classification of a system may seem redundant. But, it is not so. This is because, sometimes, it may be necessary to design systems for given specifications. When a system design problem is attempted, it becomes necessary to test the causality of the system, which if not satisfied, cannot be realized by any means. **Hypothetical examples** of non-causal systems are given in figure below.



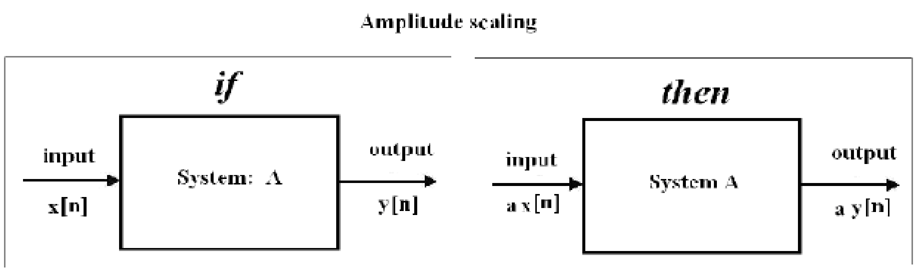
Invertibility:

A system is invertible if,

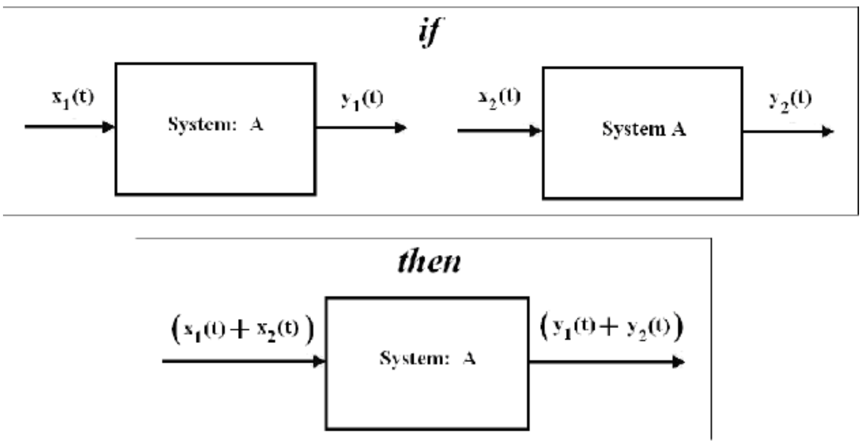


Linearity:

The system is a device which accepts a signal, transforms it to another desirable signal, and is available at its output. We give the signal to the system, because the output is s

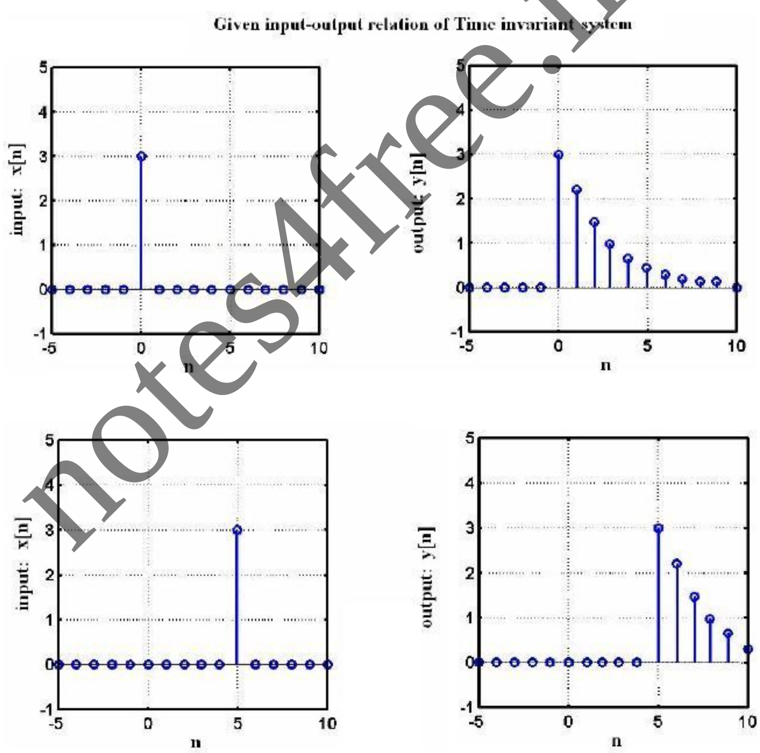


Superposition principle



Time invariance:

A system is time invariant, if its output depends on the input applied, and not on the time of application of the input. Hence, time invariant systems, give delayed outputs for delayed inputs.



Recommended Questions

1. What are even and Odd signals
2. Find the even and odd components of the following signals
   a $x(t) = \cos t + \sin t + \sin t \cos t$
   b. $x(t) + 1 + 3t^2 + 5t^3 + 9t^4$
   c. $x(t) + (1 + t^3)\cos t^3 10t$
3. What are periodic and A periodic signals. Explain for both continuous and discrete cases.
4. Determine whether the following signals are periodic. If they are periodic find the fundamental period.
   a. $x(t) = (\cos(2\pi t))^2$
   b. $x(n) = \cos(2n)$
   c. $x(n) = \cos 2\pi n$
5. Define energy and power of a signal for both continuous and discrete case.
6. Which of the following are energy signals and power signals and find the power or energy of the signal identified.
   a. $x(t) = \begin{cases} t, & 0 \le t \le 1 \\ 2 - t, & 1 \le t \le 2 \\ 0 & \text{otherwise} \end{cases}$
   b. $x(n) = \begin{cases} n, & 0 \le n \le 5 \\ 10 - n, & 5 \le n \le 10 \\ 0 & \text{otherwise} \end{cases}$
   c $x(t) = \begin{cases} 5\cos \pi t & -0.5 \le t \le 0.5 \\ 0 & \end{cases}$
   d $x(n) = \begin{cases} \sin \pi n, & -4 \le n \le 4 \\ 0 & \text{otherwise} \end{cases}$

Module 2: Time-domain representations for LTI systems – 1

Time-domain representations for LTI systems – 1: Convolution, impulse response representation, Convolution Sum and Convolution Integral.

TEXT BOOK

Simon Haykin and Barry Van Veen "Signals and Systems", John Wiley & Sons, 2001.Reprint 2002

REFERENCE BOOKS :

1. **Alan V Oppenheim, Alan S, Willsky and A Hamid Nawab,** "Signals and Systems" Pearson Education Asia / PHI, 2nd edition, 1997. Indian Reprint 2002
2. **H. P Hsu, R. Ranjan**, "Signals and Systems", Scham‟s outlines, TMH, 2006
3. **B. P. Lathi**, "Linear Systems and Signals", Oxford University Press, 2005
4. **Ganesh Rao and Satish Tunga**, "Signals and Systems", Sanguine Technical Publishers, 2004

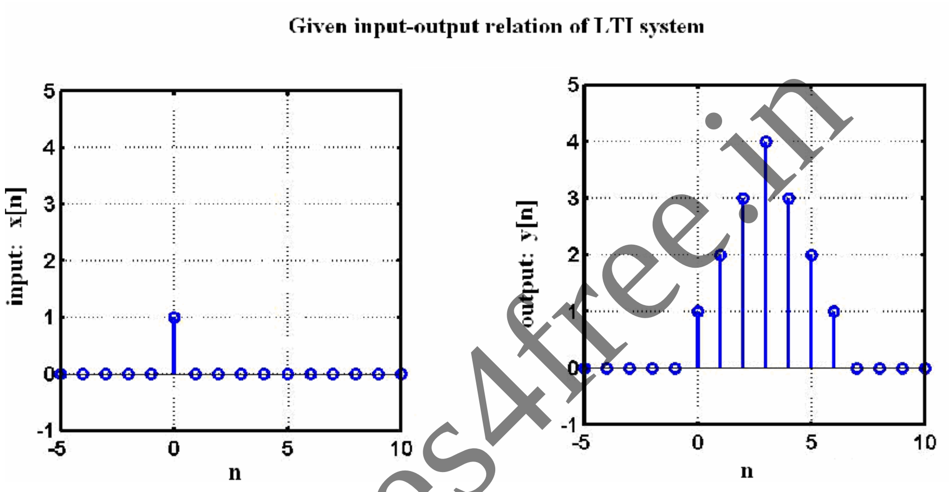
Module 2
Time-domain representations for LTI systems – 1

2.1 Introduction:

The Linear time invariant (LTI) system:

Systems which satisfy the condition of linearity as well as time invariance are known as linear time invariant systems. Throughout the rest of the course we shall be dealing with LTI systems. If the output of the system is known for a particular input, it is possible to obtain the output for a number of other inputs. We shall see through examples, the procedure to compute the output from a given input-output relation, for LTI systems.

Example – I:



2.1.1 <u>Convolution</u>:

A continuous time system as shown below, accepts a continuous time signal x(t) and gives out a transformed continuous time signal y(t).

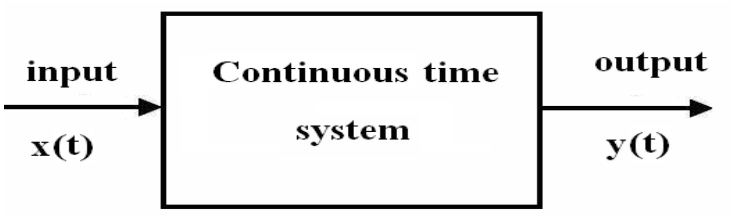


Figure 1: The continuous time system

Some of the different methods of representing the continuous time system are:

i) Differential equation
ii) Block diagram
iii) Impulse response
iv) Frequency response
v) Laplace-transform

vi) Pole-zero plot

It is possible to switch from one form of representation to another, and each of the representations is complete. Moreover, from each of the above representations, it is possible to obtain the system properties using parameters as: stability, causality, linearity, invertibility etc. We now attempt to develop the convolution integral.

2.2 Impulse Response

The impulse response of a continuous time system is defined as the output of the system when its input is an unit impulse, $\delta(t)$. Usually the impulse response is denoted by $h(t)$.

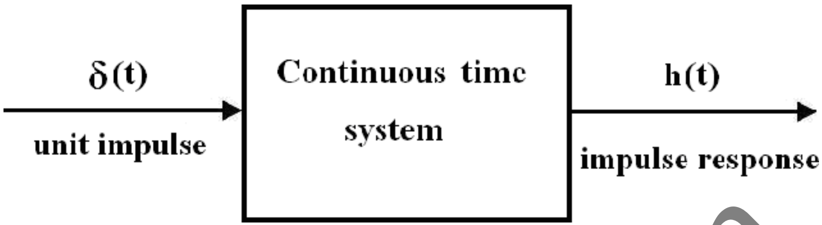
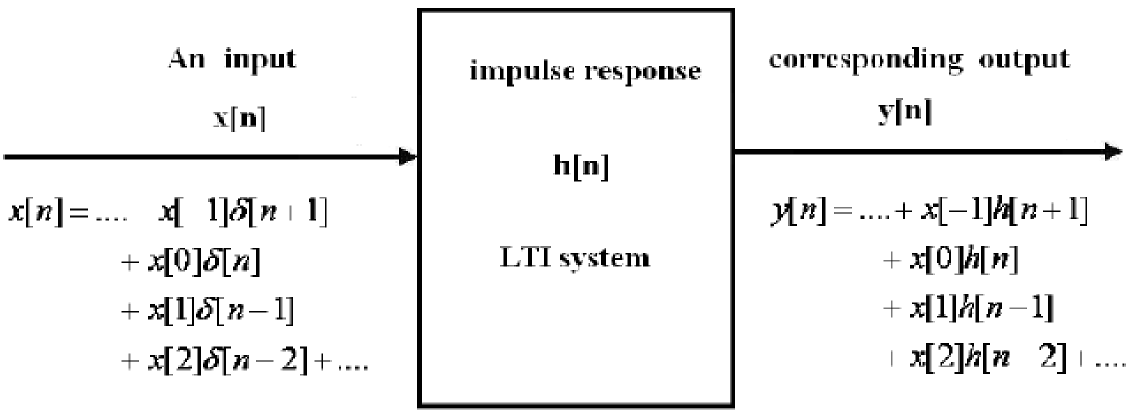
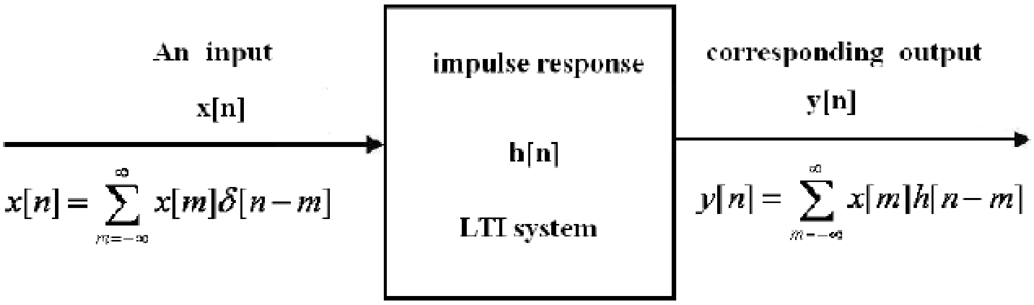


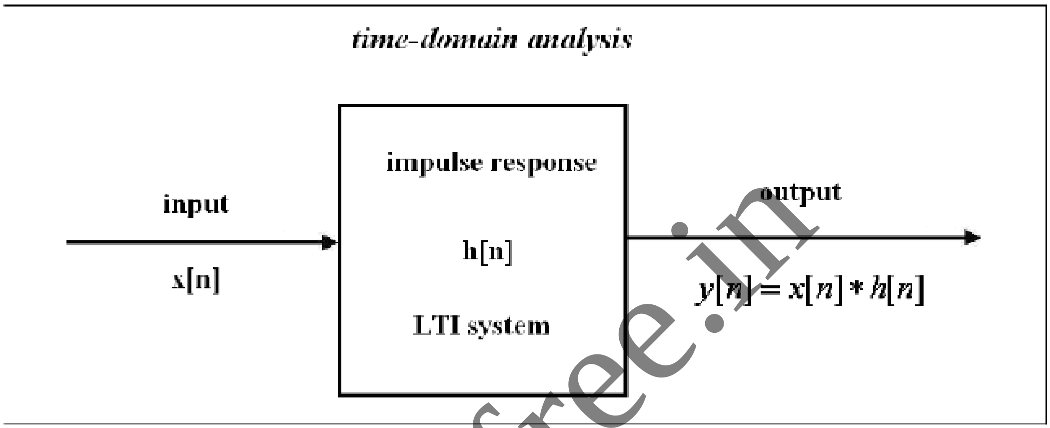
Figure 2: The impulse response of a continuous time system

2.3 Convolution Sum:

We now attempt to obtain the output of a digital system for an arbitrary input x[n], from the knowledge of the system impulse response h[n].







$$y[n]=x[n]*h[n]$$

Methods of evaluating the convolution sum:

