MODULE 1

OSCILLATIONS AND SHOCK WAVES

Oscillations and vibrations are most frequently occurring phenomena. Oscillation is a repeating motion that occurs when a time varying force acts on the system. Oscillations are periodic motions. If the oscillations occur in the system without the action of an external force, then the oscillation is said to be "free oscillation". Simple harmonic motion is a special kind of periodic motion.

IMPORTANT TERMINOLOGIES TO BE KNOWN TO UNDERSTAND SHM OF A BODY:

- **Displacement (x):** The distance of location of a body from its mean position at a particular instant of time.
- **Amplitude (a):** The maximum value of displacement that a body can undergo on either side from its mean position during the oscillation.
- **Frequency (ν):** The number of oscillations executed by an oscillating body in unit time (ν) = $\frac{1.}{T}$
- **Angular frequency (ω) (or) angular velocity:** It is the angle covered in unit time by a body moving in circular motion at that instant of time. $\omega = \pi = 2\pi\nu$
- **Period (T):** It is the time taken by the body to complete one oscillation. $T = \frac{2\pi}{\omega}$

Restoring force: When a body is oscillating in a medium, the action of force whose magnitude is proportional to displacement and acting in a direction opposite to displacement w.r.t equilibrium position. This force is called restoring force and is basically responsible for the oscillation of the body.

If F is the restoring force, and "x" is the displacement then ,

$$F \propto -x,$$

$$\mathbf{F = -k\,x}$$

Where, k is force constant.

The above equation is called Hooke's law. This law states that, "the restoring force in oscillating body is directly proportional to displacement and acting in a direction opposite to displacement".

Simple Harmonic Motion (SHM): The motion of a body is said to be SHM if the (restoring force) acceleration is directly proportional to the displacement and acts in a direction opposite to that of motion from the equilibrium position".

(OR)

It is a periodic motion executed by a body such that its acceleration is proportional to its displacement from the equilibrium position and always directed towards it under the action of restoring force.

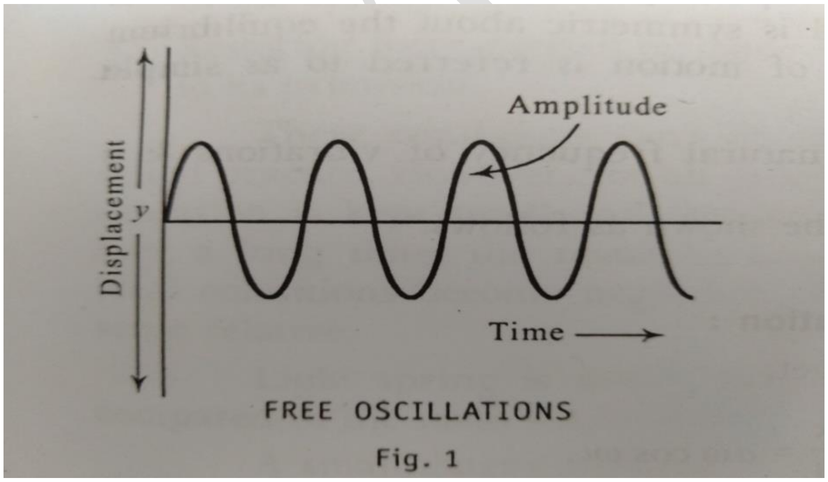
Characteristics of SHM:

- It is a particular type of periodic motion.
- There is a constant restoring force continuously acting on the body.
- The acceleration developed during the motion is directly proportional to displacement.
- Restoring force acts in a direction opposite to displacement. i.e., $F \propto -x$
- It is represented by sine or cosine functions. i.e. $x = a \sin \omega t$

Examples of SHM:

1. A pendulum set for oscillation.
2. Excited tuning fork.
3. A shock absorber after being bumped.
4. A mass suspended to a spring and left free to oscillate.
5. Oscillations of LC circuits.

Differential equation of motion of SHM:



FREE OSCILLATIONS

Fig. 1

Consider a body of mass “m”executing SHM. Let “x” be the displacement of the body under the action of restoring force.

For an oscillating body, from Hooke’s law

$$F = -k\,x \qquad \ldots.. (1)$$

From Newton’s second law of motion, the force experienced by the body under motion is given by

$$F = m\,a = m\,\frac{d^2x}{dt^2} \qquad \ldots.. (2) \qquad \text{Where ‘a’ is acceleration}$$

Equating equation (1) and (2),

$$m\frac{d^2x}{dt^2} = -k\,x$$

$$\frac{d^2x}{dt^2} = -\frac{k}{m}\,x$$

$$\frac{d^2x}{dt^2} + \frac{k}{m}\,x = 0$$

.

OR

$$\frac{d^2x}{dt^2} + \omega^2 x = 0 \quad \ldots..(3) \qquad \text{where } \omega = \sqrt{\frac{k}{m}} \text{ (angular frequency)}$$

This equation is called, equation of motion of SHM

The solution of equation (3) is given by

$$x = a\sin\omega t$$

Differentiate w.r.t "t" twice

$$\frac{dx}{dt} = \omega\, a\cos\omega t$$

$$\frac{d^2x}{dt^2} = -\,\omega^2 a\sin\omega t$$

$$\frac{d^2x}{dt^2} = -\omega^2 x$$

$$\frac{d^2x}{dt^2} + \omega^2 x = 0 \quad \ldots.(4)$$

Comparing equations (3) and (4)

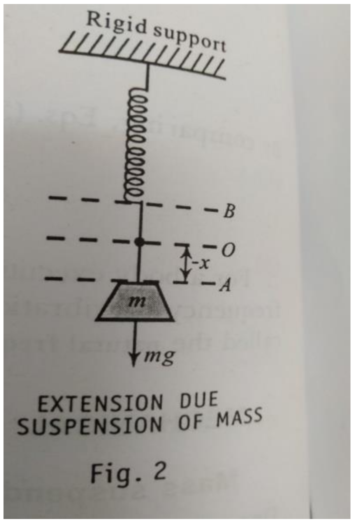
$$\omega^2 = \frac{k}{m}$$

The angular frequency is given by $\boldsymbol{\omega = \sqrt{\frac{k}{m}}}$

Mechanical Simple Harmonic Oscillator:

Mass suspended to a spring [vertical vibrations]

Consider a spring fixed at one end by the rigid support. Let a body of mass "m" is suspended at the lower end of the spring due to which force "mg" acts on the spring vertically downwards. Let "O" be equilibrium position of the spring with the application of mass "m". The body of mass "m" is pulled down and left free to oscillate with respect to "O" and "x" be the displacement of the body under the action of restoring force.



EXTENSION DUE SUSPENSION OF MASS

Fig. 2

Therefore, from Hooke's law,

$$F = -k\,x$$

$$k = -\frac{F}{x}$$

Consider the magnitude of force. $|F| = |-F|$

Then for x = 1, **k = F**

Thus force constant k is defined as, "the magnitude of the applied force that produces unit extension in the spring while it is loaded within the elastic limit."

Physical significance of force constant (k):

Force constant is a measure of stiffness of the material. In case of spring it represents the amount of force required to stretch the spring by unit length. The springs with larger value of force constant will be stiffer. It is also called spring constant (or) stiffness factor.

Period of oscillation (T): The period of oscillation of mass spring system is given by

$$T = 2\pi\sqrt{\frac{m}{k}}\ \text{second}$$

Where, m is suspended mass and k is force constant.

Frequency of oscillation (ν): The frequency is the reciprocal of time period and is given by

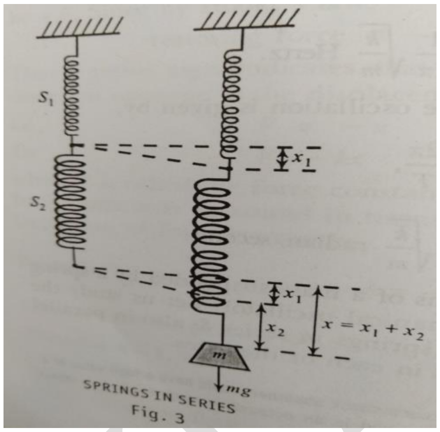
$$\nu = \frac{1}{T} \quad , \quad \nu = \frac{1}{2\pi}\sqrt{\frac{k}{m}}\ \text{Hz}$$

Angular frequency (ω): Angular frequency of oscillation is given by

$$\omega = 2\pi \nu = \frac{2\pi}{T}$$

$$\omega = \sqrt{\frac{k}{m}} \text{ radian/second}$$

Expression for force constant (Spring constant) for series combination of springs:



SPRINGS IN SERIES
Fig. 3

Consider the spring “S_1” with force constant “k_1” suspended by mass “m” and displaced through “x_1” under the action of restoring force.

Therefore, from Hooke’s law,

$$F = -k_1 x_1$$

$$x_1 = -\frac{F}{k1} \qquad \ldots\ldots. (1)$$

Similarly for the spring s_2

$$F = -k_2\ x_2$$

$$x_2 = -\frac{F}{k2} \qquad \ldots\ldots(2)$$

x_2 is the displacement of s_2, k_2 is the force constant of S_2. When the springs are connected in series, k_s is the force constant and displaced through “x” suspended by a same mass “m”.

From Hooke’s law,

$$F = -\ k_s\ x$$

$$x = -\frac{F}{ks} \qquad \ldots\ldots.(3)$$

The displacement for series combination is $x = x_1+x_2$ ………….(4)

Substituting eqn (1), (2) and (3) in equation (4)

$$-\frac{F}{ks} = -\frac{F}{k1} - \frac{F}{k2}$$

$$\frac{1}{ks} = \frac{1}{k1} + \frac{1}{k2}$$

$$k_s = \frac{k1\,k2}{k1+k2}$$

This is an expression for spring constant of two springs connected in series combination. If "n" number of springs connected in series, then

$$\frac{1}{ks} = \frac{1}{k1} + \frac{1}{k2} + \frac{1}{k3} + \ldots\ldots.. + \frac{1}{kn}$$

$$\text{Time period, } T = 2\pi\sqrt{\frac{m}{ks}}$$

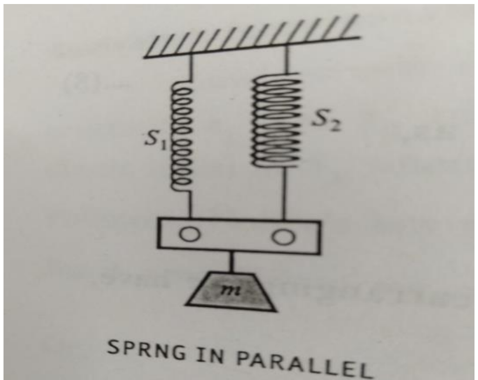
Expression for force constant (spring constant) for parallel combination of two springs:

Consider two springs S_1 and S_2 with spring constants k_1 and k_2 respectively. Let x_1 and x_2 be displacement (extension) of the individual springs when they suspended by a mass "m".

Hence, $F_1 = -k_1\ x_1$ ….(1)

Similarly for spring S_2 $F_2 = -k_2\,x_2$ .....(2)

where F_1 and F_2 are restoring force of the springs S_1 and S_2 respectively.



SPRNG IN PARALLEL

Fig. 4

When the springs are connected in parallel and suspended by the same load, with force constant (K_p) and displaced through distance "x" under the action of restoring force (F_p).

From Hooke's law, $F_p = k_p\ x$ ….(3)

The restoring force (F_p) is equally shared by two springs when they are connected in parallel with same extension $x_1 = x_2 = x$.

Then $F_p = F_1 + F_2$

From (1), (2) and (3)

$$-\ k_p\ x = -k_1\ x_1 - k_2\ x_2 \quad (\text{since } x_1 = x_2 = x.$$

$$k_p = k_1 + k_2$$

This is the expression for equivalent force constant for springs in parallel combination.

If ' n' number of spring are connected in parallel then,

$$K_p = k_1 + k_2 + k_3 + \ldots k_n$$

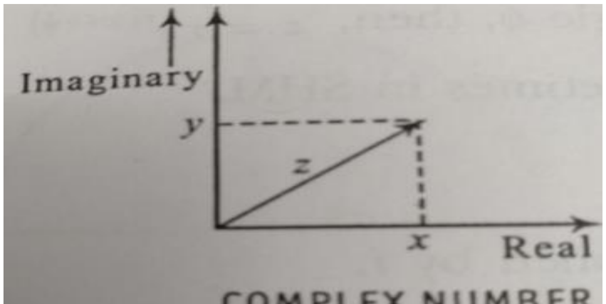
The time period for parallel combination of springs is given by

$$T = 2\pi\sqrt{\frac{m}{kp}}$$

Complex Notation of SHM:

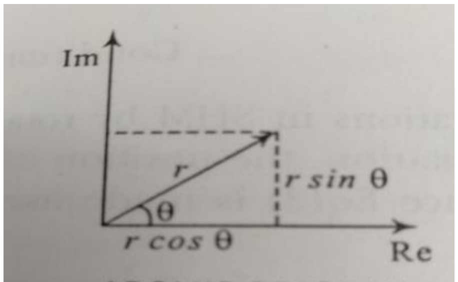
Complex numbers are a convenient tool to mathematically analyze sinusoidal functions. It can be used to represent amplitude and phase of a periodically varying function.
A complex number Z is of the form Z= x + iy, where x is the real part, which is the projection of Z on x-axis and y is imaginary which is the projection of Z on y-axis.

The method of representation of complex number in a co-ordinate system called Argand diagram.



COMPLEX NUMBER REPRESENTATION IN CARTESIAN FORM

Fig. 5



ARGAND DIAGRAM
IN POLAR COORDINATES
Fig. 6

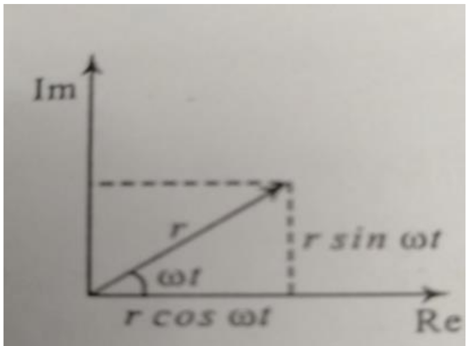


Fig. 7

In polar coordinates, $Z = r\cos\theta + i\sin\theta$, where' r' is magnitude of Z.

since, $e^{i\theta} = \cos\theta + i\sin\theta$

$$Z = r\,e^{i\theta}$$

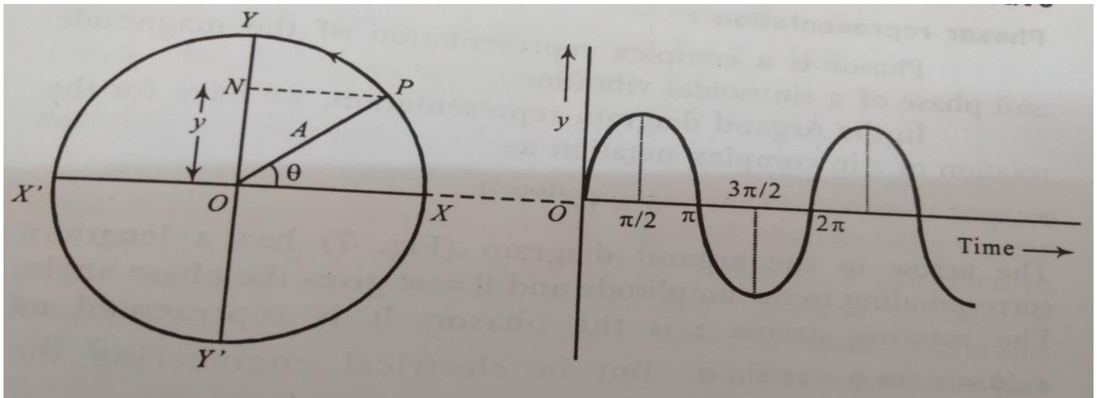
If 'θ' changes W.r.t. time, then $\theta = \omega t$

$$\text{Therefore, } Z = r\,e^{i\omega t}$$

At t = 0, if already Z makes an angle φ, then

$Z = r\,e^{i(\omega t+\varphi)}$, is the **complex notation of SHM.**

Phasor representation of SHM:



SIMPLE HARMONIC MOTION
Fig. 8

Phasor is a complex representation of the magnitude and phase of sinusoidal vibrations (SHM).

Consider a particle moving on a circular path of radius “R” in the anticlockwise direction with angular velocity ω.

Let “P” be the position of the particle at particular instant of time “t”. Draw perpendicular PN to the diameter. As the particle move in circular path , the point ‘N’ moves to and fro motion on the diameter W.r.t origin ‘O’ executing SHM.

Where, O N→Displacement, OY→ maximum displacement = amplitude.

The angle ∠POX = θ gives the phase,

Therefore, we have $Y = A \sin\theta$

Phasor representation:

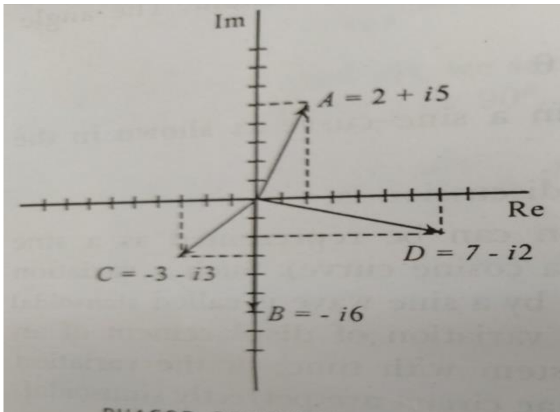
$$Z = r\,e^{i(wt+\varphi)}$$

$$r\angle\phi = r\cos\phi + i\,r\sin\phi$$

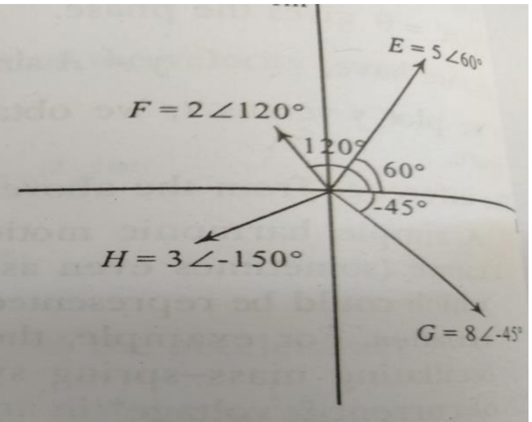
the current and voltage phasor in electrical circuit is

$$I = I\angle\phi = I\cos\phi + j\,I\sin\phi$$

$$V = V\angle\phi = V\cos\phi + j\,V\sin\phi$$



PHASOR REPRESENTATION
(RECTANGULAR FORM)
Fig. 9



PHASOR REPRESENTATION
(POLAR FORM)
Fig. 10

Free oscillations: "The oscillatory body oscillates with undiminished amplitude with its own natural frequency of vibrations for infinite length of time under the action of restoring force, until an external force affects its motion" are called free oscillations.

Examples:

- Oscillation of mass suspended to spring with negligible damping and small displacement.
- Oscillations of simple pendulum.
- LC oscillations.

Equation of Motion of Free Oscillations:

If "m" is the mass of oscillating body with a force constant "k" and 'x' is the displacement at the instant "t" of the oscillating body, then

$$\frac{d^2x}{dt^2} + \frac{k}{m}x = 0$$

$$\frac{d^2x}{dt^2} + \omega^2 x = 0$$

Where ω is the angular frequency , $\omega = \sqrt{\frac{k}{m}}$

Natural frequency of oscillations: It is a characteristic with which a body (system) body oscillates under the action of restoring force after it gets displaced from its equilibrium position and left force.

Damped oscillations: An oscillatory body oscillates such that its amplitude gradually decreases and comes to rest at equilibrium position in a finite interval of time due to the action of resistive force.

Examples:

- Mechanical oscillations of simple pendulum.
- Electrical oscillations of LC circuit
- A swing left free to oscillate after being pushed once.

Theory of Damped Oscillations:

Consider a body of mass "m" executing damped oscillations in a resistive medium. The resistive force is proportional to velocity of the body and acting in opposite directions.

$$\text{Therefore, Resistive force} = -r\frac{dx}{dt} \qquad \ldots..(1)$$

Where, r → damping constant, $\frac{dx}{dt} \rightarrow$ velocity.

The restoring force acting on the body set for oscillations given by

$$\text{Restoring force} = -\,k\,x \qquad \text{…. (2)}$$

Where, k→ force constant, x→displacement

The resultant force acting on the body is,

Resultant force = resistive force + restoring force

From (1) and (2)

$$\text{Resultant force} = -\,r\,\frac{dx}{dt} - k\,x \qquad \text{…. (3)}$$

From Newton's second law of motion, the resultant force experienced by the body under motion is given by

$$\text{Resultant force} = m\,a = m\,\frac{d^2x}{dt^2} \qquad \text{….(4)}$$

where a→ acceleration

Equating equations (3) and (4)

$$m\,\frac{d^2x}{dt^2} = -\,r\,\frac{dx}{dt} - kx$$

$$\frac{d2x}{dt2} = -\,\frac{r}{m}\,\frac{dx}{dt} + \frac{k}{m}\,x = 0$$

$$\frac{d2x}{dt2} + 2b\,\frac{dx}{dt} + \omega^2 x = 0 \quad \text{……(5)} \qquad \text{where,}\quad 2b = \frac{r}{m}\ ,\ \omega = \sqrt{\frac{k}{m}}$$

this is the equation of motion for damped oscillations

the solution of equation (5) is given by

$$x = C\,e^{(-b+\sqrt{b^2\,-\omega^2})t} + D\,e^{(-b-\sqrt{b^2\,-\omega^2})t} \quad \text{……(6)}$$

where C and D are constants given by,

$$C = \frac{x_0}{2}\left[1 + \frac{b}{\sqrt{b^2\,-\omega^2}}\right], \qquad D = \frac{x_0}{2}\left[1 - \frac{b}{\sqrt{b^2\,-\omega^2}}\right]$$

where, x is displacement at t = 0

the general solution of equation (5) is given by substituting the values of C and D in equation (6)

$$x = \frac{x_0}{2}\ \{(1+ \frac{b}{\sqrt{b^2 - \omega^2}})\, e^{(-b+\sqrt{b^2 - \omega^2})t} + (1 - \frac{b}{\sqrt{b^2 - \omega^2}})\, e^{(-b-\sqrt{b^2 - \omega^2})t}\} \quad \ldots\ldots(7)$$

This is the General solution of damped oscillations.

In the above equation, as “t” varies, x also varies. But the nature of variation depends on $\sqrt{b^2 - \omega^2}$.

Case (i): If $b^2 > \omega^2$, over damping (or) dead beat case:

When $b^2 > \omega^2$, $b^2 - \omega^2$ is positive, but $\sqrt{b^2 - \omega^2} < b$

Therefore, the coefficient of “t” in both terms of equation (7) is negative, this indicates exponential decay of displacement W.r.t time, and the body after maximum displacement comes to rest at equilibrium position with the decay of time.

Over damping is the condition such that the body comes to rest at equilibrium position under the action of restoring force and resistive force with long interval.

Example: motion of pendulum in highly viscous liquid.

Case (ii): when $b^2 = \omega^2$, critical damping:

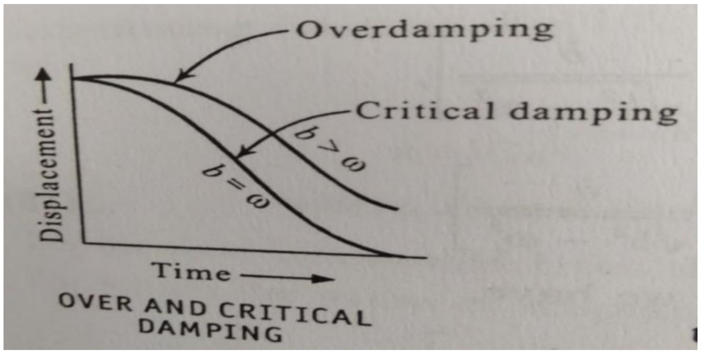
If $b^2 = \omega^2$, then $\sqrt{b^2 - \omega^2} \approx \xi$ (small quantity),

Then equation (6) reduces to **$x = e^{-bt} [(C+D) + (C-D) \xi t]$**

As “t” varies, the displacement decreases exponentially and become zero for short interval of time.

“It is a condition such that the body comes to half at equilibrium position under the action of restoring and resistive force in a short interval of time.”

Example: Pointer galvanometer, voltmeter, current meter. Shock absorbers designed with springs.



Case(iii): when $b^2 < \omega^2$, under damping:

When $b^2 > \omega^2$, $b^2 - \omega^2$ is negative. But $\omega^2 - b^2$ is positive

Therefore, $\sqrt{b^2 - \omega^2} = \sqrt{-(\omega^2 - b^2)} = i\sqrt{\omega^2 - b^2}$ $(i = \sqrt{-1})$

Let $\omega^2 - b^2 = n$

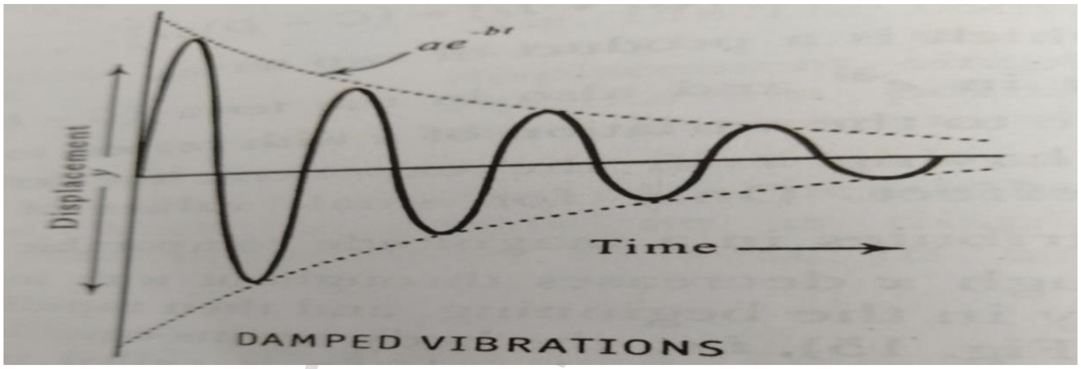
Then, equation (7) reduces to,

$$\mathbf{x = a\, e^{-bt} \sin(nt+\varphi)}, \qquad \text{where } a = x_0\,(\alpha/n)\ ,\ \varphi = \tan^{-1}(n/b)$$

As time "t" increases, the displacement 'x' decreases exponentially.

"It is a condition such that, the body vibrates with diminishing amplitude with the decay of time under the action of restoring and resistive forces and comes to rest at equilibrium position".

Examples: Any real physical system like oscillations of simple pendulum, vibrations of tuning fork, mass suspended to spring.



Quality factor (Q):

The amount of damping is described by the quantity called quality factor and is given by,

$$Q = \frac{\omega}{2b},\ \omega \rightarrow \text{angular frequency}\ ,\ 2b \rightarrow \frac{r}{m}\ ,\ r \rightarrow \text{damping factor.}$$

"Quality factor is the number of cycles required for the energy to fall off by a factor $e^{2\pi}$ (≈535).

If Q value is more indicated the sustained oscillations overcoming the resistive forces.

Q factor describes how much under damped is the oscillatory system.

Forced oscillations (or) forced vibrations:

"It is a steady state sustained vibrations of a body vibrating in a resistive medium under the action of external periodic force which acts independently of the restoring force."

Examples:

- Oscillations of swing pushed periodically by a person
- oscillations of LC circuit with the applied ac source
- Motion of diaphragm in a telephone receiver.

Theory of forced oscillations (OR) Expression for amplitude and phase of the forced vibrations:

Consider a body of mass "m" executing vibrations in a damping medium under the application of external periodic force. F sin p t,

If 'x' is the displacement of the body at any instant of time "t".

The restoring force = $-$ k x ..…(1), where k → force constant.

The resistive force acts opposite to velocity given by

Resistive force = $-r \frac{dx}{dt}$ …..(2), where r → damping constant, $\frac{dx}{dt}$ → velocity

The applied periodic force = F sin pt …(3), where p → frequency of external force.

Therefore, the resultant force experienced by the body is given by

From equations (1), (2) & (3)

$$\text{Resultant force} = -r\frac{dx}{dt} - kx + F\sin pt \qquad \ldots\ldots(4)$$

From Newton second law of motion, the resultant force experienced by moving body is given by

$$\text{Resultant force} = ma = m\frac{d^2x}{dt^2} \qquad \ldots(5), \qquad \text{where } a \rightarrow \text{acceleration.}$$

Equating (4) and (5)

$$m\frac{d^2x}{dt^2} = -r\frac{dx}{dt} - kx + F\sin pt$$

$$\frac{d^2x}{dt^2} + \frac{r}{m}\frac{dx}{dt} + \frac{k}{m}x = \frac{F}{m}\sin pt \qquad \ldots\ldots(6)$$

This is the equation of motion of forced oscillations.

$$\frac{d^2x}{dt^2} + 2b\frac{dx}{dt} + \omega^2 x = \frac{F}{m}\sin pt \qquad \ldots..(7) \quad \text{where, } 2b = \frac{r}{m},\ \omega = \sqrt{\frac{k}{m}}$$

The solution of equation (7) is given by

$$x = a\sin(pt - \alpha)$$

Differentiate W.r.t "t" twice,

$$\frac{dx}{dt} = \text{p a} \cos(pt - \alpha)$$

$$\frac{d^2x}{dt^2} = -\text{p}^2 \text{ a} \sin(pt - \alpha)$$

equation (7) becomes,

$$-\text{p}^2 \text{ a} \sin(pt - \alpha) + 2\text{b p a} \cos(pt - \alpha) + \omega^2 \text{ a} \sin(pt - \alpha) = \frac{F}{m} \sin[(\text{pt} - \alpha) + \alpha]$$

$-\text{p}^2 \text{ a} \sin(pt - \alpha) + 2\text{ b p a} \cos(pt - \alpha) + \omega^2 \text{ a} \sin(pt - \alpha) = \frac{F}{m} [sin(\text{pt} - \alpha)\, cos\alpha + cos(\text{pt-}\alpha)\, sin\alpha]$

Equating the coefficients of cos(pt-a) & sin(pt-a) on both sides of the equation

$$2 \text{ b p a} = \frac{F}{m} sin\ \alpha \qquad \text{……(8)}$$

$$\text{a}(\omega^2 - \text{p}^2) = \frac{F}{m} cos\ \alpha \qquad \text{…..(9)}$$

squaring and adding equations (8) and (9)

$$\text{a}^2 [4\text{b}^2 \text{p}^2 + (\omega^2 - \text{p}^2)^2] = \left(\frac{F}{m}\right)^2 (\sin^2\alpha + \cos^2\alpha)$$

$$\text{a}^2 = \frac{\left(\frac{F}{m}\right)^2}{(4b^2 p^2 - (\omega^2 - p^2)^2)}$$

$$\text{a} = \frac{\frac{F}{m}}{\sqrt{4b^2 p^2 - (\omega^2 - p^2)^2}}$$

This is the equation for amplitude of the forced oscillations.

Equation (8) divided by (9) gives,

$$\frac{2bp}{\omega^2 - b^2} = \frac{sin\alpha}{cos\alpha} = tan\alpha$$

$$\tan\alpha = \left(\frac{2bp}{\omega^2 - b^2}\right)$$

$$\alpha = \tan^{-1}\left(\frac{2bp}{\omega^2 - b^2}\right)$$

this is the equation for phase of forced oscillations.

Dependence of amplitude and phase on the frequency of the applied force:

Case (i): $p<<\omega$, p → angular frequency of applied force, ω → natural frequency of the vibrating body. When $p<<\omega$, p^2 is very small.

Thus, $\omega^2 - p^2 = \omega^2$ and $2bp = 0$. Since b is small,

therefore the amplitude a = $\frac{F}{m\omega^2}$

The phase, $\alpha = \tan^{-1}(\frac{0}{\omega 2})$ =0. The displacement and force will be in the same phase.

Case(ii): $p = \omega$, (resonance)

$\omega^2 - p^2 = 0$

Amplitude a = $\frac{F/m}{2bp} = \frac{F/r\omega}{r}$ the amplitude have highest value. Since a α ($\frac{1}{\omega}$) , phase

a= $\tan^{-1}$ (2bp / 0) =tan-1(∞) = $\frac{\pi}{2}$

The displacement has a phase log of ($\frac{\pi}{2}$) w.r.t phase of the applied force.

