FLUID MECHANICS NOTES

MODULE-1

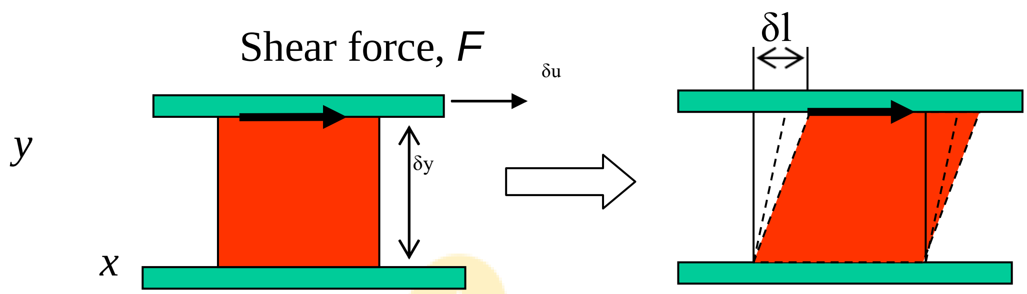
- **Fluids & Their Properties**
- **Fluid Pressure and Its Measurements**

Module -1: Fluids & Their Properties:
Concept of fluid, Systems of units. Properties of fluid; Mass density, Specific weight, Specific gravity, Specific volume, Viscosity, Cohesion, Adhesion, Surface tension& Capillarity. Fluid as a continuum, Newton's law of viscosity (theory &problems).Capillary rise in a vertical tube and between two plane surfaces (theory & problems). Vapor pressure of liquid, compressibility and bulk modulus, capillarity, surface tension, pressure inside a water droplet, pressure inside a soap bubble and liquid jet. Numerical problems

1.0 INTRODUCTION: In general matter can be distinguished by the physical forms known as solid, liquid, and gas. The liquid and gaseous phases are usually combined and given a common name of fluid. Solids differ from fluids on account of their molecular structure (spacing of molecules and ease with which they can move). The intermolecular forces are large in a solid, smaller in a liquid and extremely small in gas.

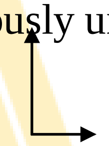
Fluid mechanics is the study of fluids at rest or in motion. It has traditionally been applied in such area as the design of pumps, compressor, design of dam and canal, design of piping and ducting in chemical plants, the aerodynamics of airplanes and automobiles. In recent years fluid mechanics is truly a 'high-tech' discipline and many exciting areas have been developed like the aerodynamics of multistory buildings, fluid mechanics of atmosphere, sports, and micro fluids.

1.1 DEFINITION OF FLUID: A *fluid* is a substance which deforms continuously under the action of shearing forces, however small they may be. Conversely, it follows that: If a fluid is at rest, there can be no shearing forces acting and, therefore, all forces in the fluid must be perpendicular to the planes upon which they act.



Fluid deforms continuously under the action of a shear force

$$\tau_{yx} = \frac{dF_x}{dA_y} = f\,(\text{Deformation Rate})$$



Shear stress in a moving fluid:

Although there can be no shear stress in a fluid at rest, shear stresses are developed when the fluid is in motion, if the particles of the fluid move relative to each other so that they have different velocities, causing the original shape of the fluid to become distorted. If, on the other hand, the velocity of the fluid is same at every point, no shear stresses will be produced, since the fluid particles are at rest relative to each other.

Differences between solids and fluids: The differences between the behaviour of solids and fluids under an applied force are as follows:

i. For a solid, the strain is a function of the applied stress, providing that the elastic limit is not exceeded. For a fluid, the rate of strain is proportional to the applied stress.

ii. The strain in a solid is independent of the time over which the force is applied and, if the elastic limit is not exceeded, the deformation disappears when the force is removed. A fluid continues to flow as long as the force is applied and will not recover its original form when the force is removed.

Differences between liquids and gases:

Although liquids and gases both share the common characteristics of fluids, they have many distinctive characteristics of their own. A liquid is difficult to compress and, for many purposes, may be regarded as incompressible. A given mass of liquid occupies a fixed volume, irrespective of the size or shape of its container, and a free surface is formed if the volume of the container is greater than that of the liquid.

A gas is comparatively easy to compress (Fig.1). Changes of volume with pressure are large, cannot normally be neglected and are related to changes of temperature. A given mass of gas has no fixed volume and will expand continuously unless restrained by a containing vessel. It will completely fill any vessel in which it is placed and, therefore, does not form a free surface.

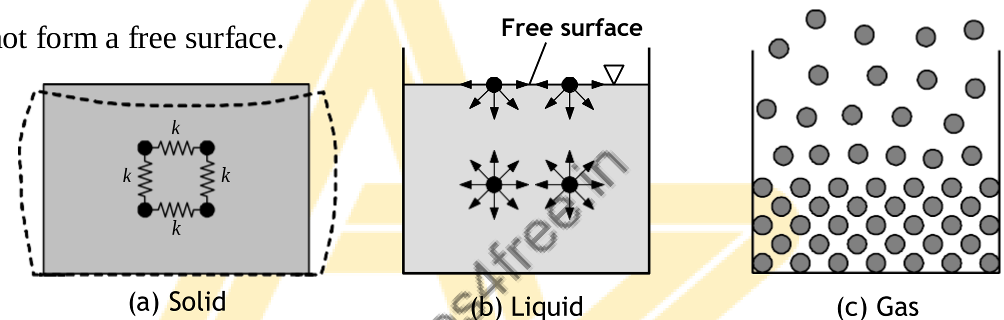


Fig.1 Comparison of Solid, Liquid and Gas

1.2 Systems of Units:

The official international system of units (System International Units). Strong efforts are underway for its universal adoption as the exclusive system for all engineering and science, but older systems, particularly the CGS and FPS engineering gravitational systems are still in use and probably will be around for some time. The chemical engineer finds many physiochemical data given in CGS units; that many calculations are most conveniently made in fps units; and that SI units are increasingly encountered in science and engineering. Thus it becomes necessary to be expert in the use of all three systems.

SI system:

Primary quantities:

Quantity	***Unit***
Mass in Kilogram	kg
Length in Meter	m
Time in Second	s or as sec
Temperature in Kelvin	K
Mole	mol

Derived quantities:

Quantity	***Unit***
Force in Newton (1 N = 1 kg.m/s^2)	N
Pressure in Pascal (1 Pa = 1 N/m^2)	N/m^2
Work, energy in Joule (1 J = 1 N.m)	J
Power in Watt (1 W = 1 J/s)	W

CGS Units:
The older centimeter-gram-second (cgs) system has the following units for derived quantities:

Quantity	***Unit***
Force in dyne (1 dyn = 1 g.cm/s^2)	dyn
Work, energy in erg (1 erg = 1 dyn.cm = 1 x 10^{-7} J)	erg
Heat Energy in calorie (1 cal = 4.184 J)	cal

Dimensions: Dimensions of the primary quantities:

Fundamental dimension	***Symbol***
Length	L
Mass	M
Time	t
Temperature	T

Dimensions of derived quantities can be expressed in terms of the fundamental dimensions.

Quantity	*Representative symbol*	*Dimensions*
Angular velocity	ω	t^{-1}
Area	A	L^2
Density	ρ	M/L^3
Force	F	ML/t^2
Kinematic viscosity	ν	L^2/t
Linear velocity	v	L/t

1.3 Properties of fluids:

1.3.1 Mass density or Specific mass (ρ):

Mass density or specific mass is the mass per unit volume of the fluid.

$$\therefore \rho = \frac{Mass}{Volume}$$

$$\rho = \frac{M}{V} \text{ or } \frac{dM}{dV}$$

Unit: kg/m^3

With the increase in temperature volume of fluid increases and hence mass density decreases in case of fluids as the pressure increases volume decreases and hence mass density increases.