Given the system impulse response h[n], and the input x[n], the system output y[n], is given by the convolution sum:

$$y[n]=\sum_{m=-\infty}^{\infty} x[m]h[n-m]$$

Problem:
To obtain the digital system output y[n], given the system impulse response h[n], and the system input x[n] as:

$$h[n]=[1,\ -1.5,\ 3]$$

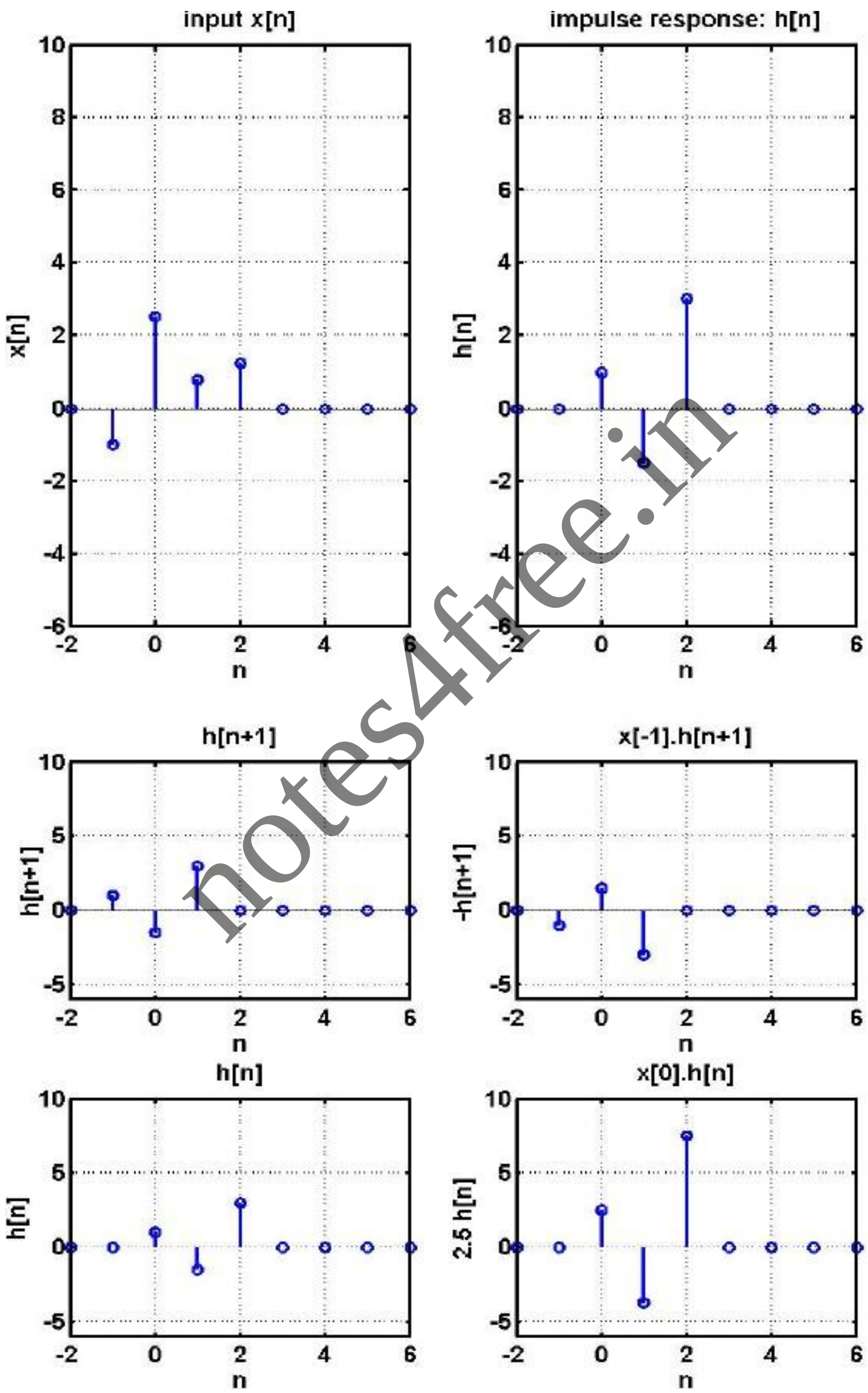
$$x[n]=[-1,\ \underset{\uparrow}{2.5},\ 0.8,\ 1.25]$$

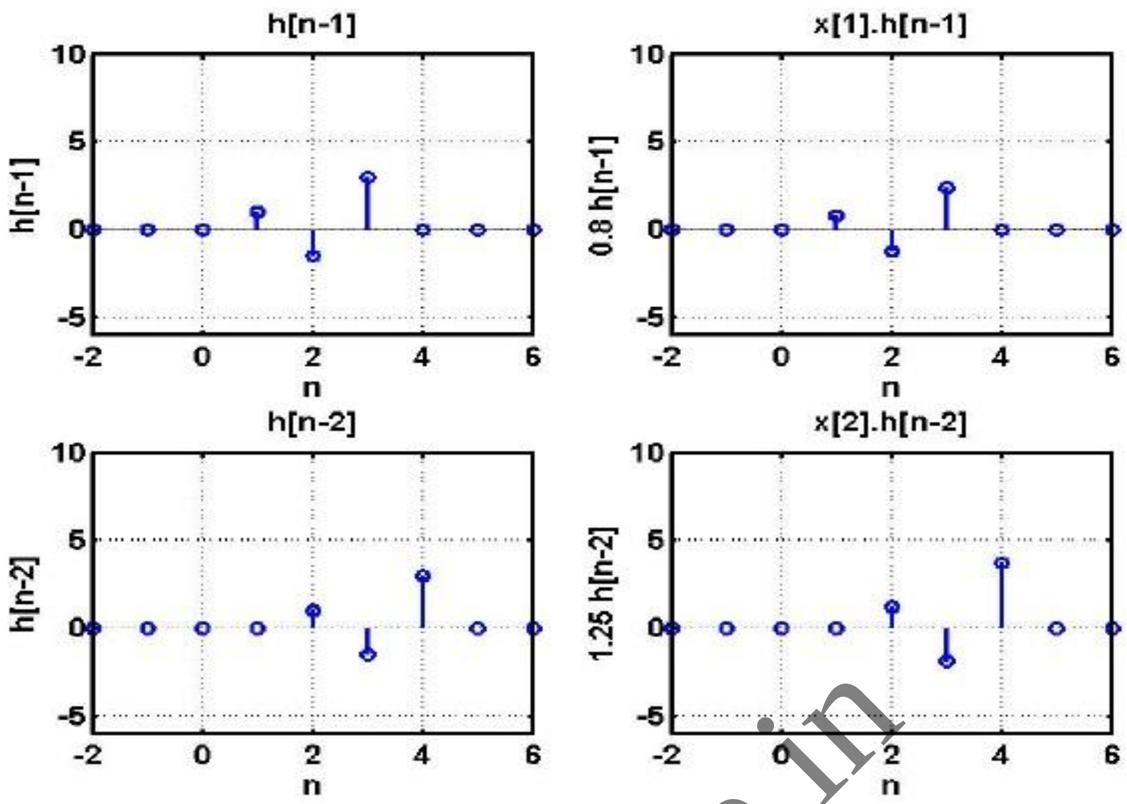
-1 4 -5.95 7.55 0.525 3.75

1. Evaluation as the weighted sum of individual responses

The convolution sum of equation (…), can be equivalently represented as:

$$y[n] = \ldots\ldots + x[-1]h[n+1] + x[0]h[n] + x[1]h[n-1] + \ldots\ldots.$$





Convolution as matrix multiplication:

Given

$$x[n] = [x_1 \quad x_2 \quad \ldots \quad x_L] \qquad \text{starting from } N_X$$

and

$$h[n] = [h_1 \quad h_2 \quad \ldots \quad h_M] \qquad \text{starting from } N_H$$

Step 1: Length of convolved sequence is NUM = (L+M-1)

Step 2: The convolved sequence starts at $i = N_X + N_H$

Step 3: The convolution is given by the following matrix multiplication

$$\begin{bmatrix} y[i] \\ y[i+1] \\ y[i+2] \\ y[i+3] \\ y[i+4] \\ y[i+5] \\ . \\ . \end{bmatrix} = \begin{bmatrix} x_1 & 0 & . & . & 0 \\ x_2 & x_1 & . & . & 0 \\ x_3 & x_2 & . & . & 0 \\ . & x_3 & . & . & 0 \\ . & . & . & . & x_1 \\ x_L & . & . & . & x_2 \\ 0 & x_L & . & . & . \\ 0 & 0 & . & . & x_L \end{bmatrix} \begin{bmatrix} h_1 \\ h_2 \\ . \\ . \\ h_M \end{bmatrix} = \begin{bmatrix} h_1 & 0 & . & . & 0 \\ h_2 & h_1 & . & . & 0 \\ h_3 & h_2 & . & . & 0 \\ . & h_3 & . & . & 0 \\ . & . & . & . & h_1 \\ h_M & . & . & . & h_2 \\ 0 & h_M & . & . & . \\ 0 & 0 & . & . & h_M \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ . \\ . \\ x_L \end{bmatrix}$$

The dimensions of the above matrices are:

$$[NUM \quad by \quad 1] = [NUM \quad by \quad M][M \quad by \quad 1] = [NUM \quad by \quad L][L \quad by \quad 1]$$

For the given example:

x[n] is of length L=4, and starts at $N_X = -1$

h[n] is of length M=3 and starts at $N_H = 0$

Step 1: Length of convolved sequence is NUM = (L+M-1)=6

Step 2: The convolved sequence starts at i=(-1+0)=(-1)

$$\begin{bmatrix} y[-1] \\ y[0] \\ y[1] \\ y[2] \\ y[3] \\ y[4] \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 2.5 & -1 & 0 \\ 0.8 & 2.5 & -1 \\ 1.25 & 0.8 & 2.5 \\ 0 & 1.25 & 0.8 \\ 0 & 0 & 1.25 \end{bmatrix} \begin{bmatrix} 1 \\ -1.5 \\ 3 \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \\ -5.95 \\ 7.55 \\ 0.525 \\ 3.75 \end{bmatrix}$$

or

$$\begin{bmatrix} y[-1] \\ y[0] \\ y[1] \\ y[2] \\ y[3] \\ y[4] \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1.5 & 1 & 0 & 0 \\ 3 & -1.5 & 1 & 0 \\ 0 & 3 & -1.5 & 1 \\ 0 & 0 & 3 & -1.5 \\ 0 & 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} -1 \\ 2.5 \\ 0.8 \\ 1.25 \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \\ -5.95 \\ 7.55 \\ 0.525 \\ 3.75 \end{bmatrix}$$

Evaluation using graphical representation:

Another method of computing the convolution is through the direct computation of each value of the output y[n]. This method is based on evaluation of the convolution sum for a single value of *n*, and varying *n* over all possible values.

$$y[n] = \sum_{m=-\infty}^{\infty} x[m]h[n-m]$$

Step 1: Sketch x[m]

Step 2: Sketch h[-m]

Step 3: Compute y[0] using:

$$y[0] = \sum_{m=-\infty}^{\infty} x[m]h[-m]$$

which is the 'sum of the product of the two signals x[m] & h[-m]'

Step 4: Sketch h[1-m], which is right shift of h[-m] by 1.

Step 5: Compute y[1] using:

$$y[1] = \sum_{m=-\infty}^{\infty} x[m]h[1-m]$$

which is the 'sum of the product of the two signals x[m] & h[1-m]'

Step 6: Sketch h[2-m], which is right shift of h[-m] by 2.

Step 7: Compute y[2] using:

$$y[2] = \sum_{m=-\infty}^{\infty} x[m]h[2-m]$$

which is the 'sum of the product of the two signals x[m] & h[2-m]'

Step 8: Proceed this way until all possible values of y[n], for positive 'n' are computed

Step 9: Sketch h[-1-m], which is left shift of h[-m] by 1.

Step 10: Compute y[-1] using:

$$y[-1] = \sum_{m=-\infty}^{\infty} x[m]h[-1-m]$$

which is the 'sum of the product of the two signals x[m] & h[-1-m]'

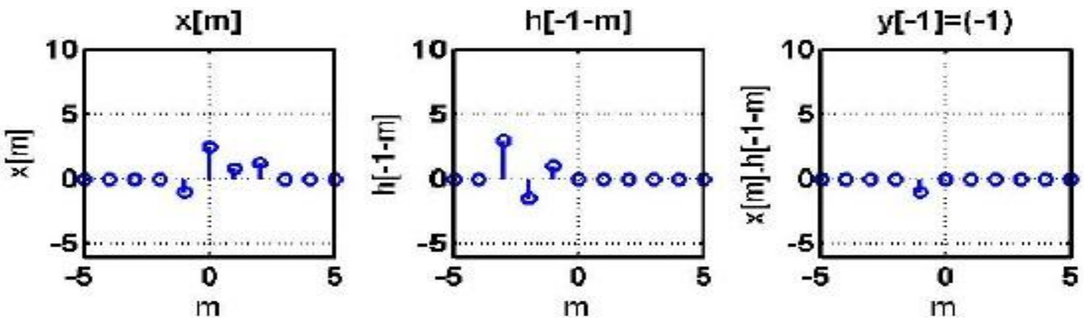
Step 11: Sketch h[-2-m], which is left shift of h[-m] by 2.