Case(iii): $p>>\omega$, it is significant when damping forces are small.

When $p>>\omega$, $(\omega^2 - p^2) \approx p^4$

Amplitude a = $\frac{\frac{F}{m}}{\sqrt{4b^2\, p^2 + p^4}}$

As ‘p’ increases ‘a’ becomes smaller. Since b is small

$4b^2p^2 << p^4$

$$\text{Amplitude } a = \frac{F/m}{p4} = \frac{F/m}{p2}$$

$$\text{Phase} \quad \alpha = \tan^{-1}\left[\frac{2bp}{\omega^2 \ -p^2}\right] = \tan^{-1}\left(\frac{2bp}{p^2}\right) = \tan^{-1}\left(\frac{2bp}{p}\right) = \pi$$

The displacement has a phase lag of ‘π’ w.r.t phase of the applied force.

Resonance: The expression for amplitude of the forced oscillations given by

$$a = \frac{F/m}{\sqrt{4b^2p^2+(\omega 2-p2)}}$$

the amplitude of vibration is maximum, this state of vibration called resonance.

Condition for resonance:

In the above equation, "a" is maximum when, "b" is minimum (or) damping caused by the medium is made minimum (since b = $\frac{r}{2m}$),

p = ω, i.e., frequency of applied force (p) is equal to natural frequency of vibration (ω) of the body.

Therefore, $a_{max} = \frac{F/m}{2bp} = \frac{F/m}{2b\omega}$

Significance of resonance:

At resonance, the vibrating body will have the ability to receive completely the energy delivered by the periodic force. Hence amplitude of vibrations are maximum. "When the frequency of a periodic force acting on a vibrating body is equal to natural frequency of vibrations of the body, the energy transfer from the periodic force to the body becomes maximum because of which the body vibrates with maximum amplitude", this phenomenon is called resonance.

Sharpness of resonance:

"sharpness of resonance is the rate at which the amplitude changes with respect to change in the frequency of the applied force at the stage of resonance".

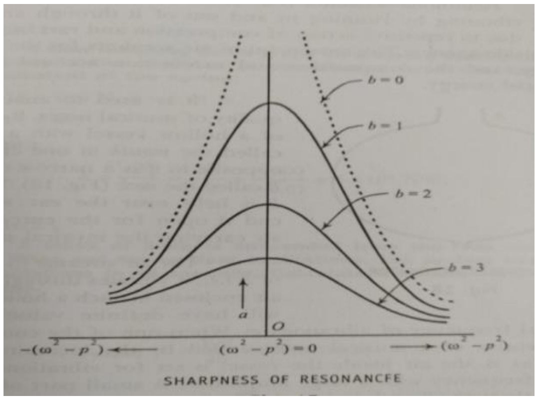
Therefore, sharpness of resonance = $\frac{change\ in\ amplitude}{change\ in\ frequency}$

Sharpness of resonance = $\frac{F/m}{2b\omega p}$

Effect of damping on sharpness of resonance:

In the above equation, sharpness of resonance α $\frac{1}{b}$

(since b = $\frac{r}{2m}$, r → damping constant)



From the graph, for small value of b exhibits higher peaks refer to sharpness of resonance, on the other for higher value of b the resonance is flat. For b = 0, the amplitude is ∞ ,can not exist in reality.

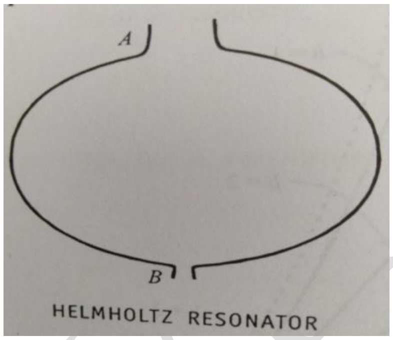
Significance of sharpness of resonance:

The rise of amplitude of amplitude will be very sharp when the damping is very small at resonance.

Examples: Helmholtz resonator, a radio receiver tuned to frequency of transmitting station, vibrations of excited tuning fork, LC oscillations with applied ac source.

Example of mechanical resonance:

Helmholtz Resonator:



Helmholtz resonator is used to analyze the quality of musical notes. It consists of a hollow vessel with a hole (A) called mouth and a narrow opening on opposite side (B) called neck. The end (A) open for the entry of the air carrying the musical note and end (B) is held near the ear. The air inside the resonator vibrates with its natural frequency of vibration (ω). When the frequency of a musical note carrying by the incoming air from the mouth "A" is equal to the natural frequency of the air enclosed in the resonator, the amplitude of vibration of air is maximum and a part of the air leaks through (neck) B and is heard. Thus the resonator is said to resonate for the note of that particular frequency.

For Helmholtz resonator it can be shown that the square of natural frequency (ω) of oscillation is inversely proportional to the volume (V) of the resonator.

$$\text{i.e., } \omega^2 \propto \frac{1}{V}$$

$$\omega^2 = \frac{k}{V}$$

$$\omega^2 V = k \quad \text{where } k \rightarrow \text{proportionality constant}$$

SHOCK WAVES

Mach number:

The ratio of speed of the object to the speed of sound in the given medium is called as a mach number.

$$\text{Mach number} = \frac{Object\ speed}{Speed\ of\ sound\ in\ the\ medium}$$

$$M = \frac{v}{a}$$

Mach number is a pure number.

Distinctions between –acoustic, ultrasonic, subsonic & supersonic waves:

Acoustic waves:

- An acoustic wave is simply a sound wave which travels with the speed of 333 m/s in air at STP.
- Amplitude of acoustic wave is very small.
- Acoustic waves are a type of longitudinal waves that propagate by means of adiabatic compression and decompression.
- Important quantities for describing acoustic waves are sound pressure, particle velocity, particle displacement and sound intensity.
- Acoustic waves travel with the speed of sound which depends on the medium they're passing through.

Ultrasonic waves:

- Ultrasonic waves are pressure waves having frequencies beyond 20000 Hz.
- At higher power levels, ultrasonic’s is useful for changing the chemical properties of substances.
- Amplitude of the ultrasonic wave is also small.

Subsonic waves:

- If the speed of mechanical wave or body moving in the fluid is lesser than that of sound, then such a speed is referred to as subsonic & the wave is a subsonic wave.
- All subsonic waves have **a mach number < 1.**
- E.g., Speed of flight of birds is subsonic, vehicles such as motor cars or trains move with subsonic speed.

Supersonic waves:

- Supersonic waves are mechanical waves which travel with speed greater than that of sound.
- They have a **mach number > 1**
- E.g.:- Fighter planes fly with supersonic speed.
- Amplitude of supersonic waves will be high & it affects the medium in which it is travelling.

SHOCK WAVES:

Any fluid that propagates at supersonic speeds, gives rise to a shock wave.

Shock waves can be produced by a sudden dissipation of mechanical energy in a medium enclosed in a small space.

A shock wave is a surface that manifests as a discontinuity in a fluid medium in which it is propagating with supersonic speed.

Shock waves are produced in nature during earth quakes (as seismic waves which travel with speed ranging from 2 km/s to 8 km/s) & when lightning strikes.

Shock waves are characterized by sudden increase in pressure, temperature & density of the gas through which it propagates.

When the speed of a source exceeds the speed of sound (v > c) the wave fronts lag behind the source in a cone –shaped region with the source at the vertex. The edge of the cone forms a supersonic wave front with unusually large amplitude called a “Shock wave”. When a shock wave reaches an observer a “sonic boom” is heard.

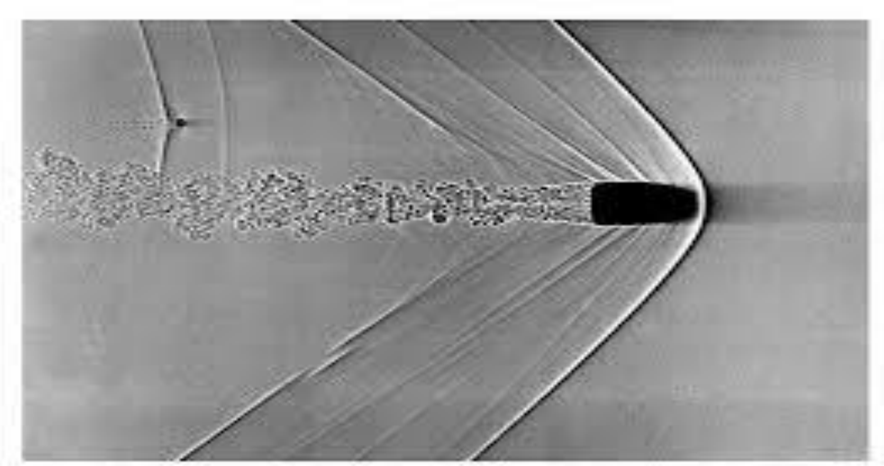
Properties of shock waves:

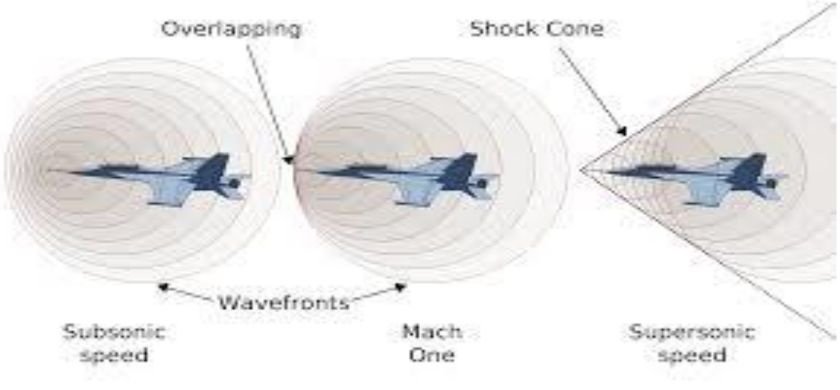
- Shock waves travel in a medium with mach number exceeds 1.
- Shock waves obey the laws of fluid dynamics.
- The change in entropy of the medium due to the generation of shock waves.
- Shock waves are confined in a medium of very thin space of thickness about 1µm.
- Shock waves are not electromagnetic in nature.
- Shock waves are not physical waves.
- Shock waves are produced in medium where the object speed is more than the speed of sound.

NOTE: pictures of shockwaves formed

1) When fighter plane moving at supersonic speed

2) when bullet moving at supersonic speed

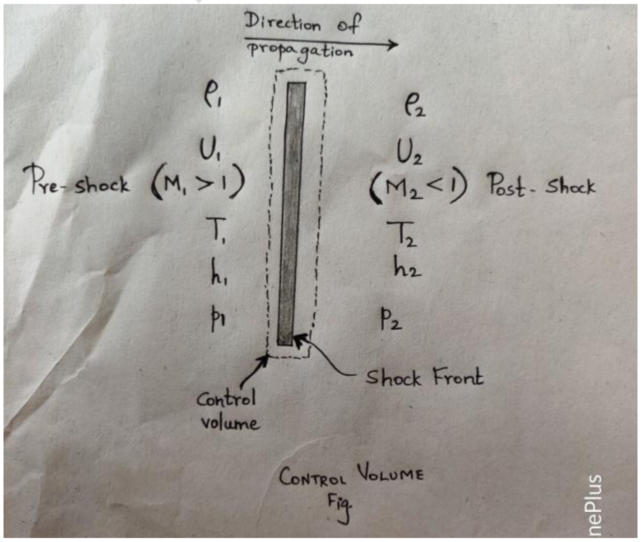




- When a shock wave passes through a matter, the total energy is preserved but the energy which can be extracted as work decreases & the entropy increases. This for example, creates additional drag force on aircraft with shocks.
- Like an ordinary wave, it carries energy & can propagate through a medium (solid, liquid, gas or plasma) or in some cases in the absence of a material medium, through a field such as an electromagnetic field.
- Contact front: in a shock wave caused by a driver gas (for example the "impact" of a high explosive on the surrounding air), the boundary between the driver (explosive products) & the driven (air) gases. The contact front trails the shock wave.
- Measurements of the thickness of shock waves in air have resulted in values around 200nm (about 10^{-5} in), which is on the same order of magnitude as the mean free gas molecule path.
- Shock waves are identified as strong or weak depending on the magnitude of the instantaneous changes in pressure & temperature in the medium that is held pressed in the space bound within the thickness of the shock front.
- Shock wave created by the Bursting of tyre & crackers is weak. Whereas those created during lightning thunder or bombing are strong. The shock waves produced during nuclear explosion are strongest ever witnessed on earth.

Control volume:

- Shock waves are analyzed by a model called "control volume". It is an imaginary thin envelope that surrounds the shock front within which; there is a sharp increase in the pressure, temperature and density in the compressed medium.
- It is a one-dimensional confinement in the medium with two surfaces. One on the pre-shock side and the other on the post-shock side. Their inter-separation is too small. The entire shock is bound between these two surfaces. Density, flow velocity, internal energy, temperature, specific enthalpy and pressure are ρ_1 , U_1,T_1, h_1, & p_1 respectively on the pre-shock side. and ρ_2 , U_2, T_2, h_2, & p_2 for the post-shock side.
- It is assumed that, the heat energy is constant i.e., no heat leaves or gets into this volume.
- The energy transfer is adiabatic. The equations for the conservation of mass, momentum and energy are the governing equations foe the control volume.



Basics of conservation of mass, momentum & energy:

Law of conservation of mass :

"The total mass of any isolated system remains unchanged & is independent of any chemical & physical changes that could occur within the system"

$$\rho_1 u_1 = \rho_2 u_2$$

Law of conservation of momentum :

"In a closed system, the total momentum remains a constant".

Or

"When two objects collide in an isolated system, the total momentum of the objects before collision is equal to the total momentum of the two objects after the collision"

$$p_1 + \rho_1 u_1^2 = p_2 + \rho_2 u_2^2$$

Law of conservation of energy :

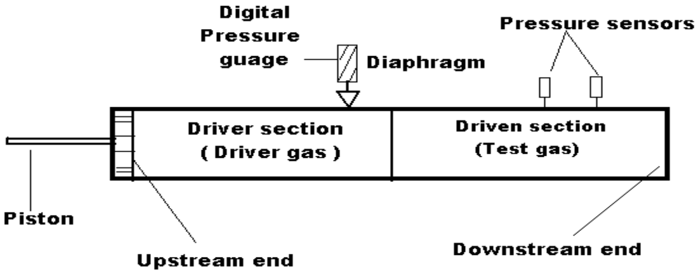
"The total energy of a closed system remains constant & is independent of any changes occurring within the system"

$$h_1 + u_1^2/2 = h_2 + u_2^2/2$$

Construction and working of Reddy Shock tube.

Reddy tube is a hand operated shock tube capable of producing shock waves by using human energy.

It is long cylindrical tube with two sections separated by a diaphragm. It's one end is fitted with a piston & the other end is closed or open to the surrounding.



Construction:

- Reddy tube consists of a cylindrical stainless steel tube of about 30mm diameter & of length nearly 1m.

- It is divided into two sections one is **driver section** & the other is **driven section** separated by a thick aluminium or Mylar or paper diaphragm of thickness 0.1mm.
- Far end of driver tube is fitted with a piston & the far end of driven tube is closed.
- A digital pressure gauge is mounted in the driver section next to the diaphragm.
- Two piezoelectric sensors S_1 and S_2 are mounted 70mm apart towards the closed end of the shock tube.
- A port is provided at the closed end of the driven section for filling the test gas to the required pressure.
- The driver section is filled with driver gas which is held at a relatively high pressure due to the compressing action of the piston. The gas in the driven section is called driven gas (test gas).

Working:

- The driver gas is compressed by pushing the piston hard into the driver tube until the diaphragm ruptures.
- Due to rupture of diaphragm the driver gas moves to the driven section, & pushes the driven gas towards the far downstream end. Hence there is sudden increase of pressure, temperature and density of driven section near the downstream end . This generates a moving shock wave that traverses the length of the driven section.
- The propagating primary shock wave is reflected back from the downstream end. After reflection, the test gas further undergoes compression produces the secondary shock waves, which increases the pressure & temperature to still higher values.
- This state of high values of pressure & temperature is sustained at the downstream end until an expansion waves reflected from the downstream end of the driver tube arrives there & neutralizes the compression partially
- The period over which the extreme temperature & pressure conditions at the downstream end is sustained, is typically in the order of milliseconds.
- The pressure rise caused by the primary & also the reflected shock wave are sensed as signals by the sensors S_1 and S_2 respectively & are recorded in a digital cathode ray oscilloscope.
- From the recording in the CRO, the shock arrival times are found out by the associated time base calculations. Using the data so obtained, Mach number, pressure & temperatures can be calculated.
-

Characteristics of Reddy tube:

a. The Reddy tube operates on the principle of free piston driven shock tube.
b. It is a hand operated shock producing device.
c. It is capable of producing mach number exceeding 1.5
d. The rupture pressure is a function of the thickness of the diaphragm.
e. Temperature exceeding 900K can be easily obtained by the Reddy tube using helium as driver gas & argon as the driven gas. This temperature is useful in chemical kinetic studies.

Applications of shock waves:

Cell information (biological application) :

By passing shock wave of appropriate strength, DNA can be pushed inside a cell. (Functionality of DNA will not be affected by the impact of the shock wave)

Wood preservation:

By using shock waves, chemical preservatives in the form of solutions could be pushed into the interior of wood samples which helps in withstanding the microbial attacks. The actual set-up used for this purpose is named as “shockwave reactor”.

This method helps in substantial increase in the life of ordinary bamboo & soft wood.

Use in pencil industry:

In the manufacture of pencil, the wood need to be softened by soaking it in a polymer for about 3 hours at 70^0 c & then dried which is time consuming.

In modern technique, the wood is placed in the liquid & a shock wave is sent through. The liquid gets into the wood almost instantaneously & takes no longer time to dry. The wood treated with shock wave is now ready for the further processing.

Kidney stone treatment:

Shock wave is used in a therapy called **‘extra–corporal lithotripsy’** to shatter the kidney stones into smaller fragments after which they are passed out of the body smoothly through the urinary tracts.

Gas dynamic studies:

The extreme conditions of pressure & temperature that can be produced in the shock tube, enables the study of high temperature gas dynamics. This knowledge is crucial in the study of supersonic motion of bodies & hypersonic re-entry of space vehicles into the atmosphere.

Shock wave assisted needleless drug delivery:

By using shock waves, drugs can be injected into the body without using needles.

The drug is filled in the cartridge which is kept pressed on the skin & the shock wave is sent into the body using high pressure. The drug enters the body directly through the porosity of the skin. Typical depth of penetration is about 100 microns.

Treatment of dry bore wells:

Water will be available in the bore wells when water from the feeder sources accumulates in the bore well through a number of seepage points which are porous. Sometimes such seepage routes are blocked by sand particles clogging the pores. It has been observed that, a shockwave sent through such dry bore well, clears the blockages & rejuvenates the bore well into a water source.

Module 2

Elastic properties of materials

Elasticity:

When the deforming forces applied to a body is removed the body tends to recover its original condition i.e. the body will recover its original shape or size. This property of material body to recover its original condition when deforming forces are removed is called elasticity.

Elastic Bodies and Plastic bodies:

The bodies which recover its original condition completely on the removal of deforming force are called perfectly elastic.

Ex. Steel, Quartz fibre, Phospor bronze, Rubber etc

The bodies which do not show any tendency to recover their original condition on the removal of deforming forces are called perfectly plastic body.

Ex. Clay, Wax, Putty.

Deforming Force:

Consider a body which is not free to move and is acted upon by external forces. Due to the action of external forces the body changes its shape or sizes changes and now body is said to be deformed. Thus the applied external force which cause deformation is called deforming force.

Restoring force:

When deforming force is applied to a body then molecules of body tend to displace from their equilibrium position. As a result of this a reaction force developed within the body which tries to bring the molecule to its equilibrium position. This reaction force which is developed in the body is called internal force or elastic force or restoring force.

Rigid body:

A rigid body can be defined as one which does not undergo any deformation under the action of various deforming forces. When forces are applied on a rigid body the distance between any two particles of the body will remain unchanged, however large the force may be. In actual practice no material body is perfectly rigid. For practical purposes solid bodies are under the influence of weak forces are taken as rigid bodies. The nearest approach to a rigid is diamond and carborundum.

Load :

It is the combination of external forces acting on a body. The effect of load is to change the form or the dimensions of the body. It is thus essentially a deforming force.

Stress:

The restoring force per unit area set up inside the body is called stress. The restoring force is equal in magnitude but opposite that of the applied force. Therefore **stress is given by the ratio of the applied force to the area. Unit of stress is Nm^{-2}.**

Strain:

It is defined as t**he ratio of change in dimension of the body to its original dimension.**

It is not having any unit.

Tensile stress (Longitudinal stress):

The stress which brings about change in length of the object is called as Longitudinal stress. It is applied normal to the body.

Ex. Load suspended normally from the wire due to which the wire undergoes change in length.

If ‘F’ is the force applied and ‘a’ is the area of cross section.

Longitudinal stress = $\frac{F}{a}$

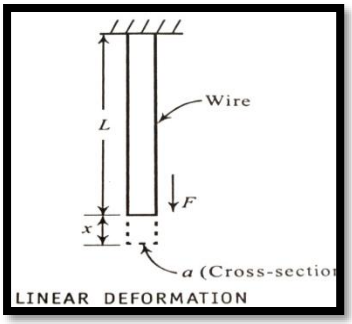
Longitudinal or Tensile strain:

If ‘*x’* is the change in length produced due to the applied stress for an original length of ‘L’ then,

$$\textbf{Longitudinal} = \frac{change\ in\ length}{original\ length} = \frac{x}{L}$$

Tangential stress or Shear stress.

The stress which brings about change in shape is called as tangential stress. It is applied parallel to the surface.



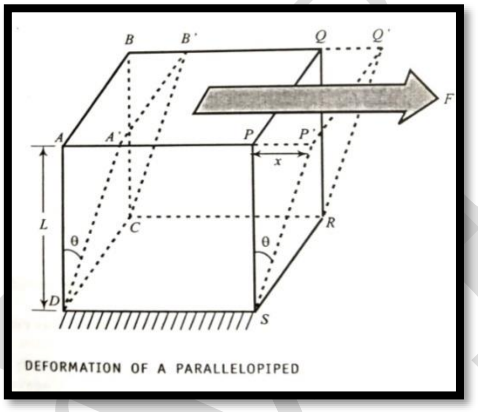
If ‘F’ is the force applied parallel to the surface and ‘a’ is the surface area.

Shear stress = $\frac{F}{a}$

Shear strain:

Shear strain is defined as the ratio of change in the shape of the object to original shape of the object.

If a force is applied tangentially to a free portion of the body whose other part is fixed then its layers slide one over the other; the body experiences a turning effect and changes its shape. This is called **shearing** and the angle through which the turning takes place is called **shearing angle (θ).**



DEFORMATION OF A PARALLELOPIPED

Within elastic limit it is measured by the ratio of relative displacement of one plane to its distance from fixed plane. It is also measured by the angle through which a line originally perpendicular to fixed plane is turned.

Shearing strain= $\theta = tan\,\theta = \frac{x}{L}$

Compressive stress or volume stress :

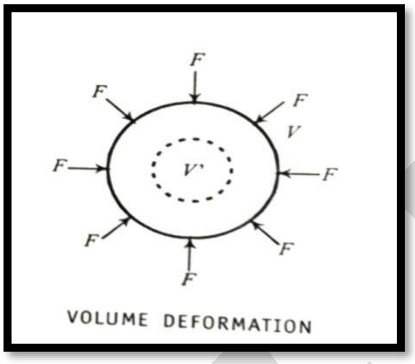
The compressive stress is restoring force developed within the body when body compressed under the action of deforming force. It brings about change in the volume of the object.

If F is the force applied uniformly and normally on a surface area ‘a’ then.

Compressive stress is = $\frac{F}{a}$

Volume strain:

If a uniform force is applied all over the surface of a body then the body undergoes a change in its volume (however the shape is retained in case of solid bodies). If *v* is the change in volume to an original volume *V* of the body then,



VOLUME DEFORMATION

Volume strain $= \frac{\text{change in volume}}{\text{original volume}} = \frac{v}{V}$

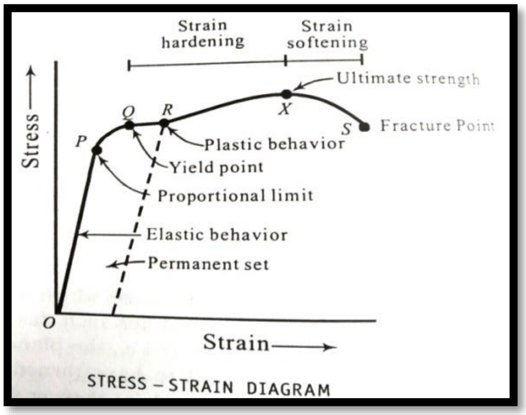
Hooke's law:

The fundamental law of elasticity was given by Robert Hook. It states that **"Stress produced in a body is directly proportional to the strain within the elastic limit".** Thus in such a case the ratio of stress to strain is a constant and it is called the modulus of elasticity or coefficient of elasticity. i.e., stress α strain,

or, $\frac{stress}{strain} = \text{a constant (E)}$

Stress – strain Graph when a wire is stressed

The relationship between stress and strain is studied by plotting a graph for various values of stress and the accompanying strain. This graph is obtained by plotting various values of stress and the accompanying strain of simple case of a bar or wire subjected to increasing tension. The graph obtained is in the general form as shown in the figure and is known as stress-strain diagram.



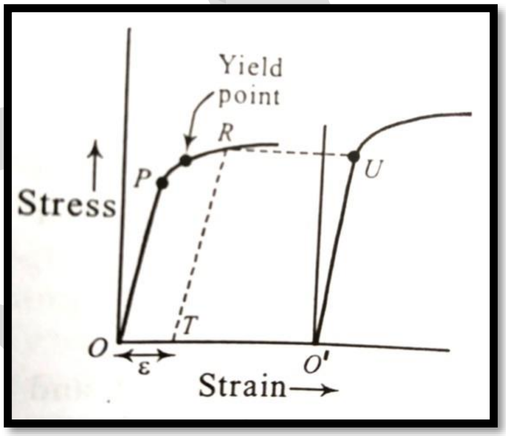
STRESS – STRAIN DIAGRAM

- In the graph the straight and slopping part OP of the curve shows that the strain produced is directly proportional to the stress or the Hook's law is obeyed perfectly up to P. In this region the material will recover its original condition of zero strain, on the removal of the stress. **The point 'P' is called as proportionality limit.**
- If the material is stressed between The region P and Q. the material will regain its original shape i.e it exhibits elasticity but it will not obey Hookes law. **The point Q is called as Yield point or elastic limit.**
- If the material is stressed between the region Q and S. it will not regain its original shape and size. it will not trace the original path instead it will trace the RT. i.e it has undergone a permanent deformation or plastic deformation.

  The point 'x' is called as the ultimate strength. It is the maximum stress that amaterial can withstand beyond which the material breaks.
- **The point 's' is called as the Breaking point.**The region 'QX' is called as **strain hardening.** The region XS is called as **Strain softening.**

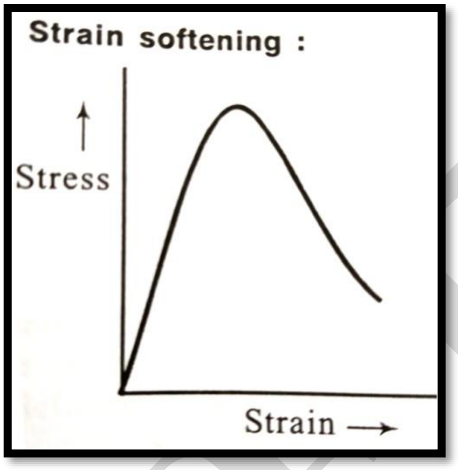
Strain Hardening and strain softening:

Certain materials that are plastically deformed earlier are stressed again, shows an **increased** yield point. This effect is called as **strain hardening.**



It is one of the process of Making a material harder by plastic deformation. It is also known as 'work hardening' or ' cold working'.

Certain materials like concrete or soil, are stressed their stress strain graph shows negative slope soon after the elastic region. The negative slope indicates the that there is softening effect of the material. This effect is called as **'strain softening'.**

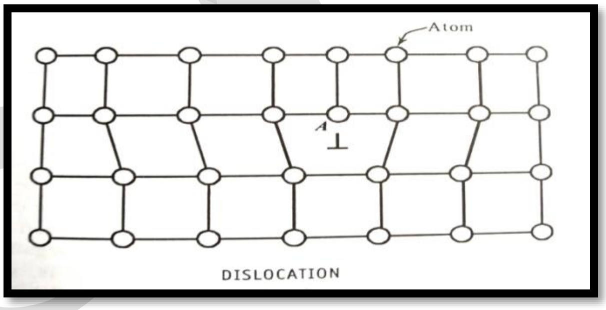


Effect of continuous stress, temperature, annealing and Impurities on the elastic properties of the body.

Effect of continuous stress.

When certain elastic materials are stressed continuously effect of creep comes in to play

Creep is the property due to which a material under steady stress undergo slow plastic deformation even below the proportionality limit.



It occurs due to the deformation caused by slip occurring along the crystallographic directions in the metal.

Ex. The warping of a shelf over time when heavy object is placed on it.

Effect of temperature:

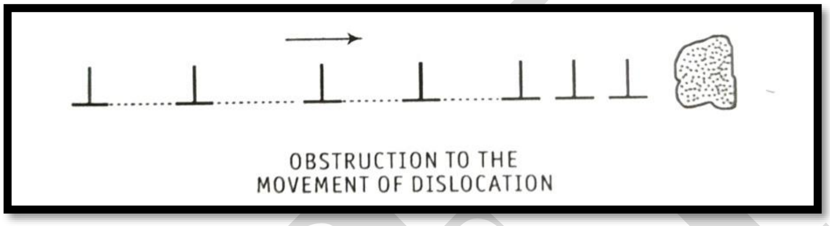
When the material is subjected stress at high temperatures. The effect of creep dominates. It is an important factor to be considered in the design of boilers , turbines , jet engines etc.

Annealing.

It is a type of heat treatment used increase the strength, hardness and toughness to meet the requirement of good machinibility, forging and casting. It improves the elasticity and increases the ductility.

In annealing the material is heated to very high temperature first to make the metal soft and then it gradually cooled down.

Effect of impurities:



Addition of impurities to metal results in either increase or decrease of elasticity depending on the type of impurity added.

If the impurity added obstructs the motion of dislocation in the lattice it increases the elastic modulus and hence the yield strength.

If the impurity added enables the movement of dislocation it causes cracks, inclusions and reduces the strength.

Types of elasticity

Corresponding to the three types of strain, we have three types of elasticity

a) Linear Elasticity or Elasticity of length or Young’s modulus
b) Elasticity of volume or Bulk modulus
c) Modulus of Rigidity (corresponding to shear strain)

a) Young’s modulus (Y).

The ratio of longitudinal stress to linear strain within elastic limit is called the coefficient of direct elasticity or Young’s modulus and is denoted by Y .

If F is the force applied normally, to a cross-sectional area a, then the stress is F/a. If L is original length and x is change in length due to the applied force, the strain is given by x/L.

$$\mathbf{Y} = \frac{\text{Normal stress}}{\text{Longitudinal strain}} = \frac{F/a}{x/L} = \frac{FL}{ax} \ \mathbf{N/m^2}$$

b) Elasticity of volume or Bulk modulus (K):

The ratio of normal stress or pressure to the volume strain without change in shape of the body within the elastic limits is called Bulk modulus.

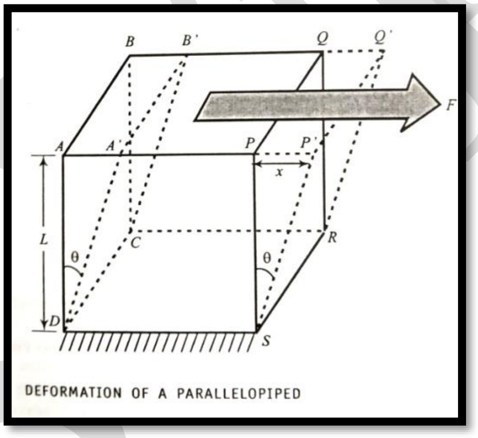
If ‘F’ is the force applied uniformly and normally on a surface area ‘a’ the stress or pressure is ‘F/a’ or P and if ‘*v*’ is the change in volume produced in an original volume V, the strain is given by ‘*v*/V’ and therefore

$$\mathbf{K} = \frac{\text{Normal stress}}{\text{Volume strain}} = \frac{F/a}{v/V} = \frac{FV}{av} = \frac{PV}{v}\ \mathbf{N/m^2}$$

Bulk modulus is referred to as incompressibility and hence its reciprocal is called **compressibility** (strain per unit stress).

c) Modulus of Rigidity (corresponding to shear strain)

In this case, while there is a change in the shape of the body, there is no change in its volume. It takes place by the movement of contiguous layers of the body, one over the other.