1.3.2 Weight density or Specific weight (γ):

Weight density or Specific weight of a fluid is the weight per unit volume.

$$\therefore \gamma = \frac{Weight}{Volume} = \frac{W}{V} \text{ or } \frac{dW}{dV}$$

Unit: N/m^3 or Nm^{-3}.

With increase in temperature volume increases and hence specific weight decreases.

With increases in pressure volume decreases and hence specific weight increases.

Note: Relationship between mass density and weight density:

We have $\gamma = \frac{Weight}{Volume}$

$$\gamma = \frac{mass \ x \ g}{Volume}$$

$$\gamma = \rho \ x \ g$$

1.3.3 Specific gravity or Relative density (S):

It is the ratio of density of the fluid to the density of a standard fluid.

$$S = \frac{\rho_{fluid}}{\rho_{standard\ fluid}}$$

Unit: It is a dimensionless quantity and has no unit.

In case of liquids water at 4°C is considered as standard liquid. $\rho_{water} = 1000$ kg/m^3

1.3.4 Specific volume (∀): It is the volume per unit mass of the fluid.

$$\therefore \quad \forall = \frac{Volume}{mass} = \frac{V}{M} \ or \ \frac{dV}{dM}$$

Unit: m^3/kg

As the temperature increases volume increases and hence specific volume increases. As the pressure increases volume decreases and hence specific volume decreases.

Solved Problems:

Ex.1 Calculate specific weight, mass density, specific volume and specific gravity of a liquid having a volume of $4m^3$ and weighing 29.43 kN. Assume missing data suitably.

$\gamma = ?$

$\rho = ?$

$\forall = ?$

$S = ?$

$V = 4\ m^3$

$W = 29.43\ kN$

$= 29.43\ x10^3\ N$

$$\gamma = \frac{W}{V}$$

$$= \frac{29.43X10^3}{4}$$

$$\gamma = 7357.58\ N/m^3$$

To find ρ - Method 1:

$$W = mg$$

$$29.43 \times 10^3 = m \times 9.81$$

$$m = 3000\ kg$$

$$\therefore \rho = \frac{m}{v} = \frac{3000}{4}$$

$$\rho = 750\ kg/m^3$$

$$i)\ \forall = \frac{V}{M}$$

$$= \frac{4}{3000}$$

$$\forall = 1.33\ x\ 10^{-3} m^3/kg$$

Method 2 :

$$\gamma = \rho\ g$$

$$7357.5 = \rho\ 9.81$$

$$\rho = 750\ kg/m^3$$

$$\rho = \frac{M}{V}$$

$$\forall = \frac{V}{M}$$

$$\forall = \frac{1}{\rho} = \frac{1}{750}$$

$$\forall = 1.33X10^{-3} m^3/kg$$

$$S = \frac{\gamma}{\gamma_{Standard}}$$

$$S = \frac{\rho}{\rho_{Standaard}}$$

$$= \frac{7357.5}{9810} \quad \text{or} \quad S = \frac{750}{1000}$$

$$S = 0.75 \qquad S = 0.75$$

Ex.2 Calculate specific weight, density, specific volume and specific gravity and if one liter of Petrol weighs 6.867N.

$$\gamma = \frac{W}{V}$$

$$V = 1\text{Litre}$$

$$= \frac{6.867}{10^{-3}}$$

$$V = 10^{-3}\,m^3$$

$$W = 6.867N$$

$$\gamma = 6867N / m^3$$

$$S = \frac{\gamma}{\gamma_{Standard}}$$

$$\rho = s\,g$$

$$= \frac{6867}{9810}$$

$$6867 = \rho x\, 9.81$$

$$S = 0.7$$

$$\rho = 700kg / m^3$$

$$\forall = \frac{V}{M}$$

$$M = 6.867 \div 9.81$$

$$= \frac{10^{-3}}{0.7}$$

$$\forall = 1.4x10^{-3}\,m^3 / kg$$

$$M = 0.7\text{ kg}$$

Ex.3 Specific gravity of a liquid is 0.7 Find i) Mass density ii) specific weight. Also find the mass and weight of 10 Liters of liquid.

$$S = \frac{\gamma}{\gamma_{Standard}}$$

$$0.7 = \frac{\gamma}{9810}$$

$$\gamma = 6867 N/m^3$$

$$\gamma = \rho g$$

$$6867 = \rho \times 9.81$$

$$\rho = 700\ kg/m^3$$

S = 0.7
V = ?
ρ = ?
M = ?
W = ?
V = 10 litre
$= 10 \times 10^{-3}\ m^3$

$$S = \frac{\rho}{\rho_{Standard}}$$

$$0.7 = \frac{\rho}{1000}$$

$$\rho = 700 kg/m^3$$

$$\rho = \frac{M}{V}$$

$$700 = \frac{M}{10 \times 10^{-3}}$$

$$M = 7kg$$

1.3.5 Viscosity: Viscosity is the property by virtue of which fluid offers resistance against the flow or shear deformation. In other words, it is the reluctance of the fluid to flow. Viscous force is that force of resistance offered by a layer of fluid for the motion of another layer over it.

In case of liquids, viscosity is due to cohesive force between the molecules of adjacent layers of liquid. In case of gases, molecular activity between adjacent layers is the cause of viscosity.

- **Newton's law of viscosity:**

Let us consider a liquid between the fixed plate and the movable plate at a distance 'Y' apart, 'A' is the contact area (Wetted area) of the movable plate, 'F' is the force required to move the plate with a velocity 'U' According to Newton's law shear stress is proportional to shear strain. (Fig.2)

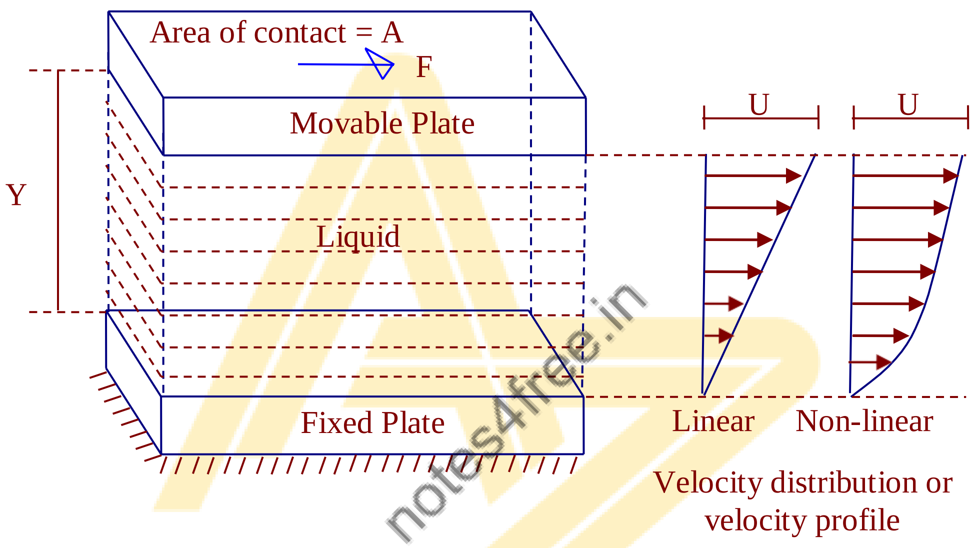


Fig.2 Definition diagram of Liquid viscosity

- $F \alpha A$
- $F \alpha \frac{1}{Y}$
- $F \alpha U$

$$\therefore F \alpha \frac{AU}{Y}$$

$$F = \mu . \frac{AU}{Y}$$

'μ' is the constant of proportionality called Dynamic Viscosity or Absolute Viscosity or Coefficient of Viscosity or Viscosity of the fluid.

$$\frac{F}{A} = \mu . \frac{U}{Y} \longrightarrow \therefore \tau = \mu \frac{U}{Y}$$

'τ' is the force required; Per Unit area called 'Shear Stress'. The above equation is called Newton's law of viscosity.