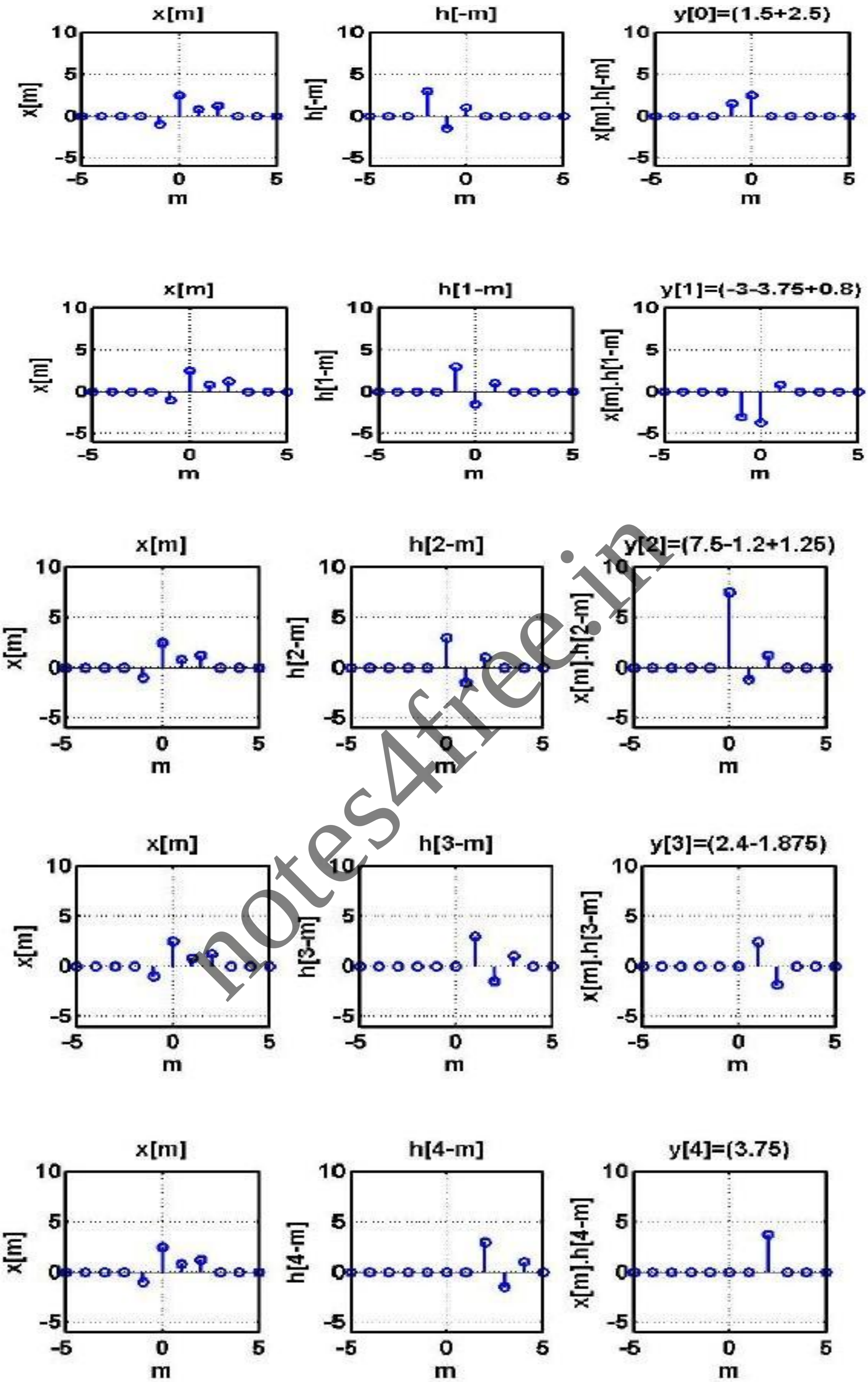
Step 12: Compute y[-2] using:

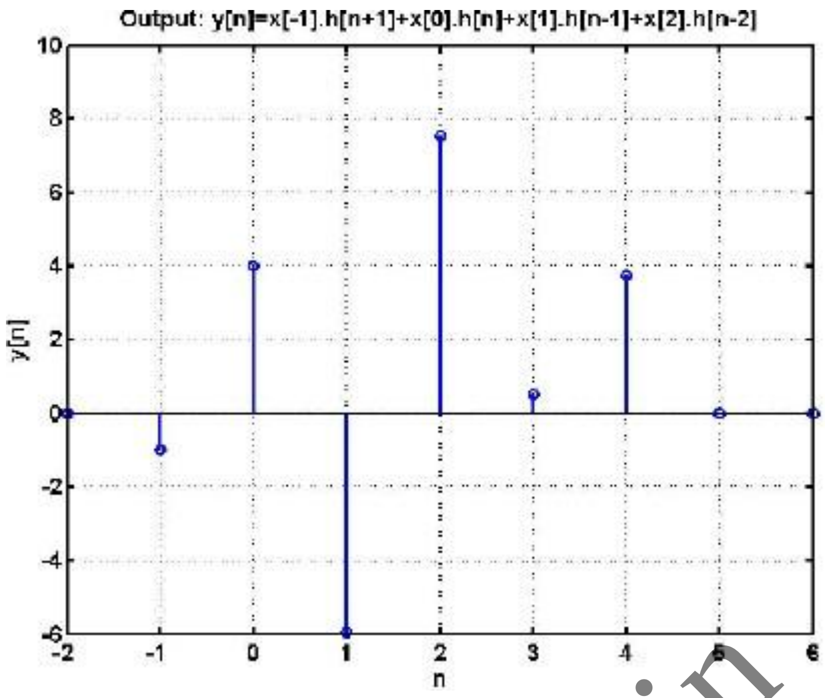
$$y[-2] = \sum_{m=-\infty}^{\infty} x[m]h[-2-m]$$

which is the 'sum of the product of the two signals x[m] & h[-2-m]'

Step 13: Proceed this way until all possible values of y[n], for negative 'n' are computed




x[m]
h[-m]
y[0]=(1.5+2.5)
x[m].h[-m]
x[m]
h[1-m]
y[1]=(-3-3.75+0.8)
x[m].h[1-m]
x[m]
h[2-m]
y[2]=(7.5-1.2+1.25)
x[m].h[2-m]
x[m]
h[3-m]
y[3]=(2.4-1.875)
x[m].h[3-m]
x[m]
h[4-m]
y[4]=(3.75)
x[m].h[4-m]
m



Evaluation from direct convolution sum:

While small length, finite duration sequences can be convolved by any of the above three methods, when the sequences to be convolved are of infinite length, the convolution is easier performed by direct use of the „convolution sum" of equation (…).

$$since: \quad u[m] = \begin{cases} 0 & for \quad m < 0 \\ 1 & for \quad m \geq 0 \end{cases}$$

$$u[n-m] = \begin{cases} 0 & for \quad (n-m) < 0 \\ 1 & for \quad (n-m) \geq 0 \end{cases}$$

$$= \begin{cases} 0 & for \quad (-m) < n \\ 1 & for \quad (-m) \geq n \end{cases}$$

$$= \begin{cases} 0 & for \quad m > n \\ 1 & for \quad m \leq n \end{cases}$$

Example: A system has impulse response $h[n] = \exp(-0.8n)u[n]$. Obtain the unit step response.

Solution:

$$y[n] = \sum_{m=-\infty}^{\infty} h[m]x[m]$$

$$= \sum_{m=-\infty}^{\infty} \{\exp(-0.8(m))u[m]\}\{u[n-m]\}$$

$$= \sum_{m=0}^{\infty} \{\exp(-0.8(m))\}\{u[n-m]\}$$

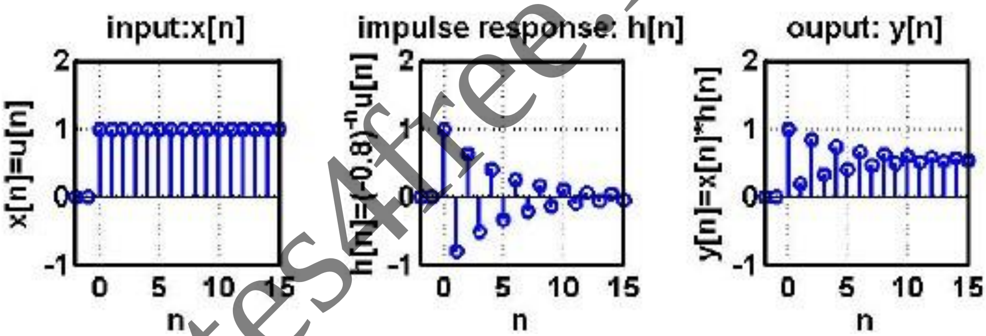
$$= \sum_{m=0}^{n} \{\exp(-0.8(m))\}$$

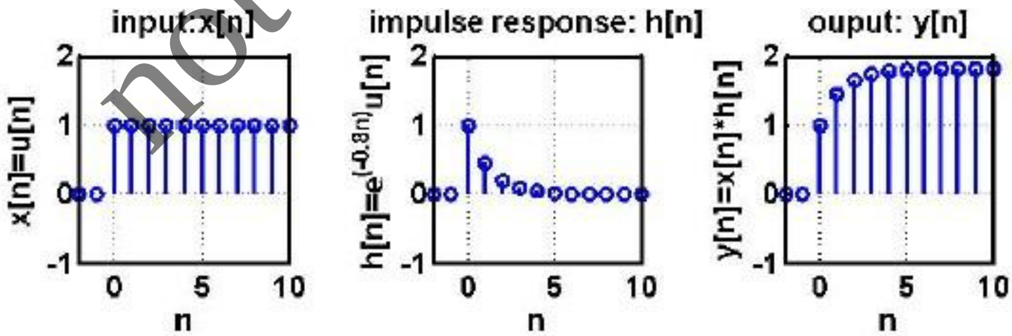
$$= \sum_{m=0}^{n} \{\exp(-0.8(m))\}$$

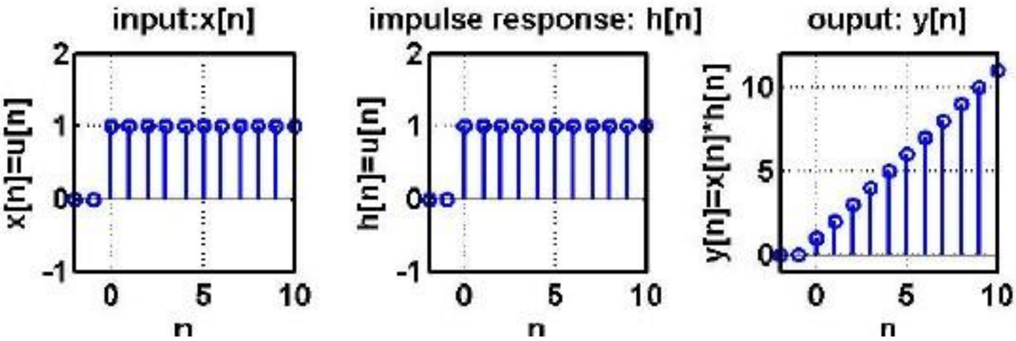
$$= \frac{\left(1-(-0.8)^{n+1}\right)}{\left(1-(-0.8)\right)}$$

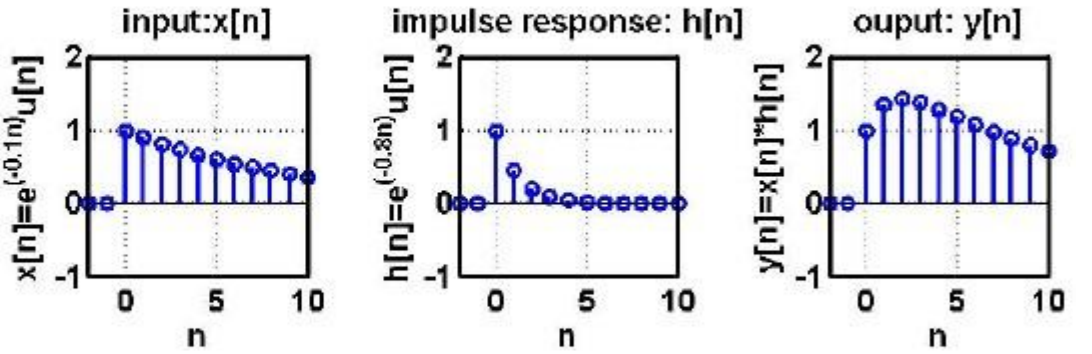
$$y[n] = \sum_{m=-\infty}^{\infty} \left\{(-0.8)^{(n-m)} u[n-m]\right\}$$

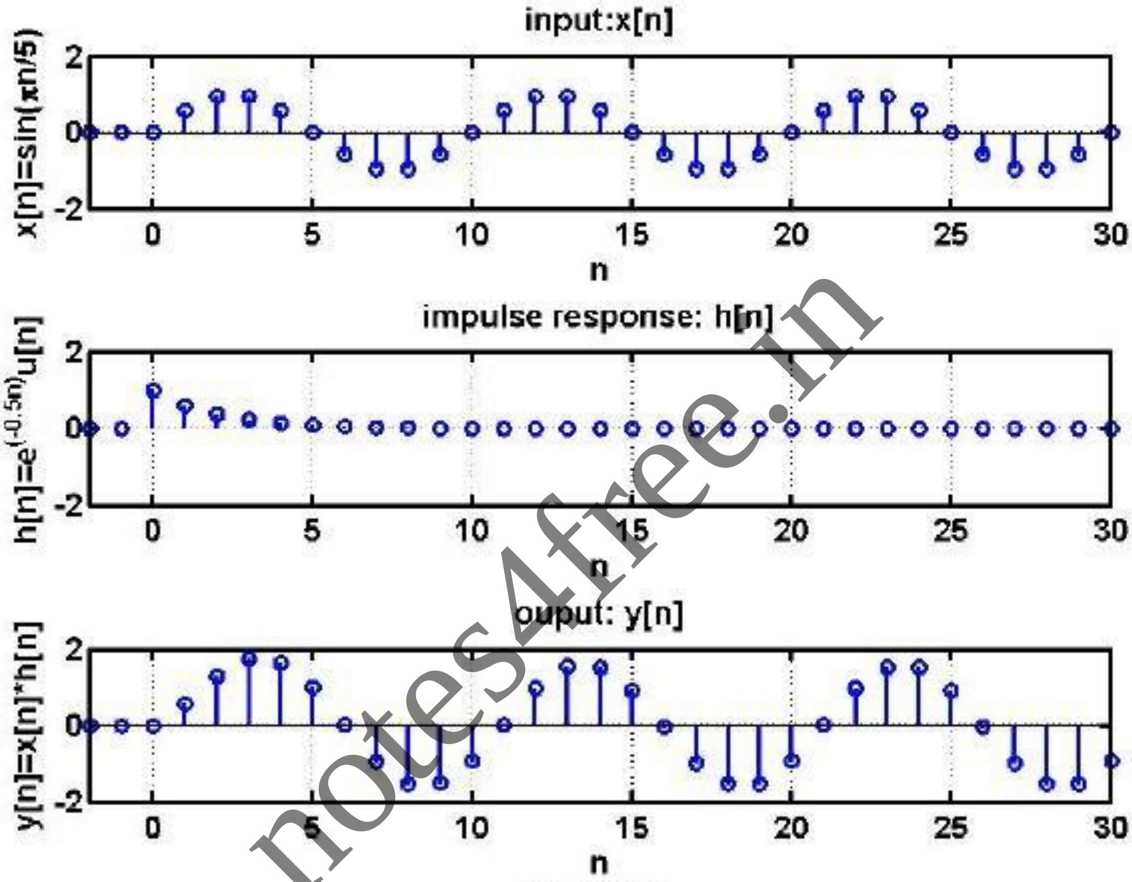
$$= \sum_{m=0}^{\infty} \{\exp(-0.8(n-m))u[n-m]\}$$