DEFORMATION OF A PARALLELOPIPED

Q^1 that is the planes of the two faces ABCD and PQRS can be said to have turned through an angle θ. This angle θ is called the angle of shear or shearing strain. Tangential stress is equal to the force F divided by area ‘a’ of the face APQB.

The rigidity modulus is defined as the ratio of the tangential stress to the shearing strain.

Hence **tangential stress** = $\frac{F}{a}$ $\qquad$ $\theta = PP^1 / PS = \frac{x}{L}$

$$\textbf{Rigidity modulus } \boldsymbol{\eta} = \frac{\text{tangential stress}}{\text{shearing strain.}} = \frac{F/a}{\theta} = \frac{F/a}{x/L} = \frac{FL}{ax}\ \mathbf{N/m^2}$$

Longitudinal Strain co efficient:

If x/L is the Longitudinal strain produced due to Longintudinal stress T

The from Hookes law

$x/L \propto T$

$x/L = \alpha T$

Where α is called as Longitudinal strain co efficient.

Logitudinal strain co efficient (α) is defined as the Longitudinal strain produced per unit stress.

Lateral Strain co efficient:

When an elastic material subjected to tensile stress it produces longitudinal strain as well as lateral strain in the direction perpendicular to longitudinal strain in the material.

Lateral strain is defined as the ratio of change in the diameter to original diameter.

If 'd' is the change in the diameter and 'D' is the original diameter then

Lateral strain = d/D

If d/D is the lateral strain produced due to Longintudinal stress T

The from Hookes law

$d/D \propto T$

$d/D = \beta T$

Where β is called as Lateral strain co efficient.

Lateral strain co efficient (β) is defined as the Lateral strain produced per unit stress.

Poisson's Ratio (σ)

It is commonly observed fact that when we stretch string or wire it becomes longer but thinner. That is an increase in length is always accompanied by a decrease in its cross section. In other words a linear or a tangential strain produced in a wire is accompanied by a transverse or lateral strain of opposite kind in a direction right angle to the direction of applied force. This change which occurs in a direction perpendicular to the direction along which the deforming force is acting is called **lateral change**.

***Within elastic limits of a body, the ratio of lateral strain (β) to the longitudinal strain (α) is a constant and is called Poisson's ratio* (σ).**

If a deforming force acting on a wire of length 'L' produces a change in length 'x' accompanied by a change in diameter of 'd' in it which has a original diameter of 'D'

$$\therefore \textbf{ Poisson's ratio, } \sigma = \frac{\textit{Lateral strain}}{\textit{Longitudinal strain}} = \frac{Ld}{xD}$$

The lateral strain coefficient $\beta = \frac{d}{D}$ and Longitudinal strain coeficient $\alpha = \frac{x}{L}$,

$$\therefore \textbf{ Poisson's ratio, } \sigma = \frac{Ld}{xD} = \frac{\beta}{\alpha}$$

Poisson's ratio is a dimensionless quantity.

Limiting value of Poisson's ratio:

W.K.T.

$$Y = 2\eta(1 + \sigma), \text{ and } Y = 3K(1-2\sigma)$$

$$\therefore 2\eta(1 + \sigma) = 3K(1-2\sigma)$$

where k & η are essentially positive quantities.

(i) If σ be positive quantity, then the right hand side and left hand side expression must be positive. i.e. $(1-2\sigma) > 0$

Or $2\sigma < 1$

$\therefore \sigma < 0.5$

(ii) If σ is A negative quantity, then the right hand side and left hand side expression must be positive. i.e. $(1 + \sigma) > 0$

Or $\sigma > -1$

Thus the limiting value of Poisson's ratio is $-1 > \sigma < 0.5$

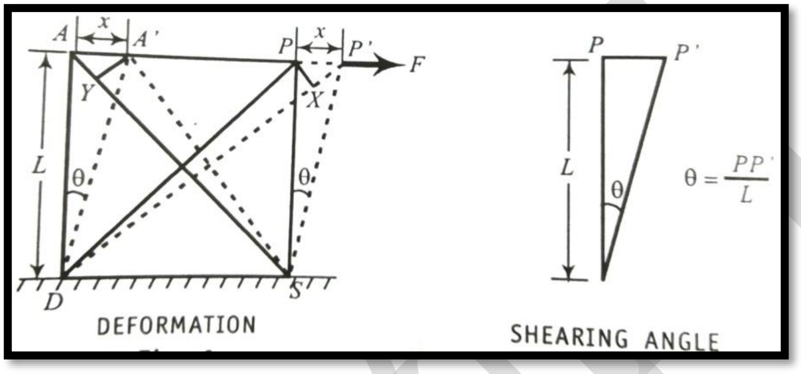
If $\sigma = 0.5$, then Bulk modulus is infinite. It means that, the substance is perfectly incompressible. Actually there is no substance which perfectly incompressible.

If $\sigma = -1$, the rigidity modulus is infinite. It means that, if a body is extended linearly, then it should also extend laterally. No substance exhibits such phenomenon.

In actual practice, the value of σ varies from 0.2 to 0.4 **(Practical limit)**

Equivalence of shear to compression and extension Strain

Consider a front face a cube ABPS, whose lower surface AS is fixed to a rigid support. Let a tangential force 'F' is applied at the upper surface along BP of the cube in a direction as shown in figure. The applied tangential force causes the relative displacements at different parts of the cube, so that, A moves to A' and P moves to P' through a small angle. Due to this the diagonal AS will be shortened to A'S and diagonal DP will be increased to a length DP'.let θ be the angle of shear which is very small in magnitude.



Let length of each side of the cube = L.

and PP'= x.

As θ is very small, from ΔSPP', we can write, $\theta = \tan\theta = x/L$.

As diagonal DP increases to DP' and diagonal AS is compressed to A'S,

We have Extension Strain along DP and compression strain along AS

Therefore extension Strain along DP

Elongation strain = P'X/DP

If L is the length of each side of the cube, then we have DP= $\sqrt{2}$ L (By Pythagoras theorem).

From Δ PP'X , P'X=PP' cos (PP'X)

$but\ \angle PP'X = \angle AP'D = \angle APD = 45^\circ$

$$\therefore\ P'X = PP'\cos 45^\circ = \frac{PP'}{\sqrt{2}}$$

Let PP' = x

we get, $P'X = PP'\cos 45^\circ = \frac{x}{\sqrt{2}}$

$\therefore$ Elongation strain along DP = P'X/DP= $\frac{x/\sqrt{2}}{L\sqrt{2}}=\frac{x}{2L}=\frac{\theta}{2}$ (1)

Since x/L = θ

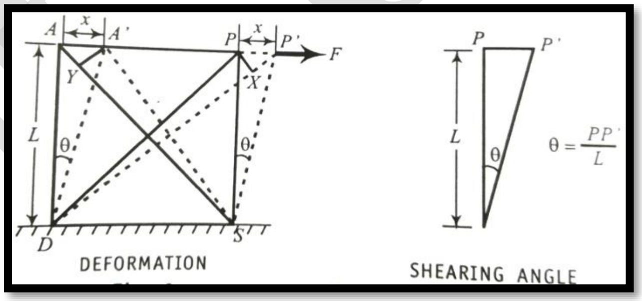
Similarly the compression strain along AS

i.e Compresion strain = $\frac{A'Y}{AS}=\frac{\theta}{2}$ (2)

From (1) and (2) it is clear that a shear strain θ is equivalent to an extension strain and compression strain at right angles to each other and each of value θ/2.

Elongation strain + compression strain = $\frac{\theta}{2}+\frac{\theta}{2}=\theta$, the shearing strain.

Relation between η, α and β



Consider a cube with each of its sides of length L under the action of tangential stress T. let tangential force F be applied to its upper face. It causes the plane of the faces perpendicular to the applied force F turn through an angle θ. as a result diagonal AC undergoes contraction and diagonal DB^{1}undergoes elongation of equal amount.

Now, shearing strain occurring along AP can be treated as equivalent to a longitudinal strain, along DP' and an equal lateral strain along the diagonal A'S i.e., perpendicular to DP. Let α and β be the longitudinal strain co efficient and lateral strain co efficients

since T is the applied stress, therefore extension produced for the length DP due to tensile stress

then Longitudinal Strain = T. DB.α

Lateral strain = T.DB.β.

$\therefore$ Total extension = Longitudinal strain along DP + Lateral strain Perpendiclar to Dp

$\therefore$ P'X = DP.T.(α+β),

P'X=PP' cos (PP'X)

but ∠ PP'X= ∠AP'D = ∠ APD = 45^o

$\therefore\ P'X = PP'\cos 45^\circ = \dfrac{PP'}{\sqrt{2}}$

Let PP'= x

$P'X = \dfrac{x}{\sqrt{2}}$

And DP = $\sqrt{2}\,L$

$\therefore$ P'X=($\sqrt{2}\,L$).T(α+β),

Rearranging the terms

$$\therefore\ (\sqrt{2}L)\,T(\alpha+\beta)=\frac{x}{\sqrt{2}},$$

$$\text{Or, }\ \frac{1}{2}\frac{1}{(\alpha+\beta)}=\frac{TL}{x}=\frac{T}{(x/L)}=\frac{T}{\theta}=\ \eta$$

$$\therefore \eta=\frac{1}{2(\alpha+\beta)},$$

$$=\frac{1}{2\alpha(1+\beta/\alpha)}\ ,$$

$$\text{Or, }\ \eta=\frac{1/\alpha}{2(1+\sigma)},\ \ (\because \sigma=\beta/\alpha),$$

$$\eta=\frac{Y}{2(1+\sigma)}$$

$$or\ Y=2\eta(1+\sigma)$$

Relation between Y and α

Consider a cube of unit side subjected to unit tension along one side. Let α be the elongation per unit length per unit tension along the direction of the force. Therefore,

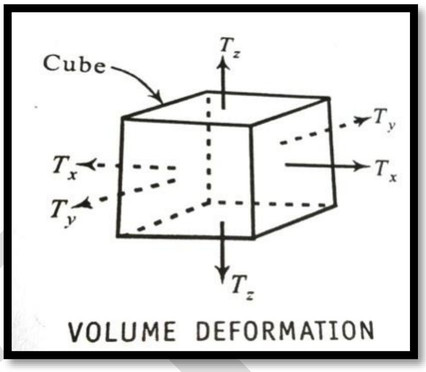
$$\text{Stress} = \frac{\text{Force}}{\text{Area}} = 1$$

Similarly, linear strain = $\dfrac{\text{change in length}}{\text{original length}} = \dfrac{\alpha}{1}$ = α

$$\therefore \quad \mathbf{Y} = \frac{\text{Normal stress}}{\text{Longitudin al strain}} = \frac{1}{\alpha}$$

Relation between K, n and Y

Consider a cube of unit volume, as shown in the diagram. Let T_x T_y *and* T_z be the stress acting on the faces along X,Y and Z.



VOLUME DEFORMATION

Each stress produces an extension in its own direction and a lateral contraction in the other two perpendicular directions. Let α be the elongation per unit length per unit stress along the direction of the forces and β be the contraction per unit length per unit stress in a direction perpendicular to the respective forces. Then stress like T_x produces an increase in length of αT_x in X-direction: but since other two stresses T_y and T_z are perpendicular to X-direction they produce a contraction of βT_y and βT_z respectively in the cube along X-direction .Hence, a length which was unity along X-direction becomes ,

$$1+ \alpha T_x - \beta T_y - \beta T_z.$$

Similarly along Y and Z directions the respective length become,

$$1+ \alpha T_y - \beta T_z - \beta T_x.$$

$$1+ \alpha T_z - \beta T_x - \beta T_y.$$

Hence the new volume of the cube is

$$= (1+ \alpha T_x - \beta T_y - \beta T_z)\,(1+ \alpha T_y - \beta T_z - \beta T_x)\,(1+ \alpha T_z - \beta T_x - \beta T_y)$$

Since α and β are very small, the terms which contain either powers of α and β, or their products can be neglected.

$\therefore$ New volume of the cube $= 1 + \alpha(T_x+ T_y+ T_z)- 2\beta(T_x+ T_y+ T_z)$,

$$= 1+ (\alpha-2\beta)(T_x+ T_y+ T_z)$$

If $T_x= T_y= T_z= T\sigma$

Then the new volume =1+ (α-2β) 3T

Since the cube under consideration is of unit volume, increase in volume = [1+3T (α-2β)]-1

= 3T (α-2β)

If instead of outward stress T, a pressure P is applied, the decrease in volume = 3P (α-2β).

$$\therefore \textbf{ Volume strain} = \frac{\text{change in volume}}{\text{original volume}} = \frac{3\text{P}(\alpha - 2\beta)}{1}$$

$$\therefore \textbf{K} = \frac{\text{Pressure}}{\text{Volume strain}} = \frac{\text{P}}{3\text{P}(\alpha - 2\beta)} = \frac{1}{3(\alpha - 2\beta)}$$

$$\text{K} = \frac{1}{3\alpha(1-2\beta/\alpha)} = \frac{(1/\alpha)}{3(1-2\beta/\alpha)} = \frac{Y}{3(1-2\sigma)} \qquad (\because \sigma = \frac{\beta}{\alpha} \text{ and Y} = \frac{1}{\alpha})$$

Relation between Y, K and η

We know that $\eta = \frac{Y}{2(1+\sigma)}$ and K $= \frac{Y}{3(1-2\sigma)}$

Rearranging which we get,

$$\frac{Y}{\eta} = 2 + 2\sigma \text{ and } \frac{Y}{3K} = 1\text{-}2\sigma$$

Adding the above we get, $\frac{Y}{\eta} + \frac{Y}{3K} = 3,$

$$\text{Or } \frac{Y(3K+\eta)}{3\eta K} = 3$$

$$\therefore Y = \left[\frac{9\eta K}{3K+\eta}\right]$$

Relation between K , η and σ

We have the relations Y = 2η (1 + σ), and Y = 3K (1-2σ)

Equating the above equations we get,

$$2\eta + 2\eta\sigma = 3K - 6K\sigma,$$

$\therefore 2\eta\sigma + 6K\sigma = 3K - 2\eta,$

Or , $\sigma(2\eta + 6K) = 3K - 2\eta,$

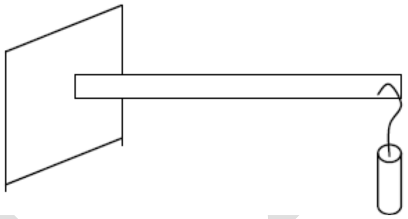
Or, $\sigma = \dfrac{3K - 2\eta}{2\eta + 6K}$

BENDING OF BEAMS

A homogenous body of uniform cross section whose length is large compared to its other dimensions is called a beam.

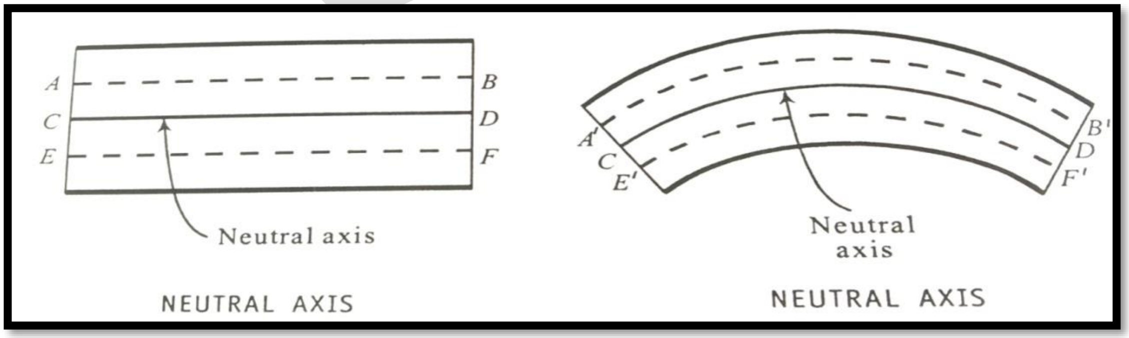
Neutral surface and neutral axis

Consider a uniform beam MN whose one end is fixed at M. The beam can be thought of as made up of a number of parallel layers and each layer in turn can be thought of as made up of a number of infinitesimally thin straight parallel longitudinal filaments or fibers arranged one closely next to the other in the plane of the layer.



If a load is attached to the free end of the beam, the beam bends. The successive layers along with constituent filaments are strained. A filament like AB of an upper layer will be elongated to A' B' and the one like EF of a lower layer will be contracted to E'F'. But there will always be a particular layer whose filaments do not change their length as shown for CD. Such a layer is called neutral surface and the line along which a filament of it is situated is called neutral axis.

Neutral Surface: It is that layer of a uniform beam which does not undergo any change in its dimensions, when the beam is subjected to bending within its elastic limit.

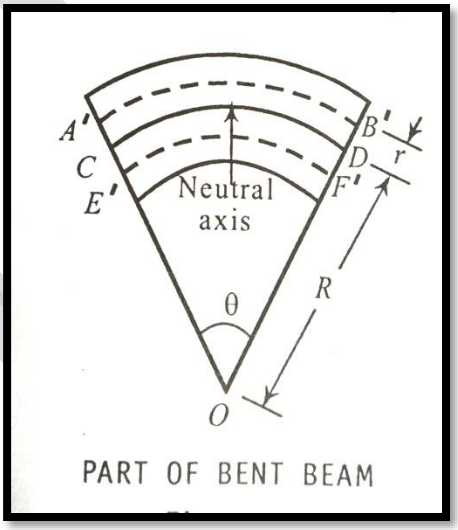


Neutral axis: It is a longitudinal line along which neutral surface is intercepted by any longitudinal plane considered in the plane of bending.

Bending moment of a beam:

Consider a uniform beam whose one end is fixed at M. If now a load is attached to the beam, the beam bends. The successive layers are now strained. A layer like AB which is above the neutral surface will be elongated to A' B' and the one like EF below neutral surface will be contracted to E'F'. CD is neutral surface which does not change its length.

The shape of each layers of the beam can be imagined to form part of concentric circles of varying radii. Let R be the radius of the circle to which the neutral surface forms a part.



PART OF BENT BEAM

$\therefore$CD=Rθ

where 'θ' is the common angle subtended by the layers at common center O of the circles. The layer AB has been elongated to A^1B^1.

$\therefore$ Change in length = A' B' -AB

But AB=CD=Rθ

If the successive layers are separated by a distance r then,

$$A'B' = (R+r)\theta$$

$\therefore$Change in length=(R+r)θ-Rθ = rθ

But original length = AB=Rθ

$$\therefore \text{Linear strain} = \frac{r\theta}{R\theta} = \frac{r}{R}$$

Youngs Modulus Y= Longitudinal stress/linear strain

Longitudinal stress = Yx Linear strain

$$= \text{Y}\times\frac{r}{R}$$

$$\text{But stress} = \frac{F}{a}$$

Where F is the force acting on the beam and a is the area of the layer AB.

$$\frac{F}{a} = \frac{Yr}{R} = \frac{Yar}{R}$$

Moment of this force about the neutral axis=F x its distance from neutral axis.

$= F \times r = Yar^2 / R$

Moment of force acting on the entire beam $= \Sigma \frac{Yar^2}{R}$

$$= \frac{Y}{R}\sum ar^2$$

The moment of inertia of a body about a given axis is given by Σmr^2, where Σm is the mass of the body. Similarly Σar^2 is called the geometric moment of Inertia Ig.

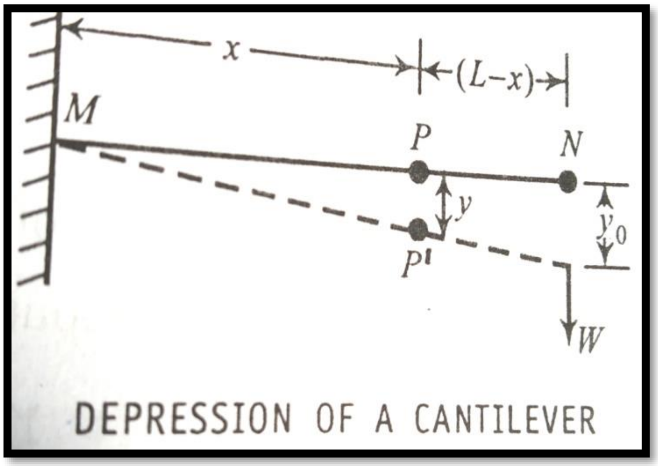
$Ig=\Sigma ar^2 = Ak^2$, where A is the area of cross section of the beam and k is the radius of gyration about the neutral axis.

$$\text{Moment of force} = \frac{Y}{R} Ig$$

1) Bending moment of rectangular beam $= \frac{Y}{R}\frac{bd^3}{12}$

2) Bending moment of circular beam $= \frac{Y}{R}\frac{\pi\rho^4}{4}$

Single Cantilever



DEPRESSION OF A CANTILEVER

If one end of beam is fixed to a rigid support and its other end is loaded, then the arrangement is called single cantilever or cantilever.

Consider a uniform beam of length L fixed at M. Let a load W act on the beam at N.

Consider a point on the free beam at a distance x from the fixed end which will be at a distance (L-x) from N. Let P' be its position after the beam is bent.

$\therefore$ Bending moment = Force x Perpendicular distance.

= W (L-x)

But bending moment of a beam is given by $\frac{Y}{R} Ig$

$$\frac{Y}{R} Ig = W(L-x) \tag{1}$$

$$\frac{1}{R} = \frac{W(L-x)}{YIg} \tag{2}$$

But if 'y' is the depression of the point P then it can be shown that

$$\frac{1}{R} = \frac{d^2 y}{dx^2} \tag{3}$$

where 'R' is the radius of circle to which the bent beam becomes a part.

Comparing equations (2)and (3)

$$\frac{d^2 y}{dx^2} = \frac{W(L-x)}{YIg}$$

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{w(L-x)}{YI_g}$$

$$d\left(\frac{dy}{dx}\right) = \frac{w(Ldx - xdx)}{YI_g}$$

Integrating both sides

$$\frac{dy}{dx} = \frac{w}{YI_g}\left[Lx - \frac{x^2}{2}\right] + C_1 \tag{4}$$

C_1 is constant of integration

But dy/dx is the slope of the tangent drawn to the bent beam at a distance x from the fixed end. When x=0, it refers to the tangent drawn at M, where it is horizontal. Hence (dy/dx)=0 at x=0. Introducing this condition in equation (4) we get $0=C_1$

Equation (4) becomes

$$\frac{dy}{dx} = \frac{W}{YIg}\left[Lx - \frac{x^2}{2}\right]$$

$$dy = \frac{W}{YIg}\left[Lx - \frac{x^2}{2}\right]dx$$

Integrating both sides we get

$$y = \frac{W}{YIg}\left[\frac{Lx^2}{2} - \frac{x^3}{6}\right] + C_2 \qquad (5)$$

where C_2 is constant of integration, y is the depression produced at known distance from the fixed end. Therefore when x=0, it refers to the depression at M, where there is obviously no depression. Hence y=0 at x=0. Introducing this condition in equation (5) we get

$$y = \frac{W}{YIg}\left[\frac{Lx^2}{2} - \frac{x^3}{6}\right]$$

At the loaded end, $y=y_0$ and x=L

Therefore $y_0 = \frac{W}{YIg}\left[\frac{L^3}{2} - \frac{L^3}{6}\right]$

Depression produced at loaded end is

$$y_0 = \frac{WL^3}{3YIg}$$

Therefore the young's modulus of the material of the cantilever is

$$Y = \frac{WL^3}{3y_0 Ig} \qquad (6)$$

Case (a):

If the beam is having rectangular cross-section, with breadth b and thickness d then,

$$Ig = \frac{bd^3}{12} \qquad (7)$$

Substituting equation (7) in equation (6) we get

$$Y = \frac{WL^3}{3y_0} x \frac{12}{bd^3}$$

$$Y = \frac{4WL^3}{Y_0bd^3}$$

Case(b):

If the beam is having a circular cross section of radius r then,

$$Ig = \frac{\Pi r^4}{4} \qquad (8)$$

Substituting equation (8) in equation (7) we get

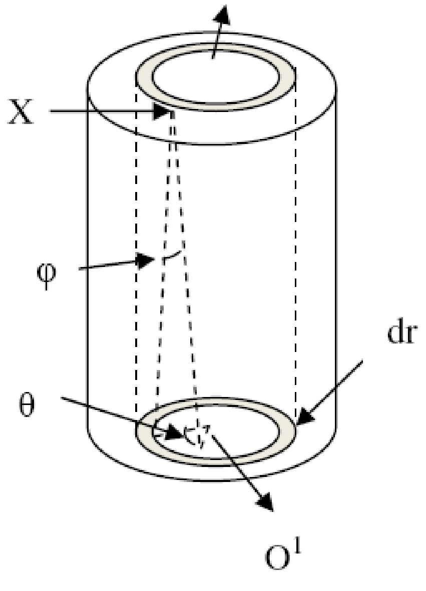
$$Y = \frac{4WL^3}{3\Pi y_0 r^4}$$

Torsion of a cylinder

A long body which is twisted around its length as an axis is said to be under torsion. The twisting is brought into effect by fixing one end of the body to a rigid support and applying a suitable couple at the other end. The elasticity of a solid, long uniform cylindrical body under torsion can be studied, by imagining it to be consisting of concentric layers of the material of which it is made up of. The applied twisting couple is calculated in terms of the rigidity modulus of the body.

Expression for the Torsion of a cylindrical rod

Consider a long cylindrical rod of length 'L' and radius 'R' rigidly fixed at its upper end. Let OO' be its axis. Imagine the cylindrical rod is made up of thin concentric hollow cylindrical layers each of thickness *'dr'*. The rod twisted at its lower end, and then the concentric layers slide one over the other. This movement will be zero at the fixed end and gradually increased along the downward direction. Let us consider one concentric circular layer of radius *'r'* and thickness *'dr'* . Any point 'X' on its



uppermost part would remain fixed and a point like ‘B’ at its bottom moves to ‘B'’.

Now, $B\hat{X}B' = \phi$ gives the angle of shear.

Since ϕ is also small, the movement length $BB' = L\phi$.

Also, if $B\hat{O}'B' = \theta$, the length B B' = rθ.

$$\therefore L\phi = r\theta$$

Now, the cross sectional area of the layer under consideration is 2πr dr . If ‘F’ is the shearing force, then the shearing stress T is given by

$$T = \frac{Force}{Area} = \frac{F}{2\pi r dr}$$

∴Shearing force F = T(2πrd$r$)

∴ Rigidity modulus n = Shearing stress/shearing strain.

$$n = \frac{T}{\phi}$$

$$\therefore T = n\phi = \frac{nr\theta}{L}$$

$$\therefore F = \frac{nr\theta}{L}\left(2\pi r dr\right) = \frac{2\pi n\theta}{L} r^2 dr$$

The moment of the force about $OO' = \left(\frac{2\pi n\theta}{L} r^2 dr\right) r = \frac{2\pi n\theta}{L} r^3 dr$

This is only for the one layer of the cylinder.

Therefore, twisting couple acting on the entire cylinder $= \int_0^R \frac{2\pi n\theta}{L} r^3 dr$

$$= \frac{2\pi n\theta}{L}\left[\frac{r^4}{4}\right]_0^R$$

$$= \frac{\pi n R^4 \theta}{2L}$$

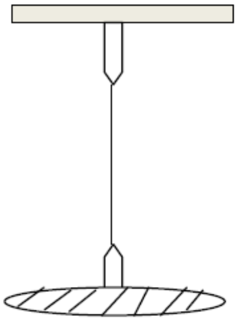
Couple per unit twist is given by C=Total twisting couple / angle of twist.

$$C = \frac{\pi n R^4 \theta / 2L}{\theta}$$

$$C = \left(\frac{\pi n R^4}{2L} \right)$$

Torsion Pendulum

Torsion pendulum consists of a heavy metal disc is suspended by means of a wire. When the disc is rotated in a horizontal plane so as to twist the wire, the various elements of the wire undergo shearing strain. The restoring couple of the wire tries to bring the wire back to the original position. Therefore disc executes torsional oscillations about the mean position.



Let θ be the angle of twist made by the wire and C be the couple per unit twist.

Then the restoring couple per unit twist = Cθ.

Therefore the angular acceleration produced by the restoring couple in the wire. a = $\frac{d^2\theta}{dt^2}$

Let I be the moment of inertia of the wire about the axis. Therefore, we have, I $\left(\frac{d^2\theta}{dt^2} \right)$ = -Cθ

The above relation shows that the angular acceleration is proportional to angular displacement and is always directed towards the mean position. The negative sign indicates that the couple tends to decrease the twist on the wire. Therefore, the motion of the disc is always simple harmonic motion (SHM).

Therefore, the time period of oscillator is given by relation,

$$T = 2\pi \sqrt{\frac{displacement}{Accleration}} = 2\pi \sqrt{\frac{\theta}{\left(\frac{C}{I} X \, \theta \right)}} = 2\pi \sqrt{\frac{I}{C}}$$

Application of torsion pendulum

1) Determination of moment of inertia of an irregular body

a) Time period (T_1) of pendulum is determined by fixing a regular body at the free end of torsion pendulum.

b) Similarly the time period (T_2) is regular body is determined.

If I_1 and I_2 are the moment of inertia of regular and irregular body respectively, then

$$T_1 = 2\pi\sqrt{\frac{I_1}{C}}$$

$$T_2 = 2\pi\sqrt{\frac{I_2}{C}}$$

$$\therefore \frac{T_1^2}{T_2^2} = \frac{I_2}{I_1}$$

The moment of inertia of the regular body about any axis can be determined by knowing its mass and dimensions. This moment of inertia of the irregular body is calculated.

$$\therefore I_2 = I_1 \frac{T_1^2}{T_2^2}$$

2) Determination of Torsion rigidity

The Time period of torsion pendulum is given by

$$T = 2\pi\sqrt{\frac{I}{C}}$$

$$\therefore\ T^2 = 4\pi^2\ \frac{I}{C}$$

$$C = 4\pi^2\ \frac{I}{T^2}$$

Where, I is the moment of inertia of the regular body and C is the couple per unit twist of the wire.

But $$C = \frac{\pi n R^4}{2L}$$

$$\therefore \quad 4\pi^2 \frac{I}{T^2} = \frac{\pi n R^4}{2L}$$

$$\text{or} \quad n = \frac{8\pi L}{R^4}\left(\frac{I}{T^2}\right)$$

MODULE III

MAXWELL'S EQUATIONS

Maxwell's equation provide a complete description of electromagnetic phenomena and includes all modern information and communication technologies. The maxwell equations are the basic tool for designing all electrical and electronics components from mobile phones to satellites.

FUNDAMENTALS OF VECTOR CALCULUS:

Vector: Any vector has both magnitude and direction. It is represented by drawing an arrow in a suitable coordinate System. (Fig 1.)

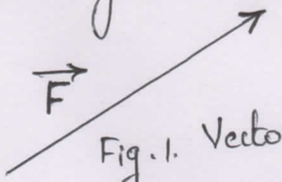


Fig. 1. Vector representation.

(i) Magnitude Of a vector: The magnitude of the vector is taken care of by making the length of the arrow (=R the distance between A and B) numerically equal to or proportional to the magnitude of the vector $\vec{F}$. (Fig 2).