Velocity gradient or rate of shear strain:

It is the difference in velocity per unit distance between any two layers.

If the velocity profile is linear then velocity gradient is given by $\frac{U}{Y}$. If the velocity profile is non – linear then it is given by $\frac{du}{dy}$.

- Unit of force (F): N.
- Unit of distance between the twp plates (Y): m
- Unit of velocity (U): m/s
- Unit of velocity gradient : $\frac{U}{Y} = \frac{m/s}{m} = /s = s^{-1}$
- Unit of dynamic viscosity (τ): $\tau = \mu . \frac{u}{y}$

$$\mu = \frac{\tau y}{U}$$

$$\Rightarrow \frac{N/m^2 . m}{m/s}$$

$$\mu \Rightarrow \frac{N - sec}{m^2} \text{ or } \mu \Rightarrow P_a - S$$

NOTE: In CGS system unit of dynamic viscosity is $\frac{dyne . S}{Cm^2}$ and is called poise (P).

If the value of μ is given in poise, multiply it by 0.1 to get it in $\frac{NS}{m^2}$.

1 Centipoises = 10^{-2} Poise.

- **Effect of Pressure on Viscosity of fluids:**

  Pressure has very little or no effect on the viscosity of fluids.

- **Effect of Temperature on Viscosity of fluids:**

1. *Effect of temperature on viscosity of liquids:* Viscosity of liquids is due to cohesive force between the molecules of adjacent layers. As the temperature increases cohesive force

decreases and hence viscosity decreases.

2. *Effect of temperature on viscosity of gases:* Viscosity of gases is due to molecular activity between adjacent layers. As the temperature increases molecular activity increases and hence viscosity increases.

- **Kinematics Viscosity:** It is the ratio of dynamic viscosity of the fluid to its mass density.

$$\therefore \text{Kinematic Viscosity} = \frac{\mu}{\rho}$$

Unit of KV:

$$KV \Rightarrow \frac{\mu}{\rho}$$

$$\Rightarrow \frac{NS/m^2}{kg/m^3}$$

$$= \frac{NS}{m^2} \times \frac{m^3}{kg}$$

$$= \left(\frac{kg\,m}{s^2}\right) \times \frac{s}{m^2} \times \frac{m^3}{kg} = m^2/s$$

$F = ma$

$N = Kg.m/s^2$

$$\therefore \text{Kinematic Viscosity} = m^2/s$$

NOTE: Unit of kinematics Viscosity in CGS system is cm^2/s and is called stoke (S)

If the value of KV is given in stoke, multiply it by 10^{-4} to convert it into m^2/s.

The Fig. 3 illustrates how μ changes for different fluids.

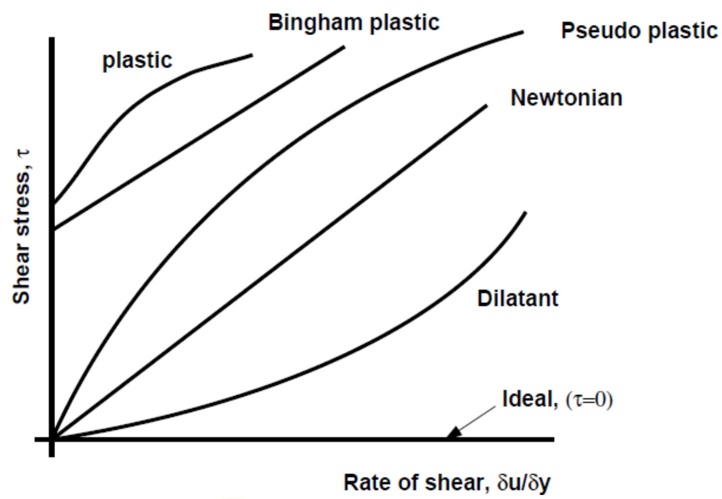
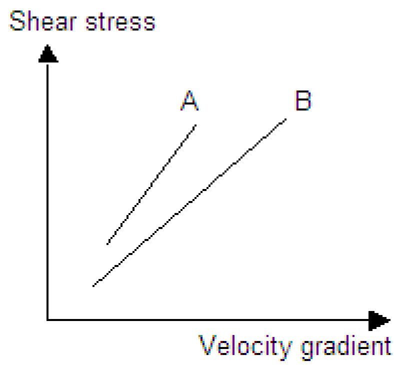


Fig.3 Variation of Viscosity based on Behaviour of Liquids

- Plastic: Shear stress must reach a certain minimum before flow commences.
- Bingham plastic: As with the plastic above a minimum shear stress must be achieved. With this classification n = 1. An example is sewage sludge.
- Pseudo-plastic: No minimum shear stress necessary and the viscosity decreases with rate of shear, e.g. colloidal substances like clay, milk and cement.
- Dilatant substances; Viscosity increases with rate of shear e.g. quicksand.
- Thixotropic substances: Viscosity decreases with length of time shear force is applied e.g. thixotropic jelly paints.
- Rheopectic substances: Viscosity increases with length of time shear force is applied
- Viscoelastic materials: Similar to Newtonian but if there is a sudden large change in shear they behave like plastic

The figure shows the relationship between shear stress and velocity gradient for two fluids, A and B. Comment on the Liquid 'A' and Liquid 'B' ?



Comments: (i) The dynamic viscosity of liquid A > the dynamic viscosity of liquid B
(ii) Both liquids follow Newton's Law of Viscosity

Solved Problems:

1. Viscosity of water is 0.01 poise. Find its kinematics viscosity if specific gravity is 0.998.

Kinematics viscosity = ?

$S = 0.998$

$$S = \frac{\rho}{\rho_{s\tan drad}}$$

$$0.998 = \frac{\rho}{1000}$$

$$\rho = 998\ \mathrm{kg/m^3}$$

$\mu = 0.01P$

$= 0.01 x 0.1$

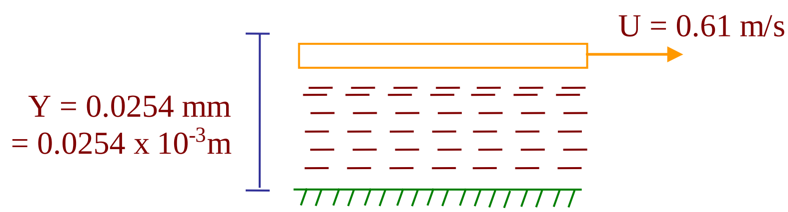
$$\mu = 0.001 \frac{NS}{m^2}$$

$$\therefore \text{Kinmetic Vis cosity} = \frac{\mu}{\rho}$$

$$= \frac{0.001}{998}$$

$$KV = 1 \text{ x } 10^{6}\ m^2/s$$

2. A Plate at a distance 0.0254mm from a fixed plate moves at 0.61m/s and requires a force of 1.962N/m^2 area of plate. Determine dynamic viscosity of liquid between the plates.