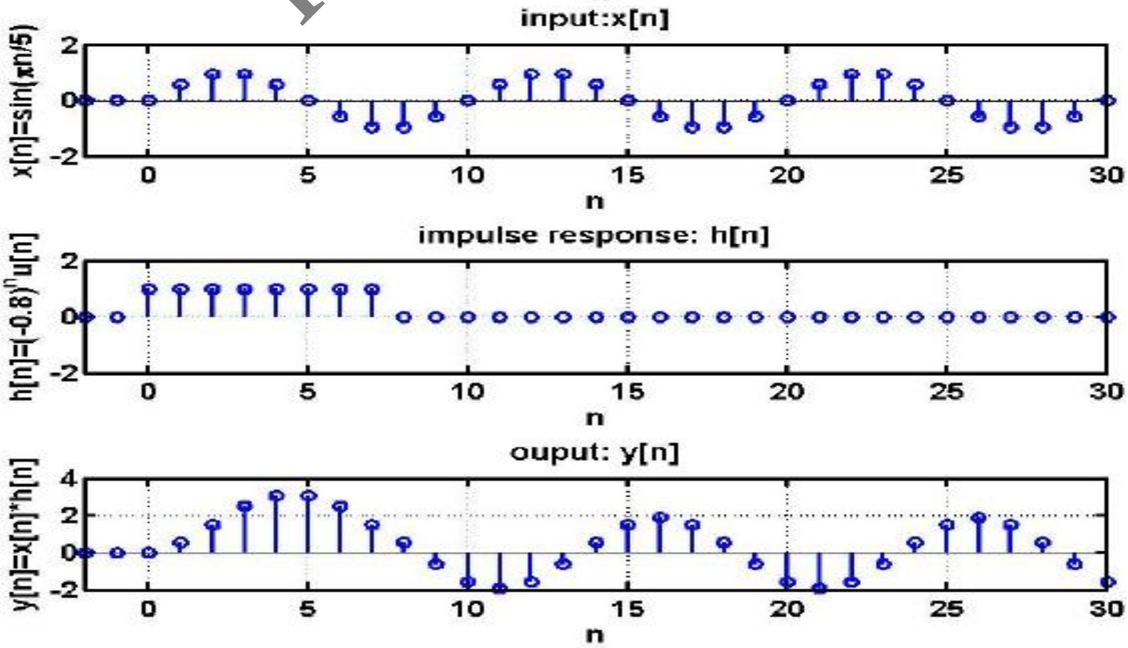







input:x[n]
impulse response: h[n]
ouput: y[n]
x[n]=e^(-0.1n)u[n]
h[n]=e^(-0.8n)u[n]
y[n]=x[n]*h[n]
n


input:x[n]
x[n]=sin(πn/5)
impulse response: h[n]
h[n]=e^(-0.5n)u[n]
ouput: y[n]
y[n]=x[n]*h[n]
n

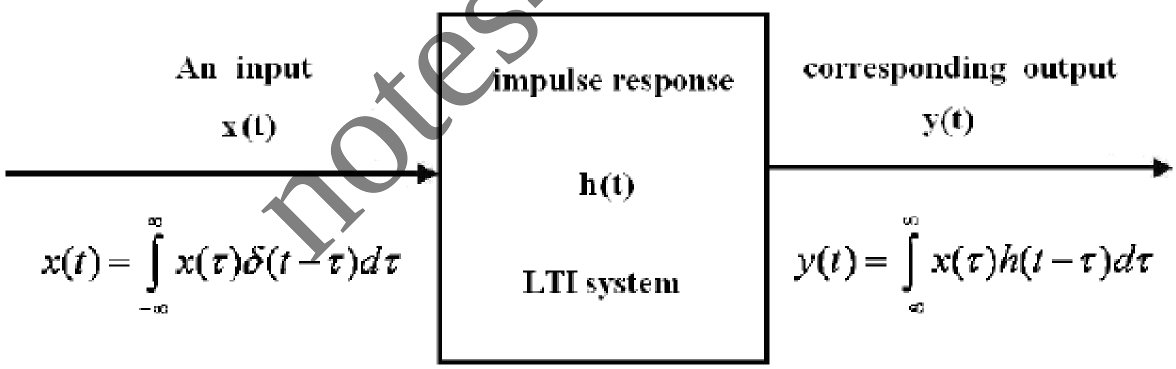

input:x[n]
x[n]=sin(πn/5)
impulse response: h[n]
h[n]=(-0.8)^n u[n]
ouput: y[n]
y[n]=x[n]*h[n]
n

2.4 Convolution Integral:

We now attempt to obtain the output of a continuous time/Analog digital system for an arbitrary input x(t), from the knowledge of the system impulse response h(t), and the properties of the impulse response of an LTI system.

The output y(t) is given by, using the notation, y(t)=R{x(t)}.

$$\begin{aligned} y(t) &= R\{x(t)\} \\ &= R\left\{\int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau\right\} \\ &= \int_{-\infty}^{\infty} x(\tau)R\{\delta(t-\tau)\}d\tau \\ &= \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau \\ &= x(t)*h(t) \end{aligned}$$



Methods of evaluating the convolution integral: (Same as Convolution sum)

Given the system impulse response h(t), and the input x(t), the system output y(t), is given by the convolution integral:

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

Some of the different methods of evaluating the convolution integral are: Graphical representation, Mathematical equation, Laplace-transforms, Fourier Transform, Differential equation, Block diagram representation, and finally by going to the digital domain.

Recommended Questions

1. Show that if x(n) is input of a linear time invariant system having impulse response h(n), then the output of the system due to x(n) is

$$y(n)=\sum_{k=-\infty}^{\infty} x(k)h(n-k)$$

2. Use the definition of convolution sum to prove the following properties
1. x(n) * [h(n)+g(n)]=x(n)*h(n)+x(n)*g(n) (Distributive Property)
2. x(n) * [h(n)*g(n)]=x(n)*h(n) *g(n) (Associative Property)
3. x(n) * h(n) =h(n) * x(n) (Commutative Property)
3. Prove that absolute summability of the impulse response is a necessary condition for stability of a discrete time system.
4. Compute the convolution y(t)= x(t)*h(t) of the following pair of signals:

$$(a)\quad x(t)=\begin{cases}1 & -a<t\le a\\ 0 & \text{otherwise}\end{cases},\quad h(t)=\begin{cases}1 & -a<t\le a\\ 0 & \text{otherwise}\end{cases}$$

$$(b)\quad x(t)=\begin{cases}t & 0<t\le T\\ 0 & \text{otherwise}\end{cases},\quad h(t)=\begin{cases}1 & 0<t\le 2T\\ 0 & \text{otherwise}\end{cases}$$

$$(c)\quad x(t)=u(t-1),\ h(t)=e^{-3t}u(t)$$

5. Compute the convolution sum y[n] =x[n]* h[n] of the following pairs of sequences:

$$(a)\quad x[n]=u[n],\ h[n]=2^n u[-n]$$

$$(b)\quad x[n]=u[n]-u[n-N],\ h[n]=\alpha^n u[n],\ 0<\alpha<1$$

$$(c)\quad x[n]=(\tfrac{1}{2})^n u[n],\ h[n]=\delta[n]-\tfrac{1}{2}\delta[n-1]$$

6. Show that if y (t) =x(t)* h(t), then

$$y'(t)=x'(t)*h(t)=x(t)*h'(t)$$

7. Let ***y[n]*** = ***x[n]* h[n].*** Then show that

$$x[n-n_1]*h[n-n_2]=y[n-n_1-n_2]$$

8. Show that

$$x_1[n]\otimes x_2[n]=\sum_{k=n_0}^{n_0+N-1} x_1[k]x_2[n-k]$$

for an arbitrary starting point ***no.***

Fourier representation for signals

Introduction:

Fourier series has long provided one of the principal methods of analysis for mathematical physics, engineering, and signal processing. It has spurred generalizations and applications that continue to develop right up to the present. While the original theory of Fourier series applies to periodic functions occurring in wave motion, such as with light and sound, its generalizations often relate to wider settings, such as the time-frequency analysis underlying the recent theories of wavelet analysis and local trigonometric analysis.

• In 1807, Jean Baptiste Joseph Fourier Submitted a paper of using trigonometric series to represent "any" periodic signal.

• But Lagrange rejected it!

• In 1822, Fourier published a book "The Analytical Theory of Heat" Fourier"s main contributions: Studied vibration, heat diffusion, etc. and found that a series of harmonically related sinusoids is useful in representing the temperature distribution through a body.

• He also claimed that "any" periodic signal could be represented by Fourier series. These arguments were still imprecise and it remained for P. L. Dirichlet in 1829 to provide precise conditions under which a periodic signal could be represented by a FS.

• He however obtained a representation for aperiodic signals i.e., Fourier integral or transform

• Fourier did not actually contribute to the mathematical theory of Fourier series.

• Hence out of this long history what emerged is a powerful and cohesive framework for the analysis of continuous- time and discrete-time signals and systems and an extraordinarily broad array of existing and potential application.

The Response of LTI Systems to Complex Exponentials:

We have seen in previous chapters how advantageous it is in LTI systems to represent signals as a linear combinations of basic signals having the following properties.

Key Properties: for Input to LTI System

1. To represent signals as linear combinations of basic signals.
2. Set of basic signals used to construct a broad class of signals.
3. The response of an LTI system to each signal should be simple enough in structure.
4. It then provides us with a convenient representation for the response of the system.
5. Response is then a linear combination of basic signal.

Eigenfunctions and Values :

• One of the reasons the Fourier series is so important is that it represents a signal in terms of eigen functions of LTI systems.

• When I put a complex exponential function like x(t) = ejωt through a linear time-invariant system, the output is y(t) = H(s)x(t) = H(s) ejωt where H(s) is a complex constant (it does not depend on time).

• The LTI system scales the complex exponential ejωt .

Historical background

There are antecedents to the notion of Fourier series in the work of Euler and D. Bernoulli on vibrating strings, but the theory of Fourier series truly began with the profound work of Fourier on heat conduction at the beginning of the century. In [5], Fourier deals with the problem of describing the evolution of the temperature of a thin wire of length X. He proposed that the initial temperature could be expanded in a series of sine functions:

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx \qquad (1)$$

$$b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx \, dx. \qquad (2)$$

The Fourier sine series, defined in Eq.s (1) and (2), is a special case of a more general concept: the Fourier series for a *periodic function*. Periodic functions arise in the study of wave motion, when a basic waveform repeats itself periodically. Such periodic waveforms occur in musical tones, in the plane waves of electromagnetic vibrations, and in the vibration of strings. These are just a few examples. Periodic effects also arise in the motion of the planets, in ac-electricity, and (to a degree) in animal heartbeats.

A function f is said to have period P if $f(x + P) = f(x)$ for all x. For notational simplicity, we shall restrict our discussion to functions of period 2π. There is no loss of generality in doing so, since we can always use a simple change of scale $x = (P/2\pi)t$ to convert a function of period P into one of period 2π.

If the function f has period 2π, then its *Fourier series* is

$$c_0 + \sum_{n=1}^{\infty} \{a_n \cos nx + b_n \sin nx\} \qquad (4)$$

with *Fourier coefficients* c_0, a_n, and b_n defined by the integrals

$$c_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \, dx \qquad (5)$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx, \qquad (6)$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx. \qquad (7)$$

The following relationships can be readily established, and will be used in subsequent sections for derivation of useful formulas for the unknown Fourier coefficients, in both time and frequency domains.

$$\int_0^T \sin(kw_0t)dt = \int_0^T \cos(kw_0t)dt \tag{1}$$
$$= 0$$

$$\int_0^T \sin^2(kw_0t)dt = \int_0^T \cos^2(kw_0t)dt \tag{2}$$
$$= \frac{T}{2}$$

$$\int_0^T \cos(kw_0t)\sin(gw_0t)dt = 0 \tag{3}$$

$$\int_0^T \sin(kw_0t)\sin(gw_0t)dt = 0 \tag{4}$$

$$\int_0^T \cos(kw_0t)\cos(gw_0t)dt = 0 \tag{5}$$

where

$$w_0 = 2\pi f \tag{6}$$

$$f = \frac{1}{T} \tag{7}$$

where f and T represents the frequency (in cycles/time) and period (in seconds) respectively. Also, k and g are integers.

A periodic function $f(t)$ with a period T should satisfy the following equation

$$f(t+T) = f(t) \tag{8}$$

Example 1

Prove that

$$\int_0^{\pi} \sin(kw_0t) = 0$$

for

$$w_0 = 2\pi f$$

$$f = \frac{1}{T}$$

and k is an integer.

Solution

Let

$$A = \int_0^T \sin(kw_0t)dt \tag{9}$$

$$= -\left(\frac{1}{kw_0}\right)\left[\cos(kw_0t)\right]_0^T$$

$$A = \left(\frac{-1}{kw_0}\right)\left[\cos(kw_0T) - \cos(0)\right] \tag{10}$$

$$= \left(\frac{-1}{kw_0}\right)\left[\cos(k2\pi) - 1\right]$$

$$= 0$$

Example 2

Prove that

$$\int_0^{\pi} \sin^2(kw_0 t) = \frac{T}{2}$$

for

$$w_0 = 2\pi f$$

$$f = \frac{1}{T}$$

and k is an integer.

Solution

Let

$$B = \int_0^T \sin^2(kw_0 t)dt \tag{11}$$

Recall

$$\sin^2(\alpha) = \frac{1-\cos(2\alpha)}{2} \tag{12}$$

Thus,

$$B = \int_0^T \left[\frac{1}{2} - \frac{1}{2}\cos(2kw_0 t)\right]dt \tag{13}$$

$$= \left[\left(\frac{1}{2}\right)t - \left(\frac{1}{2}\right)\left(\frac{1}{2kw_0}\right)\sin(2kw_0 t)\right]_0^T$$

$$B = \left[\frac{T}{2} - \frac{1}{4kw_0}\sin(2kw_0 T)\right] - [0] \tag{14}$$

$$= \frac{T}{2} - \left(\frac{1}{4kw_0}\right)\sin(2k * 2\pi)$$

$$= \frac{T}{2}$$

Example 3

Prove that

$$\int_0^{\pi} \sin(gw_0 t)\cos(kw_0 t) = 0$$

for

$$w_0 = 2\pi f$$

$$f = \frac{1}{T}$$

and k and g are integers.