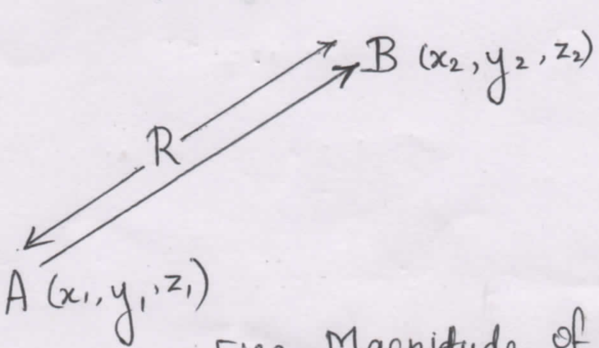
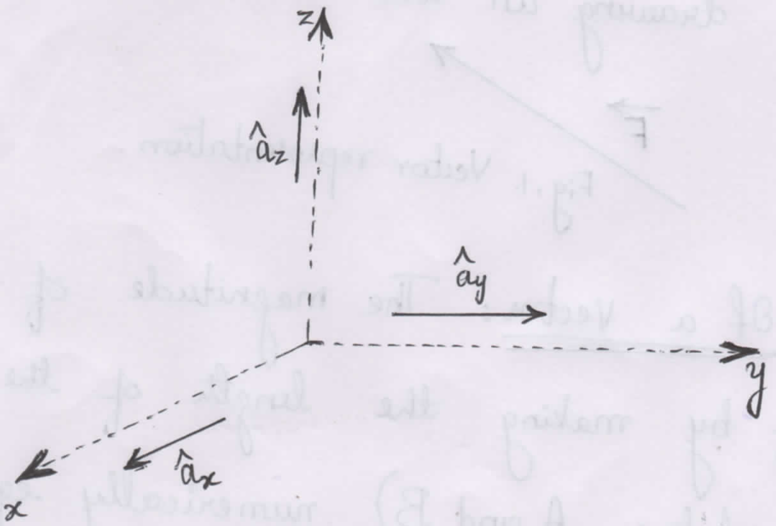


Fig 2. Magnitude of the vector.

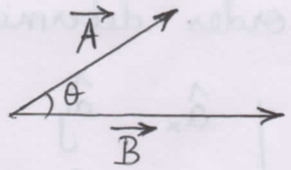
(ii) Direction Of the vector: The orientation Of $\vec{F}$ with respect to the coordinates is taken at the same inclination as described in the given situation. However, representing the direction of $\vec{F}$ is actually achieved by assigning its direction to what we call a unit vector.

(iii) Base Vectors: Base vectors are same as unit vectors but oriented strictly along the coordinates in the given coordinate system and pointing away from the origin. In rectangular coordinate system we can represent the base vectors as $\hat{a}_x$, $\hat{a}_y$ and $\hat{a}_z$, along the x, y and z coordinates. (Fig 5).



BASE VECTOR
(Fig. 5)

Scalar Product or Dot Product: The scalar produt or dot product of two vectors is defined as the produt of their magnitudes and of the cosine of the smaller angle between them.

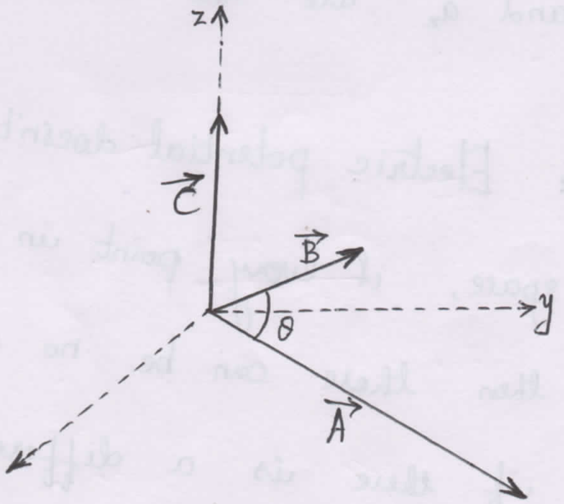


(Fig.6) Dot produt

If $\vec{A}$ and $\vec{B}$ are two vectors inclined at an angle θ (Fig.6) their dot product is given as,

$$\vec{A}\cdot\vec{B} = |\vec{A}|\,|\vec{B}|\cos\theta.$$

Cross Produut or Vector Product: Given two vectors $\vec{A}$ and $\vec{B}$, their cross produt is a single vector $\vec{C}$ whose magnitude is equal to the product of the magnitude of $\vec{A}$ and the magnitude of $\vec{B}$ multiplied by the sine of the smaller angle between them. The direction of $\vec{C}$ is perpendicular to the plane which has both $\vec{A}$ and $\vec{B}$ such that, $\vec{A}$, $\vec{B}$ and $\vec{C}$ form a right handed system (Fig7).



Cross Product.
(Fig.7)

$$\therefore \quad \vec{A} \times \vec{B} = |\vec{A}||\vec{B}| \sin\theta$$

$$\& \quad \vec{A} \times \vec{B} = \vec{C}$$

In terms of the components of $\vec{A}$ and $\vec{B}$, the vector $\vec{C}$ can be expressed as a third order determinant expressed as,

$$\vec{C} = \vec{A} \times \vec{B} = \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

Vector Operator ∇ : ∇ is a mathematical operator. It is called del (sometimes called nabla) and is meant to carry out a speific vector calulus operation. If it is a Cartesian coordinate problem, then the operation is as per the equation given below:

$$\nabla = \frac{\partial}{\partial x}\hat{a}_x + \frac{\partial}{\partial y}\hat{a}_y + \frac{\partial}{\partial z}\hat{a}_z$$

$\hat{a}_x$, $\hat{a}_y$ and $\hat{a}_z$ are the base vectors.

Gradient, and ∇ : Electric potential doesn't have a direction. In certain region of space, if every point in it is at the same electric potential, then there can be no electric field. On the other hand, if there is a difference of potential between any two points in the region then, an electric field does exist between them. The actual direction of the field will be in the direction in which maximum decrease of potential is established. The rate of change of potential decides the strength of the field $\vec{E}$. The relation is given as,

$$\vec{E} = -\frac{\partial V}{\partial r}\hat{a}_r .$$

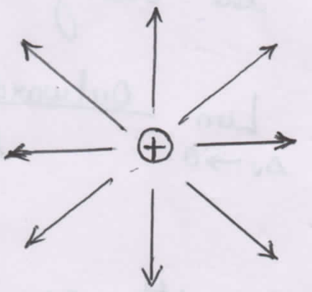
$$\vec{E} = -\left(\frac{\partial V}{\partial x}\hat{a}_x + \frac{\partial V}{\partial y}\hat{a}_y + \frac{\partial V}{\partial z}\hat{a}_z\right),$$

$$= -\left(\frac{\partial}{\partial x}\hat{a}_x + \frac{\partial}{\partial y}\hat{a}_y + \frac{\partial}{\partial z}\hat{a}_z\right)V$$

Thus, $\vec{E} = -\nabla V$ (grad V).

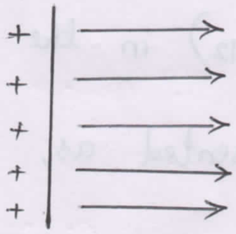
Convention for Directions Of Field in Electrostatics:

In electrostatics we assume that the field diverges radially from a positive charge (Fig 8). For a positively charged plane, the field points away from the plane. (Fig 9). normally.



Field lines due to a positive charge

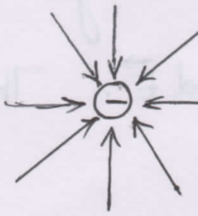
Fig. 8.



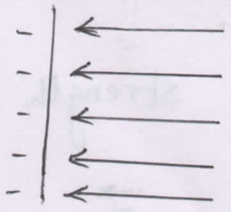
Field due to a positively charged plane.

Fig. 9.

In the case of negative charges, it is just the opposite. (Figs. 10 and 11).



Field lines due to a negative charge
Fig. 10.



Field lines due to a negatively charged plane
Fig. 11.

The direction of the field line at any given point is the direction along which a positive charge would experience the force when placed in the field at that point.

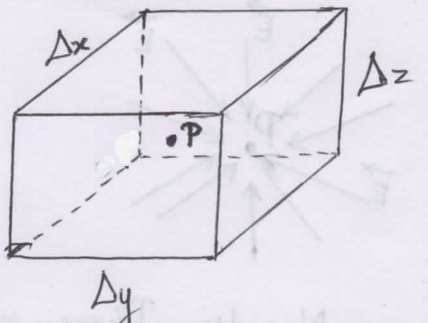
<u>Divergence, $\nabla \cdot \vec{A}$</u>: The divergence of a vector field $\vec{A}$ at a given point P means, it is the outward flux per unit volume as the volume shrinks to zero about P.

Considering an elementary volume Δv around a point P (Fig. 12) in the given space, the divergence at P can be represented as,

$$\text{div } \vec{A} = \lim_{\Delta v \to 0} \frac{\text{Outward flux of } \vec{A}}{\Delta v}$$

Mathematically, it could be rewritten as,

$$\text{div } \vec{A} = \lim_{\Delta v \to 0} \frac{\oint_S \vec{A} \cdot \vec{ds}}{\Delta v} \quad \text{———} ①$$



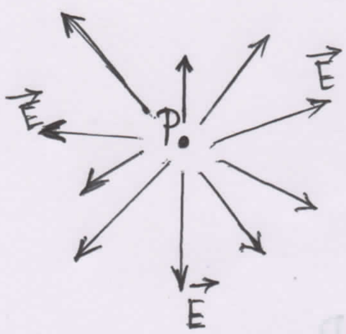
Elementary Volume Around P
Fig. 12

Now, by considering a rectangular parellelopiped around the given point P, (Fig.12) as the elementary volume Δv, and working the total outward flux from all its six faces, it is possible to show that,
$$\lim_{\Delta v \to 0} \frac{\oint_s \vec{A}\cdot d\vec{s}}{\Delta v} = \left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}\right)\Bigg|_{at\ P}$$

Since as per equation (1) the left side is divergence, we can write
$$\text{divergence of } \vec{A} = \left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}\right).$$

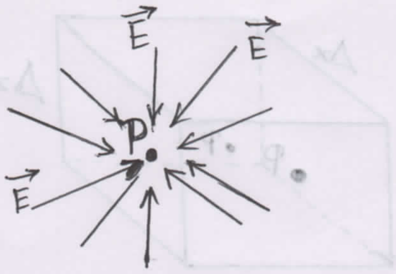
In other words, divergence of $\vec{A} = \nabla\cdot\vec{A}$

Physical Significance Of Divergence: The divergence of the vector field $\vec{A}$ is a measure of how much the field diverges or emanates from that point. If there are positive charges densely packed at a point, then a large no. of field lines diverge from that point. In other words, there is more divergence. The field $\vec{E}$ indicates a source of positive charges* at P negative divergence indicating the presence of negative charges at P. A vector field whose divergence is zero, is called solenoidal field.



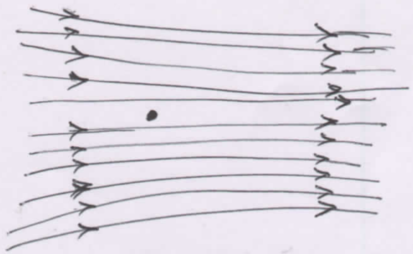
Positive Divergence

Fig. 13A



Negative Divergence

Fig. 13 B



Zero Divergence

Fig 13. C

<u>Curl $\nabla \times \vec{A}$</u>: The curl of a vector field $\vec{A}$ at a given point P, means, it is the maximum circulation of $\vec{A}$ per unit area as the area shrinks to zero about P. Curl $\vec{A}$ is represented as a vector whose direction is normal to the area around P when the area is oriented to make the circulation maximum.

It can be represented as,

$$\text{curl } \vec{A} = \lim_{\Delta s \to 0} \frac{\text{Max. circulation around P}}{\Delta S}$$

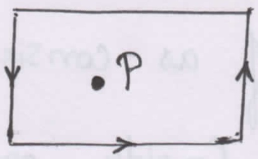
Mathematically we can write,

$$\text{Curl } \vec{A} = \left(\lim_{\Delta s \to 0} \frac{\oint \vec{A} \cdot dl}{\Delta S} \right) \hat{a}_n$$

where, the elementary area ΔS is bounded by the curve $L = \oint dl$, and $\hat{a}_n$ is the unit vector normal to ΔS.

By considering a rectangular elementary area across the point P as ΔS (Fig. 14) and working the closed line integral about the 4 sides of the boundary line,

$$\left(\lim_{\Delta s \to 0} \frac{\oint \vec{A} \cdot dl}{\Delta s} \right) \hat{a}_n = \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$

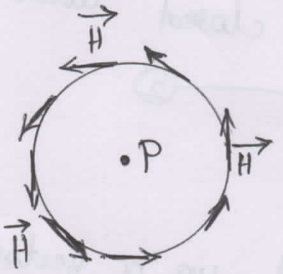


Δy

$$\text{curl of } \vec{A} = \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$

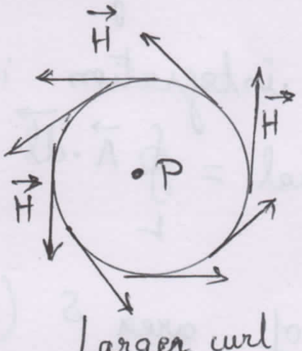
$$\text{curl of } \vec{A} = \nabla \times \vec{A}$$

Physical Significance Of curl: The curl of a vector field $\vec{A}$ at a point P is a measure of how much the field curls (circulates) around P. Let the field be a magnetic field $\vec{H}$ around the point P. A vector field whose curl is zero is called irrotational.



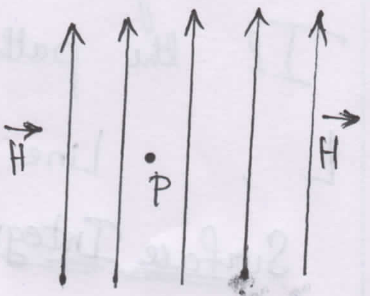
Curl of $\vec{H}$

Fig. 15A



Larger curl

Fig. 15B

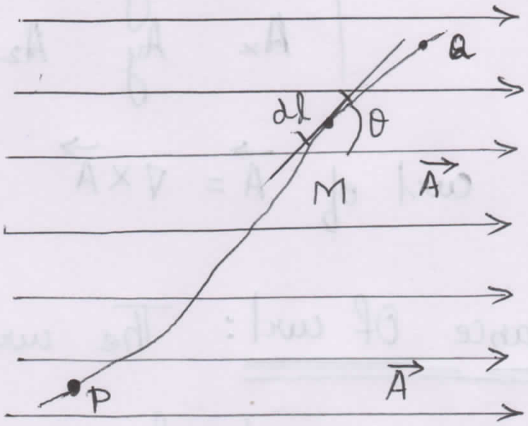


A field of zero curl

Fig. 15C

Linear Integral (or Line Integral):

Consider a linear path of length L from P to Q in a vector field $\vec{A}$ (Fig 16). The line could be thought of as consisting of small elementary lengths dl. Consider one such element at M. At M, draw a tangent. Let the tangent make an angle θ with $\vec{A}$. Then we have $\vec{A}\cdot\vec{dl} = A\,dl\cos\theta$.



Line Integration Fig.16

The dot product $\vec{A}\cdot\vec{dl}$ between P to Q becomes the line integral. ∴ Line integral $= \int_{P}^{Q} \vec{A}\cdot\vec{dl}$ ——— ①

If the path of integration is a closed curve of length L, Line integral $= \oint_{L} \vec{A}\cdot\vec{dl}$ ——— ②

Surface Integral:

Consider a surface of area S (Fig.18) in a vector field $\vec{A}$. The surface could be thought of as made up of a number of elementary surfaces each of area dS. Let $\hat{a}_N$ be the unit normal to a dS at M. In a vector field, the elementary surface dS acts as a vector $\vec{dS}$ given as,

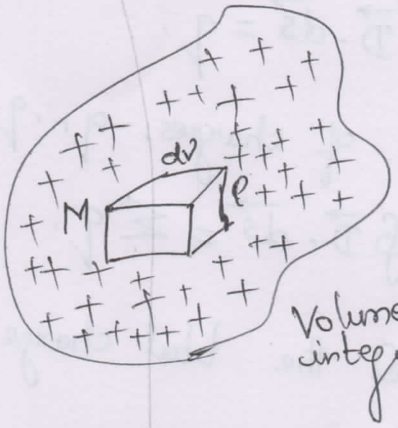
$$\vec{dS} = dS.\,\hat{a}_N$$

$\vec{A}\cdot\vec{dS}$, it represents the flux of the vector field $\vec{A}$ through $\vec{dS}$. The flux of the field $\vec{A}$ through the surface S can be obtained by integrating $\vec{A}\cdot\vec{dS}$ over the entire surface S.

$$\psi = \int_S \vec{A}\cdot\vec{dS} \tag{3}$$

$$\boxed{\psi = \oint_S (\vec{A}\cdot\hat{a}_N)\, dS}$$

<u>Volume Integral</u>: Consider a volume V in which the charges are distributed uniformly, it is referred as volume charge distribution. Consider an elementary volume dv at M. Let the charge density at M be ρ_v. Then the volume integral of ρ_v over the volume v is $\oint_v \rho_v\, dv$.



Volume integration.

LAWS OF ELECTROSTATICS, MAGNETISM & FARADAY'S LAW

Gauss' law in electrostatics, Gauss' law for magnetic field, Ampere's law and Faraday's law form the foundations of Maxwell's equations. Bio-Savart's law which is the fundamental law in magnetostatics.

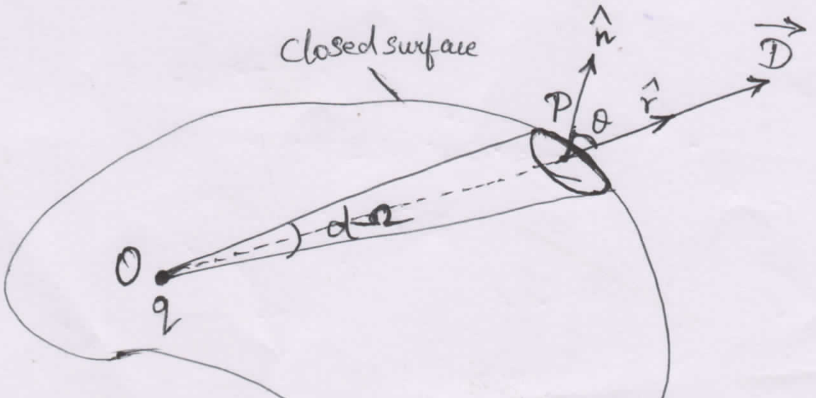
Gauss' Law in electrostatics: Consider electric charges in a certain region. A closed surface of any shape can be imagined surrounding those charges. Such a surface is called "Gaussian surface". The closed surface to be made up of a number of elementary surfaces each of area ds. If $\vec{D}$ is the flux density at ds. The total electric flux over the entire surface.

$$\psi = \oint_S \vec{D} \cdot \vec{ds} = q.$$

If there are a number of charges, $q_1, q_2 \ldots$

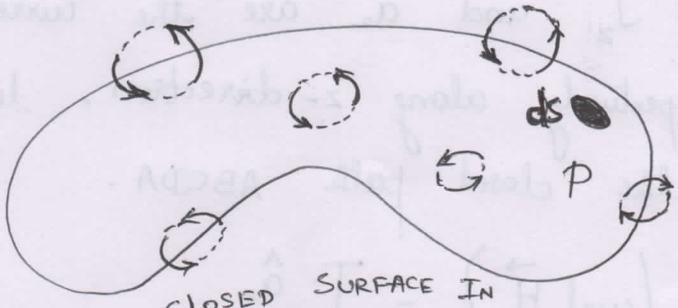
$$\psi = \oint \vec{D} \cdot \vec{ds} = \sum q.$$

or, $\psi = Q$ the total charge enclosed.



Flux through closed surface.

Gauss' Law for Magnetic Fields :

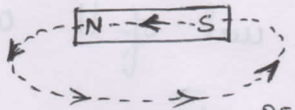


CLOSED SURFACE IN A MAGNETIC FIELD

The flux over the entire closed surface,

total outward flux = total inward flux

+ve -ve



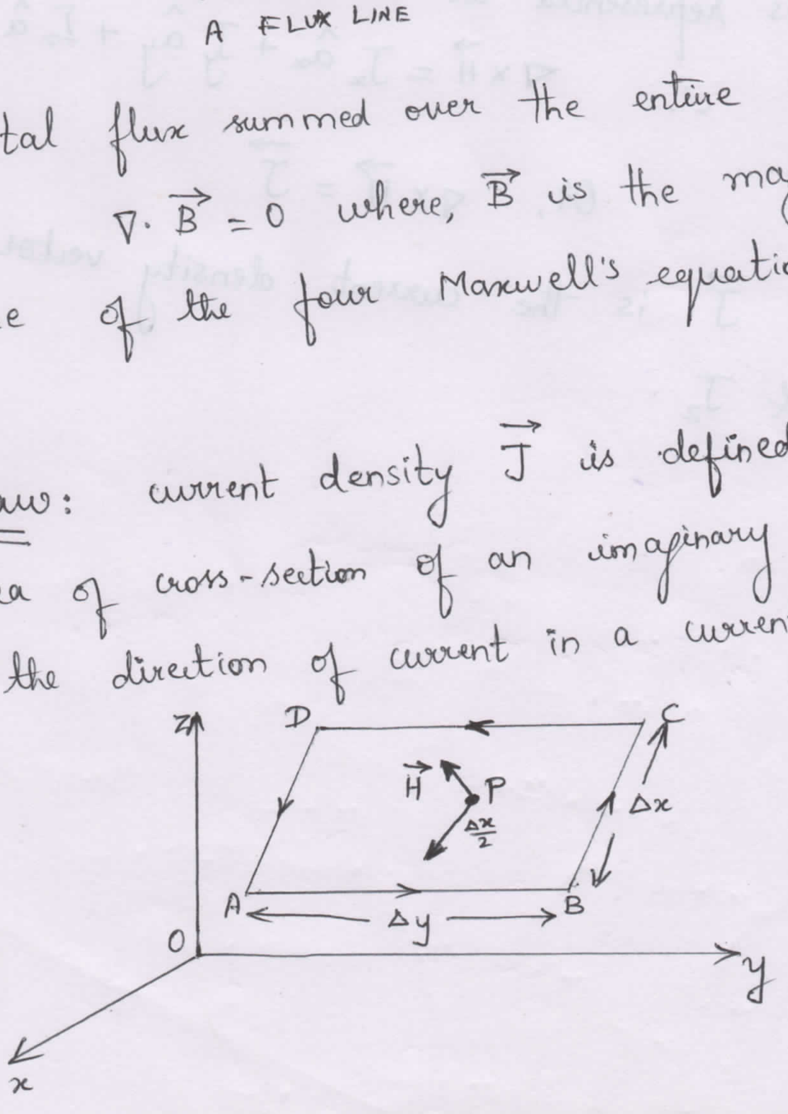
CLOSED PATH OF A FLUX LINE

∴ The total flux summed over the entire Gaussian surface = 0.

$\nabla \cdot \vec{B} = 0$ where, $\vec{B}$ is the magnetic flux density.

This is one of the four Maxwell's equations.

Ampere's Law: current density $\vec{J}$ is defined as the current per unit area of cross-section of an imaginary plane held normal to the direction of current in a current carrying conductor.



Consider a point P in a magnetic field $\vec{H}$. Curl $\vec{H}$ of the rectangular loop ABCD around P in a plane parallel to x-y plane. If J_z and $\hat{a}_z$ are the current density and unit vector respectively along z-direction, line integral of $\vec{H} \cdot \vec{dl}$ over the closed path ABCDA.

$$\left(\text{curl } \vec{H}\right)_1 = J_z \, \hat{a}_z$$

$$\left(\text{curl } \vec{H}\right)_2 = J_x \, \hat{a}_x$$

$$\left(\text{curl } \vec{H}\right)_3 = J_y \, \hat{a}_y$$

are the components of curl of $\vec{H}$ around the point P.

curl $\vec{H}$ is represented as $\nabla \times \vec{H}$

$$\therefore \nabla \times \vec{H} = J_x \hat{a}_x + J_y \hat{a}_y + I_z \hat{a}_z$$

$$\text{Or, } \nabla \times \vec{H} = \vec{J}$$

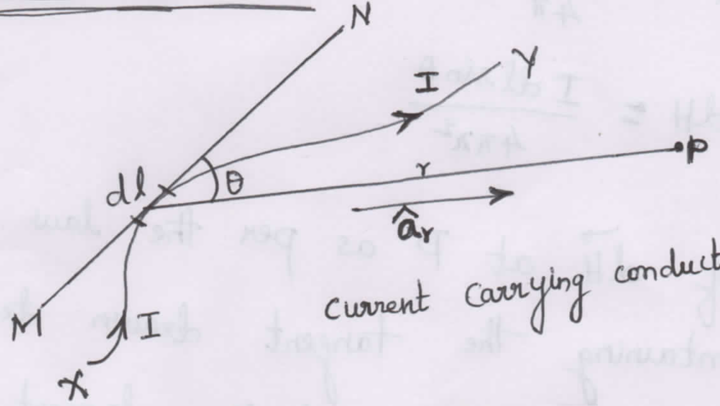
where, $\vec{J}$ is the current density vector with components J_x, J_y & J_z.

BIOT-SAVART'S LAW:

Magnetic field Intensity $\vec{H}$ and Magnetic flux Density $\vec{B}$:

The magnetic field at a point is dealt with in terms of two vectors $\vec{H}$ and $\vec{B}$. The vector $\vec{H}$ refers to the magnetic field intensity or magnetic field. $\vec{B}$ also refers to the magnetic field, and is referred to as magnetic flux density or magnetic induction. They are related by $\vec{B} = \mu \vec{H}$, where μ is the permeability of the medium in vacuum, $\vec{B} = \mu_0 \vec{H}$, where μ_0 is the permeability of vaccum $= 4\pi \times 10^{-7}$ H/m.

Biot-Savart's Law:



Let XY be a conductor carrying a current I. Consider a differential element of length dl. Let P be a point chosen arbitrarily. Let the distance OP be r, and the angle between MN and OP be θ.

Statement Of Biot-Savart's law: The magnitude of the magnetic field intensity $\overrightarrow{dH}$ at a point due to the current in the differential element is directly proportional to the product of the current I, the magnitude of the length of the differential element dl, and the sine of the angle between the tangent drawn to the element, and the line joining the point and the element, (ie., $\sin\theta$), and it is inversely proportional to the square of the distance between them.

$$\therefore dH \propto \frac{I\, dl \sin\theta}{r^2},$$

$$dH = k \frac{I\, dl \sin\theta}{r^2},$$

$$k = \frac{1}{4\pi}$$

$$dH = \frac{I\, dl \sin\theta}{4\pi r^2}$$

The direction of $\overrightarrow{dH}$ at P as per the law is perpendicular to the plane containing the tangent drawn to the element and the line joining the point and the element.

In vector notation, $\overrightarrow{dH} = \dfrac{I\, \overrightarrow{dl} \times \hat{a}_r}{4\pi r^2}$,

where, the vector $\overrightarrow{dl}$ is directed along the direction of current, and $\hat{a}_r$ is the unit vector.

Faraday's Law Of Electromagnetic Induction :

Faraday's experiments revealed that whenever there is a change of magnetic flux linkage in a circuit, an emf is induced in it which results in the flow of current. The flow of current itself sets up a magnetic field. Lenz found that, the direction of magnetic field due to the current flow is such that it opposes the change causing it. i.e., it decreases the magnetic field if the magnetic flux is increasing and increases the magnetic field if the magnetic flux is decreasing. Lenz stated that, the induced e.m.f. is in such a direction as to oppose the change causing it. Faraday's law can be stated as "the magnitude of the induced e.m.f in a circuit is equal to the rate of change of the magnetic flux through it, and its direction opposes the flux change".

∴ The induced emf e is expressed as,

$$e = -\frac{d\phi}{dt}.$$

If we consider a coil with N turns, then the emf induced across the coil is, $e = -N\frac{d\phi}{dt}$.

Integral and Point (or Differential) Forms Of Faraday's Law:

Consider a loop of a conducting material which is stationary.

As per Faraday's law, $e = -\frac{d\phi}{dt}$, where $e \rightarrow$ induced emf

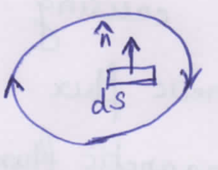
$\phi \rightarrow$ magnetic flux.

$$\frac{d\phi}{dt} = \int_S \frac{\partial \vec{B}}{\partial t} \cdot \vec{ds}$$

induced emf in the loop is given by

$$e = \oint \vec{E} \cdot dL$$

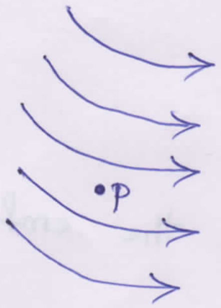
$$\oint \vec{E} \cdot dL = -\int_S \frac{\partial \vec{B}}{\partial t} \cdot \vec{ds}$$



This is Faraday's Law in integral form.

GAUSS DIVERGENCE THEOREM

Divergence of $\vec{D}$: consider a point P in a region where there are charges. Let ρ_v be the charge density at P. Due to the charges there will be electric field around P. Then,



$$\left(\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial z}{\partial z} \right) = \rho_v \, ,$$

$$\nabla \cdot \vec{D} = \rho_v$$

The above equation is known as Maxwell's first equation.

$\nabla \cdot \vec{D}$ is the divergence of $\vec{D}$.

Statement Of Gauss' Divergence Theorem: The integral of the normal component of the flux density over any closed surface in an electric field is equal to the volume integral of the divergence of the flux throughout the space enclosed by the surface.

$$\oint_S \vec{D} \cdot d\vec{S} = \int_V \nabla \cdot \vec{D} \, dv$$

Proof: Consider a gaussian surface in a region with certain charge density.

Let ΔQ be the charge within the element. If ρ is the charge density, and since ρ can vary continuously in the volume, we have,

$$\rho_v = \lim_{\Delta v \to 0} \left[\frac{\Delta Q}{\Delta v} \right] = \frac{dQ}{dv}$$

$$dQ = \rho_v \, dv .$$

If Q is the total charge enclosed by the gaussian surface, then,

$$Q = \int dQ = \int_V \rho_v \, dv .$$

But from Maxwell's first equation

$$Q = \int_V \nabla \cdot \vec{D} \, dv .$$

From Gauss' law,

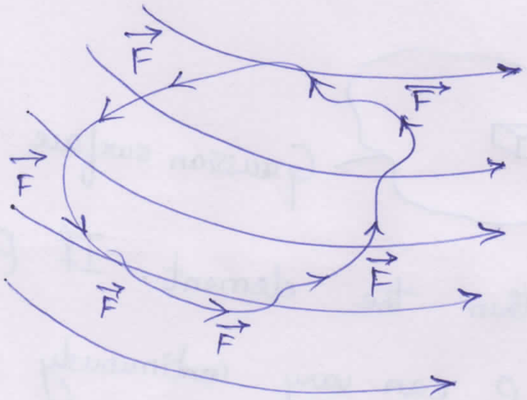
$$\oint_S \vec{D} \cdot d\vec{S} = Q .$$

$$\oint_S \vec{D}\cdot\vec{ds} = \int_v \nabla\cdot\vec{D}\, dv \cdot$$

This is,
Gauss' divergence theorem.

STOKE'S THEOREM: This provides a way of relating line integral to a surface integral in cases where curl of a vector exists in a vector field.

$$\int(\nabla\times\vec{F})\cdot\vec{ds} = \oint \vec{F}\cdot\vec{dl}$$



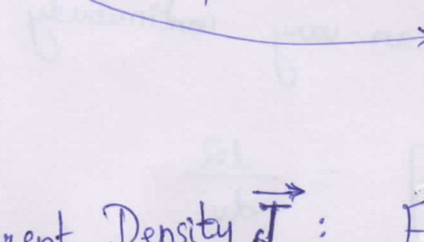
The curl of $\vec{F}$ and sum of curl within any chosen surface then, the sum will be equal to just the circulation of $\vec{F}$ around the boundary of the chosen surface.

Current Density $\vec{J}$: From the discussion of Ampere's law, the current density is the current per unit area of cross-section. If ρ_v is the charge density & $\vec{v}$ is the velocity of charge flow, then

$$\vec{J} = \rho_v \vec{v}$$

Equation Of Continuity: For direct current, if we consider a closed surface at some part of it, the charge flow into the surface is equal to the charge outflow, which means there is no net charge within the surface. (Kirchoff's law)

$$\nabla\cdot\vec{J} = 0.$$



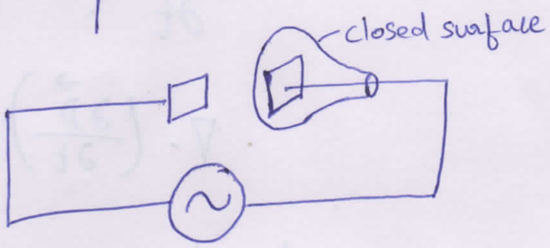
Consider the ampere's law under time varying conditions for the field. $\nabla \times \vec{H} = \vec{J}$. Taking the divergence

$$\nabla \cdot (\nabla \times \vec{H}) = \nabla \cdot \vec{J}$$

But, as per rules of vector analysis, the divergence of a curl for any field is always zero.

$$\nabla \cdot \vec{J} = 0.$$

This condition holds good under static conditions. Consider an ac circuit with a capacitor. Consider a closed surface which encloses



only one plate of the capacitor. Whenever there is any current flow into the closed surface, it is not accompanied by a simultaneous current outflow through any part of the surface and vice versa. $\nabla \cdot \vec{J} = 0$ fails in ac circuit.