$\tau = 1.962 \text{ N / m}^2$

$\mu = ?$

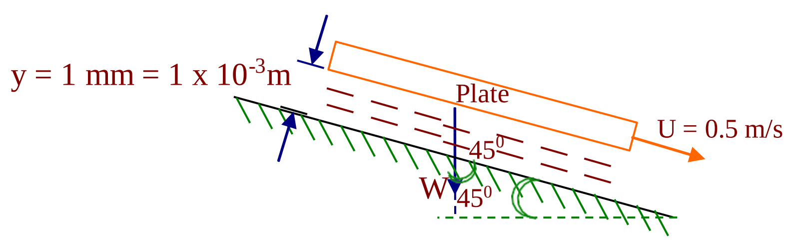
Assuming linear velocity distribution

$$\tau = \mu \frac{U}{Y}$$

$$1.962 = \mu \times \frac{0.61}{0.0254 \times 10^{-3}}$$

$$\mu = 8.17 \times 10^{-5} \frac{NS}{m^2}$$

3. A plate having an area of 1m^2 is dragged down an inclined plane at 45^0 to horizontal with a velocity of 0.5m/s due to its own weight. Three is a cushion of liquid 1mm thick between the inclined plane and the plate. If viscosity of oil is 0.1 PaS find the weight of the plate.



A =1m^2

U = 0.5m/s

Y = 1x10^{-3}m

μ = 0.1NS/m^2

$W = ?$

$F = W \times \cos 45^0$

$= W \times 0.707$

$F = 0.707W$

$$\tau = \frac{F}{A}$$

$$\tau = \frac{0.707W}{1}$$

$\tau = 0.707W N / m^2$

Assuming linear velocity distribution,

$$\tau = \mu . \frac{U}{Y}$$

$$0.707W = 0.1 \times \frac{0.5}{1 \times 10^{-3}}$$

$W = 70.72$ N

4. A flat plate is sliding at a constant velocity of 5 m/s on a large horizontal table. A thin layer of oil (of absolute viscosity = 0.40 N-s/m^2) separates the plate from the table. Calculate the thickness of the oil film (mm) to limit the shear stress in the oil layer to 1 kPa,

Given : τ = 1 kPa = 1000 N/m2; U = 5m/s; μ = 0.4 N-s/m^2

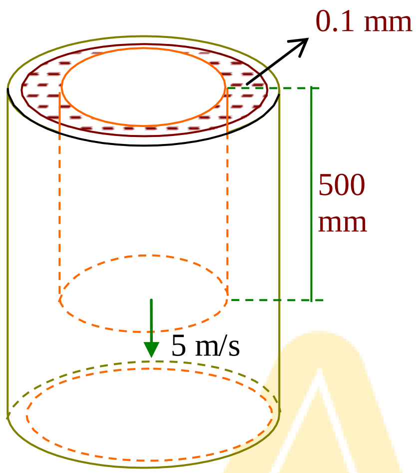
Applying Newton's Viscosity law for the oil film -

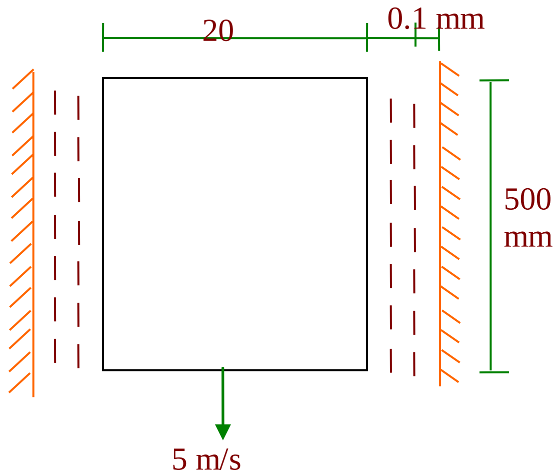
$$\tau = \mu \frac{du}{dy} = \mu \frac{U}{y}$$

$$1000 = 0.4 \frac{5}{y}$$

$$y = 2 \times 10^{-3} = 2 \text{ mm}$$

5. A shaft of ϕ 20mm and mass 15kg slides vertically in a sleeve with a velocity of 5 m/s. The gap between the shaft and the sleeve is 0.1mm and is filled with oil. Calculate the viscosity of oil if the length of the shaft is 500mm.





$D = 20mm = 20 \times 10^{-3} m$

$M = 15$ kg

$W = 15 \times 9.81$

$\underline{W = 147.15N}$

$y = 0.1mm$

$\underline{y = 0.1 \times 10^{-3} mm}$

$U = 5m/s$

$F = W$

$\underline{F = 147.15N}$

$\mu = ?$

$A = \Pi D L$

$A = \Pi \times 20 \times 10^{-3} \times 0.5$

$\underline{A = 0.031\ m^2}$

$$\tau = \mu.\frac{U}{Y}$$

$$4746.7 = \mu\, x \frac{5}{0.1x10^{-3}}$$

$$\mu = 0.095 \ \frac{NS}{m^2}$$

$$\tau = \frac{F}{A}$$

$$= \frac{147.15}{0.031}$$

$$\tau = 4746.7 N / m^2$$

6. If the equation of velocity profile over 2 plate is $V = 2y^{2/3}$. in which 'V' is the velocity in m/s and 'y' is the distance in 'm' . Determine shear stress at (i) y = 0 (ii) y = 75mm. Take $\mu = 8.35P$.

   a. at y = 0
   b. at y = 75mm

$= 75 \times 10^{-3}$m

$\tau = 8.35$ P

$$= 8.35 \times 0.1 \ \frac{NS}{m^2}$$

$$= 0.835 \ \frac{NS}{m^2}$$

$V = 2y^{2/3}$

$\frac{dv}{dy} = 2x\ \frac{2}{3} y^{2/3-1}$

$= \frac{4}{3} y^{-1/3}$

$at,\ y = 0, \frac{dv}{dy} = 3\frac{4}{\sqrt[3]{0}} = \infty$

$at,\ y = 75x10^{-3}\,m, \frac{dv}{dy} = 3\frac{4}{\sqrt[3]{75x10^{-3}}}$

$\frac{dv}{dy} = 3.16\ /\ s$

$\tau = \mu.\frac{dv}{dy}$

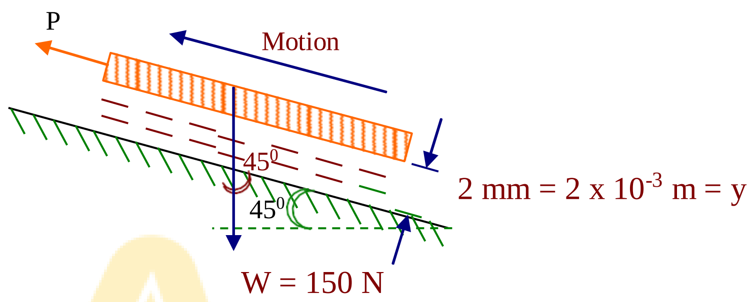
$at,\ y = 0,\ \tau = 0.835x\infty$

$\tau = \infty$

$at,\ y = 75\ x10^{-3}\,m, \tau = 0.835\ x\ 3.16$

$\tau = 2.64\ N\ /\ m^2$

7. A circular disc of 0.3m dia and weight 50 N is kept on an inclined surface with a slope of 45^0. The space between the disc and the surface is 2 mm and is filled with oil of dynamics viscosity $\frac{1NS}{m^2}$. What force will be required to pull the disk up the inclined plane with a velocity of 0.5m/s.