Solution

Let

$$C = \int_0^T \sin(gw_0 t)\cos(kw_0 t)dt \tag{15}$$

Recall that

$$\sin(\alpha+\beta)=\sin(\alpha)\cos(\beta)+\sin(\beta)\cos(\alpha) \quad (16)$$

Hence,

$$C=\int_0^T\left[\sin\left[(g+k)w_0t\right]-\sin(kw_0t)\cos(gw_0t)\right]dt \quad (17)$$

$$=\int_0^T\sin\left[(g+k)w_0t\right]dt-\int_0^T\sin(kw_0t)\cos(gw_0t)dt \quad (18)$$

From Equation (1),

$$\int_0^T[\sin(g+k)w_0t]dt=0$$

then

$$C=0-\int_0^T\sin(kw_0t)\cos(gw_0t)dt \quad (19)$$

Adding Equations (15), (19), $2C=\int_0^T\sin(gw_0t)\cos(kw_0t)dt-\int_0^T\sin(kw_0t)\cos(gw_0t)dt$

$$=\int_0^T\sin\left[(gw_0t)-(kw_0t)\right]dt=\int_0^T\sin\left[(g-k)w_0t\right]dt \quad (20)$$

$2C=0$, since the right side of the above equation is zero (see Equation 1). Thus,

$$C=\int_o^T\sin(gw_0t)\cos(kw_0t)dt=0 \quad (21)$$

$$=0$$

Example 4

Prove that

$$\int_0^T\sin(kw_0t)\sin(gw_0t)dt=0$$

for

$w_0=2\pi f$

$f=\dfrac{1}{T}$

$k, g=$ integers

Solution

Let $D=\int_0^T\sin(kw_0t)\sin(gw_0t)dt \quad (22)$

Since

$$\cos(\alpha+\beta)=\cos(\alpha)\cos(\beta)-\sin(\alpha)\sin(\beta)$$

or

$$\sin(\alpha)\sin(\beta)=\cos(\alpha)\cos(\beta)-\cos(\alpha+\beta)$$

Thus,

$$D=\int_0^T\cos(kw_0t)\cos(gw_0t)dt-\int_0^T\cos\left[(k+g)w_0t\right]dt \quad (23)$$

From Equation (1)

$$\int_0^T \cos[(k+g)w_0 t]dt = 0$$

then

$$D = \int_0^T \cos(kw_0 t)\cos(gw_0 t)dt - 0$$

Adding Equations (23), (26)

$$2D = \int_0^T \sin(kw_0 t)\sin(gw_0 t) + \int_0^T \cos(kw_0 t)\cos(gw_0 t)dt$$

$$= \int_0^T \cos[kw_0 t - gw_0 t]dt$$

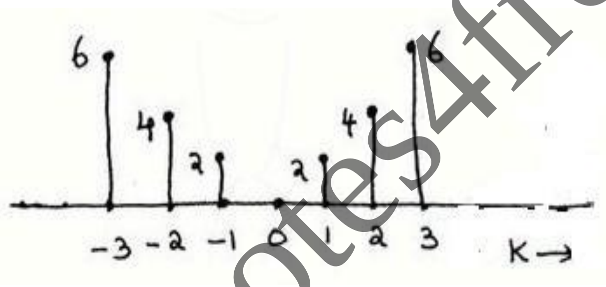
$$= \int_0^T \cos[(k-g)w_0 t]dt$$

2D = 0, since the right side of the above equation is zero (see Equation 1). Thus,

$$D \equiv \int_0^T \sin(kw_0 t)\sin(gw_0 t)dt = 0$$

Recommended Questions

1. Find x(t) if the Fourier series coefficients are shown in fig. The phase spectrum is a null spectrum.



2. Prove the following properties of Fourier series. i) Convolution property ii) Parsevals relationship.

3. Find the DTFS harmonic function of x(n) = A Cos (2πn/No). Plot the magnitude and phase spectra.

4. Determine the complex Fourier coefficients for the signal. X(t)= {t+1 for -1 < t< 0; 1-t for 0 < t < 1 which repeats periodically with T=2 units. Plot the amplitude and phase spectra of the signal.

5. State and prove the following of Fourier transform. i) Time shifting property ii) Time differentiation property iii) Parseval's theorem.

Fourier representation for signals – 2

Fourier representation for signals – 2: Discrete and continuous Fourier transforms(derivations of transforms are excluded) and their properties.

TEXT BOOK

Simon Haykin and Barry Van Veen "Signals and Systems", John Wiley & Sons, 2001.Reprint 2002

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1. **Alan V Oppenheim, Alan S, Willsky and A Hamid Nawab,** "Signals and Systems" Pearson Education Asia / PHI, 2nd edition, 1997. Indian Reprint 2002

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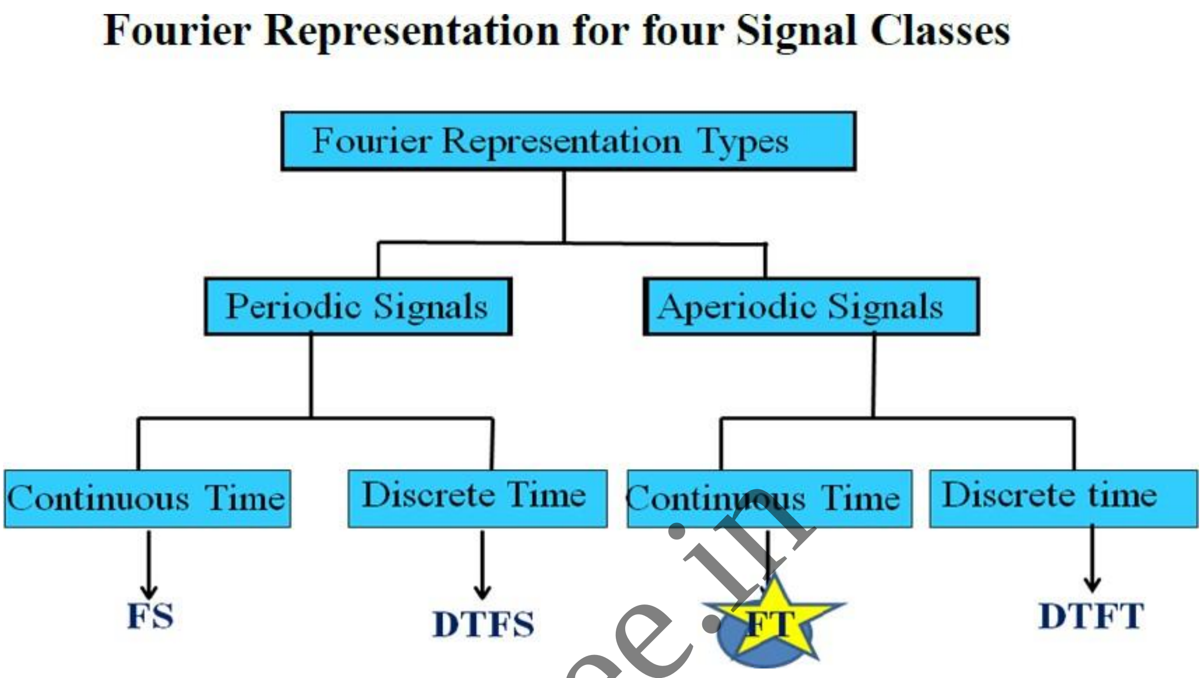
Fourier representation for signals

Introduction:



5.1 The Fourier transform

5.1.1 From Discrete Fourier Series to Fourier Transform:
Let ***x [n]*** be a nonperiodic sequence of finite duration. That is, for some positive integer ***N ,***

$$x[n] = 0 \qquad |n| > N_1$$

Such a sequence is shown in Fig. ***6-l(a).*** Let ***x,Jn]*** be a periodic sequence formed by repeating ***x [n]*** with fundamental period ***No*** as shown in Fig. 6-l(b). If we let ***No*** -, m, we have

$$\lim_{N_0 \to \infty} x_{N_0}[n] = x[n]$$

The discrete Fourier series of ***xNo[n]*** is given ***by***

$$x_{N_0}[n] = \sum_{k=\langle N_0 \rangle} c_k e^{jk\Omega_0 n} \qquad \Omega_0 = \frac{2\pi}{N_0}$$

$$c_k = \frac{1}{N_0} \sum_{n=\langle N_0 \rangle} x_{N_0}[n]\, e^{-jk\Omega_0 n}$$

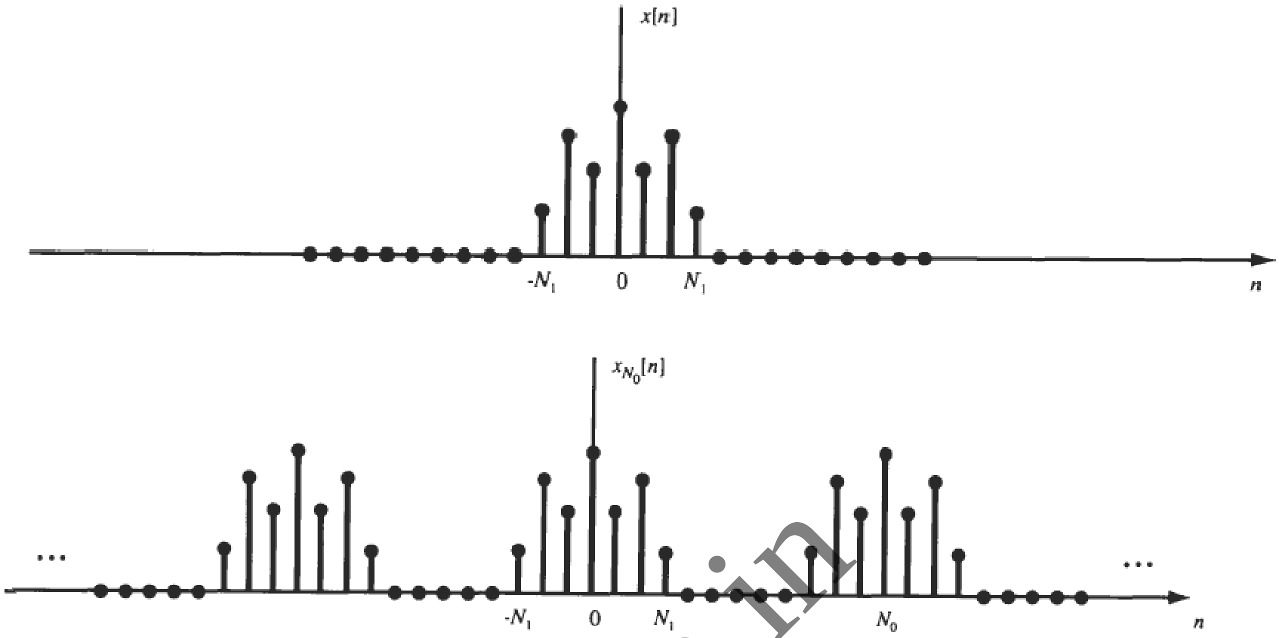


Fig. 6-1 (*a*) Nonperiodic finite sequence $x[n]$; (*b*) periodic sequence formed by periodic extension of $x[n]$.

$$c_k = \frac{1}{N_0} \sum_{n=-N_1}^{N_1} x[n]\, e^{-jk\Omega_0 n} = \frac{1}{N_0} \sum_{n=-\infty}^{\infty} x[n]\, e^{-jk\Omega_0 n}$$

$$X(\Omega) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\Omega n}$$

the Fourier coefficients c_k can be expressed as

$$c_k = \frac{1}{N_0} X(k\Omega_0)$$

$$x_{N_0}[n] = \sum_{k=\langle N_0 \rangle} \frac{1}{N_0} X(k\Omega_0)\, e^{jk\Omega_0 n}$$

$$x_{N_0}[n] = \frac{1}{2\pi} \sum_{k=\langle N_0 \rangle} X(k\Omega_0)\, e^{jk\Omega_0 n}\Omega_0$$

Properties of the Fourier transform

Periodicity

As a consequence of Eq. (6.41), in the discrete-time case we have to consider values of R(radians) only over the range0 < Ω < 2π or π < Ω < π**,** while in the continuous-time case we have to consider values of ***0*** (radians/second) over the entire range –∞ < ω < ∞.

$$X(\Omega + 2\pi) = X(\Omega)$$

Linearity:

$$a_1 x_1[n] + a_2 x_2[n] \longleftrightarrow a_1 X_1(\Omega) + a_2 X_2(\Omega)$$

Time Shifting:

$$x[n-n_0] \leftrightarrow e^{-j\Omega n_0}X(\Omega)$$

Frequency Shifting:

$$e^{j\Omega_0 n}x[n] \leftrightarrow X(\Omega - \Omega_0)$$

Conjugation:

$$x^*[n] \leftrightarrow X^*(-\Omega)$$

Time Reversal:

$$x[-n] \leftrightarrow X(-\Omega)$$

Time Scaling:

$$x(at) \leftrightarrow \frac{1}{|a|}X\left(\frac{\omega}{a}\right)$$

Duality:

The duality property of a continuous-time Fourier transform is expressed as

$$X(t) \leftrightarrow 2\pi x(-\omega)$$

There is no discrete-time counterpart of this property. However, there is a duality between the discrete-time Fourier transform and the continuous-time Fourier series. Let

$$x[n] \leftrightarrow X(\Omega)$$

$$X(\Omega) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\Omega n}$$

$$X(\Omega + 2\pi) = X(\Omega)$$

Since Ω is a continuous variable, letting $\Omega = t$ and $n = -k$

$$X(t) = \sum_{k=-\infty}^{\infty} x[-k]\, e^{jkt}$$

Since X(t) is periodic with period $T_0 = 2\pi$ and the fundamental frequency $\omega_0 = 2\pi/T_0 = 1$, Equation indicates that the Fourier series coefficients of X(t) will be x[-k]. This duality relationship is denoted by

$$X(t) \overset{FS}{\longleftrightarrow} c_k = x[-k]$$

where FS denotes the Fourier series and ***c,*** are its Fourier coefficients.

Differentiation in Frequency:

$$nx[n] \leftrightarrow j\frac{dX(\Omega)}{d\Omega}$$

Differencing:

$$x[n]-x[n-1] \leftrightarrow (1-e^{-j\Omega})X(\Omega)$$

The sequence x[n] -x[n – 1] is called the first difference sequence. Equation is easily obtained from the linearity property and the time-shifting property .

Accumulation:

$$\sum_{k=-\infty}^{n} x[k] \leftrightarrow \pi X(0)\,\delta(\Omega)+\frac{1}{1-e^{-j\Omega}}X(\Omega) \qquad |\Omega| \le \pi$$

Note that accumulation is the discrete-time counterpart of integration. The impulse term on the right-hand side of Eq. ***(6.57)*** reflects the dc or average value that can result from the accumulation.

Convolution:

$$x_1[n] * x_2[n] \leftrightarrow X_1(\Omega)X_2(\Omega)$$

As in the case of the z-transform, this convolution property plays an important role in the study of discrete-time LTI systems.

Multiplication:

$$x_1[n]x_2[n] \leftrightarrow \frac{1}{2\pi}X_1(\Omega) \otimes X_2(\Omega)$$

where @ denotes the periodic convolution defined by

$$X_1(\Omega) \otimes X_2(\Omega) = \int_{2\pi} X_1(\theta)X_2(\Omega-\theta)\,d\theta$$

The multiplication property (6.59) is the dual property of Eq. (6.58).

Parseval's Relations:

$$\sum_{n=-\infty}^{\infty} x_1[n]x_2[n] = \frac{1}{2\pi}\int_{2\pi} X_1(\Omega)X_2(-\Omega)\,d\Omega$$

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi}\int_{2\pi} |X(\Omega)|^2\,d\Omega$$

Summary

Property	$x(t), y(t)$	$X(j\omega), Y(j\omega)$
Linearity	$ax(t) + by(t)$	$aX(j\omega) + bY(j\omega)$
Time Shifting	$x(t - t_0)$	$e^{-j\omega t_0}X(j\omega)$
Frequency Shifting	$e^{j\omega_0 t}x(t)$	$X(j(\omega - \omega_0))$
Conjugation	$x^*(t)$	$X^*(-j\omega)$
Time Reversal	$x(-t)$	$X(-j\omega)$
Time and Frequency Scaling	$x(at)$	$\frac{1}{\|a\|}X(\frac{j\omega}{a})$
Convolution	$x(t) ** y(t)$	$X(j\omega)Y(j\omega)$
Multiplication	$x(t)y(t)$	$X(j\omega) ** Y(j\omega)$
Differentiation in Time	$\frac{d}{dt}x(t)$	$j\omega X(j\omega)$
Integration	$\int_{-\infty}^{t} x(t)dt$	$\frac{1}{j\omega}X(j\omega) + \pi X(0)\delta(\omega)$
Differentiation in Frequency	$tx(t)$	$j\frac{d}{d\omega}X(j\omega)$

Recommended Questions

1. Obtain the Fourier transform of the signal e^{-at}. u(t) and plot spectrum.

2. Determine the DTFT of unit step sequence x(n) = u(n) its magnitude and phase.

3. The system produces the output of yet) = e^{-t} u(t), for an input of x(t) = e-2t.u(t). Determine impulse response and frequency response of the system.
4. The input and the output of a causal LTI system are related by differential equation
   $\frac{d2y(t)}{dt2} + \frac{6dy(t)}{dt} + 8y(t) = 2x(t)$
   i) Find the impulse response of this system
   ii) What is the response of this system if x(t) = te^{-at} u(t)?