Equation Of Continuity: Consider a closed surface enclosing certain amount of charge. According to equation of continuity, if there is any charge outflow through this surface, it must be accompanied by a simultaneous reduction of charge within the surface. $\boxed{\nabla \cdot \vec{J} = -\frac{\partial \rho_v}{\partial t}}$, where, $\vec{J}$ is the current density & ρ_v is the charge density.

This is equation of continuity.

Displacement Current:

(Modification Of Ampere's Law to suit the Time Varying condition):

To make the ampere's law to work under time varying field conditions, Maxwell suggested the following treatment.

We know from Gauss' Law, $\nabla \cdot \vec{D} = \rho_v$.

Differentiate w.r.t. time, $\frac{\partial}{\partial t}\left(\nabla \cdot \vec{D}\right) = \frac{\partial \rho_v}{\partial t}$

$$\nabla \cdot \left(\frac{\partial \vec{D}}{\partial t}\right) = \frac{\partial \rho_v}{\partial t}$$

from equation of continuity,

$$\nabla \cdot \vec{J} = -\frac{\partial \rho_v}{\partial t},$$

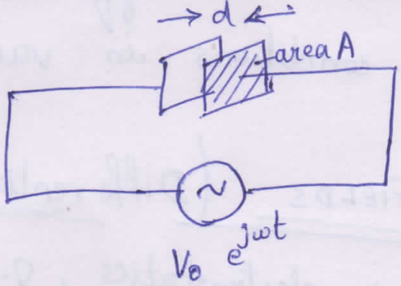
$$\nabla \cdot \left(\vec{J} + \frac{\partial \vec{D}}{\partial t}\right) = 0.$$

For the time varying case, $\nabla \cdot \left(\vec{J} + \frac{\partial \vec{D}}{\partial t}\right) = 0$, must be considered instead of $\nabla \cdot \vec{J} = 0$

$$\boxed{\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}}$$

This equation is known as Maxwell - Ampere's law.

Expression for Displacement Current: Consider a parallel plate capacitor connected across an a.c. source.



(i) **Condution current:** Let the potential be, $V = V_0\, e^{j\omega t}$

$$E = \frac{V}{d}$$

$$D = \frac{\varepsilon V}{d} = \frac{\varepsilon}{d} V_0\, e^{j\omega t}$$

Displacement current density is given by $\left(\frac{\partial D}{\partial t}\right)$.

If I_D is the displacement current, then $\left(\frac{\partial D}{\partial t}\right) = \frac{I_D}{A}$.

$$\therefore\ I_D = \left(\frac{\partial D}{\partial t}\right) A = \frac{\partial}{\partial t}\left(\frac{\varepsilon}{d} V_0 e^{j\omega t}\right) A$$

$$I_D = \frac{j\omega \varepsilon A}{d} V_0\, e^{j\omega t}$$

Displacement current is the correction factor in Maxwell's equation that appears in time-varying condition but doesn't describe any movement of charges though it has an associated magnetic field.

MAXWELL'S EQUATIONS

Four Maxwell's equations in differential form for time-varying fields and in conditions in vaccum as follows.

TIME-VARYING FIELDS (Differential form or Point form):

1. From Gauss' Law in electrostatics, $\nabla \cdot \vec{D} = \rho_v$
2. From Faraday's law, $\nabla \times \vec{E} = -\frac{\partial B}{\partial t}$
3. From Gauss' law for magnetic fields, $\nabla \cdot \vec{B} = 0$
4. From Ampere's law, $\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$

STATIC FIELDS: (Differential form or Point form):

1. $\nabla \cdot \vec{D} = \rho_v$
2. $\nabla \times \vec{E} = 0$
3. $\nabla \cdot \vec{B} = 0$
4. $\nabla \times \vec{H} = \vec{J}$

Wave Equations in Differential Form in Free Space in terms Of Field: Consider the two curl equations of Maxwell,

$$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}.$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

But, $D = \varepsilon E$, and $B = \mu H$.

$$\therefore \nabla \times \vec{H} = \vec{J} + \varepsilon \frac{\partial \vec{E}}{\partial t}. \quad \text{———} ①$$

$$\nabla \times \vec{E} = -\mu \frac{\partial H}{\partial t} \text{ ——— ②}$$

Taking curl for both sides of Eq. ②.

$$\nabla \times \nabla \times \vec{E} = -\mu \frac{\partial}{\partial t}\left(\nabla \times \vec{H}\right) \text{ ——— ③}$$

$$\text{Since, } \nabla \times \nabla \times \vec{E} = \nabla\left(\nabla \cdot \vec{E}\right) - \nabla^2 E$$

$$= \nabla\left(\frac{\rho_v}{\varepsilon}\right) - \nabla^2 \vec{E}$$

(Since, as per Maxwell's equation, $\nabla \cdot \vec{D} = \rho_v$ or $\nabla \cdot \vec{E} = \frac{\rho_v}{\varepsilon}$).

∴ From Eqs (3) and (4),

$$\nabla\left(\frac{\rho_v}{\varepsilon}\right) - \nabla^2 \vec{E} = -\mu \frac{\partial}{\partial t}\left(\nabla \times \vec{H}\right)$$

Using Eq (1) in the right side, we have.

$$\nabla\left(\frac{\rho_v}{\varepsilon}\right) - \nabla^2 \vec{E} = -\mu \frac{\partial}{\partial t}\left(\vec{J} + \varepsilon \frac{\partial \vec{E}}{\partial t}\right)$$

$$\nabla^2 \vec{E} - \mu\varepsilon \frac{\partial^2 \vec{E}}{\partial t^2} = \mu \frac{\partial \vec{J}}{\partial t} + \nabla\left(\frac{\rho_v}{\varepsilon}\right)$$

The above equation represents the wave equation in $\vec{E}$ for a medium with constant μ and ε, i.e, a homogeneous and isotropic medium. If we consider free space, i.e, space where there are no charges, or currents, then $\rho_v = 0$, and $J = 0$

$$\boxed{\nabla^2 E - \mu\varepsilon \frac{\partial^2 \vec{E}}{\partial t^2} = 0}$$

Plane Electromagnetic waves in vacuum:

Consider a plane electromagnetic wave along x-direction of wavelength λ. Let the electric field associated with the wave be oriented parallel to y-direction and be denoted as E_y. Let A be its amplitude.

$$\vec{E_y} = A \cos\left[\frac{2\pi}{\lambda}(x-ct)\right]\hat{a}_y .$$

$$\text{and} \quad \vec{B_z} = \frac{1}{c} A \cos\left[\frac{2\pi}{\lambda}(x-ct)\right]\hat{a}_z .$$

electromagnetic wave in vacuum.

Since $\vec{E_y}$ and $\vec{B_z}$ perpendicular to each other in the above two equations, we have,

$$c = \frac{E_y}{B_z} . \quad \text{where, } E_y = |\vec{E_y}| \ \& \ B_z = |\vec{B_z}| .$$

The classical wave equation for an oscillating physical quantity $\vec{F}$ is represented as,

$$\nabla^2 F - \frac{1}{v^2}\frac{\partial^2 \vec{F}}{\partial t^2} = 0$$

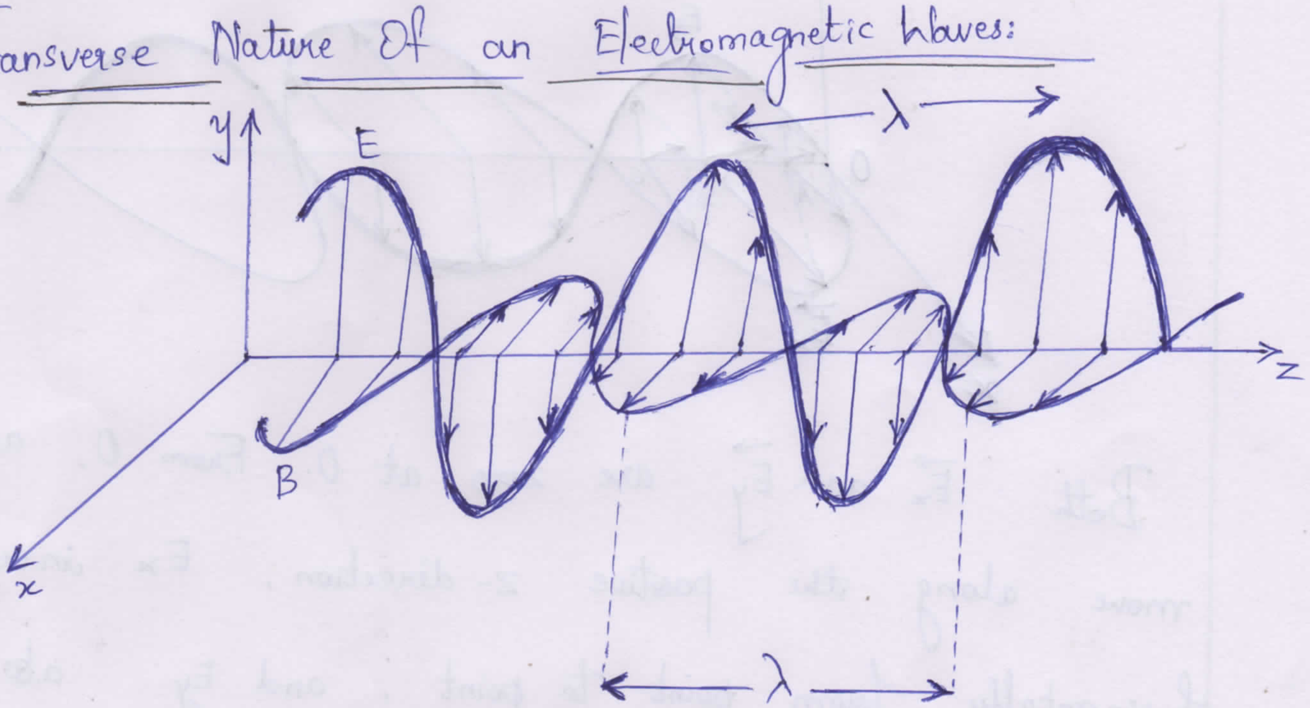
$$\nabla^2 E - \mu t \frac{\partial^2 E}{\partial t^2} = 0$$

$$v = \frac{1}{\sqrt{\mu \varepsilon}}$$

$$c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}}$$

POLARIZATION OF EM WAVES

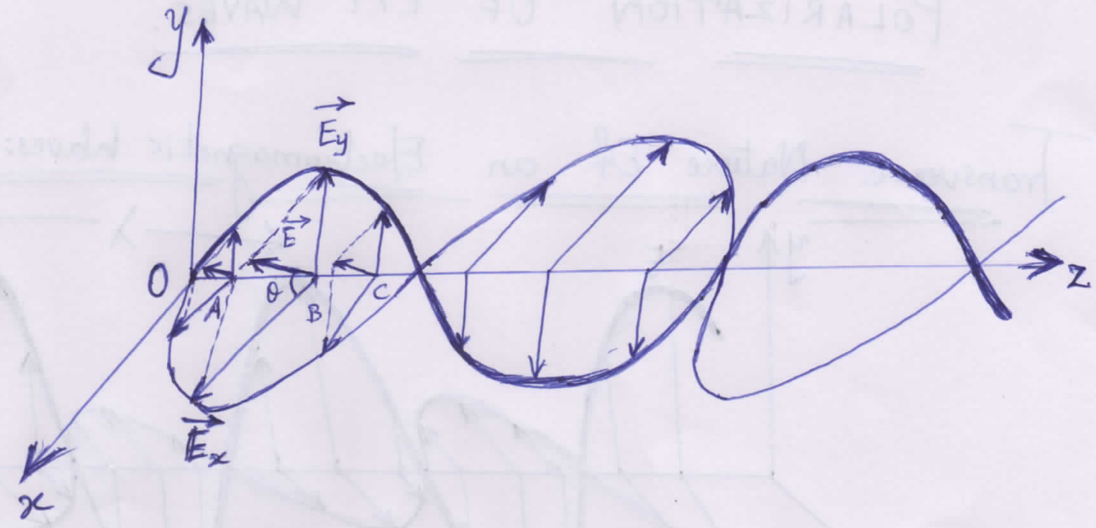
Transverse Nature Of an Electromagnetic Waves:



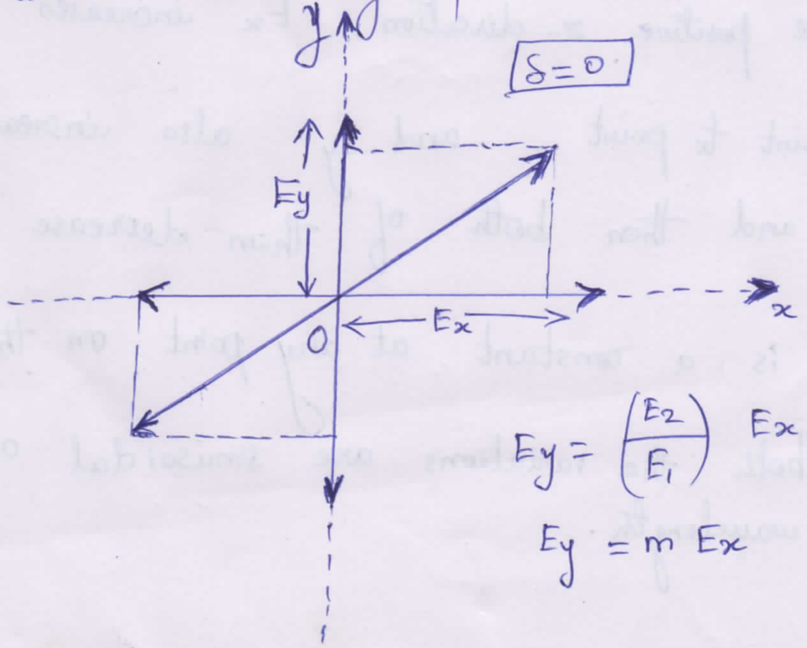
The velocity of movement of any electromagnetic wave, irrespective of wavelength or intensity, is same in a given medium.

Linear Polarization: Let us consider E_x and E_y components which are in identical phase, but of different amplitudes. Both $\vec{E_x}$ and $\vec{E_y}$ are zero at O. From O, as we move along the positive z-direction, E_x increases horizontally from point to point, and E_y also increases vertically upto B, and then both of them decrease simultaneously.

$\left|\frac{E_y}{E_x}\right|$ is a constant at any point on the z-axis since, both the variations are sinusoidal and of same wavelength.



Both $\vec{E_x}$ and $\vec{E_y}$ are zero at O. From O, as we move along the positive z-direction, E_x increases horizontally from point to point, and E_y also increases vertically upto B, and then both of them decrease simultaneously. $\left|\frac{E_y}{E_x}\right|$ is a constant at any point on the z-axis since, both the variations are sinusoidal and of same wavelength. The resultant $\vec{E}$ also varies along with $\vec{E_x}$ and $\vec{E_y}$ but always oriented at an inclination of θ to E_x. Thus, we say that the wave is linearly polarized.

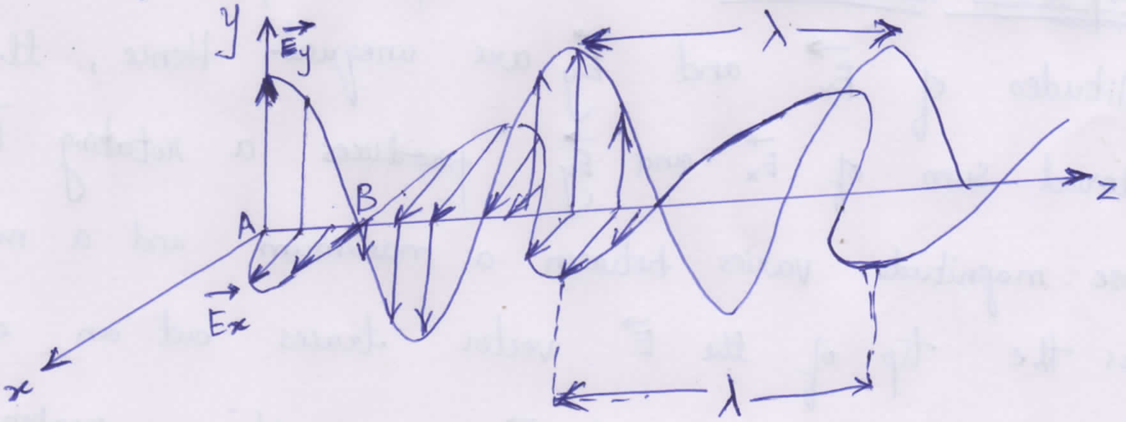


$$E_y = \left(\frac{E_2}{E_1}\right) E_x$$

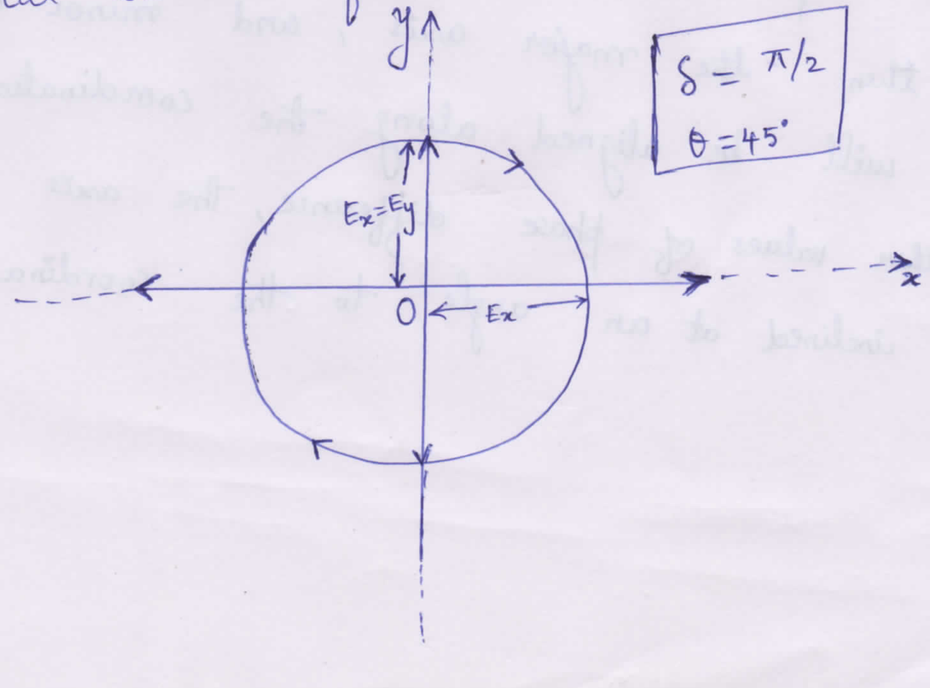
$$E_y = m\, E_x$$

Circular Polarization:

Let us consider the two components $\vec{E_x}$ and $\vec{E_y}$ to be of equal amplitudes i.e., $E_1 = E_2$. But, Let one of them, say $\vec{E_y}$, be ahead of $\vec{E_x}$ by a phase of 90°.



At point A, E_y is maximum but E_x is zero. Hence the resultant field $\vec{E}$ is vertically upwards. At the point B, it becomes horizontal. Between A and B, it is vice-versa, and $\vec{E_y}$ and $\vec{E_x}$ make the field $\vec{E}$ tilt progressively towards horizontal direction from A to B.

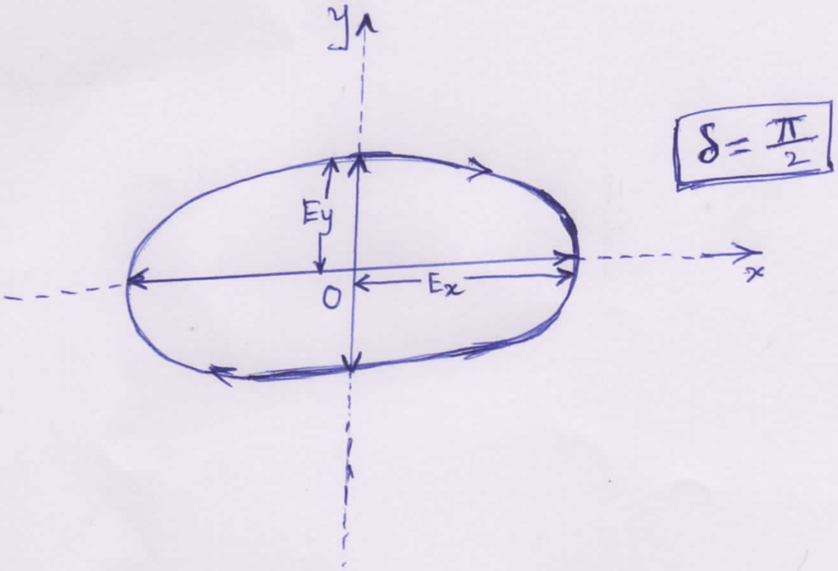


$$\boxed{\begin{aligned}\delta &= \pi/2 \\ \theta &= 45^\circ\end{aligned}}$$

The condition for circular polarization for a wave travelling in z-direction is that, the components $\vec{E_x}$ and $\vec{E_y}$ must have a constant phase difference of 90° and their amplitudes are equal.

Elliptical Polarization: For elliptical polarization, the amplitudes of $\vec{E_x}$ and $\vec{E_y}$ are unequal. Hence, the vectorial sum of $\vec{E_x}$ and $\vec{E_y}$ produces a rotating $\vec{E}$-vector whose magnitude varies between a maximum and a minimum. Thus the tip of the $\vec{E}$ vector traces out an ellipse in the projected image of $\vec{E}$ on a plane perpendicular to the z-axis. Thus the wave is elliptically polarized.

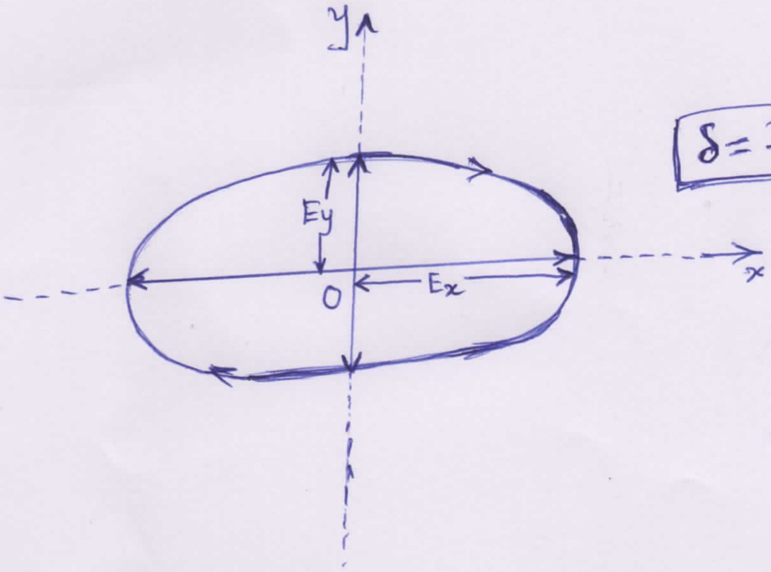
Hence, the condition for elliptical polarization for a wave travelling in z-direction is that, the components $\vec{E_x}$ and $\vec{E_y}$ must have a constant non-zero phase difference, and their amplitudes are unequal. If the phase difference is 90°, then the major axis, and minor axis of the ellipse will be aligned along the coordinates axes. For other values of phase difference, the axis of the ellipse will be inclined at an angle to the coordinate axes.



For elliptical polarization, the two components have unequal amplitudes ie., $E_x \neq E_y$. The phase difference $\delta \neq 0$. Here, Let us assume $\delta = \pi/2$. For these conditions, it can be shown that,

$$\boxed{\left(\frac{E_x}{E_1}\right)^2 + \left(\frac{E_y}{E_2}\right)^2 = 1.}$$

This is the equation for an ellipse. Hence the wave is elliptically polarized.



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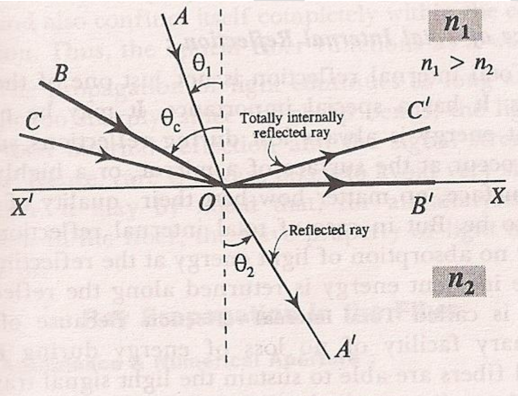
OPTICAL FIBERS

Total Internal Reflection:

When a ray of light travels from denser to rarer medium it bends away from the normal. As the angle of incidence increases in the denser medium, the angle of refraction also increases. For a particular angle of incidence called the "*critical angle*", the refracted ray grazes the surface separating the media or the angle of refraction is equal to 90°. If the angle of incidence is greater than the critical angle, the light ray is reflected back to the same medium. This is called *"Total Internal Reflection".*

XX^1 is the surface separating medium of refractive index n_1 and medium of refractive index n_2, $n_1 > n_2$. AO and OA^1 are incident and refracted rays. θ_1 and θ_2 are angle of incidence and angle of refraction, $\theta_2 > \theta_1$. For the ray BO, θ_c is the critical angle. OB^1 is the refracted ray which grazes the interface. The ray CO incident with an angle greater than θ_c is totally reflected back along OC^1.



TOTAL INTERNAL REFLECTION

From Snell's law,

$$n_1 \sin\theta_1 = n_2 \sin\theta_2$$

For total internal reflection,

$$\theta_1 = \theta_c \text{ and } \theta_2 = 90°$$

$$n_1 \sin\theta_c = n_2 \quad (\text{because } \sin 90° = 1)$$

$$\theta_c = \sin^{-1}(n_2/n_1)$$

In total internal reflection there is no loss or absorption of light energy.

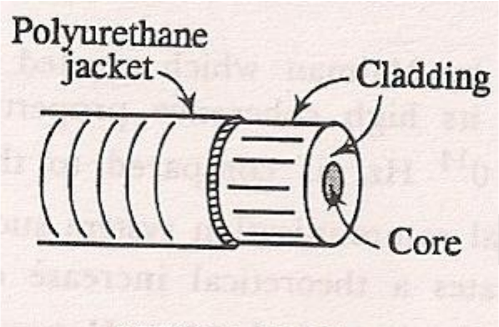
The entire energy is returned along the reflected light. Thus is called Total internal reflection.

Optical Fibers:

Optical fiber is made from transparent dielectrics. It works on the principle of Total internal reflection (TIR).

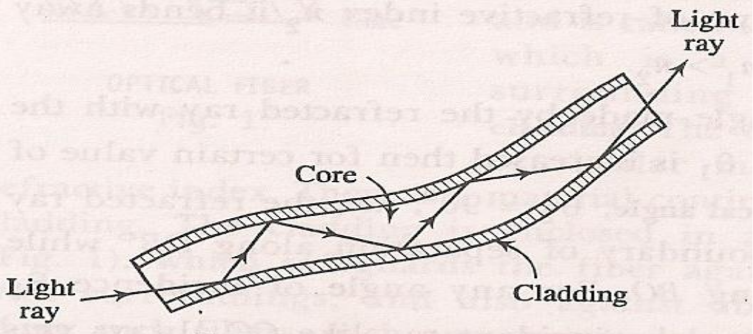
Construction and propagation mechanism:

Optical fiber is cylindrical in shape. The inner cylindrical part is called as core of refractive index n_1. The outer part is called as cladding of refractive index n_2, $n_1>n_2$. There is continuity between core and cladding. Cladding is enclosed inside a polyurethane jacket. Number of such fibers is grouped to form a cable.



OPTICAL FIBER

The light entering through one end of core strikes the interface of the core and cladding with angle greater than the critical angle and undergoes total internal reflection. After series of such total internal reflection, it emerges out of the core. Thus the optical fiber works as a waveguide. Care must be taken to avoid very sharp bends in the fiber because at sharp bends, the light ray fails to undergo total internal reflection.

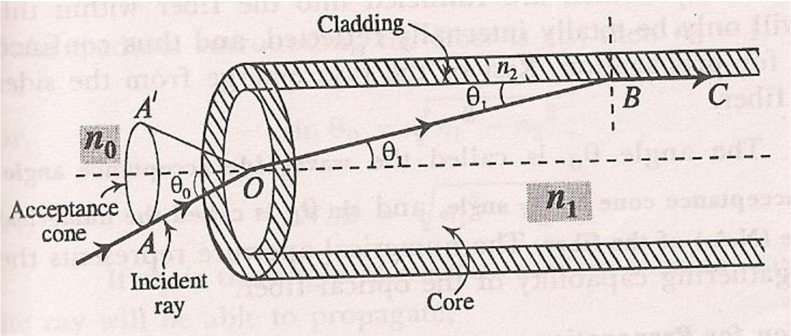


FIBER AS A WAVEGUIDE

Expression for Numerical Aperture of an optical fiber:

Consider a light ray AO incident at an angle ‘θ_0’ enters into the fiber. Let ‘θ_1’ be the angle of refraction for the ray OB. The refracted ray OB incident at a critical angle ($90^\circ - \theta_1$) at B grazes the interface between core and cladding along BC. If the angle of incidence is greater than critical angle, it undergoes total internal reflection.

Now if AO is rotated around the fiber axis by keeping θ_0 same then it describes a conical surface through which the beam converges at wide angles into the core and will only be totally internally reflected. Thus θ_0 is called the waveguide acceptance angle and $\sin\theta_0$ is called the numerical aperture.



ACCEPTANCE CONE

Let n_0, n_1 and n_2 be the refractive indices of the medium, core and cladding respectively.

From Snell’s law,

$$n_o \sin\theta_0 = n_1 \sin\theta_1 \rightarrow (1)$$

At B the angle of incidence is ($90 - \theta_1$)

From Snell’s law,

$$n_1 \sin(90-\theta_1) = n_2 \sin 90$$

$$n_1 \cos\theta_1 = n_2$$

$$\cos\theta_1 = n_2 / n_1 \rightarrow (2)$$

From eqn (1)

$$Sin\,\theta_0 = \frac{n_1}{n_0} \sin\,\theta_1$$

$$= \frac{n_1}{n_o}\sqrt{1 - cos^2\theta_1} \rightarrow (3)$$

Using eqn (2) in (3)

$$Sin\theta_0 = \frac{n_1}{n_0}\sqrt{1-\frac{n_2^2}{n_1^2}}$$

$$= \frac{\sqrt{n_1^2-n_2^2}}{n_o} \rightarrow (4)$$

If the surrounding medium is air, $n_o = 1$

$$Sin\theta_0 = \sqrt{n_1^2 - n_2^2}$$

But $\sin\theta_0$ is called numerical aperture.

$$\text{N.A} = \sqrt{n_1^2 - n_2^2}$$

Therefore for any angle of incidence θ_i equal to or less than θ_0, the incident ray is able to propagate if,

$$\theta_i < \theta_0$$

$$\text{Sin}\,\theta_i < Sin\,\theta_0$$

$$sin\theta_i < \sqrt{n_1^2 - n_2^2}$$

$sin\theta_i < N.A$ is the condition for propagation

Acceptance angle is defined as the maximum angle that a light ray can have relative to the axis of the fiber and propagate through the fiber.

Numerical aperture indicates the ability of the optical fiber to accept light i.e the light gathering capability of the optical fiber. The sign of the acceptance angle also called numerical aperture.