D = 0.3m

$$A = \frac{\prod x\, 0.3m^2}{4}$$

A = 0.07m^2

W = 50N

$$\mu = 1\frac{NS}{m^2}$$

$$F = P - 50\cos 45$$

$$F = (P - 35,35)$$

$$\frac{y = 2x10^{-3}m}{U = 0.5m/s}$$

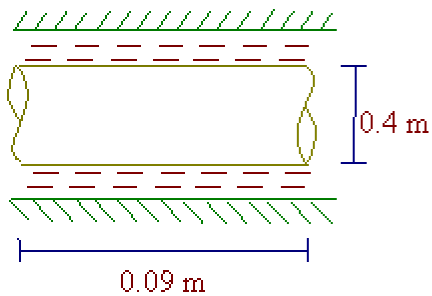
$$\upsilon = \frac{(P - 35.35)}{0.07} N/m^2$$

$$\tau = \mu.\frac{U}{Y}$$

$$\left(\frac{P - 35,35}{0.07}\right) = 1x\frac{0.5}{2x10^{-3}}$$

$$P = 52.85N$$

8. Dynamic viscosity of oil used for lubrication between a shaft and a sleeve is 6 P. The shaft is of diameter 0.4 m and rotates at 190 rpm. Calculate the power lost in the bearing for a sleeve length of 0.09 m .Thickness of oil is 1.5 mm.



$\mu = 6 = 0.6 \frac{NS}{m^2}$

N = 190 rpm

Power lost = ?

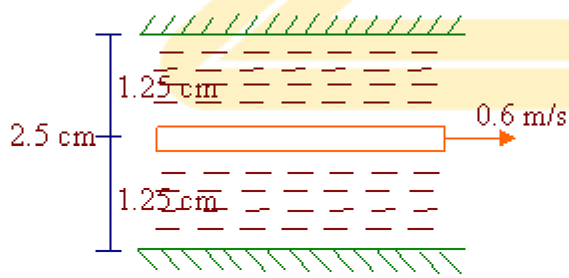
A = Π D L

= Π x 0.4 x0.09   A = $0.11m^2$

Y = $1.5x10^{-3}$ m

9. Two large surfaces are 2.5 cm apart. This space is filled with glycerin of absolute viscosity 0.82 NS/m^2. Find what force is required to drag a plate of area $0.5m^2$ between the two surfaces at a speed of 0.6m/s. (i) When the plate is equidistant from the surfaces, (ii) when the plate is at 1cm from one of the surfaces.

Case (i) When the plate is equidistant from the surfaces,



$$U = \frac{\Pi DN}{60}$$

$$= \frac{\Pi \text{ x } 0.4 \text{ x } 190}{60}$$

$$U = 3.979 \text{ m / s}$$

$$\tau = \mu.\frac{U}{Y}$$

$$= 0.6 \text{ x } \frac{3.979}{1.5 \text{ x } 10^{-3}}$$

$$\tau = 1.592 \text{ x } 10^{3} \text{ N / m}^{2}$$

$$\frac{F}{A} = 1.59 \text{ x } 10^{3}$$

$$F = 1.591 \text{x } 10^{3} \text{ x } 0.11$$

$$F = 175.01 \text{ N}$$

$$\boldsymbol{T} = \boldsymbol{F} \text{ x } \boldsymbol{R}$$

$$= 175.01 x 0.2$$

$$\boldsymbol{T} = 35 \boldsymbol{Nm}$$

$$\boldsymbol{P} = \frac{2\Pi \boldsymbol{NT}}{60{,}000}$$

$$\boldsymbol{P} = 0.6964 \boldsymbol{KW}$$

$$\boldsymbol{P} = 696.4 \boldsymbol{W}$$

Let F_1 be the force required to overcome viscosity resistance of liquid above the plate and F_2 be the force required to overcome viscous resistance of liquid below the plate. In this case $F_1 = F_2$. Since the liquid is same on either side or the plate is equidistant from the surfaces.

$$\tau_1 = \mu_1 \frac{U}{Y}$$

$$\tau_1 = 0.82x \frac{0.6}{0.0125}$$

$$\tau_1 = 39.36N / m^2$$

$$\frac{F_1}{A} = 39.36$$

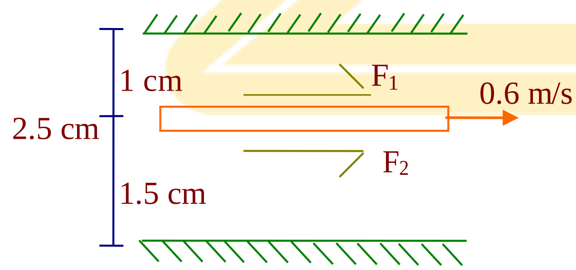
$$F_1 = 19.68N$$

$\therefore$ Tatal force required to drag the plate $= F_1 + F_2 = 19.68 + 19.68$

$$F = 39.36N$$

Case (ii) when the plate is at 1cm from one of the surfaces.

Here $F_1 \neq F_2$



$$\frac{F_1}{A} = 49.2$$

$$F_1 = 49.2x0.5$$

$$F_1 = 24.6N$$

$$\frac{F_2}{A} = 32.8$$

$$F_2 = 32.8x0.5$$

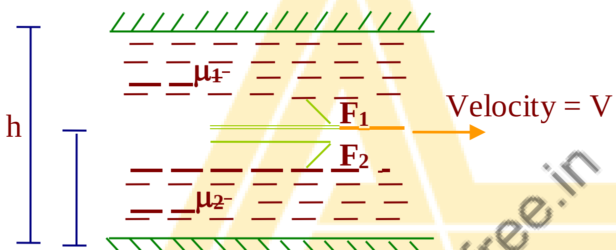
$$F_2 = 16.4N$$

Total Force $F = F_1 + F_2 = 24.6 + 16.4$

$F = 41N$

10. Through a very narrow gap of ht a thin plate of large extent is pulled at a velocity `V'. On one side of the plate is oil of viscosity μ_1 and on the other side there is oil of viscosity μ_2. Determine the position of the plate for the following conditions.
    i. Shear stress on the two sides of the plate is equal.
    ii. The pull required, to drag the plate is minimum.