5. Discuss the effects of a time shift and a frequency shift on the Fourier representation.

6. Use the equation describing the DTFT representation to determine the time-domain signals corresponding to the following DTFTs :
   i) $X(e^{j\Omega})$= Cos(Ω)+j Sin(Ω)
   ii) $X(e^{j\Omega})$={1, for π/2<Ω< π; 0 otherwise and $X(e^{j\Omega})$=-4 Ω

7. Use the defining equation for the FT to evaluate the frequency-domain representations for the following signals:
   i) X(t)= e^{-3t}u(t-1)
   ii) X(t)=e^{-t} Sketch the magnitude and phase spectra.

8. Show that the real and odd continuous time non periodic signal has purely imaginary Fourier transform. (4 Marks)

Fourier Series and LTI System

- Fourier series representation can be used to construct any periodic signals in discrete as well as continuous-time signals of practical importance.
- We have also seen the response of an LTI system to a linear combination of complex exponentials taking a simple form.
- Now, let us see how Fourier representation is used to analyze the response of LTI System.

Consider the CTFS synthesis equation for x(t) given by
Suppose we apply this signal as an input to an LTI System with impulse respose h(t). Then, since each of the complex exponentials in the expression is an eigen function of the system. Then, with $s_k = jk\omega_0$, it follows that the output is

$$y(t) = \sum_{k=-\infty}^{+\infty} a_k H(e^{jk\omega_0}) e^{jk\omega_0 t}$$

Thus y(t) is periodic with frequency as x(t). Further, if ak is the set of Fourier series coefficients for the input x(t), then $\{a_k H(e^{jk\omega_0})\}$ is the set of coefficient for the y(t). Hence in LTI, modify each of the Fourier coefficient of the input by multiplying by the frequency response at the corresponding frequency.

Example:

Consider a periodic signal x(t), with fundamental frequency 2π, that is expressed in the form

$$x(t) = \sum_{k=-3}^{+3} a_k e^{jk2\pi t} \quad \text{.........(1)}$$

$$\text{where,} \; a_0=1, \quad a_1=a_{-1}=1/4, \; a_2=a_{-2}=1/2, \; a_3=a_{-3}=1/3,$$

Suppose that the this periodic signal is input to an LTI system with impulse response To calculate the FS Coeff. Of o/p y(t), lets compute the frequency response.The impulse response is therefore,

$$H(j\omega) = \int_0^{\infty} e^{-\tau} e^{-j\omega\tau} d\tau \qquad = -\frac{1}{1+j\omega} e^{-\tau} e^{-j\omega\tau} \Big|_0^{\infty}$$

and

$$H(j\omega) = \frac{1}{1+j\omega}$$

Y(t) at ωo = 2π . We obtain,

$$y(t) = \sum_{k=-3}^{+3} b_k e^{jk2\pi t} \qquad \text{with } b_k = a_k H(jk2\pi), \text{so that}$$

$$b_1 = \frac{1}{4}\left(\frac{1}{1 + j2\pi}\right) b_2 = \frac{1}{2}\left(\frac{1}{1 + j4\pi}\right) b_3 = \frac{1}{3}\left(\frac{1}{1 + j6\pi}\right)$$

$$b_{-1} = \frac{1}{4}\left(\frac{1}{1 - j2\pi}\right) b_{-2} = \frac{1}{2}\left(\frac{1}{1 - j4\pi}\right) \quad b_{-3} = \frac{1}{3}\left(\frac{1}{1 - j6\pi}\right)$$

$$b_0 = 1$$

The above o/p coefficients. Could be substituted in

$$y(t) = \sum_{k=-3}^{+3} b_k e^{jk2\pi t}$$

Finding the Frequency Response

We can begin to take advantage of this way of finding the output for any input once we have H(ω).

To find the frequency response H(ω) for a system, we can:

1. Put the input $x(t) = e^{i\omega t}$ into the system definition
2. Put in the corresponding output $y(t) = H(\omega)\, e^{i\omega t}$
3. Solve for the frequency response H(ω). (The terms depending on t will cancel.)

Example:

Consider a system with impulse response

$$h(t) = \begin{cases} \frac{1}{5} & \text{for } t \in [0,5] \\ 0 & \text{otherwise} \end{cases}$$

Find the output corresponding to the input x(t) = cos(10 t).

$$y(t) = \int_{\tau=-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau = \int_{\tau=0}^{5} \frac{1}{5} \cos(10(t-\tau))\, d\tau$$

$$y(t) = \frac{1}{5}\left(-\frac{1}{10}\sin(10(t-\tau))\right)\Bigg|_{\tau=0}^{5} = \frac{1}{50}(\sin(10t) - \sin(10(t-5)))$$

Differential and Difference Equation Descriptions

Frequency Response is the system's steady state response to a sinusoid. In contrast to differential and difference-equation descriptions for a system, the frequency response description cannot represent initial conditions, it can only describe a system in a steady state condition. The differential-equation representation for a continuous-time system is

$$\sum_{k=0}^{N} a_k \frac{d^k}{dt^k} y(t) = \sum_{k=0}^{N} b_k \frac{d^k}{dt^k} x(t)$$

$$since, \frac{d}{dt} g(t) \xleftrightarrow{FT} j\omega G(j\omega)$$

Rearranging the equation we get

$$\frac{Y(j\omega)}{X(j\omega)} = \frac{\sum_{k=0}^{M} b_k (j\omega)^k}{\sum_{k=0}^{N} a_k (j\omega)^k}$$

The frequency of the response is

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{\sum_{k=0}^{M} b_k (j\omega)^k}{\sum_{k=0}^{N} a_k (j\omega)^k}$$

Hence, the equation implies the frequency response of a system described by a linear constant-coefficient differential equation is a ratio of polynomials in jω.

The difference equation representation for a discrete-time system is of the form.

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

Take the DTFT of both sides of this equation, using the time-shift property.

$$g[n-k] \xleftrightarrow{DTFT} e^{-jk\omega} G(e^{j\omega})$$

To obtain

$$\sum_{k=0}^{N} a_k \left(e^{-j\omega}\right)^k Y\left(e^{j\omega}\right) = \sum_{k=0}^{N} a_k \left(e^{-j\omega}\right)^k X\left(e^{j\omega}\right)$$

- Rewrite this equation as the ratio

$$\frac{Y\left(e^{j\omega}\right)}{X\left(e^{j\omega}\right)} = \frac{\sum_{k=0}^{M} b_k \left(e^{j\omega}\right)^k}{\sum_{k=0}^{N} a_k \left(e^{j\omega}\right)^k}$$

- The frequency response is the polynomial in $e^{j\omega}$

$$H(e^{j\omega}) = \frac{Y\left(e^{j\omega}\right)}{X\left(e^{j\omega}\right)} = \frac{\sum_{k=0}^{M} b_k \left(e^{j\omega}\right)^k}{\sum_{k=0}^{N} a_k \left(e^{j\omega}\right)^k}$$

Differential Equation Descriptions

Ex: Solve the following differential Eqn using FT.

$$\frac{d^2}{dt^2} y(t) + 4\frac{d}{dt} y(t) + 5y(t) = 3\frac{d}{dt} x(t) + x(t)$$

For all t where, $x(t) = (1 + e^{-t})u(t)$

Soln:we have

$$\frac{d^2}{dt^2} y(t) + 4\frac{d}{dt} y(t) + 5y(t) = 3\frac{d}{dt} x(t) + x(t)$$

FT gives,

$$[(j\omega)^2 + 4(j\omega) + 5]Y(j\omega) = (3j\omega + 1)X(j\omega)$$

$$and\ x(t) = (1+e^{-t})u(t) \qquad x(t) = u(t) + (e^{-t})u(t)$$

$$X(j\omega) = \left(\frac{1}{j\omega} + \pi\delta(\omega)\right) + \frac{1}{(j\omega+1)}\ since\ u(t) \xleftrightarrow{FT} \pi\delta(\omega) + \frac{1}{j\omega}$$

$$and (e^{-t})u(t) \xleftrightarrow{FT} \frac{1}{j\omega+1}$$

Hence we have

$$X(j\omega) = \left(\frac{1}{j\omega} + \pi\delta(\omega)\right) + \frac{1}{(j\omega+1)}$$

And $[(j\omega)^2 + 4(j\omega) + 5]Y(j\omega) = (3j\omega+1)X(j\omega)$

i.e

$$Y(j\omega) = \frac{(3j\omega+1)}{[(j\omega)^2 + 4(j\omega) + 5]}X(j\omega)$$

$$Y(j\omega) = \frac{(3j\omega+1)}{[(j\omega+2)^2 + 1]}\left[\frac{1}{j\omega} + \pi\delta(\omega) + \frac{1}{(j\omega+1)}\right]$$

$$Y(j\omega) = \frac{(3j\omega+1)}{[(j\omega)^2 + 4(j\omega) + 5]}\left[\left(\frac{1}{j\omega} + \pi\delta(\omega)\right) + \frac{1}{(j\omega+1)}\right]$$

$$\boxed{Y(j\omega) = Y(1) + Y(2) + Y(3)}$$

$$Y(j\omega) = \frac{(3j\omega+1)}{[(j\omega+2)^2+1]j\omega} + \frac{\pi}{5}\delta(\omega) + \frac{(3j\omega+1)}{[(j\omega+2)^2+1](j\omega+1)}$$

$$Y(j\omega) = \frac{(3j\omega+1)}{[(j\omega+2)^2+1]j\omega} + \frac{(3j(\omega=0)+1)\pi[\delta(0)=1]}{[(j(\omega=0)+2)^2+1]j(\omega=0)} + \frac{(3j\omega+1)}{[(j\omega+2)^2+1](j\omega+1)}$$

$$Y(1) = \frac{(3j\omega+1)}{[(j\omega+2)^2+1]j\omega} \quad Y(1) = \frac{A}{j\omega} + \frac{Bj\omega + C}{[(j\omega+2)^2+1]}$$

$Performing\ partial\ fraction\ we\ get$ $A = \frac{1}{5}, B = -\frac{1}{5}, C = \frac{11}{5}$

$$\boxed{Y(1) = \frac{1/5}{j\omega} + \frac{-1/5 j\omega + 11/5}{[(j\omega+2)^2+1]}}$$

Similarly

$$Y(3) = \frac{(3j\omega+1)}{[(j\omega+2)^2+1](j\omega+1)}$$

$$Y(3) = \frac{R}{(j\omega+1)} + \frac{Pj\omega + Q}{[(j\omega+2)^2+1]}$$

$Performing\ partial\ fraction\ we\ get$ $R = -1, P = 1, Q = 6$

$$Y(3) = \frac{-1}{(j\omega + 1)} + \frac{j\omega + 6}{[(j\omega + 2)^2 + 1]}$$

$$Y(3) = \frac{-1}{(j\omega + 1)} + \frac{j\omega + 6}{[(j\omega + 2)^2 + 1]}$$

$$Y(j\omega) = Y(1) + Y(2) + Y(3)$$

Hence, we have

$$Y(1) = \frac{1/5}{j\omega} + \frac{-1/5 j\omega + 11/5}{[(j\omega + 2)^2 + 1]}$$

$$Y(2) = \frac{\pi}{5}\delta(\omega)$$

Readjusting

$$Y(j\omega) = \frac{1/5}{j\omega} + \frac{-1/5 j\omega + 11/5}{[(j\omega + 2)^2 + 1]} + \frac{\pi}{5}\delta(\omega) + \frac{-1}{(j\omega + 1)} + \frac{j\omega + 6}{[(j\omega + 2)^2 + 1]}$$

$$Y(j\omega) = \frac{1}{5}\left[\frac{1}{j\omega} + \pi\delta(\omega)\right] - \frac{1}{(j\omega + 1)} + \frac{1}{5}\left[\frac{4j\omega + 41}{[(j\omega + 2)^2 + 1]}\right]$$

$$Y(j\omega) = \frac{1/5}{j\omega} + \frac{\pi}{5}\delta(\omega) + \frac{11/5 - 1/5 j\omega}{[(j\omega + 2)^2 + 1]} + \frac{j\omega + 6}{[(j\omega + 2)^2 + 1]} - \frac{1}{(j\omega + 1)}$$

we know that,

$$e^{-\beta t}\cos\omega_o t u(t) \xleftrightarrow{FT} \frac{\beta + j\omega}{[(\beta + j\omega)^2 + \omega_o^2]}$$

$$e^{-\beta t}\sin\omega_o t u(t) \xleftrightarrow{FT} \frac{\omega_o}{[(\beta + j\omega)^2 + \omega_o^2]}$$

Readjusting the last term, we get

$$Y(j\omega) = \frac{1}{5}\left[\frac{1}{j\omega} + \pi\delta(\omega)\right] - \frac{1}{(j\omega + 1)} + \frac{4}{5}\left[\frac{j\omega + 2}{[(j\omega + 2)^2 + 1]}\right] + \frac{33}{5}\left[\frac{1}{[(j\omega + 2)^2 + 1]}\right]$$

Now, taking the inverse Fourier Transform, we get

$$y(t) = \frac{1}{5}u(t) - e^{-t}u(t) + \frac{4}{5}e^{-2t}\cos t\, u(t) + \frac{33}{5}e^{-2t}\sin t\, u(t)$$

Differential Equation Descriptions

- Ex: Find the frequency response and impulse response of the system described by the differential equation.

$$\frac{d^2}{dt^2}y(t) + 3\frac{d}{dt}y(t) + 2y(t) = 2\frac{d}{dt}x(t) + x(t)$$

Here we have N=2, M=1. Substituting the coefficients of this differential equation in

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{\sum_{k=0}^{M} b_k (j\omega)^k}{\sum_{k=0}^{N} a_k (j\omega)^k}$$

Differential Equation Descriptions

We obtain

$$H(j\omega) = \frac{2j\omega + 1}{(j\omega)^2 + 3j\omega + 2}$$

The impulse response is given by the inverse FT of H(jω). Rewrite H(jω) using the partial fraction expansion.

$$H(j\omega) = \frac{A}{j\omega + 1} + \frac{B}{j\omega + 2}$$

Solving for A and B we get, A=-1 and B=3. Hence

$$H(j\omega) = \frac{-1}{j\omega + 1} + \frac{3}{j\omega + 2}$$

The inverse FT gives the impulse response

$$\boxed{h(t) = 3e^{-2t}u(t) - e^{-t}u(t)}$$

Difference Equation

Ex: Consider an LTI system characterized by the following second order linear constant coefficient difference equation.

$$y[n] = 1.3433y[n-1] - 0.9025y[n-2] + x[n] - 1.4142x[n-1] + x[n-2]$$

Find the frequency response of the system.