Fractional Index Change:

"It is the ratio of the refractive index difference between the core and cladding to the refractive index of the core of an optical fiber". $\Delta = \dfrac{n_1 - n_2}{n_1}$

Relation between N.A and Δ:

$$\text{Consider } \Delta = \frac{n_1 - n_2}{n_1}$$

$$n_1 - n_2 = \Delta n_1$$

We have

$$\text{N.A} = \sqrt{n_1^2 - n_2^2}$$

$$= \sqrt{(n_1 + n_2)(n_1 - n_2)}$$

Considering $n_1 \approx n_2$

$$= \sqrt{(n_1 + n_2)\Delta n_1}$$

$$N.A = \sqrt{2n_1^2 \Delta}$$

$$N.A = n_1\sqrt{2\Delta}$$

Increase in the value of Δ increases N.A

It enhances the light gathering capacity of the fiber. Δ value cannot be increased very much because it leads to intermodal dispersion intern signal distortion.

Modes of propagation and V-number:

The number of paths supported for propagation in the fiber is known as modes of propagation and is determined by a parameter called V-number.

If the surrounding medium is air, then v- number is given by

$$V = \frac{\pi d}{\lambda}\sqrt{n_1^2 - n_2^2}$$

Where 'd' is the core diameter, n_1 and n_2 are refractive indices of core and cladding respectively, 'λ' is the wavelength of light propagating in the fiber.

$$V = \frac{\pi d}{\lambda}(NA)$$

If the fiber is surrounded by a medium of refractive index n_0, then,

$$V = \frac{\pi d}{\lambda} \frac{\sqrt{n_1^2 - n_2^2}}{n_0}$$

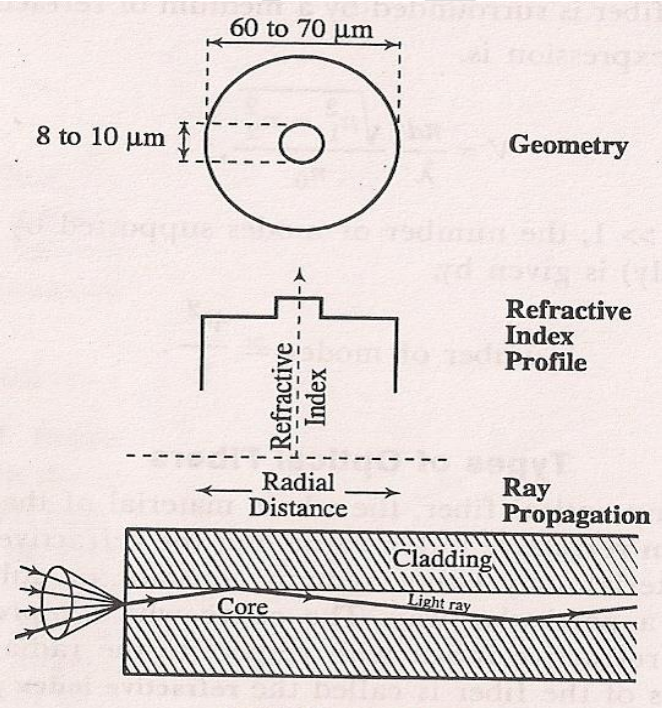
For V >1, the number of modes supported by the fiber is given by, number of modes ≈ $V^2/2$.

Types of optical fibers:

In an optical fiber the refractive index of cladding is uniform and the refractive index of core may be uniform or may vary in a particular way. The curve which represents the variation of refractive index with respect to the radial distance from the axis of the optical fiber is called refractive index profile.

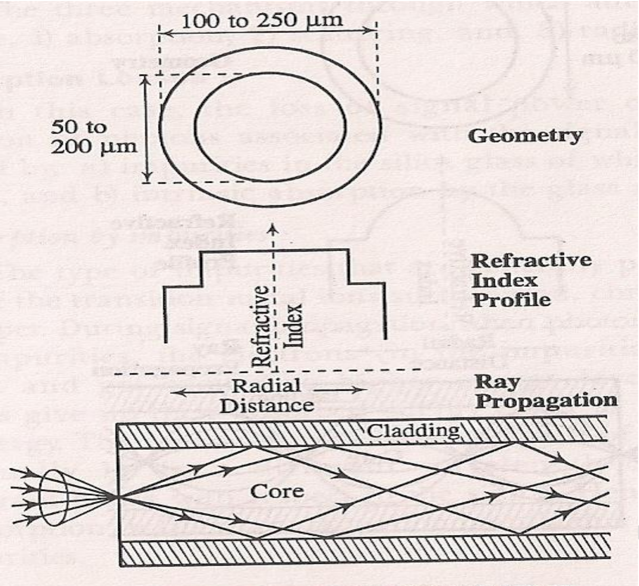
Following are the different types of fibers:

***1) Single mode fiber*:** Refractive index of core and cladding has uniform value. the refractive index of core is greater than that of cladding, hence R I profile takes the form of a step .the diameter of the core is about 8-10μm and that of cladding is 60-70μm.Because of its narrow core it can guide just a single mode. LASER is used as source. They are used in submarine.



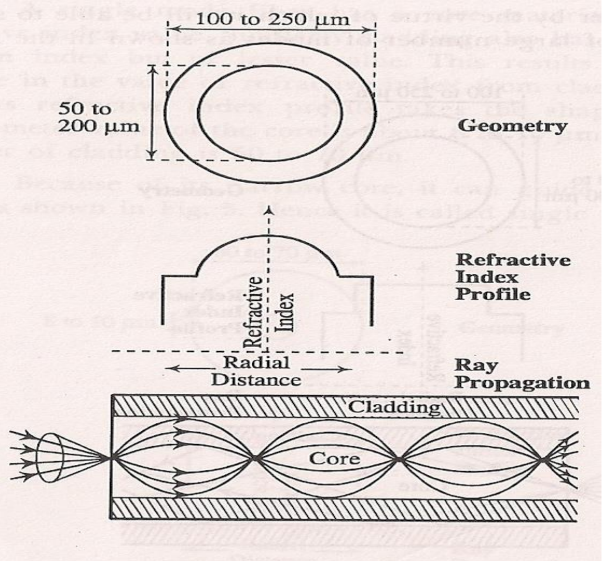
SINGLE MODE FIBER

2) Step index multimode fiber: It is similar to single mode fiber but core has large diameter. It can guide large number of modes hence it is called as multimode fiber. In this type of fiber the path lengths for different modes are different. Therefore they reach the end with time delay. Laser or LED is used as a source of light. It has an application in data links.



STEP - INDEX MULTIMODE FIBER

3)Graded index multimode fiber: It is also called GRIN. The refractive index of core decreases from the axis towards the core cladding interface. The refractive index profile is shown in figure. In this type of fiber all the modes will have same optical path length and reach the other end at the same time. Laser or LED is used as a source of light. It is the expensive of all. It is used in telephone trunk between central offices.



GRADED - INDEX MULTIMODE FIBER

Attenuation:

Attenuation is the loss of optical power as light signal travels through a fiber. It is expressed in decibel/kilometer [db/km]. If P_{in} is the input power and P_{out} is the output power after passing through a fiber of length 'L', the mean attenuation constant or coefficient 'α' of the fiber, in units of db/km is given by

$$\alpha = -\frac{10}{L}\log_{10}\left(\frac{p_{out}}{p_{in}}\right) \qquad \text{dB/km}$$

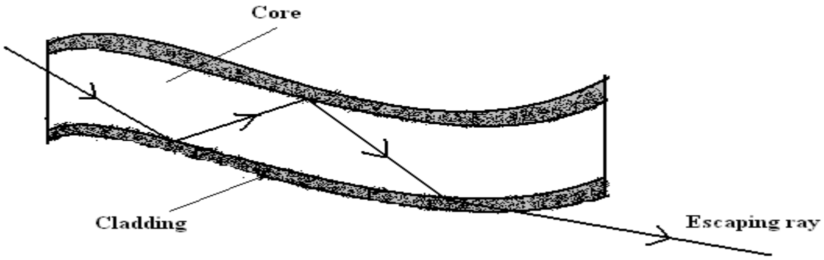
Attenuation can be caused by three mechanisms.

1. **Absorption loss: -** Absorption of photons by impurities like metal ions such as iron, chromium, cobalt and copper in the silica glass of which the fiber is made of. During signal processing photons interact with electrons of impurity atoms. The atoms are excited and de-excite by emitting photons of different wavelengths. Hence it is a loss of energy. The other impurity such as hydroxyl ions (OH) causes significant absorption loss. The absorption of photons by fiber material itself is called intrinsic absorption.

2. **Scattering loss:** When the wavelength of the photon is comparable to the size of the particle then the scattering takes place known as Rayleigh scattering. This is because of the non uniformity in manufacturing, the refractive index of the material changes leads to a scattering of light when light enters such non-uniform regions. It is inversely proportional to the fourth power of wavelength. Scattering of photons also takes place due to trapped gas bubbles which are not dissolved at the time of manufacturing.

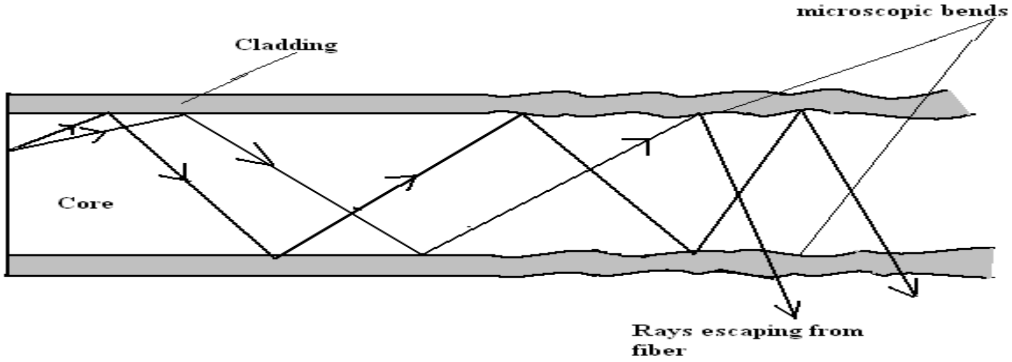
3. **Radiation losses:** Radiation losses occur due to macroscopic bends and microscopic bends.

   Macroscopic bending: All optical fibers are having critical radius of curvature provided by the manufacturer. If the fiber is bent below that specification of radius of curvature, the light ray incident on the core cladding interface will not satisfy the condition of TIR. This causes loss of optical power.



RADIATION LEAKS

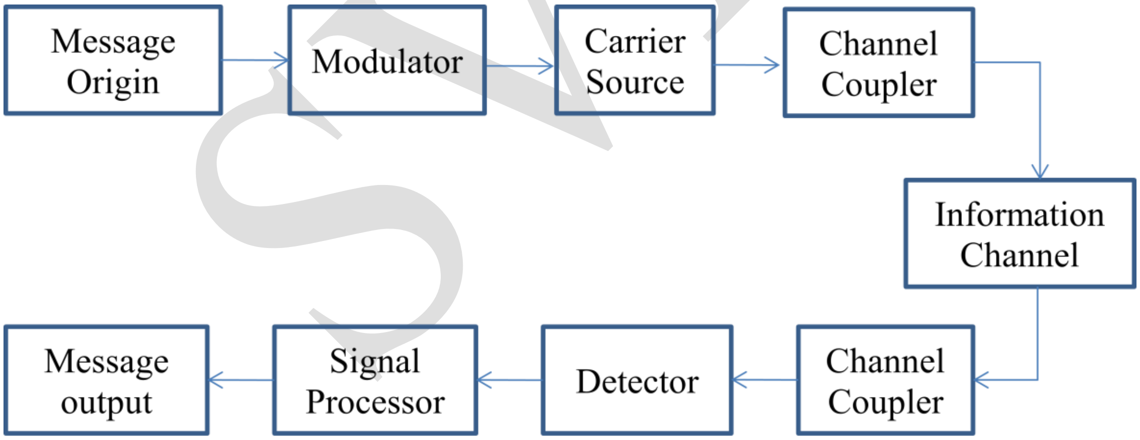
Microscopic bending: Optical power loss in optical fibers is due to non-uniformity of the optical fibers when they are laid. Non uniformity is due to manufacturing defects and also lateral pressure built up on the fiber. The defect due to non uniformity (micro endings) can be overcome by introducing optical fiber inside a good strengthen polyurethane jacket.



MICROSCOPIC BENDING LOSS

Applications:

Point to Point Communication using optical fibers :



Optical fiber communication system consists of transmitter, information channel and receiver. Transmitter converts an electrical signal into optical signal. Information channel carries the signal from transmitter to receiver. The receiver converts optical signal to electrical form. The block diagram of optical fiber communication system is shown in fig.

Message origin: It converts a non electrical message into an electrical signal.

Modulator: It converts the electrical message into proper format and it helps to improve the signal onto the wave which is generated by the carrier source.

There are two types of format. They are Analog and digital. Analog signal is continuous and it doesn't make any change in the original format. But digital signal will be either in ON or OFF state.

Carrier source: It generates the waves on which the data is transmitted. These carrier waves are produced by the electrical oscillator. Light emitting diodes (LED) and laser diodes (LD) are the different sources.

Channel Coupler: (Input) The function of the channel coupler is to provide the information to information channel. It can be an antenna which transfers all the data.

Information channel: It is path between transmitter and receiver. There are two types of information channel. They are guided and unguided. Atmosphere is the good example for unguided information channel. Co-axial cable, two-wire line and rectangular wave guide are example for guided channel.

Channel Coupler: (Output) The output coupler guides the emerged light from the fiber on to the light detector.

Detector: The detector separates the information from the carrier wave. Here a photo-detector converts optical signal to electronic signal.

Signal processor: Signal processor amplifies the signals and filters the undesired frequencies.

Message output: The output message will be in two forms. Either person can see the information or hear the information. The electrical signal can be converted into sound wave or visual image by using CRO.

Advantages of optical communication system:

1) It carries very large amount of information in either digital or analog form due to its large bandwidth.
2) The materials used for making optical fiber are dielectric nature. So, it doesn't produces or receives any electromagnetic and R-F interferences.
3) Fibers are much easier to transport because of their compactness and lightweight.
4) It is easily compatible with electronic system.
5) It can be operated in high temperature range.
6) It does not pick up any conducted noise.
7) Not affected by corrosion and moisture.
8) It does not get affected by nuclear radiations.
9) No sparks are generated because the signal is optical signal.

- Note: - Optical fibers are used in sensors like pressure sensor, voltage sensor and current sensors.
  Optical fibers are used in local networks like data link purpose.

MODULE 4

QUANTUM MECHANICS AND LASERS

Dual nature of matter (de-Broglie Hypothesis)

Dual nature of light:

The concept of photoelectric effect and Compton Effect gives the evidence for particle nature of light. Where as in physical optics the phenomenon like interference, diffraction, superposition was explained by considering wave nature of light. This is wave particle duality of light.

Dual nature of matter:

On the basis of above concept (dual nature of light), in 1923, Louis de Broglie gave a hypothesis

"Since nature loves symmetry, if the radiation behaves as particles under certain conditions and as waves under certain conditions, then one can expect that, the entities which ordinarily behaves as particles (ex. Like electrons, protons, neutrons) must also exhibit properties attributable to waves under appropriate circumstances" . This is known as **deBroglie hypothesis**

Matter is made up of discrete constituent particles like atoms, molecules, protons, neutrons and electrons, hence matter has particle nature. Wave nature of matter is explained by Davisson and Germer experiment. Hence matter also exhibit wave particle duality.

The waves associated with the moving particles are called de Broglie waves or matter waves or pilot waves.

Characteristics of matter waves:

1. Waves associated with moving particles are called matter waves. The wavelength 'λ' of a de-Broglie wave associated with particle of mass 'm' moving with velocity 'v' is
$$\lambda = h/(mv)$$
2. Matter waves are not electromagnetic waves because the de Broglie wavelength is independent of charge of the moving particle.
3. The amplitude of the matter wave depends on the probability of finding the particle in that position.
4. The speed of matter waves depends on the mass and velocity of the particle associated with the wave.

Debroglie's Wavelength:

A particle of mass 'm' moving with velocity 'c' possess energy given by

$E = mc^2$ → **(**Einstein's Equation) (1)

According to Planck's quantum theory the energy of quantum of frequency 'υ' is

$E = h\upsilon \rightarrow (2)$

From (1) & (2)

$mc^2 = h\upsilon = hc/\lambda$ since $\upsilon = c/\lambda$

$\lambda = hc/mc^2 = h/mc$

$\lambda = h/mv$ since $v \approx c$

de-Broglie wavelength of a free particle in terms of its kinetic energy

Consider a particle, since the particle is free, the total energy is same as

$$E = \frac{1}{2}mv^2 = \frac{p^2}{2m}$$

Where 'm' is the mass, 'v' is the velocity and 'p' is the momentum of the particle.

$$p = \sqrt{2mE}$$

The expression for de-Broglie wavelength is given by

$$\lambda = \frac{h}{p} = \frac{h}{\sqrt{2mE}}$$

Debroglie Wavelength of an Accelerated Electron:

If an electron accelerated with potential difference 'V' the work done on the 'eV', which is converted to kinetic energy.

Then $eV = \frac{1}{2}mv^2 \rightarrow (1)$

If 'p' is the momentum of the electron, then p=mv

Squaring on both sides, we have

$p^2 = m^2v^2$

$mv^2 = p^2/m$

Using in equation (1) we have

$$eV = p^2/(2m)$$

or $p = \sqrt{2meV}$

According to de-Broglie $\lambda = h/p$

Therefore $\lambda = \left[\frac{h}{\sqrt{2meV}}\right]$

$$\lambda = \frac{1}{\sqrt{V}}\left[\frac{6.626\times10^{-34}}{\sqrt{2\times9.11\times10^{-31}\times1.602\times10^{-19}}}\right] = \frac{1.226\times10^{-9}}{\sqrt{V}}\text{ m}, \quad \lambda = \frac{1.226}{\sqrt{V}}\text{ nm}$$

Heisenberg's Uncertainty Principle:

According to classical mechanics a particle occupies a definite place in space and possesses a definite momentum. If the position and momentum of a particle is known at any instant of time, it is possible to calculate its position and momentum at any later instant of time. The path of the particle could be traced. This concept breaks down in quantum mechanics leading to Heisenberg's Uncertainty Principle.

Heisenberg's Uncertainty Principle states that "It is impossible to measure simultaneously both the position and momentum of a particle accurately. If we make an effort to measure very accurately the position of a particle, it leads to large uncertainty in the measurement of momentum and vice versa".

If Δx and ΔP_x are the uncertainties in the measurement of position and momentum of the particle then the uncertainty can be written as

$$\Delta x \, . \, \Delta P_x \geq (h/4\pi)$$

In any simultaneous determination of the position and momentum of the particle, the product of the corresponding uncertainties inherently present in the measurement is equal to or greater than h/4π.

Similarly 1) $\Delta E . \Delta t \geq h/4\pi$ 2) $\Delta L . \Delta\theta \geq h/4\pi$

Significance of Heisenberg's Uncertainty Principle:

Heisenberg's Uncertainty Principle asserts that it is impossible to measure simultaneously both the position and momentum of a particle accurately. If we make an effort to measure very accurately the position of a particle, it leads to large uncertainty in the measurement of momentum and vice versa. Therefore one should think only of the probability of finding the particle at a certain position or of the probable value for the momentum of the particle.

Application of Uncertainty Principle:

Impossibility of existence of electrons in the atomic nucleus:

The energy of a particle is given by

$$E=\frac{1}{2}mv^2=\frac{p^2}{2m} \qquad (1)$$

Heisenberg's uncertainty principle states that

$$\Delta x\,.\,\Delta P_x \geq \frac{h}{4\pi} \rightarrow (4)$$

The diameter of the nucleus is of the order 10^{-14}m. If an electron is to exist inside the nucleus, the uncertainty in its position Δx must not exceed 10^{-14}m.

i.e. $\Delta x \leq 10^{-14}$m

The minimum uncertainty in the momentum

$$(\Delta P_x)_{\min} \geq \frac{h}{4\pi\,(\Delta x)_{\max}} \geq \frac{6.63\times10^{-34}}{4\pi\times10^{-14}} \geq 0.5\times10^{-20}\text{ kg. m/s}$$

By considering minimum uncertainty in the momentum of the electron

i.e., $(\Delta P_x)_{\min} \geq 0.5\times10^{-20}$ kg.m/s = p → (2)

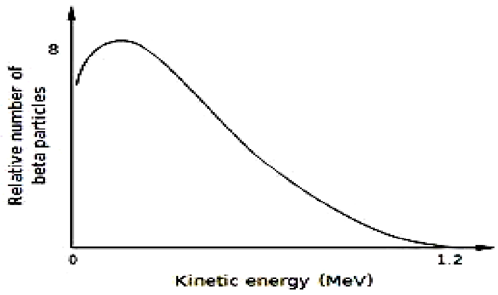
Consider eqn (1)

$$E=\frac{p^2}{2m}=\frac{(0.5\times10^{-20})^2}{2\times9.1\times10^{-31}}=1.373\times10^{-11} = 85\text{ MeV}$$

Where $m_o = 9.11\times10^{-31}$ kg

If an electron exists in the nucleus its energy must be greater than or equal to 85 MeV.. It is experimentally measured that the beta particles ejected from the nucleus during beta decay have energies of about 3 to 4 MeV. This shows that electrons cannot exist in the nucleus.

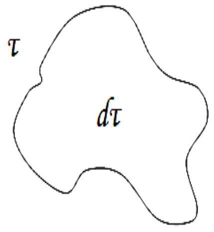
[**Beta decay:** In beta decay process, from the nucleus of an atom, when neutrons are converting into protons in releasing an electron (beta particle) and an antineutrino. When proton is converted into a neutron in releasing a positron (beta particle) and a neutrino. In both the processes energy sharing is statistical in nature. When beta particles carry maximum energy neutrino's carries minimum energy and vice-versa. In all other processes energy sharing is in between maximum and minimum energies. The maximum energy carried by the beta particle is called as the end point energy (E_{max}).



Wave Function:

A physical situation in quantum mechanics is represented by a function called wave function. It is denoted by 'ψ'. It accounts for the wave like properties of particles. Wave function is obtained by solving Schrodinger equation.

Physical significance of wave function:



Probability density: If ψ is the wave function associated with a particle, then $|\psi|^2$ is the probability of finding a particle in unit volume. If 'τ' is the volume in which the particle is present but where it is exactly present is not known. Then the probability of finding a particle in certain elemental volume dτ is given by $|\psi|^2 d\tau$. Thus $|\psi|^2$ is called probability density. The probability of finding an event is real and positive quantity. In the case of complex wave functions, the probability density is $|\psi|^2 = \psi * \psi$ where ψ^* is Complex conjugate of ψ.

Normalization:

The probability of finding a particle having wave function 'ψ' in a volume 'dτ' is '$|\psi|^2 d\tau$'. If it is certain that the particle is present in finite volume 'τ', then

$$\int_0^\tau |\psi|^2 d\tau = 1$$

If we are not certain that the particle is present in finite volume, then

$$\int_{-\infty}^{\infty} |\psi|^2 d\tau = 1$$

In some cases $\int |\psi|^2 d\tau \neq 1$ and involves constant.

The process of integrating the square of the wave function within a suitable limits and equating it to unity the value of the constant involved in the wave function is estimated. The constant value is substituted in the wave function. This process is called as normalization. The wave function with constant value included is called as the normalized wave function and the value of constant is called normalization factor.

Time independent Schrodinger wave equation

Consider a particle of mass ‘m’ moving with velocity ‘v’. The de-Broglie wavelength ‘λ’ is

$$\lambda = \frac{h}{m\text{v}} = \frac{h}{P} \rightarrow (1)$$ Where ‘mv’ is the momentum of the particle.

The wave eqn is

$$\psi = Ae^{i(kx-\omega t)} \rightarrow (2)$$

Where ‘A’ is a constant and ‘ω’ is the angular frequency of the wave.

Differentiating equation (2) with respect to ‘t’ twice

$$\frac{d^2\psi}{dt^2} = -A\omega^2 e^{i(kx-\omega t)} = -\omega^2\psi \quad \rightarrow (3)$$

The equation of a travelling wave is

$$\frac{d^2y}{dx^2} = \frac{1}{\text{v}^2}\frac{d^2y}{dt^2}$$

Where ‘y’ is the displacement and ‘v’ is the velocity.

Similarly for the de-Broglie wave associated with the particle

$$\frac{d^2\psi}{dx^2} = \frac{1}{\text{v}^2}\frac{d^2\psi}{dt^2} \rightarrow (4)$$

where ‘ψ’ is the displacement at time ‘t’.

From eqns (3) & (4)

$$\frac{d^2\psi}{dx^2} = -\frac{\omega^2}{\mathrm{v}^2}\psi$$

But ω = 2πυ and v =υ λ where ‘υ’ is the frequency and ‘λ’ is the wavelength.

$$\frac{d^2\psi}{dx^2} = -\frac{4\pi^2}{\lambda^2}\psi \text{ or } \frac{1}{\lambda^2} = -\frac{1}{4\pi^2\psi}\frac{d^2\psi}{dx^2} \rightarrow (5)$$

$$K.E = \frac{1}{2}m\mathrm{v}^2 = \frac{m^2\mathrm{v}^2}{2m} = \frac{P^2}{2m} \rightarrow (6)$$

$$= \frac{h^2}{2m\lambda^2} \rightarrow (7)$$

Using eqn (5)

$$K.E = \frac{h^2}{2m}\left(-\frac{1}{4\pi^2\psi}\right)\frac{d^2\psi}{dx^2} = -\frac{h^2}{8\pi^2 m\psi}\frac{d^2\psi}{dx^2} \rightarrow (8)$$

Total Energy E = K.E + P.E

$$E = -\frac{h^2}{8\pi^2 m\psi}\frac{d^2\psi}{dx^2} + V$$

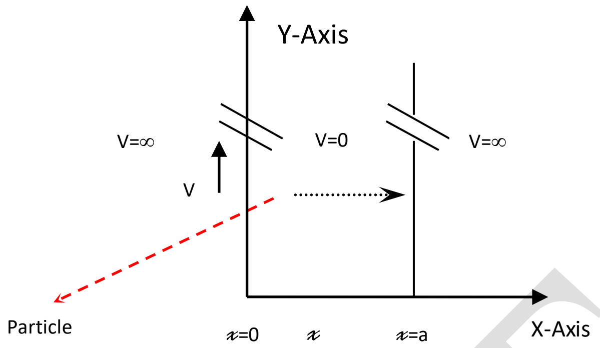
$$E - V = -\frac{h^2}{8\pi^2 m\psi}\frac{d^2\psi}{dx^2}$$

$$\frac{d^2\psi}{dx^2} = -\frac{8\pi^2 m}{h^2}(E - V)\psi$$

$$\boxed{\frac{d^2\psi}{dx^2} + \frac{8\pi^2 m}{h^2}(E - V)\psi = 0}$$ This is the time independent Schrodinger wave equation.

Application of Schrodinger wave equation:

Energy Eigen values of a particle in one dimensional, infinite potential well (potential well of infinite depth) or of a particle in a box



Consider a particle of a mass 'm' free to move in one dimension along positive x-direction between x=0 to x=a. The potential energy outside this region is infinite and within the region is zero. The particle is in bound state. Such a configuration of potential in space is called infinite potential well. It is also called particle in a box. The Schrödinger equation outside the well is

$$\frac{d^2\psi}{dx^2}+\frac{8\pi^2 m}{h^2}(E-\infty)\psi=0 \rightarrow (1) \qquad \because V=\infty$$

For outside, the equation holds good if ψ = 0 & |ψ|² = 0. That is particle cannot be found outside the well and also at the walls

The Schrodinger's equation inside the well is:

$$\frac{d^2\psi}{dx^2}+\frac{8\pi^2 m}{h^2}E\psi=0 \rightarrow (2) \qquad \because V=0$$

Let $\frac{8\pi^2 m}{h^2}E=k^2 \quad \rightarrow (3)$

$$\frac{d^2\psi}{dx^2}+k^2\psi=0$$

The solution of this equation is:

$$\psi = C\cos kx + D\sin kx \rightarrow (4)$$

at $x=0 \rightarrow \psi=0$

$0 = C\cos 0 + D\sin 0$

$\therefore\ C = 0$

Also $x = a \rightarrow \psi = 0$

$0 = C\cos ka + D\sin ka$

But $C = 0$

$\therefore D\sin ka = 0 \quad \longrightarrow \quad (5)$

$D \neq 0$ (because the wave concept vanishes)

i.e. $ka = n\pi$ where $n = 0, 1, 2, 3, 4\ldots$ (Quantum number)

$k = \frac{n\pi}{a} \rightarrow (6)$

sub eqn (5) and (6) in (4)

$$\psi_n = D\sin\frac{n\pi}{a}x \rightarrow (7)$$

This gives permitted wave functions.

The Energy Eigen value given by

Substitute equation (6) in (3)

$$\frac{8\pi^2 m}{h^2}E = k^2 = \frac{n^2\pi^2}{a^2}$$

$$E = \frac{n^2h^2}{8ma^2}$$

For n = 0 is not acceptable inside the well because $\psi_n = 0$. It means that the electron is not present inside the well which is not true. Thus the lowest energy value for $n = 1$ is called zero point energy value or ground state energy.

i.e. $E_{\text{zero-point}} = \frac{h^2}{8ma^2}$

The states for which n >1 are called exited states.

To find out the value of D, normalization of the wave function is to be done.

i.e. $\int_0^a \left|\psi_n^2\right| dx = 1 \rightarrow (8)$

using the values of ψ_n from eqn (7)

$$\int_0^a D^2 \sin^2 \frac{n\pi}{a} x dx = 1$$

$$D^2 \int_0^a \left[\frac{1-\cos(2n\pi/a)x}{2}\right] dx = 1 \qquad \because \sin^2\theta = \left(\frac{1-\cos 2\theta}{2}\right)$$

$$\frac{D^2}{2}\left[\int_0^a dx - \int_0^a \cos\frac{2n\pi}{a} x dx\right] = 1$$

$$\frac{D^2}{2}\left[x - \frac{a}{2n\pi}\sin\frac{2n\pi}{a}x\right]_0^a = 1$$

$$\frac{D^2}{2}[a-0] = 1$$

$$\frac{D^2}{2}a = 1$$

$$D = \sqrt{\frac{2}{a}}$$

Substitute D in equation (7)

the normalized wave functions of a particle in one dimensional infinite potential well is:

$$\psi_n = \sqrt{\frac{2}{a}} \sin\frac{n\pi}{a} x \rightarrow (9)$$

<u>Wave functions, probability densities and energy levels for particle in an infinite potential well</u>:

Let us consider the most probable location of the particle in the well and its energies for first three cases.

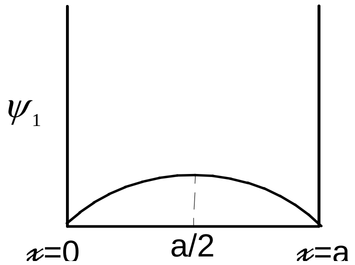
<u>Case I → n=1</u> It is the ground state and the particle is normally present in this state.

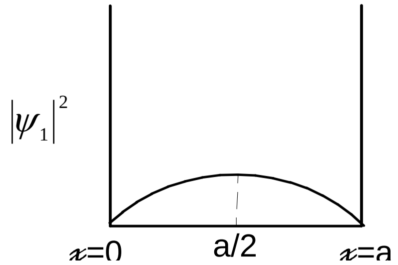
The Eigen function is

$\psi_1 = \sqrt{\frac{2}{a}} Sin\frac{\pi}{a} x \quad \because$ from eqn (7)

$\psi_1 = 0$ for $x = 0$ and $x = a$

But ψ_1 is maximum when $x = a/2$.





The plots of ψ_1 versus x and $|\psi_1|^2$ verses x are shown in the above figure.

$|\psi_1|^2 = 0$ for $x = 0$ and $x = a$ and it is maximum for $x = a/2$. i.e. in ground state the particle cannot be found at the walls, but the probability of finding it is maximum in the middle.

The energy of the particle at the ground state is

$$E_1 = \frac{h^2}{8ma^2} = E_0$$

<u>Case II → n=2</u>

In the first excited state the Eigen function of this state is

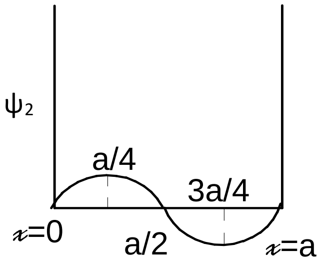
$$\psi_2 = \sqrt{\frac{2}{a}} Sin \frac{2\pi}{a} x$$

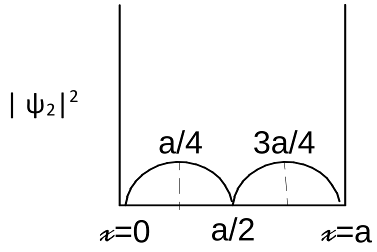
$\psi_2 = 0$ for the values $x = 0, a/2, a$.