Condition 1: Shear stress on the two sides of the plate is equal $F_1 = F_2$



$y = ?$ for $F_1 = F_2$

$$\tau = \mu.\frac{U}{Y}$$

$$\frac{F}{A} = \mu.\frac{U}{Y}$$

$$F = A\mu.\frac{U}{Y}$$

$$F_1 = \frac{A\mu_1 V}{(h-y)}$$

$$F_2 = \frac{A\mu_2 V}{y}$$

$$F_1 = F_2$$

$$\frac{A\mu_1 V}{h-y} = \frac{A\mu_2 V}{y}$$

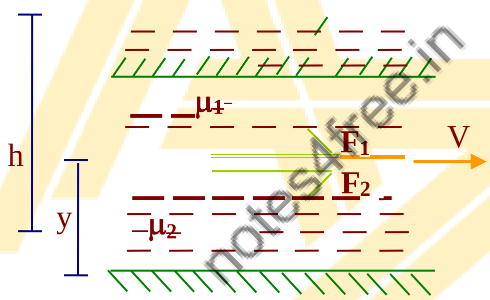
$$\mu_1 y = \mu_2 (h-y)$$

$$\mu_1 y + \mu_2 y = \mu_2 h$$

$$y = \frac{\mu_2 h}{\mu_1 + \mu_2} \text{ or } y = \frac{h}{\frac{\mu_1}{\mu_2} + 1}$$

Condition 2: The pull required, to drag the plate is minimum (i.e. $[\frac{dF}{dy}]_{minimum}$)

$\therefore$ Total drag forced required



$y = ?$ *if*, $F_1 + F_2$ *is to be* min *imum*

$$F_1 = \frac{A\mu_1 V}{h\ y}$$

$$F_2 = \frac{A\mu_2 V}{y}$$

$$F = F_1 + F_2$$

$$F = \frac{A\mu_1 V}{h \quad y} + \frac{A\mu_2 V}{y}$$

$$\text{For } F \text{ to be } \min . \frac{dF}{dy} = 0$$

$$\frac{dF}{dy} = 0 = +A\mu_1 V \equiv (h \quad y)^2 \quad A\mu_2 V y^2$$

$$= \frac{V\mu_1 A}{(h \quad y)^2} \quad \frac{V\mu_2 A}{y^2}$$

$$\frac{(h \quad y)^2}{y^2} = \frac{\mu_1}{\mu_2}$$

$$\frac{h \quad y}{y} = \sqrt{\frac{\mu_1}{\mu_2}}$$

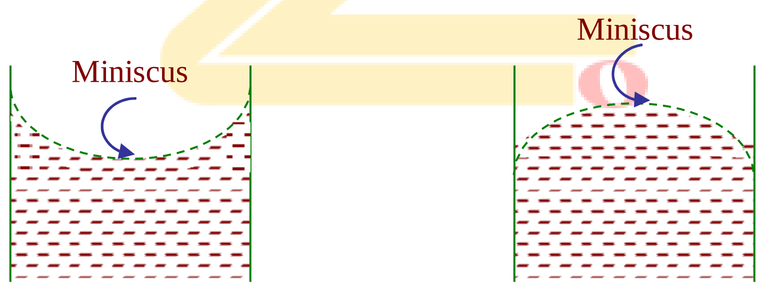
$$(h \quad y) = y\sqrt{\frac{\mu_1}{\mu_2}}$$

$$h = y\sqrt{\frac{\mu_1}{\mu_2}} + y$$

$$h = y \quad 1 + \sqrt{\frac{\mu_1}{\mu_2}}$$

$$\therefore y = \frac{h}{1 + \sqrt{\frac{\mu_1}{\mu_2}}}$$

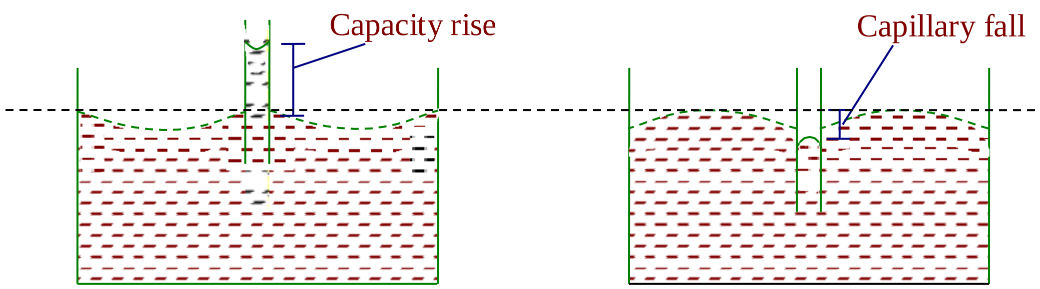
1..3.6 Capillarity :



Cohesion < Adhesion
Eg: Water

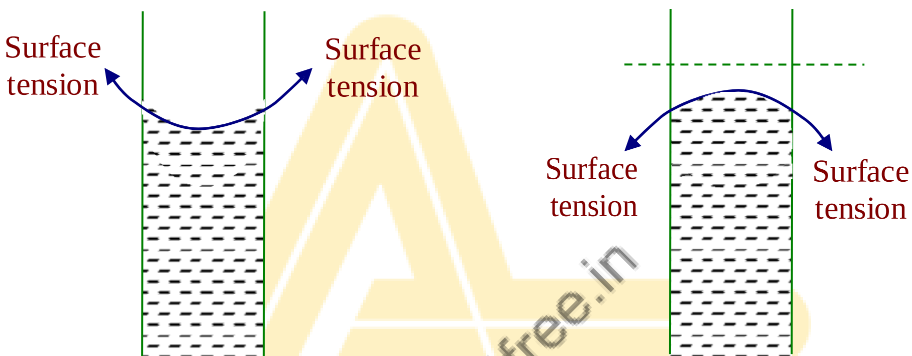
Cohesion > Adhesion
Eg: Mercury

Any liquid between contact surfaces attains curved shaped surface as shown in figure. The curved surface of the liquid is called Meniscus. If adhesion is more than cohesion then the meniscus will be concave. If cohesion is greater than adhesion meniscus will be convex.



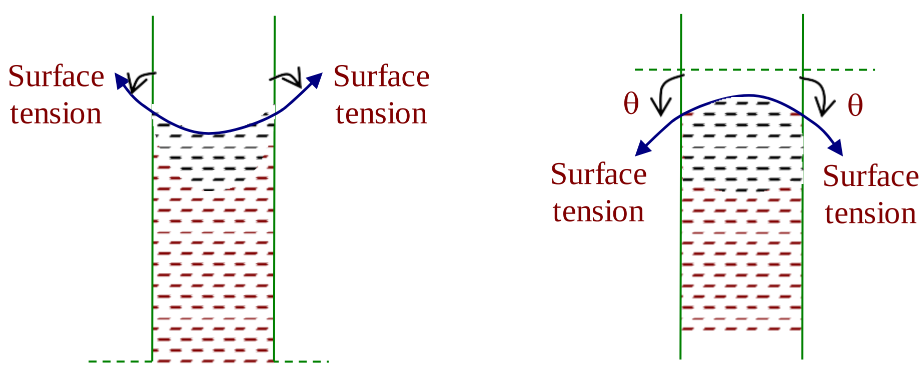
Cohesion < Adhesion
Eg: Water

Cohesion > Adhesion
Eg: Mercury



Capillarity is the phenomena by which liquids will rise or fall in a tube of small diameter dipped in them. Capillarity is due to cohesion adhesion and surface tension of liquids. If adhesion is more than cohesion then there will be capillary rise. If cohesion is greater than adhesion then will be capillary fall or depression. The surface tensile force supports capillary rise or depression.