Soln:

$$y[n] = 1.3433y[n-1] - 0.9025y[n-2] + x[n] - 1.4142x[n-1] + x[n-2]$$

$$Y(e^{j\omega}) = 1.3433(e^{-j\omega})Y(e^{j\omega}) - 0.9025(e^{-j2\omega})Y(e^{j\omega}) + X(e^{j\omega}) - 1.4142(e^{-j\omega})X(e^{j\omega}) + (e^{-j2\omega})X(e^{j\omega})$$

$$we\ know, y[n-k] \xleftrightarrow{DTFT} e^{-jk\omega}Y(e^{j\omega})$$

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})}$$

$$= \frac{1 - 1.4142e^{-j\omega} + e^{-j2\omega}}{1 - 1.3433e^{-j\omega} + 0.9025e^{-j2\omega}}$$

Ex: If the unit impulse response of an LTI System is $h(n)=\alpha^{n}u[n]$, find the response of the system to an input defined by $x[n] = \beta^{n}u[n]$, where $\beta, \alpha < 1 \text{ and } \alpha \neq \beta$

Soln:

$$y[n] = h[n] * x[n]$$

Taking DTFT on both sides of the equation, we get

$$Y(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega}) \qquad Y(e^{j\omega}) = \frac{1}{1 - \alpha e^{-j\omega}} \text{x} \frac{1}{1 - \beta e^{-j\omega}}$$

$$Y(e^{j\omega}) = \frac{1}{1 - \alpha e^{-j\omega}} \text{x} \frac{1}{1 - \beta e^{-j\omega}} = \frac{A}{1 - \alpha e^{-j\omega}} \text{x} \frac{B}{1 - \beta e^{-j\omega}}$$

where A and B are constants to be found by using partial fractions

$$Let,\ e^{-j\omega} = v \qquad Then, Y(e^{j\omega}) = \frac{A}{1 - \alpha v} \text{x} \frac{B}{1 - \beta v}$$

$$By\ performing\ partial\ fractions, we\ get\ A = \frac{\alpha}{\alpha - \beta},\ B = \frac{-\beta}{\alpha - \beta}$$

$$Therefore, Y(e^{j\omega}) = \frac{\frac{\alpha}{\alpha - \beta}}{1 - \alpha\, e^{-j\omega}} \text{x} \frac{\frac{-\beta}{\alpha - \beta}}{1 - \beta\, e^{-j\omega}}$$

Taking inverse DTFT, we get

$$y[n] = \left[\frac{\alpha}{\alpha - \beta}\alpha^{n} - \frac{\beta}{\alpha - \beta}\alpha^{n}\right]u[n]$$

Sampling

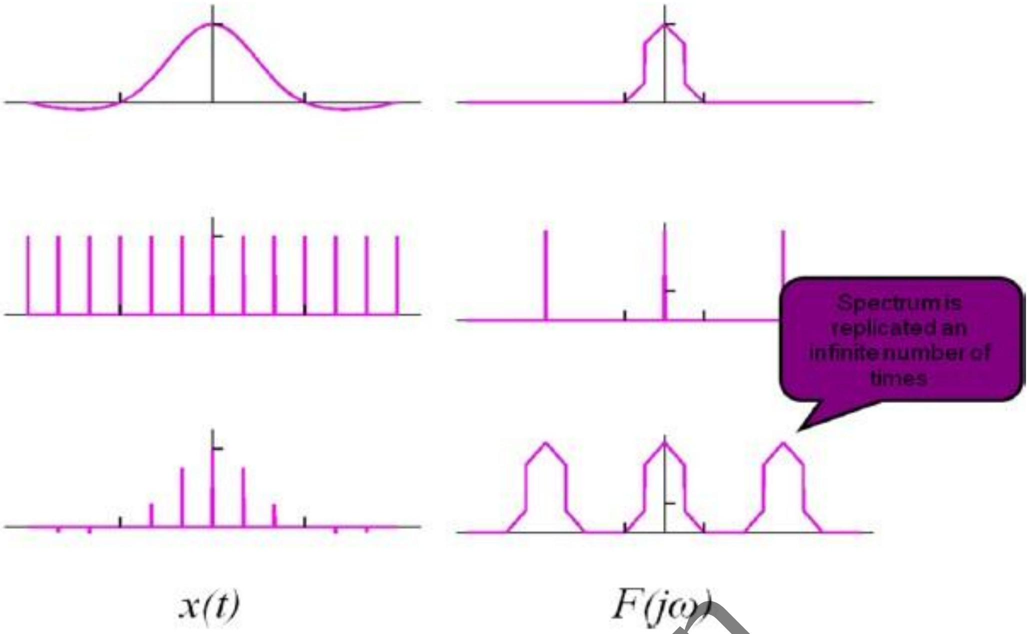
In this chapter let us understand the meaning of sampling and which are the different methods of sampling. There are the two types. Sampling Continuous-time signals and **Sub-sampling**. In this again we have *Sampling Discrete-time signals*.

Sampling Continuous-time signals

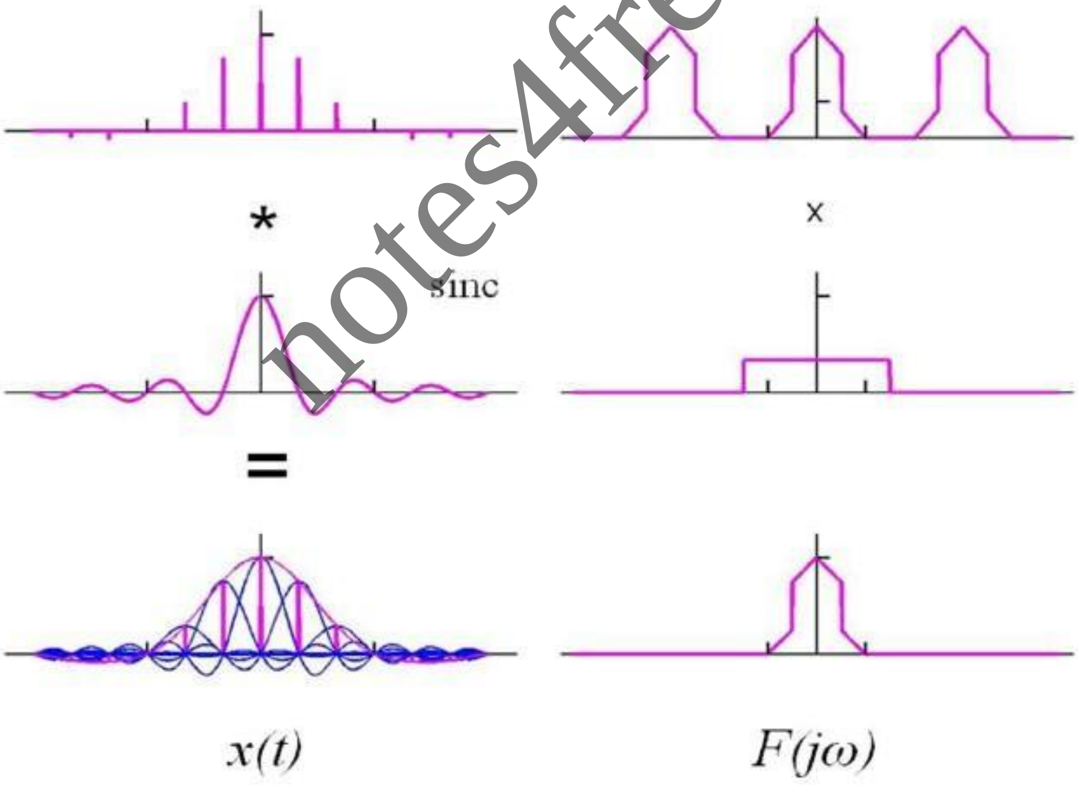
Sampling of continuous-time signals is performed to process the signal using digital processors. The sampling operation generates a discrete-time signal from a continuous-time signal.DTFT is used to analyze the effects of uniformly sampling a signal.Let us see, how a DTFT of a sampled signal is related to FT of the continuous-time signal.

- Sampling: Spatial Domain: A continuous signal x(t) is measured at fixed instances spaced apart by an interval 'T'. The data points so obtained form a discrete signal x[n]=x[nT]. Here, ΔT is the sampling period and 1/ ΔT is the sampling frequency.Hence, sampling is the multiplication of the signal with an impulse signal.

- **Sampling theory**



- **Reconstruction theory**



Sampling: Spatial Domain

From the Figure we can see

Where x[n] is equal to the samples of x(t) at integer multiples of a sampling interval T

$$x_\delta(t) = \sum_{n=-\infty}^{+\infty} x(n)\delta(t - n\tau)$$

Now substitute $x(nT)$ *for* $x[n]$ *to obtain*

$$x_\delta(t) = \sum_{n=-\infty}^{+\infty} x(n\tau)\ \delta(t - n\tau)$$

since $x(t)\delta(t - n\tau) = x(n\tau)\delta(t - n\tau)$

we may rewrite $x_\delta(t)$ *as a product of time functions*

$$x_\delta(t) = x(t)p(t) \quad \text{where,} \quad p(t) = \delta(t - n\tau)$$

Hence, Sampling is the multiplication of the signal with an impulse train.

The effect of sampling is determined by relating the FT of $x_\delta(t)$ to the FT of $x(t)$. Since Multiplication in the time domain corresponds to convolution in the frequency domain, we have

$$X_\delta(j\omega) = \frac{1}{2\pi} X(j\omega) * P(j\omega)$$

Substituting the value of $P(j\omega)$ as the FT of the pulse train i.e

$$p(t) = \sum_{n=-\infty}^{+\infty} \delta(t - nT)$$

We get,

$$P(j\omega) = \frac{2\pi}{\tau} \sum_{n=-\infty}^{+\infty} \delta(\omega - k\omega_s)$$

$$\text{where, } \omega_s = \frac{2\pi}{\tau}, \text{ is the sampling frequency. Now}$$

$$X_\delta(j\omega) = \frac{1}{2\pi} X(j\omega) * \frac{2\pi}{\tau} \sum_{n=-\infty}^{+\infty} \delta(\omega - k\omega_s)$$

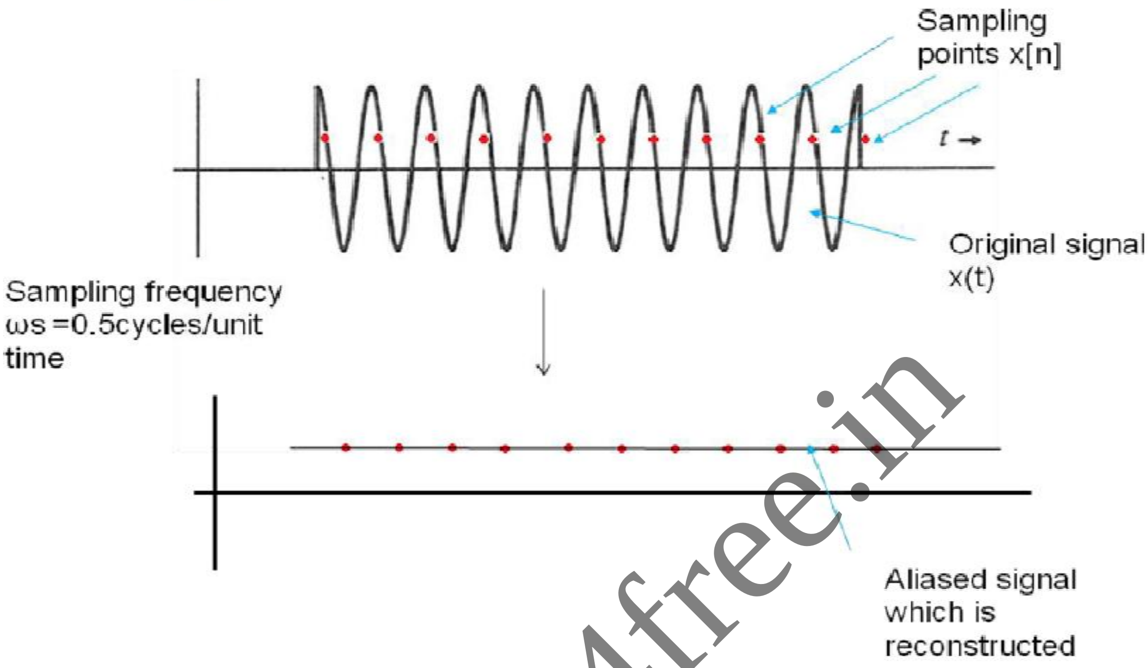
$$X_\delta(j\omega) = \frac{1}{\tau} \sum_{n=-\infty}^{+\infty} X(j(\omega - k\omega_s))$$

The FT of the sampled signal is given by an infinite sum of shifted version of the original signals **FT** and the offsets are integer multiples of ω_s.

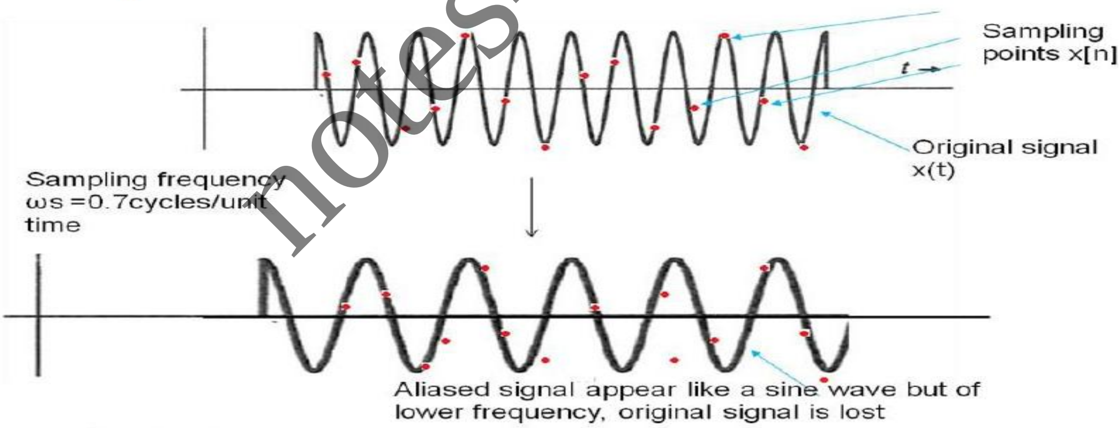
Aliasing : an example

Frequency of original signal is 0.5 oscillations per time unit). Sampling frequency is also 0.5 oscillations per time unit). Original signal cannot be recovered.