Also ψ_2 is maximum for the values $x = a/4$ and $3a/4$.

These are represented in the graphs.

$|\psi_2|^2 = 0$ at $x = 0, a/2, a$, i.e. particle cannot be found either at the walls or at the centre. $|\psi_2|^2 = \text{maximum} \;\; for \;\; x = \frac{a}{4}, \; x = \frac{3a}{4}$





The energy of the particle in the first excited state is $E_2 = 4E_0$.

Case III → n=3

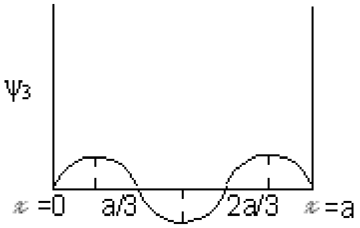
In the second excited state,

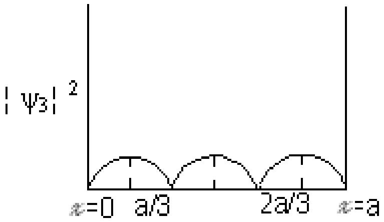
$$\psi_3 = \sqrt{\frac{2}{a}} Sin \frac{3\pi}{a} x$$

$\psi_3 = 0$, for x = 0, a/3, 2a/3 and a.

ψ_3 is maximum for x = a/6, a/2, 5a/6.

These are represented in the graphs.





$|\psi_3|^2 = 0$ for x = 0, a/3, 2a/3 and a. $|\psi_3|^2 = \text{maximum}$ *for* $x = \frac{a}{6}, x = \frac{a}{2}, x = \frac{5a}{6}$

The energy of the particle in the second excited state is $E_3 = 9\ E_0$.

LASERS

The word Laser stands for **L**ight **A**mplification by **S**timulated **E**mission of **R**adiation. It is a device which amplifies light. It has properties like Coherence, Unidirectional, Monochromatic, Focus ability, etc.

Interaction of an electromagnetic wave with matter leads to transition of an atom or a molecule from one energy state to another. If the transition is from lower state to higher state it absorbs the incident energy. If the transition is from higher state to lower state it emits a part of its energy.

Emission or Absorption takes through quantum of energy called photons. $h\nu$ is called quantum energy or photon energy.

$h = 6.626\times10^{-34}$ Joules Second is Planck's constant and 'ν' is the frequency.

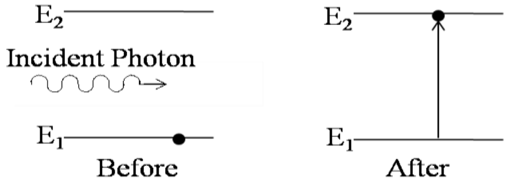
If ΔE is the difference between the two energy levels,

Then $\Delta E = (E_2 - E_1)$ Joule

According to Max Planck, $\Delta E = h\nu = (E_2 - E_1)$

$\nu = (E_2 - E_1)/h$ Hz.

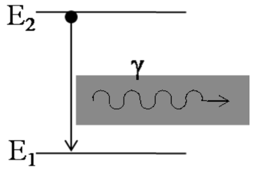
Three types of interactions, which are possible, are as follows:



1)*Induced Absorption:*

Induced absorption is the absorption of an incident photon by system as a result of which the system is elevated from a lower energy state to a higher state, wherein the difference in energy of the two states is the energy of the photon.

Consider the system having two energy states E_1 and E_2, $E_2 > E_1$. When a photon of energy $h\nu$ is incident on an atom at level E_1, the atom goes to a higher energy level by absorbing the energy.



When an atom is at ground level (E_1), if an electromagnetic wave of frequency ν is applied to the atom, there is possibility of getting excited to higher level (E_2). The incident photon is absorbed. It is represented as

$$\text{Atom} + \text{Photon} \rightarrow \text{Atom}^*$$

2) Spontaneous Emission: The emission of a photon by the transition of a system from a higher energy state to a lower energy state without the aid of an external energy is called spontaneous emission.

Let 'E_1' and 'E_2' be two energy levels in a material, such that $E_2>E_1$. E_1 is ground level and E_2 is the higher level. $h\nu=E_2-E_1$ is the difference in the energy. The atom at higher level (E_2) is more unstable as compared to that at lower level (E_1).

The life time of an atom is less in the excited state, In spontaneous emission atom emits the photon without the aid of any external energy. It is called spontaneous emission. The process is represented as

$$\text{Atom}^* \rightarrow \text{Atom} + \text{Photon}$$

The photons emitted in spontaneous emission may not have same direction and phase similarities. It is incoherent.

Ex: Glowing electric bulbs, Candle flame etc.

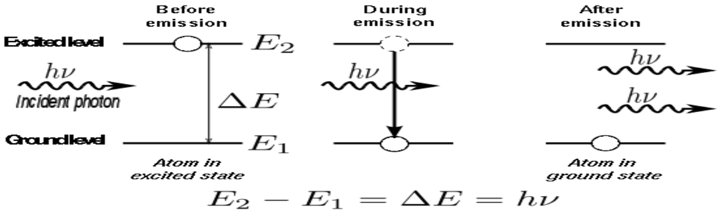
3) Stimulated Emission:

Stimulated emission is the emission of a photon by a system under the influence of a passing photon of just the right energy due to which the system transits from a higher energy state to a lower energy state.

The photon thus emitted is called stimulated photon and will have the same phase, energy and direction of movement as that of the passing photon called the stimulation photon.

Initially the atom is at higher level E_2. The incident photon of energy hν forces the atom to get de-excited from higher level E_2 to lower level E_1.

i.e. $h\nu=E_2–E_1$ is the change in energy.



The incident photon stimulates the excited atom to emit a photon of exactly the same energy as that of the incident photons. The emitted two photons have same phase, frequency, direction and polarization with the incident photon and results in coherent beam of radiation. This kind of action is responsible for lasing action.

$$\text{Atom}^{*} + \text{Photon} \rightarrow \text{Atom} + (\text{Photon} + \text{Photon})$$

Expression for energy density in terms of Einstein's Coefficients

Consider two energy levels E_1 and E_2 of a system of atoms with N_1 and N_2 are population of energy levels respectively.

Let U_v be the energy density of incident beam of radiation of frequency γ. Let us consider the absorption and two emission process

1) Induced absorption:

Induced absorption is the absorption of an incident photon by system as a result of which the system is elevated from a lower energy state to a higher state.

The rate of absorption is proportional to N_1U_v

Rate of absorption = $B_{12}N_1U_v$ …………….. (1)

Where 'B_{12}' is the proportionality constant called Einstein Coefficient of induced absorption.

2) Spontaneous emission:

The emission of a photon by the transition of a system from a higher energy state to a lower energy state without the aid of an external energy is called spontaneous emission.

Spontaneous emission depends on N_2 and independent of energy density.

The rate of spontaneous emission = $A_{21}N_2$ …………….(2)

Where 'A_{21}' is called proportionality constant called Einstein coefficient of spontaneous emission.

3) Stimulated emission:

Stimulated emission is the emission of a photon by a system under the influence of a passing photon of just the right energy due to which the system transits from a higher energy state to a lower energy state

The rate of stimulated emission is directly proportional to N_2U_γ.

The rate of stimulated emission = $B_{21}N_2U_\nu$ …………….. (3)

Where 'B_{21}' is the proportionality constant called Einstein's Coefficient of stimulated emission.

At thermal equilibrium,

Rate of absorption = (Rate of spontaneous emission + Rate of stimulated emission)

$$B_{12}N_1U_\nu = A_{21}N_2 + B_{21}N_2U_\nu$$

$$U_\nu (B_{12}N_1 - B_{21}N_2) = A_{21}N_2$$

$$U_\nu = \frac{A_{21}N_2}{(B_{12}N_1 - B_{21}N_2)}$$

$$\text{i.e. } U_\nu = \frac{A_{21}}{B_{21}}\left[\frac{N_2}{\left(\frac{B_{12}}{B_{21}}N_1 - N_2\right)}\right]$$

$$U_\nu = \frac{A_{21}}{B_{21}}\left[\frac{1}{\left(\frac{B_{12}N_1}{B_{21}N_2}\right) - 1}\right] \qquad \rightarrow (4)$$

By Boltzmann's law, $N_2 = N_1 e^{-\left(\frac{E_2 - E_1}{KT}\right)} = N_1\, e^{-h\nu/KT}$

i.e., $N_1/N_2 = e^{h\nu/KT}$

Eqn. (4) becomes $$U_\upsilon = \frac{A_{21}}{B_{21}}\left[\frac{1}{\left(\frac{B_{12}}{B_{21}}e^{\left(\frac{h\upsilon}{kT}\right)} - 1\right)}\right] \qquad \rightarrow (5)$$

By Planck's law, $$U_\upsilon = \frac{8\pi h\upsilon^3}{c^3}\left[\frac{1}{\left(e^{\left(\frac{h\upsilon}{kT}\right)} - 1\right)}\right] \qquad \rightarrow (6)$$

Comparing equation (5) & (6)

$\frac{A_{21}}{B_{21}} = 8\pi h\nu^3/c^3$ & $\frac{B_{12}}{B_{21}} = 1$ i.e. $B_{12} = B_{21}$

The probability of induced adsorption is equal to the stimulated emission.

Therefore $A_{12}$is written as A and B_{12}, B_{21} written as B.

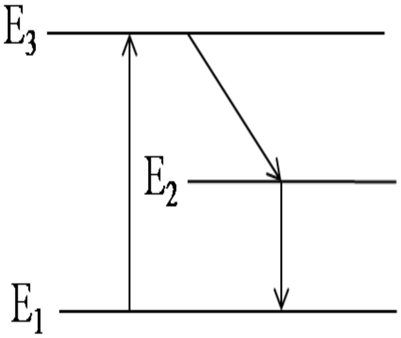
Equation (5) becomes

$$U_\upsilon = \frac{A}{B}\left[\frac{1}{\left(e^{\left(\frac{h\upsilon}{kT}\right)} - 1\right)}\right]$$

Above equation is the expression for energy density

Condition for laser action:

1) ***Meta Stable State:***
It is the special type of excited state where in the life time of atom is more than the normal excited state.
This state plays an important role in lasing action. In metastable state, atoms stay of the order of 10^{-3} to 10^{-2} second. In normal excited state other than metastable atom stay of order of 10^{-8} to10^{-9} seconds. It is possible to achieve population inversion condition in certain system which possesses a metastable state.



2) ***Population Inversion:*** It is the state of the system at which the population of a higher energy level is greater than that of the lower energy level.

Let E_1, E_2, E_3 be the energy levels of the system $E_3 > E_2 > E_1$. E_2 is the metastable state of the system. Atoms get excited from the state E_1 to E_3 by means of external source and stay there for short time. These atoms undergo spontaneous transitions to E_2 and E_1. The atoms at the state E_2 stay for longer time. A stage is reached in which the number of atoms at state E_2 is more than the number of atoms at E_1 which is known as population inversion.

Requisites of a Laser System:

1) **The pumping process:**

It is the process of supplying energy to the medium in order to transfer it to the state of population inversion is known as pumping process

Optical Pumping: It is the process of exciting atoms from lower energy level to higher energy level by using high intensity light or by operating flash tube as an external source called optical pumping.

Electrical pumping: It is the process of exciting atoms from lower energy level to higher energy level by using dc power supply as an external source called electrical pumping.

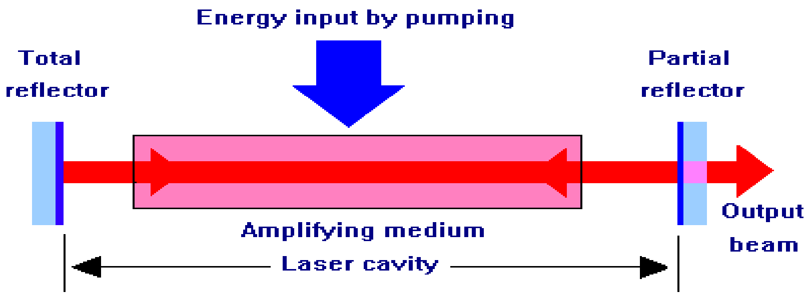
2) **Active medium:**

It is a medium which supports population inversion and promotes stimulated emission leading to light amplification

Active centers: In a medium consisting of different species of atoms only small fraction of the atoms of a particular type are responsible for stimulated emission and consequent light amplification they are known as Active centers

3) **Laser cavity**.

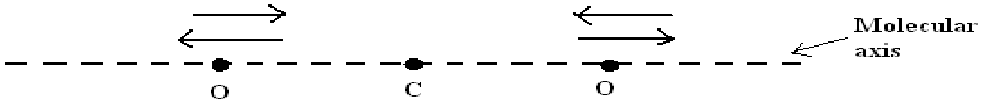
An active medium bounded between two mirrors is called as a laser cavity.



Vibrational modes of CO2 molecule:

A carbn dioxide molecule has two oxygen atoms between which there is a carbon atom. It has 3 different modes of vibration.

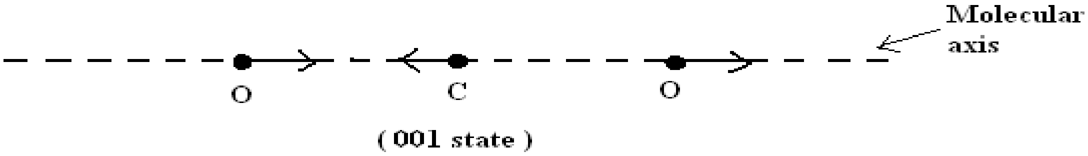
1. **Symmetric stretching mode** : In this mode, carbon atom is stationary and the oxygen atoms oscillate to and fro along the molecular axis. This state is referred as (100) state.



(100 State)

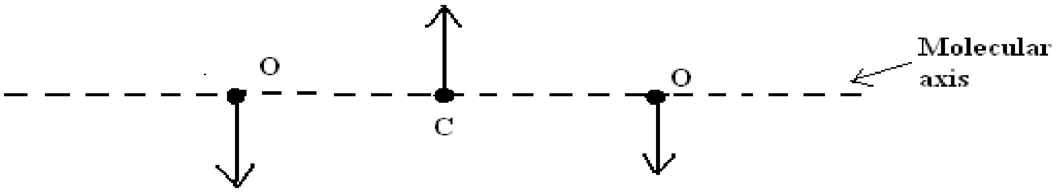
Symmetric stretching mode

2. **Asymmetric stretching mode**: In this mode, both the oxygen atoms moves in one direction while the carbon atom moves in opposite direction along the molecular axis. This state is referred as (001) state.



(001 state)
Asymmetric stretching mode

3. **Bending mode**: In this mode, the carbon atom and oxygen atoms moves perpendicular to molecular axis in the opposite direction. This state is referred to as (010)state.

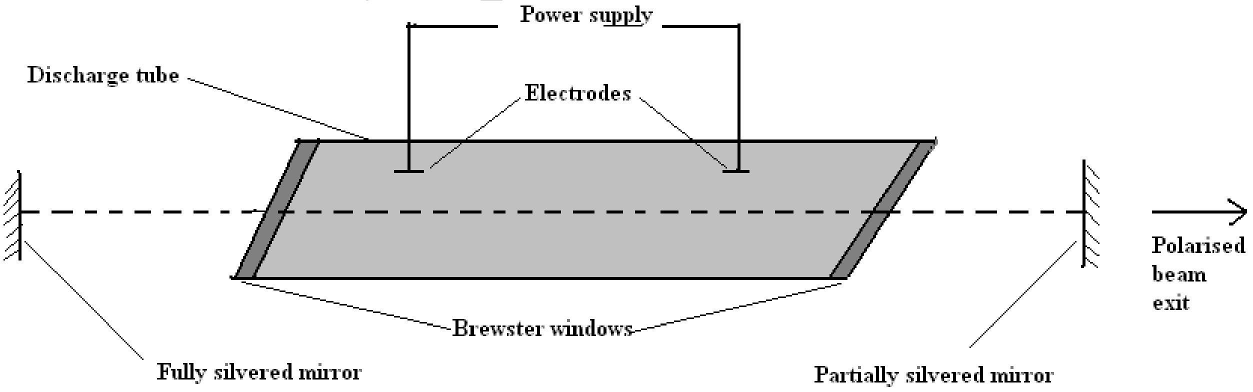


(010 state)

Bending mode

CO_2 LASER: It was devised by C.K.N Patel in 1924. CO_2 laser is molecular gas laser which operates in the IR region involving a set of vibrational – rotational transitions. It is a four level laser producing both continuous and pulsed laser.

Construction:

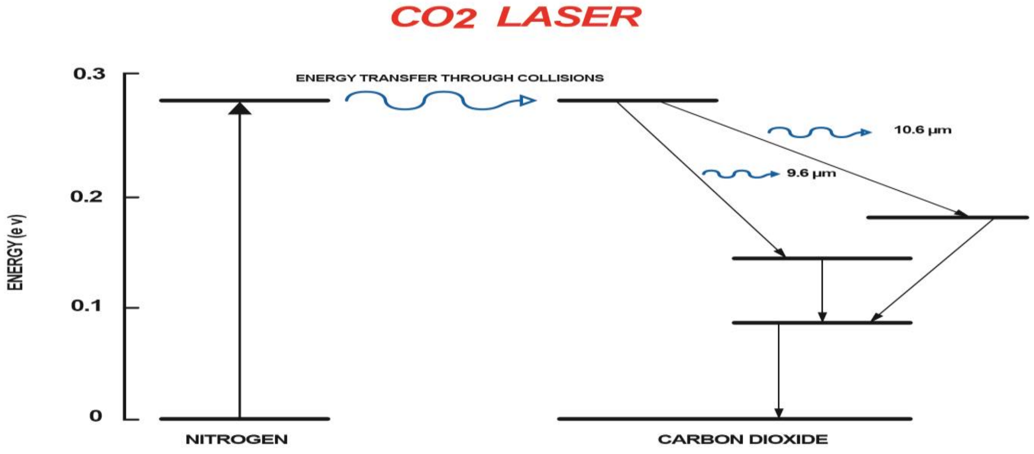


CARBON-DIOXIDE LASER

- It consists of discharge tube (quartz) of diameter 2.5cm and length of 5m.

- The tube is filled with a mixture of CO_2, N_2 and He gas in the ratio 1:2:3.
- Sometimes water vapour is added because during discharge CO_2 molecule breaks up into CO and O. The water vapour additives help in deoxidizing CO to CO_2.
- Brewster window made up of flat quartz are sealed to the tube at both of its ends to give polarized light.
- The tube has got two parallel mirrors. One is partially silvered and the other is fully silvered to function as laser cavity

Working:



CO_2 laser Energy level diagram

- When An electric field is applied electrons rendered free from atoms, collide with N_2 & CO_2 molecules in their path towards positive electrodes due to which N_2 atoms are excited to the higher energy level v_1 .
- Likewise it happens to the CO_2 molecule. This is collision of first kind

$$e_1 + N_2 \rightarrow e_2 + N_2^*$$

$$e_1 + CO_2 \rightarrow e_2 + Co_2^*$$

Where, e_1 and e_2 are the energies of electrons before and after collisions.

- Let the ground state, (010) state, (020) state,(100) state and (001) are represented as E_1,E_2,E_3,E_4 and E_5 levels respectively
- Because of matching energy levels, v = 1 state of N_2 is equal to (001) state of CO_2 , N_2 molecule in the metastable state collide with the CO_2 in the ground state and transfer of energy takes place from N_2 to CO_2 . As a result of which CO_2 molecule moved to (001) state where as the N_2 molecule moved to ground state. This is the collision of second kind.

$$N_2^* + CO_2 \rightarrow N_2 + CO_2^*$$

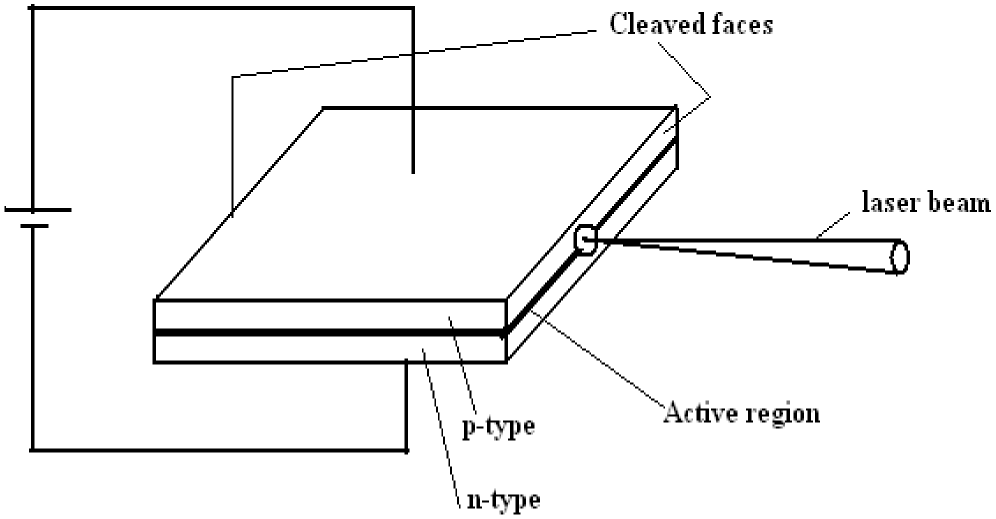
Where, CO_2 and CO_2^* are the energies of CO_2 in ground state and excited states.

- Because of the resonant transfer of energy, the population inversion is achieved in (001) state with respect to (100) and (020)
- The transition from E_5 to E_4 levels gives wavelength of 10.6μm (in far IR region)
- The transitions from E_5 to E_3 level gives wavelength of 9.6 μm (in far IR region)
- Following these transitions the CO_2 molecules in E_4 and E_3 collide with the ground state CO_2 molecules (because of the matching energy levels) and arrive at E_2 state.
- The molecules in the E_2 state collide with He and water vapour molecules, so that come down to the ground state.
- The cycle of operation gives both continuous and pulsed laser.

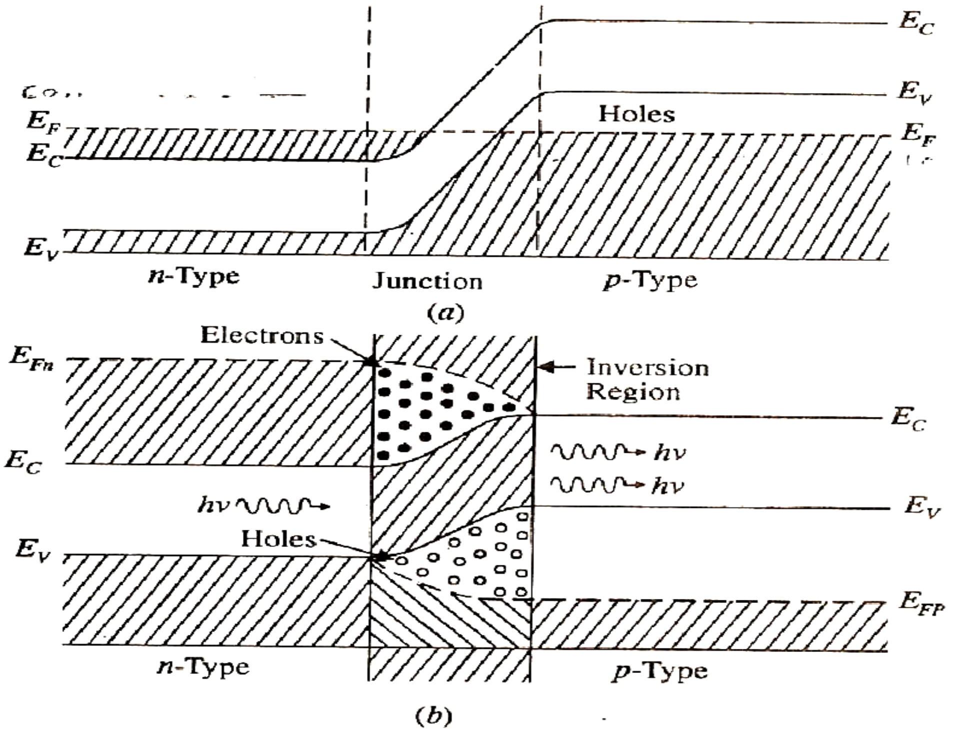
Gallium-Arsenide Laser Semiconductor laser:

A Semiconductor diode laser is one in which the active medium is formulated by semiconducting materials.

Construction: Gallium-Arsenide Laser is a single crystal of GaAs consists of heavily doped n-type and p-type. The diode is very small size with sides of the order of 1mm. The width of the junction varies from 1-100μm. The top and bottom surfaces are metalized and Ohmic contacts are provided for external connection. The front and rear faces are polished. The polished faces functions as the resonant cavity. The other two faces are roughened to prevent lasing action in that direction.



Working:



Energy level diagram of p-n junction Ga-As semi conductor diode laser
(a) Before biasing (b) After biasing

- The energy band diagram of heavily doped p-n junction is as shown. At thermal equilibrium the Fermi level is uniform.
- Because of very high doping on **n- side**, the Fermi level is pushed in to the conduction band and electrons occupy the portions of the conduction band that lies below the Fermi level and on **p-side**, the Fermi level lies within the valence band and holes occupy the portions of the valence band that lies above the Fermi level.

- A suitable forward bias is applied to overcome the potential barrier. As a result electrons from n-region and holes from p-region injected into the junction.

- The current begins to flow following which there will be a region in junction in which the population inversion can be achieved.

- Initially concentration of electrons in the energy levels at the bottom of the conduction band will be less than that of energy levels at top of valence band. So that the recombination of electrons and holes result only in spontaneous emission.

- When the current exceeds the threshold value, population inversion is achieved in the active region which is formulated in the junction.

- At this stage the photons emitted by spontaneous emission triggers stimulated emission, over a large number of recombination leading to build up laser.

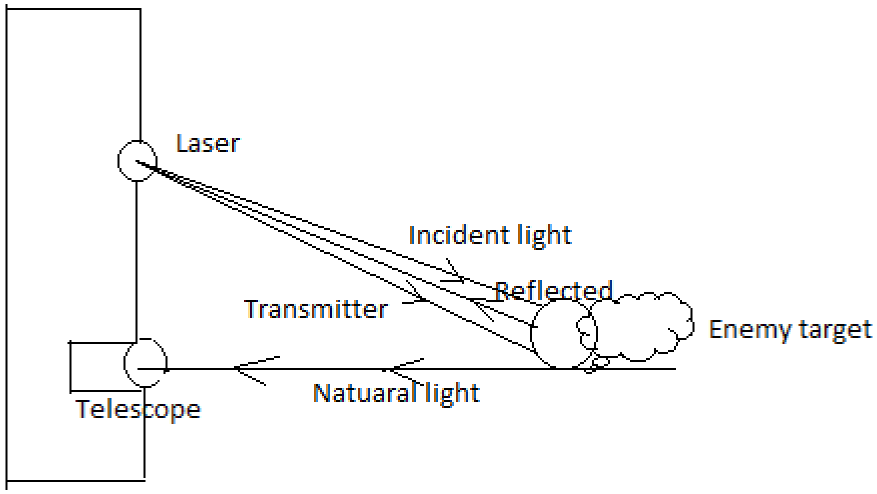
- Since the energy gap of GaAs is 1.4eV, the wavelength of emitted light is 8400 A^o .

Properties of laser:

1. ***Coherence*****:** The emitted radiation after getting triggered is in phase with the incident radiation.
2. ***Monochromaticity*****:** The laser beam is highly monochromatic than any other radiations.
3. ***Unidirectionality*****:** Laser beam travels in only one direction. It can travel long distance without spreading.
4. ***Focusability*****:** A laser beam can be focused to an extremely fine spot.

Applications of laser:

1) Laser range finder in defense

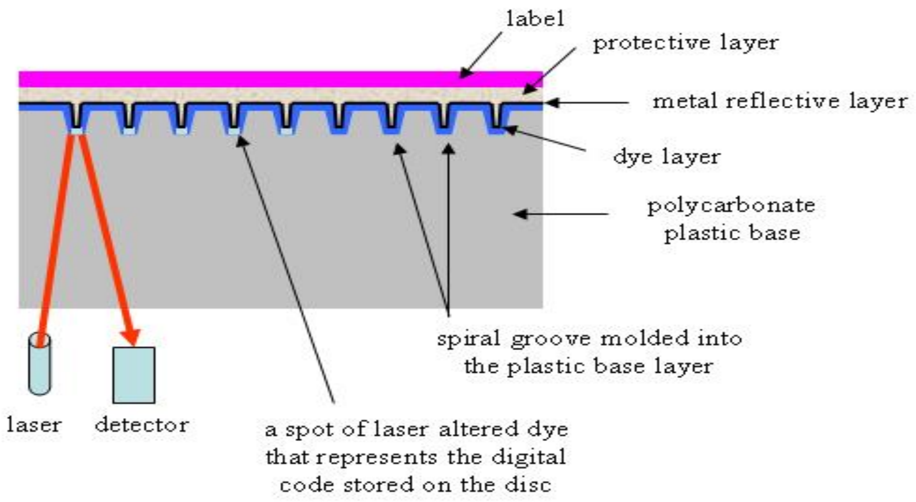


A high power pulsed laser (Nd-YAG) beam is directed towards the enemy target from the transmitter. The beam bounces back from the surface of the target as a reflection. A part of the reflected beam is received as a signal by the receiver. The unwanted noise signal will be filtered by the optical filter and pure signal is amplified by the photomultiplier in the receiver. The range finders high speed clock measures the exact time of incident and reflection of the pulse and then convert it in to distance.

2) Laser in compact disc for data storage

A compact disc (CD) is a thin circular disc of about 12cm diameter and its shining side is made of metal and plastic consisting of 3 layers.

At the bottom is a layer of polycarbonate which is tough and brittle plastic. above that is a layer of aluminum coated with plastic and lacquer (dye layer).The information is created in digital form in the CD by using laser.



The laser beam burns and etches bumps on its surface at certain specifies intervals on a track. These bumps are called pits. The presence of bump in a fixed length in the track indicates a zero.An unburnt space in a specific length of the track remains flat on the length called land and represent one. Thus laser beam can store the information by burning some lengths and leaving some lengths unburned in the binary language.

While reading CD , the laser beam scans the track. As it is bounced, it follows the pattern of pits and lands. A photocell converts these into electric pulses in the same order. In turn electronic circuit generates zeros and ones. A decoder converts these binary numbers into a changing pattern of electric currents and the analog form which is made of for required application. A CD can store a data around 700 MB.

MODULE 5

MATERIAL SCIENCE

Review of Classical free electron theory

Assumptions of classical free electron theory:

- The metal consists of free electrons; the current flowing through a metal is due drift velocity of electrons.
- The free electrons are treated as equivalent to gas molecules and they are assumed to obey the laws of kinetic theory of gases. In the absence of the field, the energy associated with each electron at a temperature T is given by 3/2 kT, where k is a Boltzmann constant. It is related to the kinetic energy.

$$3/2\ kT = \frac{1}{2} m v_{th}^2$$

Where v_{th} is the thermal velocity same as root mean square velocity.

- The electric potential due to the ionic cores is assumed to be constant throughout the metal
- The attraction between the lattice ions and electrons, the repulsion between the electrons are neglected.

Failures of classical free electron theory:

Electrical and thermal conductivities can be explained from classical free electron theory. It fails to account the facts such as specific heat, temperature dependence of conductivity and dependence of electrical conductivity on electron concentration.

i) Specific heat: The molar specific heat of a gas at constant volume is

$$C_v = \frac{3}{2} R$$

As per the classical free electron theory, free electrons in a metal are expected to behave just as gas molecules. Thus the above equation holds good equally well for the free electrons also.

But experimentally it was found that, the contribution to the specific heat of a metal by its conduction electrons was $C_V = 10^{-4} RT$

which is for lower than the expected value. Also according to the theory the specific heat is independent of temperature whereas experimentally specific heat is proportional to temperature.

ii)Temperature dependence of electrical conductivity:

Experimentally, electrical conductivity σ is inversely proportional to the temperature T.

i.e. $\sigma_{exp} \ \alpha \ \frac{1}{T} \rightarrow (1)$

According to the assumptions of classical free electron theory

or $\sigma \propto \frac{1}{\sqrt{T}}$ →(2)

From equations (1) & (2) it is clear that the experimental value is not agreeing with the theory.

iii)Dependence of electrical conductivity on electron concentration:

According to classical free electron the theory

$\sigma = \frac{ne^2\tau}{m}$ i.e., $\sigma \propto n$, where n is the electron concentration,

Consider copper and aluminum. Their electrical conductivities are $5.88\times10^7/\Omega m$ and $3.65x10^7/\Omega m$. The electron concentrations for copper and aluminum are $8.45\times10^{28}/m^3$and $18.06x10^{28}/m^3$. Hence the classical free electron theory fails to explain the dependence of σ on electron concentration.