Angle of contact:

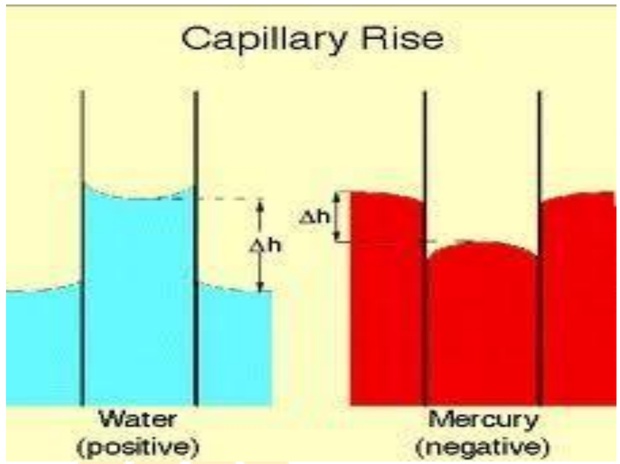


θ → Angle of contact
→ Acute

θ → Angle of contact
→ Obtuse

Note:

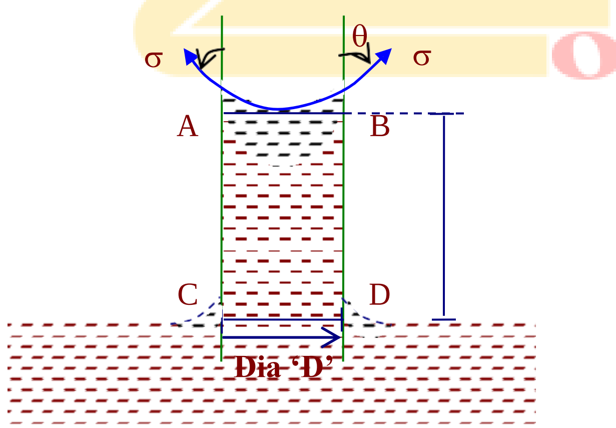
The angle between surface tensile force and the vertical is called angle of contact. If adhesion is more than cohesion then angle of contact is obtuse.



- **To derive an expression for the capillary rise of a liquid in small tube dipped in it:**

Let us consider a small tube of diameter 'D' dipped in a liquid of specific weight γ. 'h' is the capillary rise. For the equilibrium,

Vertical force due to surface tension = Weight of column of liquid ABCD



$$[\sigma(\Pi D)] \cos\theta = \gamma \text{ x volume}$$

$$[\sigma(\Pi D)]\cos\theta = \gamma \text{ x } \frac{\Pi D^2}{4} \text{ x h}$$

$$h = \frac{4\,\sigma\cos\theta}{\gamma\,D}$$

It can be observed that the capillary rise is inversely proportional to the diameter of the tube.

Note:

The same equation can be used to calculate capillary depression. In such cases ‘ θ ’ will be obtuse ‘h’ works out to be –ve.

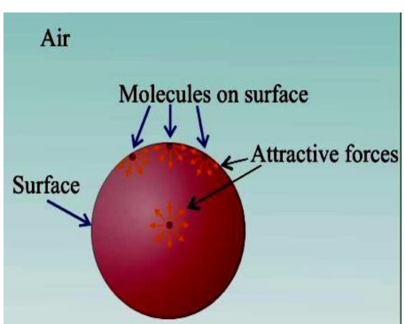
Excess Pressure inside a Water Droplet:

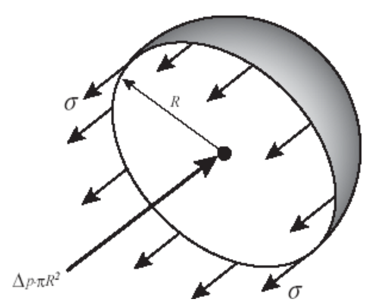
Pressure inside a Liquid droplet: Liquid droplets tend to assume a spherical shape since a sphere has the smallest surface area per unit volume.

The pressure inside a drop of fluid can be calculated using a free-body diagram of a spherical shape of radius R cut in half, as shown in Figure below and the force developed around the edge of the cut sphere is 2πRσ. This force must be balance with the difference between the internal pressure pi and the external pressure Δp acting on the circular area of the cut. Thus,

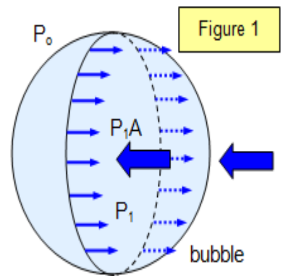
$$2\pi R\sigma = \Delta p \pi R^2$$

$$\Delta p = (p_{internal} - p_{external}) = \frac{2\times\sigma}{R} = \frac{4\times\sigma}{D}$$





The excess pressure within a Soap bubble:
The fact that air has to be blown into a drop of soap solution to make a bubble should suggest that the pressure within the bubble is greater than that outside. This is in fact the case: this excess pressure creates a force that is just balanced by the inward pull of the soap film of the bubble due to its surface tension.



Consider a soap bubble of radius r as shown in Figure 1. Let the external pressure be P_o and the internal pressure P_1. The excess pressure ΔP within the bubble is therefore given by: Excess pressure $\Delta p = (P_1 - P_0)$

Consider the left-hand half of the bubble. The force acting from right to left due to the internal excess pressure can be shown to be PA, where A is the area of a section through the centre of the bubble. If the bubble is in equilibrium this force is balanced by a force due to surface tension acting from left to right. This force is $2x2\pi r\sigma$ (the factor of 2 is necessary because the soap film has two sides) where 'σ' is the coefficient of surface tension of the soap film. Therefore

$2x2\pi r\sigma = \Delta pA = \Delta p\pi r^2$ giving:
Excess pressure in a soap bubble (P) = $4\sigma/r$

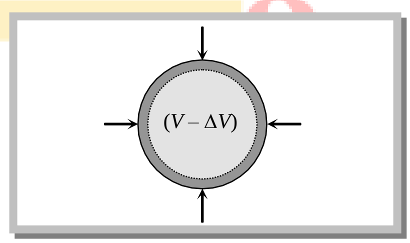
Bulk Modulus (K):

When a solid or fluid (liquid or gas) is subjected to a uniform pressure all over the surface, such that the shape remains the same, then there is a change in volume.

Then the ratio of normal stress to the volumetric strain within the elastic limits is called as Bulk modulus. This is denoted by K.

$$K = \frac{\text{Normal stress}}{\text{volumetric strain}}$$

$$K = \frac{F/A}{-\Delta V/V} = \frac{-pV}{\Delta V}$$



where p = increase in pressure; V = original volume; ΔV = change in volume

The negative sign shows that with increase in pressure p, the volume decreases by ΔV i.e. if p is positive, ΔV is negative. The reciprocal of bulk modulus is called compressibility.

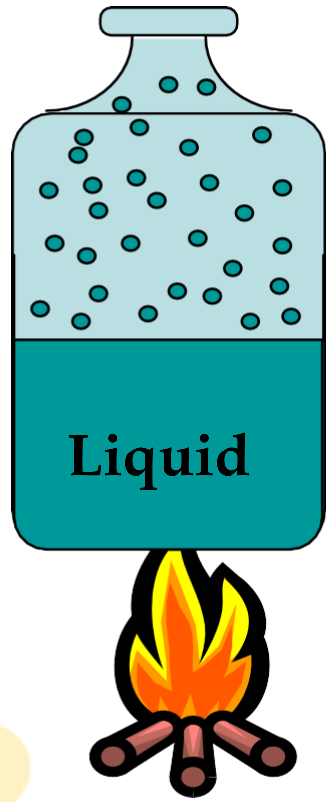
$$C = Compressibility = \frac{1}{K} = \frac{\Delta V}{pV}$$

S.I. unit of compressibility is N–1m2 and C.G.S. unit is dyne–1 cm2.

Vpour Pressure:

Vapor pressure is defined as the pressure at which a liquid will boil (vaporize) and is in equilibrium with its own vapor (fig). Vapor pressure rises as temperature rises. For example, suppose you are camping on a high mountain (say 3,000m in altitude); the atmospheric pressure at this elevation is about 70 kPa and the boiling temperature is around 90°C. This has consequences for cooking. For example, eggs have to be cooked longer at elevation to become hard-boiled since they cook at a lower temperature.