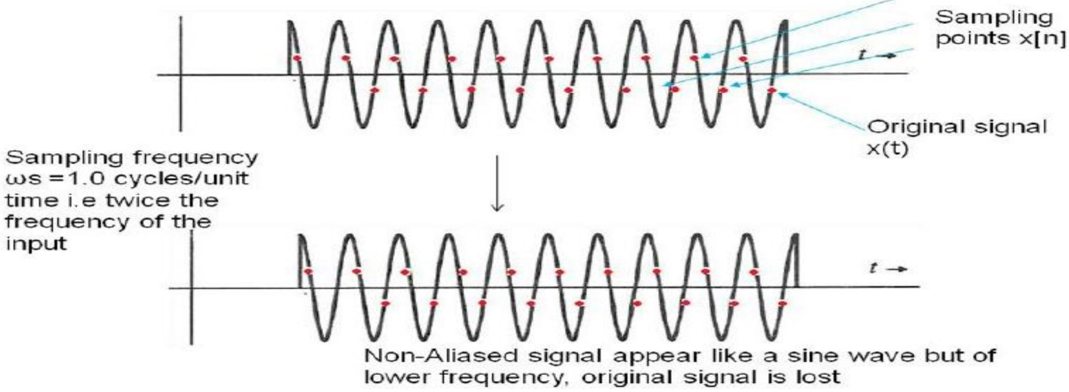
Aliasing Ex:1



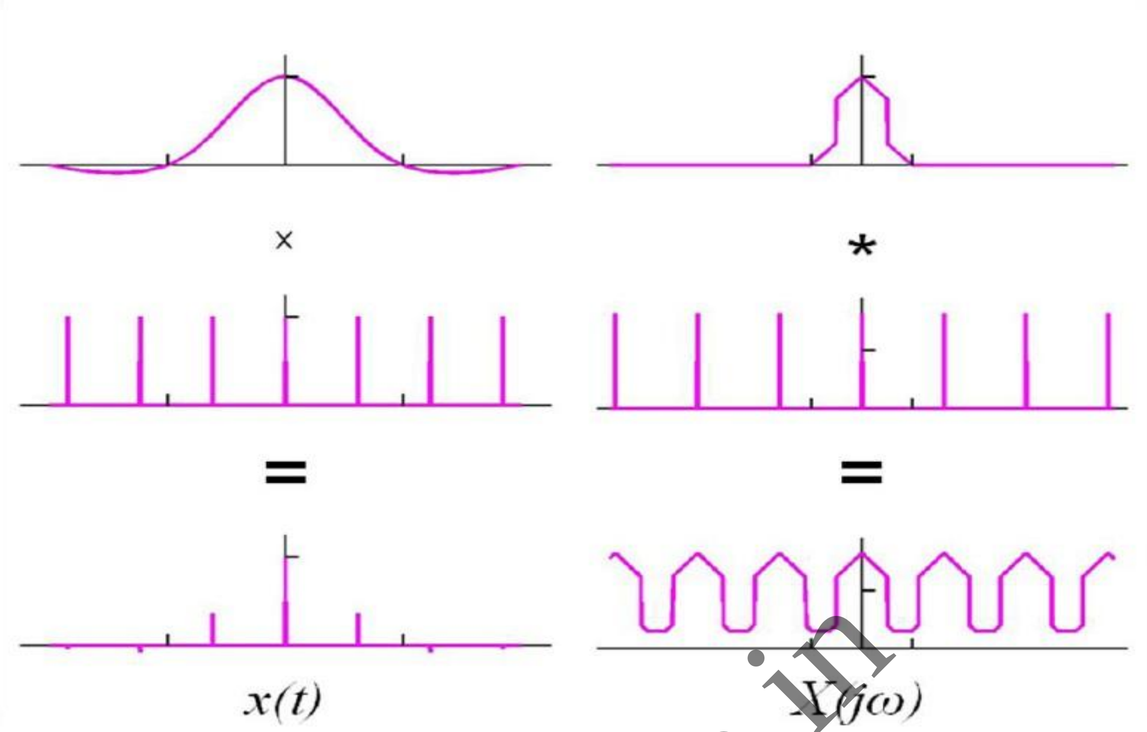
Aliasing Ex:2



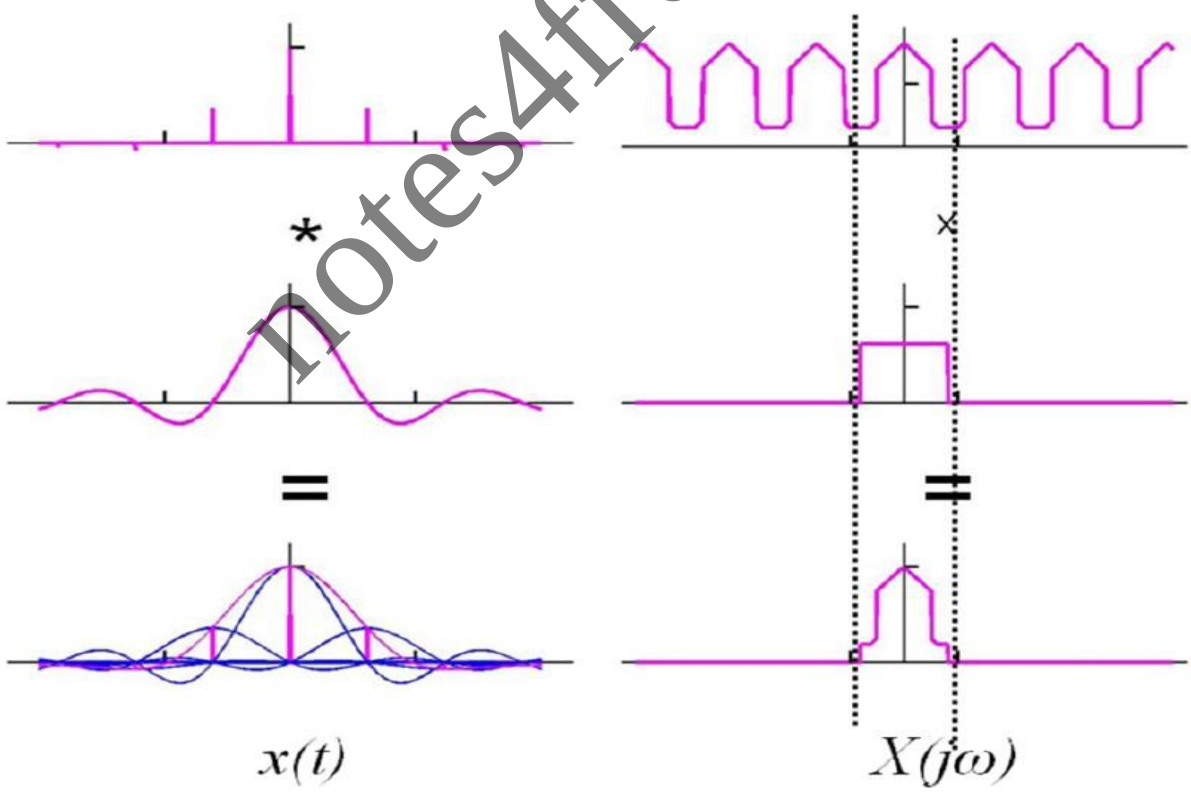
Non-Aliasing: Ex 3



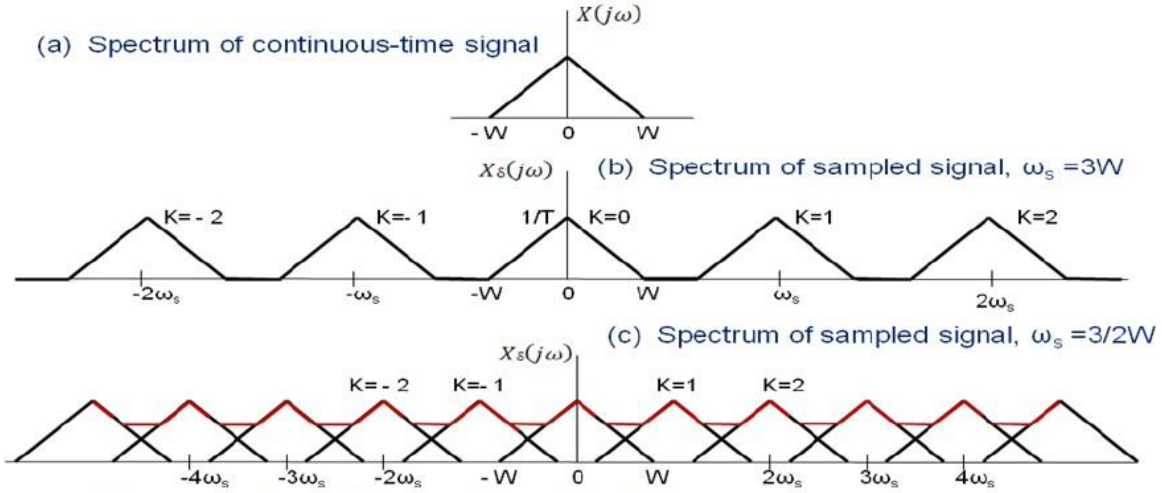
Sampling below the Nyquist rate



Reconstruction below the Nyquist rate



FT of sampled signal for different sampling frequency



- Reconstruction problem is addressed as follows.
- Aliasing is prevented by choosing the sampling interval T so that $\omega_s > 2W$, where W is the highest frequency component in the signal.
- This implies we must satisfy $T < \pi/W$.
- Also, DTFT of the sampled signal is obtained from $X_\delta(j\omega)$ using the relationship $\Omega = \omega T$, that is

$$x[n] \xleftrightarrow{DTFT} X(e^{j\omega}) = X_\delta(j\omega)\,|_{\omega = \Omega/\tau}$$

- This scaling of the independent variable implies that $\omega = \omega_s$ corresponds to $\Omega = 2\pi$

Subsampling: Sampling discrete-time signal

- FT is also used in discrete sampling signal.
- Let $y[n] = x[qn]$ be a subsampled version x[n], where q is a positive integer.
- Relating DTFT of y[n] to the DTFT of x[n], by using FT to represent x[n] as a sampled versioned of a continuous time signal x(t).
- Expressing now y[n] as a sampled version of the sampled version of the same underlying CT x(t) obtained using a sampling interval q that associated with x[n]
- We know to represent the sampling version of x[n] as the impulse sampled CT signal with sampling interval T.

$$x_\delta(t) = \sum_{n=-\infty}^{+\infty} x(n)\delta(t - n\tau)$$

- Suppose, x[n] are the samples of a CT signal x(t), obtained at integer multiples of T. That is, x[n]=x[nT]. Let $x(t) \xleftrightarrow{FT} X(j\omega)$ and applying it to obtain

$$X_\delta(j\omega) = \frac{1}{\tau}\sum_{k=-\infty}^{+\infty} X(j(\omega - k\omega_s))$$

- Since y[n] is formed using every qth sample of x[n], we may also express y[n] as a sampled version of x(t).we have $y[n] = x[qn] = x(nq\tau)$

- Hence, active sampling rate for y]n] is T'=qT. Hence

$$y_\delta(t) = x(t) \sum_{n=-\infty}^{\infty} \delta(t - n\tau') \overset{FT}{\longleftrightarrow} Y_\delta(j\omega) = \frac{1}{\tau'} \sum_{k=-\infty}^{+\infty} X(j(\omega - k\omega_s'))$$

- Hence substituting T'=q T, and $\omega_s' = \omega_s/q$

$$Y_\delta(j\omega) = \frac{1}{q\tau} \sum_{k=-\infty}^{+\infty} X(j(\omega - \frac{k}{q}\omega_s))$$

- We have expressed both $Y_\delta(j\omega)$ and $X_\delta(j\omega)$ as a function of .
- Expressing $X(j\omega)$ as a function of $X_\delta(j\omega)$. Let us write k/q as a proper function, we get

$$\frac{k}{q} = l + \frac{m}{q},$$

where l is the integer portion of $\frac{k}{q}$, *and m is the remainder*

allowing k to range from $-\infty$ *to* $+\infty$ *corresponds*

to having l range from $-\infty$ *to* $+\infty$ *and m from 0 to* $q-1$

$$Y_\delta(j\omega) = \frac{1}{q} \sum_{m=0}^{q-i} \left\{ \frac{1}{\tau} \sum_{l=-\infty}^{+\infty} X_\delta\left(j\left(\omega - l\omega_s - \frac{m}{q}\omega_s\right)\right)\right\}$$

$$Y_\delta(j\omega) = \frac{1}{q} \sum_{m=0}^{q-i} X_\delta\left(j\left(\omega - \frac{m}{q}\omega_s\right)\right)$$

which represents a sum of shifted versions of

$X_\delta(j\omega)$ *normalized by q.*

Converting from the FT representation back to DTFT

and substituting $\Omega = \omega\tau'$ *above*

and also $X(e^{j\Omega}) = X_\delta(j\Omega/\tau)$, *we write this result as*

$$Y_\delta(e^{j\Omega}) = \frac{1}{q} \sum_{m=0}^{q-i} X_q(e^{j(\Omega - m2\pi)})$$

where, $X_q(e^{j\Omega}) = X(e^{j\Omega/q})$ — *a scaled DTFT version*

<u>**Recommended Questions**</u>

1. Find the frequency response of the RLC circuit shown in the figure. Also find the impulse response of the circuit

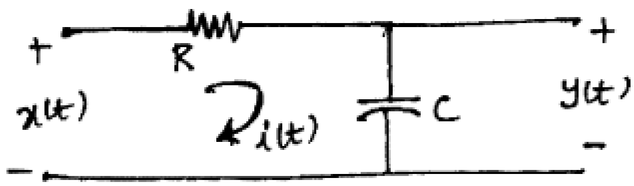


Fig.Q6(b)

2.

The input and output of causal LTI system are described by the differential equation.

$$\frac{d^2y(t)}{dt^2} + 3\frac{dy(t)}{dt} + 2y(t) = x(t)$$

i) Find the frequency response of the system

ii) Find impulse response of the system

iii) What is the response of the system if $x(t) = te^{-t}u(t)$. **(10 Marks)**

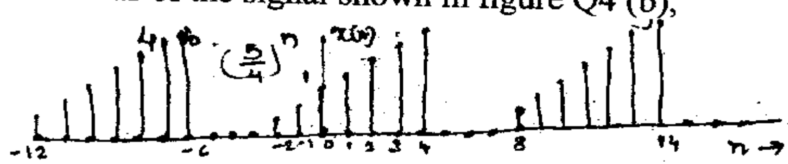
3. If $x(t) \leftrightarrow X(f)$. Show that $x(t)\text{Cos}w_0t \leftrightarrow 1/2[X(f-f_0)+X(f-f_0)]$ where $w0=2\pi f_0$

4.

The input $x(t) = e^{-3t}u(t)$ when applied to a system, results in an output $y(t) = e^{-t}u(t)$. Find the frequency response and impulse response of the system. **(07 Marks)**

5.

Find the DTFS co-efficients of the signal shown in figure Q4 (b),



6. State sampling theorem. Explain sampling of continuous time signals with relevant expressions and figures.

7. Find the Nyquist rate for each of the following signals:
i) $x(t) = \text{sinc}(200t)$ ii) $x(t) = \text{sinc}^2(500t)$