Experimental results:

Metals	Electron concentration(n)	conductivity (σ)
Copper	$8.45\times10^{28}/m^3$	$5.88x10^7/\Omega m$
Aluminium	$18.06x10^{28}/m^3$	$3.65x10^7/ \Omega m$

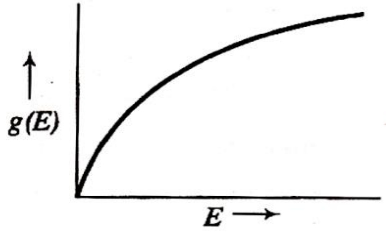
Quantum free electron theory:

Assumptions of quantum free electron theory:

- The energy values of the conduction electrons are quantized. The allowed energy values are realized in terms of a set of energy values.
- The distribution of electrons in the various allowed energy levels occur as per Pauli's exclusion principle.
- The electrons travel with a constant potential inside the metal but confined within its boundaries.
- The attraction between the electrons and the lattice ions and the repulsion between the electrons themselves are ignored.

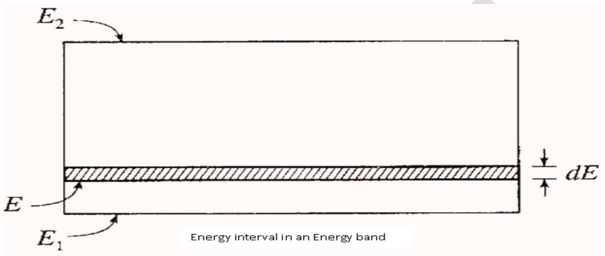
Density of states g(E):

Density of states is defined as the number of allowed energy states per unit energy range per unit volume in the valance band of a material. It is denoted as g(E).
A graph of g(E) verses E is shown below.



DENSITY OF STATES

Consider an energy band spread in an energy interval between E_1 and E_2. Below E_1 and above E_2 there are energy gaps. g(E) represents the density of states at E. As dE is small, it is assumed that g(E) is constant between E and E+dE. The density of states in range E and (E+dE) is denoted by g(E)dE.



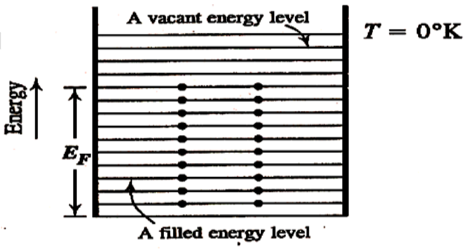
Energy interval in an Energy band

i.e. $$g(E)dE = \left[\frac{8\sqrt{2\pi m^{\frac{3}{2}}}}{h^3}\right] E^{\frac{1}{2}}\, dE$$

It is clear g(E) is proportional to $\sqrt{E}$ in the interval dE

Fermi energy **and Fermi level:**

The energy of electrons corresponding to the highest occupied energy level at absolute $0°K$ is called Fermi energy and the energy level is called Fermi level.



FERMI ENERGY

Fermi-Dirac statistics:

- At thermal equilibrium the free electrons acquire energy obeying a statistical rule called Fermi-Dirac statistics.

- Fermi-Dirac statistics is applicable to the assembly of particles which obeys Pauli's exclusion principle; they must also be identical particles of spin ½ and are indistinguishable.
- Fermi-Dirac statistics permits to evaluation of probability of finding the electrons occupying energy levels in a certain energy range and is given by a function called Fermi factor f(E).

Fermi factor:

Fermi factor is the probability of occupation of a given energy state by the electrons in a material at thermal equilibrium.

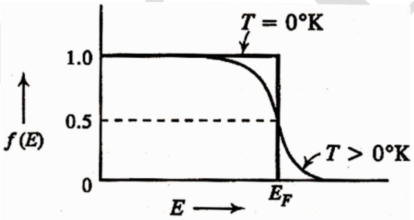
The probability f(E) that a given energy state with energy E is occupied by the electrons at a steady temperature T is given by

$$f(E)=\frac{1}{e^{\frac{(E-E_F)}{kT}}+1}$$

f(E) is called the Fermi factor.

Dependence of Fermi factor with temperature and energy:

The dependence of Fermi factor on temperature and energy is as shown in the figure.



VARIATION OF f(E) WITH E

i) **Probability of occupation for $E<E_F$ at T=0K:**

When T=0K and $E<E_F$

$$f(E)=\frac{1}{e^{-\infty}+1}=\frac{1}{0+1}=1$$

The probability of occupation of energy state is 100%

f(E)=1 for $E<E_F$.

f(E)=1 means the energy level is certainly occupied and $E<E_F$ applies to all energy levels below E_F. Therefore at T=0K all the energy levels below the Fermi level are occupied.

ii) Probability of occupation for E>E_F at T=0K:

When T=0K and E>E_F

$$f(E) = \frac{1}{e^{\infty}+1} = \frac{1}{\infty} = 0$$

The probability of occupation of energy state is 0%

f(E)=0 for E>E_F

∴ At T=0K, all the energy levels above Fermi levels are unoccupied. Hence at T=0K the variation of f(E) for different energy values, becomes a step function as shown in the above figure.

ii) The probability of occupation at ordinary temperature(for E≈E_F at T>0K)

At ordinary temperatures though the value of probability remains 1, for E<E_F it starts reducing from 1 for values of E close to but lesser than E_F as in the figure.

The values of f(E)becomes ½ at E=E_F

This is because for E=E_F

$$e^{(E-E_f)/kT} = e^0 = 1$$

$$\therefore f(E) = \frac{1}{e^{(E-E_f)/kT}+1} = \frac{1}{1+1} = \frac{1}{2}$$

The probability of occupation of energy state is 50%

Further for E>E_F the probability value falls off to zero rapidly.

Hence, the Fermi energy is the most probable or the average energy of the electrons across which the energy transitions occur at temperature above zero degree absolute.

<u>Expression for Fermi Energy at Absolute Zero temperature (T=0K)</u>

According to Fermi-Dirac statistics the distribution of electrons among the various allowed energy levels is given by

N(E) dE = g(E) f(E) dE …………….(1)

The number of free electrons distributed /unit volume of the material up to Fermi level is given by

$$n = \int_0^{E_{F_0}} N(E)dE = \int_0^{E_{F_0}} g(E)\, f(E)\, dE$$

But for the energy levels from E=0 to E= E_{F_0}

f (E) = 1 at T= 0K

n = $\int_0^{E_{F_0}} g(E)\ dE$

g(E) is given by $g(E)= \frac{8\pi\sqrt{2}\, m^{3/2}}{h^3}\ E^{1/2}\ dE$

where ‘m’ is the mass of electron, ‘h’ is the planck’s constant

$$n = \frac{8\pi\sqrt{2}\, m^{3/2}}{h^3} \int_0^{E_{F_0}} E^{1/2}\ dE = \frac{8\pi\sqrt{2}\, m^{3/2}}{h^3} \left[\frac{2}{3} E^{\frac{3}{2}}\right]_0^{E_{F0}}$$

$$n = \frac{8\pi\sqrt{2}\, m^{3/2}}{h^3} \times \frac{2}{3} (E_{F_0})^{3/2} = \left[\frac{8\pi\ 2^{3/2} m^{3/2}}{h^3}\right]\left[\frac{\pi}{3}\right](E_{F_0})^{3/2}$$

$$(E_{F_0})^{3/2} = \frac{h^3}{(8m)^{3/2}}\left(\frac{3n}{\pi}\right)$$

$$E_{F_0} = \left(\frac{h^2}{8m}\right)\left(\frac{3n}{\pi}\right)^{2/3}$$

$$E_{F_0} = \left(\frac{h^2}{8m}\right)\left(\frac{3}{\pi}\right)^{2/3} n^{2/3} = 5.85 \times 10^{-38}\, n^{2/3}\ J$$

***<u>Merits of Quantum free electron theory</u>*:**

Quantum free electron theory has successfully explained following observed experimental facts where as the classical free electron theory failed.

i) Specific heat:

According to quantum free electron theory, it is only those electron that are occupying energy levels close to E_F , which are capable of absorbing the heat energy to get excited to higher energy levels .

Hence only a small percentage of the conduction electrons are capable of receiving the thermal energy input, thus the specific heat value becomes very small for the metal.

According to quantum free electron theory, it can be shown

$$C_V = \left(\frac{2k}{E_F}\right)RT$$

Considering E_F= 5 eV$\left(\frac{2k}{E_F}\right)\approx 10^{-4}$

$\therefore C_V = 10^{-4}\, RT$ which is close to experimental value.

ii) Temperature depends on electrical conductivity:

Experimentally electrical conductivity $\sigma_{\text{expt}}\, \alpha \frac{1}{T}$

According to quantum free electron theory electrical conductivity is given by

$$\sigma_{QFT} = \frac{ne^2\lambda}{m^* \mathrm{v}_F} \quad \text{------------------ (1)}$$

Where m^* is called effective mass of an electron.

According to quantum free electron theory E_F and V_F are independent of temperature. The dependence of λ & T is as follows

Conduction electrons are scattered by the vibrating ions of the lattice. As the temperature increases the vibrational cross sectional areas (πr^2) increases and hence mean free path decreases.

The mean free path of the electrons is given by

$\therefore \lambda\, \alpha \frac{1}{\pi r^2}$ Where 'r' is the amplitude of vibrations of lattice ions

Considering the facts the energy of vibrating body is proportional to the square of amplitude and the energy of ions is due to the thermal energy.

The thermal energy is proportional to the temperature 'T'.

We can write $r^2\, \alpha\, T$

$\therefore \lambda\, \alpha\, 1/T$ substituting for λ in eqn (1)

Hence $\sigma_{QFT}\, \alpha\, \frac{1}{T}$

Thus $\sigma_{QFT}\, \alpha\, \frac{1}{T}$ is correctly explained by quantum free electron theory.

iii) Electrical conductivity and electron concentration:

Aluminium and gallium which have three free electrons per atom have lower electrical conductivity than that of copper and silver, which have only one free electron per atom.

As per quantum free electron the electrical conductivity is

$$\sigma_{QFT} = \frac{ne^2\lambda}{m^* v_F}$$

i.e $\sigma \; \alpha \; (n)$ and $\sigma \; \alpha \; (\lambda/v_F)$

The value of (n) for aluminium is 2.13 times higher than that of copper. But the value of (λ/v_f)for copper is about 3.73 times higher than that of aluminium. Thus the conductivity of copper exceeds that of aluminium.

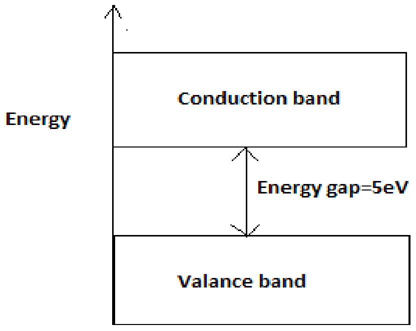
Physics of Semiconductors:

Semiconductors are materials whose electrical conductivity lies between conductors and insulators. The conductivity of semiconductors is in the order of 10^4 to 10^{-4}mho/m.

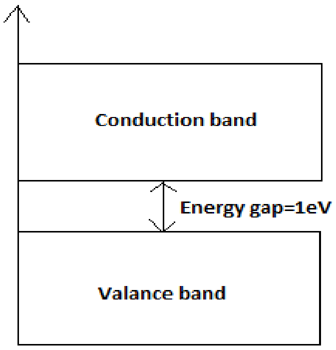
Valance band: Energy level occupied by valance electrons is called valance band. Valance band may be fully filled or partially filled at O k

Conduction band: Energy level occupied by conduction electrons (free electrons) is called conduction band. The conduction band is an empty band at O k

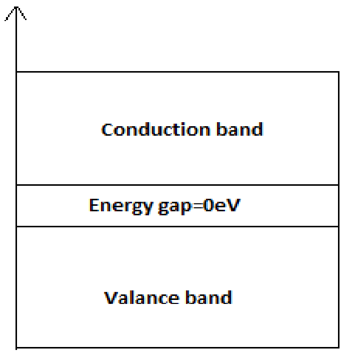
Forbidden energy gap: The separation between valance band and conduction band is known as forbidden energy gap. The energy gap is measured in electron volt (eV).



Insulator



Semiconductor



Conductor

Fermi level in intrinsic semiconductor

Fermi level represent the average energy of conduction electrons.

For a semiconductor at T=0K all the energy levels in the valance band are completely filled and all energy levels in conduction band are empty. At room temperature due to thermal excitation some of the electron will jump to bottom of conduction and return soon to the energy levels in top of valance band. Therefore conduction electrons are distributed between top of valance band and bottom of conduction band. Hence the average energy of electrons will be equal to (1/2) Eg. Thus Fermi level lies at the middle of the forbidden energy gap for an intrinsic semiconductor.

Concentration of electrons and holes in intrinsic semiconductors:

The number of electrons in conduction band per unit volume of a material is called electron concentration and is given by

$$N_e = \frac{4\sqrt{2}}{h^3} (\pi\, m_e^* KT)^{3/2}\, e^{\frac{Ef - \mathrm{Eg}}{KT}}$$

Where, m^* is the effective mass of electrons

k is the Boltzmann constant

E_f is the Fermi level

E_g is the energy gap

h is Planck's constant

Similarly, the number of holes in valance band per unit volume of a material is called holes concentration and is given by

$$N_h = \frac{4\sqrt{2}}{h^3} (\pi\, m_h^* KT)^{3/2}\, e^{\frac{-Ef}{KT}}$$

m_h^* is the effective mass of holes

Relation between fermi energy and energy gap in an intrinsic semiconductor

For intrinsic semiconductor the number of electrons in conduction band is equal to the number of holes in valance band

i.e Ne = Nh

$$N_e = \frac{4\sqrt{2}}{h^3} (\pi\, m_e^* KT)^{3/2}\, e^{\frac{Ef - \mathrm{Eg}}{KT}} \quad = \quad N_h = \frac{4\sqrt{2}}{h^3} (\pi\, m_h^* KT)^{3/2}\, e^{\frac{-Ef}{KT}}$$

$$\left((m^*{}_e)^{3/2}\right) e^{\frac{E_F - E_g}{KT}} = \left((m^*{}_h)^{3/2}\right) e^{\frac{-E_F}{KT}}$$

$$e^{\frac{2E_F - E_g}{KT}} = \left(\frac{m^*{}_e}{m^*{}_h}\right)^{3/2}$$

By taking natural log on both sides

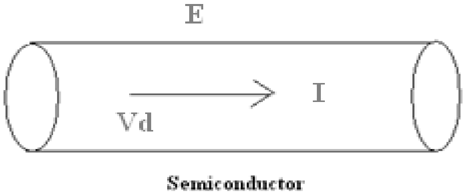
$$\frac{2E_F - E_g}{KT} = \frac{\mathbf{3}}{\mathbf{2}} \boldsymbol{ln}\left(\frac{m^*_e}{m^*_h}\right)$$

$$\textbf{Or } E_F = \frac{3}{4} KT \ln\left(\frac{m^*_e}{m^*_h} + \frac{\boldsymbol{E_g}}{\mathbf{2}} \right)$$

Since $m^*_e = m^*_h$ and $\ln 1 = 0$

$$E_F = \frac{E_g}{2}$$

Expression for electrical conductivity of semiconducting materials:



Consider a semiconductor of area of cross section A, carrying a current I. Let V_d be the drift velocity of the charge carriers.

In the presence of electric field, both electrons and the holes contribute to the conductivity of the semiconductor. Let us consider first the conductivity in a semiconductor is due to the flow of electrons only.

The current due to flow of electrons is $I = N e A V_d$ ………… (1)

$N = N_e$ = concentration of charge carriers (electrons)

The current density of electrons is given by $J_e = N_e e V_d$ …..(2) [J= I/A]

The quantity $\frac{Vd}{E}$ is a constant, specific for a particular type of charge carriers and is called mobility of charge carriers. i.e., $\frac{Vd}{E} = \mu$

$V_d = \mu_e E$ in the above equation

$$J_e = N_e e \mu_e E \quad \text{……..(3)}$$

But current density is given by $J_e = \sigma_e E$ ………(4)

$$\sigma_e = \frac{J}{E}$$

$$\sigma_e = N_e e \mu_e \quad \ldots\ldots\ldots \quad (5)$$

eqn(5) represents the conductivity due to electrons in conduction band.

similarly the conductivity due holes in valance band is given by

$$\sigma_h = N_h e \mu_h \ldots\ldots\ldots\ldots \quad (6)$$

Total conductivity $\sigma = \sigma_e + \sigma_h = e\,(N_e \mu_e + N_h \mu_h)$

$$\sigma = e\,(N_e \mu_n + N_h \mu_h)$$

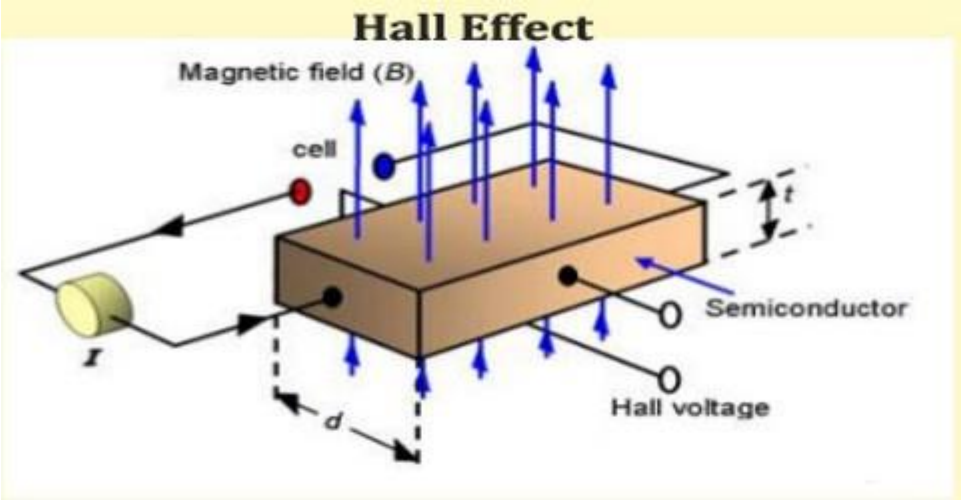
For intinsic semiconductors $N_e = N_h = n_i$

Hence the expression for conductivity in intrinsic semiconductor is given by

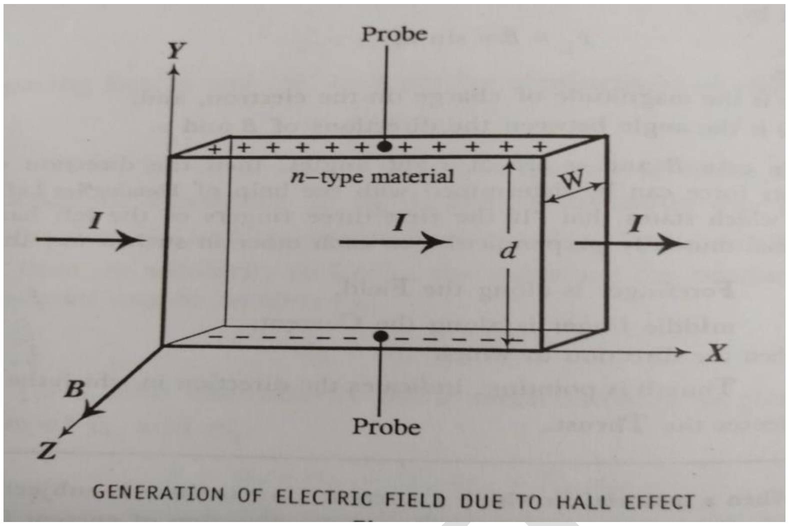
$$\boxed{\boldsymbol{\sigma = n_i e\,(\mu_e + \mu_h)}}$$

Hall Effect:

"When a current carrying conductor / semiconductor subjected to external magnetic field perpendicular to the direction of current. The electric field induced (produced) across the material in a direction perpendicular to both the magnetic field and direction of current flow". This phenomenon is called Hall Effect.



Expression for Hall coefficient and Hall voltage:



GENERATION OF ELECTRIC FIELD DUE TO HALL EFFECT

Consider a rectangular n –type semiconductor material in which current is flowing in +ve x -direction.

Let the magnetic field (B) applied along z-direction.

Therefore, the electrons experience the Lorentz force (F_L) along –ve y-direction given by,

$$F_L = -Bev \quad \ldots\ldots(1)$$

Therefore, the electrons density increases in lower surface and in the upper surface become positively charged due to deficiency of electrons. Hence the potential developed between two surfaces called Hall voltage, which produces an electric field E_H called Hall field (E_H).

The electrons exerts an upward force FH due to the Hall field and is given by,

$$F_H = -eE_H \quad \ldots\ldots..(2)$$

Equating equations (1) and (2)

$$-eE_H = -BeV_d$$

$$E_H = BV_d \quad \ldots\ldots(3) \quad V_d \rightarrow \text{drift velocity.}$$

If "d" is the distance between upper and lower surface then

$$E_H = V_H/d$$

$$V_H = E_H d \qquad \text{(from 3)}$$

$$V_H = BV_d d \quad \ldots\ldots..(4)$$

The current density $J = I / A = neVd = \rho V_d$ ……(5)

Therefore, $\rho = ne$

For the given semiconductor, $J = I/A = \frac{I}{\omega d}$ …… (6)

$\omega \rightarrow$ thickness of the material . therefore $(A = \omega\, d)$

equating (5) & (6) $\rho\, V_d = \frac{I}{\omega d}$

$V_d = \frac{I}{\rho \omega d}$ …… (7)

Substituting eqn (7) in (4)

$V_H = \frac{B\, d\, I}{\rho \omega d} = \frac{B\, I}{\rho \omega}$ ……. (8)

$\rho = \frac{B\, I}{VH\omega}$

Expression for Hall coefficient:

Since, $E_H \propto J\, B$

$E_H = R_H\, J\, B$ ($R_H \rightarrow$ Hall coefficient)

$R_H = E_H / J\, B$ [From eqn (3) and (5)]

$R_H = BV_d / \rho V_d\, B = 1 / \rho = 1 / ne$

$\mathbf{R_H = 1 / ne}$ ……(9)

The above expression represents **Hall coefficient**.

Expression for Hall Voltage:

From (8) $V_H = \frac{B\, I}{\rho \omega} = \frac{RH\, B\, I}{\omega}$ (since, $R_H = \frac{1}{\rho} = \frac{1}{ne}$)

$V_H = \frac{RH\, B\, I}{\omega}$

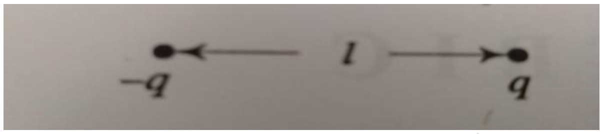
The above expression represents **Hall voltage**.

DIELECTRIC MATERIALS

Dielectric materials are nothing but insulators which have the ability to get electrically polarized with the application of external electric field.

Examples: Glass, rubber, plastic, mica.

Electric dipole:



Two equal and opposite charges separated by a distance is called dipole.

The product of one of the charge and distance of separation between them is called dipole moment (μ).

$$\mu = q\, l$$

Polarization: The displacement of charges in the atoms or molecules of a dielectric under the action of an applied electric field loads to the development of dipole moment is called polarization.

Types of dielectric materials:

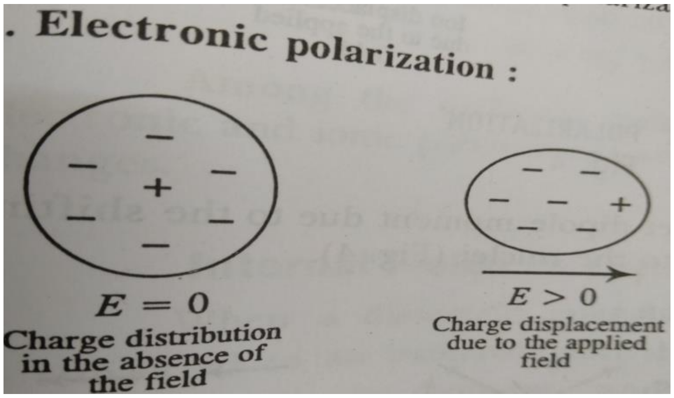
Polar dielectrics: In some dielectric materials, the effective centers of the negative and positive charges in the molecules do not coincide with each other and exhibit permanent dipoles in the absence of electric field are called polar dielectrics. **Example**: H_2O.

Non polar dielectrics: In some dielectric materials, the effective centers of the negative and positive charges in the molecules do not coincide with each other and do not exhibit permanent dipoles in the absence of electric field are called non polar dielectrics. **Examples**: paper, wood, glass.

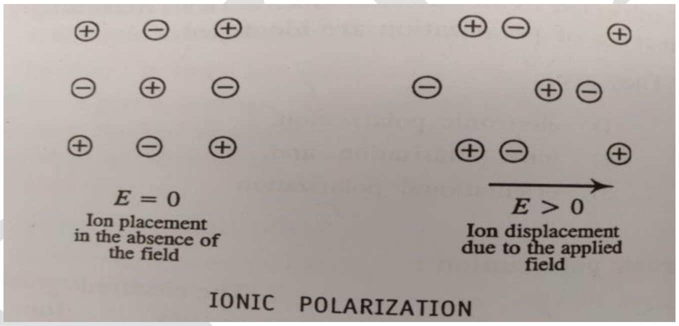
TYPES OF POLARIZATION MECHANISMS:

Electronic Polarization: The electronic polarization occurs due to displacement of the positive and negative charges in a dielectric material with the application of external electric field, which leads to development of dipole moment. The electronic polarizability is given by,

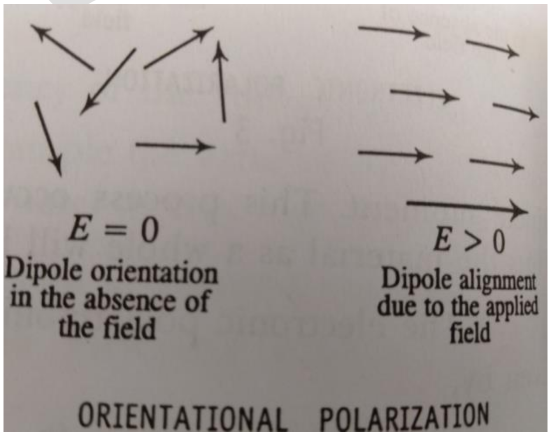
$$\alpha e = \epsilon_0 (\epsilon r - 1) / N .$$



- **Ionic Polarization:** This is kind of polarization occurs in some dielectric materials which have ionic bonds like Nacl. When ionic solids are subjected to an external electric field, the adjacent ions of opposite sign undergo displacement. The distance of separation between the ion pair depends on the location of atoms in the lattice results in the development of dipole moment



- **Oriental Polarization:** This type of polarization occurs in polar dielectric material. (Permanent dipoles).



The orientation of molecules is random in the absence of electric field; therefore net dipole moment in the material is zero. But under the influence of an applied electric field each dipole reorient along the field direction. Thus the material develops an electrical polarization. This type of polarization is temperature dependent. In polar dielectrics the orientation polarizability α_0 is given by $\alpha_0 = \mu^2 / 3KT$

- **Space charge Polarization:** It occurs in multiphase dielectric materials in which change in resistivity between different phases. When such materials subjected to external electric field at high temperatures, the charges get accumulated at the interface leads to change in conductivity across the material.

Relation between Polarization and Dielectric constant:

For isotropic materials, the applied electric field (E), flux density (D) are related by

$$D = \epsilon_0 \epsilon_r E$$

$\epsilon_0 \rightarrow$ dielectric constant of vacuum

$\epsilon \rightarrow$ relative dielectric constant of the material

The relation between dielectric er and polarization p is given by $P = \epsilon_0(\epsilon_r - 1) E$

Internal fields in solids: When dielectric material subjected to an external electric field, each atom develops a dipole moment and acts as electric dipole. Hence the field at any given atom will be the applied electric field and the electric field due to the surrounding dipoles. “The internal field is the electric field that acts at the site of any atom of a solid subjected to an external external electric field and is the resultant of the applied field and the field due to all the surrounding dipoles”.

In one dimension the internal field is given by

$$E_i = E_{applied} + E_{dipoles}$$

$$E_i = E + \left(\frac{\nu}{\epsilon_0}\right)P$$

where $\nu \rightarrow$ internal field constant, $p \rightarrow$ polarization

For, 3D the internal field is called Lorentz field and is given by

$$\mathbf{E_{Lorentz} = E + \frac{P}{3\epsilon_0}}$$

Clausius – Mossotti Equation:

Consider a solid dielectric material with dielectric constant Er. If ‘n’ is the number of atoms/unit volume and ‘μ’ is the dipole moment of the atoms in the material.

Therefore, the dipole moment / unit volume = N μ ………(1)

The field experienced by the atom is an internal field Ei. αe is the electronic polarizability of the atoms.

Then dipole moment / unit volume = $N\alpha_e E_i$

The dipole moment $\mu = \alpha_e E_i$ ………(2)

Dipole moment / unit volume is nothing but polarization (P)

$$P = N \alpha_e E_i$$

$$E_i = \frac{P}{N\alpha e} \quad \text{……….(3)}$$

we have $P = \epsilon_0 (\epsilon_r - 1) E$

$$E = \frac{P}{\epsilon_0 (\epsilon_r - 1)} \quad \text{………(4)}$$

The expression for internal field for 3D material is given by $E_i = E + \frac{P}{3\epsilon_0}$

Substituting (3) and (4) in the above equation

$$\frac{P}{N\alpha e} = \frac{P}{\epsilon_0(\epsilon_r - 1)} + \frac{P}{3\epsilon_0}$$

$$\frac{1}{N\alpha e} = \frac{1}{\epsilon_0(\epsilon_r - 1)} + \frac{1}{3\epsilon_0} = \frac{1}{\epsilon_0}\left[\frac{1}{(\epsilon_r - 1)} + \frac{1}{3}\right]$$

$$\boldsymbol{\frac{\epsilon_r - 1}{\epsilon_r + 2} = \frac{N\alpha e}{3\,\epsilon_0}}$$

This is called Clausius Mossotti equation.

Solid, Liquid and Gaseous Dielectrics: Use of appropriate dielectric materials suitable for their working so that they have both long life, reliability and trouble free working abilities, the dielectrics are broadly classified into:

- **Solid insulating materials:** Solid dielectrics are available extremely large in number like, Mica, porcelain, glass, plastic. Paper is hygroscopic, hence it is dried and dipped with mineral oil, and high density papers are preferred in dc and energy storage capacitors.
- **Liquid insulating materials:** Mainly liquid insulating materials are used in transformers switched and circuit breakers. During the working conditions, the windings in an electrical device get heated due to eddy heating and joule heating. Liquid insulators allow the winding to cool faster by conveying heat efficiency to the surroundings. Examples: transformer oil, silicon fluids, viscous Vaseline, fluro organic fluids etc.
- **Gaseous insulating materials:** Gases are good insulators and work as heat transforming media. Example: Air, Nitrogen, inert gases, hydrogen, $CO2$,. In general, air provides insulation between the overhead transmission power lines etc.

Application of dielectrics in transformers:

A transformer consist of two insulated conducting coils wound on a core, the core is also insulated. In high voltage transformers further insulation is required to be provide between individual windings in the coils and also between the core and the coils. Hence the size of the transformers increases and also the size increases with operational ac frequency. Normally the insulation is provided by using paper, mica or cloth.

"If there is air gap between the windings, as the permittivity of air is less, ionization of air occurs at high voltage leads to excessive heating which damages the insulation. This effect is called corona".

Generally mica is used to guard against corona, when the operating voltage is up to 3KV. However, if the operating voltage is above 3KV up to 100KV, a kind of oil called transformer oil is used between individual windings and the core. Apart from guarding against corona, the oil helps to keep the transformer cool. It remains stable at high temperature. For large size transformers, water circulation round the body of the transformer is provided to cool the transformer.