A pressure cooker has the opposite effect. Namely, the tight lid on a pressure cooker causes the pressure to increase above the normal atmospheric value. This causes water to boil at a temperature even greater than 100°C; eggs can be cooked a lot faster in a pressure cooker!



Vapor pressure is important to fluid flows because, in general, pressure in a flow decreases as velocity increases. This can lead to ***cavitation***, which is generally destructive and undesirable. In particular, at high speeds the local pressure of a liquid sometimes drops below the vapor pressure of the liquid. In such a case, ***cavitation*** occurs. In other words, a "cavity" or bubble of vapor appears because the liquid vaporizes or boils at the location where the pressure dips below the local vapor pressure.

Cavitation is not desirable for several reasons. First, it causes noise (as the cavitation bubbles collapse when they migrate into regions of higher pressure). Second, it can lead to inefficiencies and reduction of heat transfer in pumps and turbines (turbo machines). Finally, the collapse of these cavitation bubbles causes pitting and corrosion of blades and other surfaces nearby. The left figure below shows a cavitating propeller in a water tunnel, and the right figure shows cavitation damage on a blade.

Problems:

1. Capillary tube having an inside diameter 5mm is dipped in water at 20^0. Determine the heat of water which will rise in tube. Take σ=0.0736N/m at 20^0C.

$$h = \frac{4\ \sigma\cos\theta}{\gamma\ D}$$

$$= \frac{4 \text{ x } 0.0736 \text{ x } \cos\theta}{9810 \text{ x } 5 \text{ x } 10^{-3}}$$

$\theta = 0^0$ (assumed)

$\gamma = 9810 N/m^3$

$$h = 6 \text{ x } 10^{-3} m$$

2. Calculate capillary rise in a glass tube when immersed in Hg at 20^0c. Assume σ for Hg at 20^0c as 0.51N/m. The diameter of the tube is 5mm. $\theta = 130^0$c.

$$h = \frac{4\ \sigma\cos\theta}{\gamma D}$$

$$h = -1.965 \text{ x} 10^{-3} m$$

$$S = \frac{\gamma}{\gamma_{S\tan dard}}$$

$$13.6 = \frac{\gamma}{9810}$$

$$\gamma = 133.416 \text{ x } 10^3 N/m^3$$

-ve sign indicates capillary depression.

3. Determine the minimum size of the glass tubing that can be used to measure water level if capillary rise is not to exceed 2.5mm. Take σ = 0.0736 N/m.

$$h = \frac{4\sigma\cos\theta}{\gamma\ D}$$

$D = ?$

$$D = \frac{4 \text{ x } 0.0736 \text{ x } \cos 0}{9810 \text{ x } 2.5 \text{ x} 10^{-3}}$$

$h = 2.5 \text{x} 10^{-3} m$

$\sigma = 0.0736 N/m$

$$D = 0.012 m$$

$$D = 12 mm$$

4. A glass tube 0.25mm in diameter contains Hg column with air above it. If σ= 0.51N/m, what will be the capillary depression? Take θ = - 40^0 or 140^0.

$$h = \frac{4\sigma\cos\theta}{\gamma D}$$

$$= \frac{4x0.51x\ \cos140}{133.146x10^{-3}\ x0.25x10^{-3}}$$

$$h = -46.851x10^{-3}\,m$$

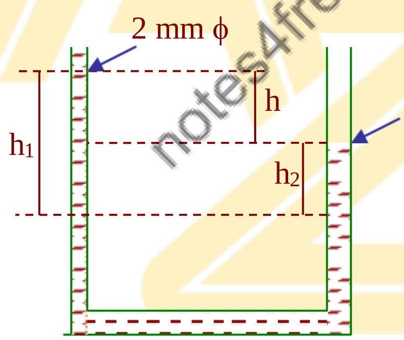
$D = 0.25x10^{-3}\,m$

$\sigma = 0.51N / m$

$\theta = 140$

$\gamma = 133.416\ x\ 10^3\ \ N / m^2$

5. If a tube is made so that one limb is 20mm in ϕ and the other 2mm in ϕ and water is poured in the tube, what is the difference in the level of surface of liquid in the two limbs. σ = 0.073 N/m for water.



$$h_1 = h = \frac{4\sigma\cos\theta}{\gamma D}$$

$$= \frac{4 \text{ x } 0.073 \text{ x coso}}{9810 \text{ x } (20 \text{ x}10^{-3})}$$

$$= 0.01488\text{m}$$

$$h_2 = \frac{4 \text{ x } 0.073\text{x coso}}{9810 \text{ x } (20 \text{ x}10^{-3})}$$

$$= 1.488 \text{ x } 10^{-3}\text{ m}$$

$$h = h_1 - h_2$$

$$= 0.01339\text{m}$$

$$h = 13.39\text{mm}$$

6. A clean glass tube is to be selected in the design of a manometer to measure the pressure of kerosene. Specific gravity of kerosene = 0.82 and surface tension of kerosene = 0.025 N/m. If the capillary rise is to be limited to 1 mm, calculate the smallest diameter (cm) of the glass tube

Soln. Given For kerosene σ = 0.025 N/m ; Sp.Gr. = 0.82; h_{max} = 1mm

Assuming contact angle θ = 0°, $\gamma_{kerosene}$ =0 .82 x 9810 = 8044.2 N/m^3

Let ‘d’ be the smallest diameter of the glass tube in Cm

Then using formula for capillary rise in (h)

$$h = \frac{4\,\sigma\,\text{Cos}\theta}{\gamma_{kerosene}\left(\frac{d_{cm}}{100}\right)} = \frac{4\text{x}0.025\ \text{Cos}0^{\circ}}{8044.2\text{x}\left(\frac{d_{cm}}{100}\right)} = \frac{1}{1000}$$

$$d_{cm} = 1.24\ \text{Cm}$$

7. The surface tension of water in contact with air at 20°C is 0.0725 N/m. The pressure inside a droplet of water is to be 0.02 N/cm^2 greater than the outside pressure. Calculate the diameter of the droplet of water.

Given: Surface Tension of Water σ =0.0725 N/m, Δp= 0.02 N/cm^2 = $0.02\times10^{-4} N/m^2$

Let 'D' be the diameter of jet

$$\Delta p = \frac{4\sigma}{D}$$

$$0.02\times10^{-4} = \frac{4\times0.0725}{D}$$

D = 0.00145m = 1.45mm

8. Find the surface tension in a soap bubble of 40mm diameter when inside pressure is 2.5 N/m^2 above the atmosphere.

Given: D = 40mm = 0.04 m, Δp = 2.5 N/m^2

Let 'σ' be the surface tension of soap bubble

$$\Delta p = \frac{8\sigma}{D}$$

$$2.5 = \frac{4\sigma}{0.04}$$

σ = 0.0125 N/m

9. Determine the bulk modulus of elasticity of a liquid, if the pressure of the liquid is Increased from 70 N/cm^2 to 130 N/cm^2 The volume of the liquid decreases by 0.15per cent

Given: Initial Pressure = 70 N/cm^2, Final Pressure = 130 N/cm^2
Decrease in Volume = 0.15%

∴ Δp = Increase in Pressure = (130-70) = 60 N/cm^2

$$K = \frac{\Delta p}{\left(-\frac{\Delta\forall}{\forall}\right)} = \frac{60}{\left(\frac{0.15}{100}\right)} = 4\times10^{4}\ N/cm^2$$

Module -1: 2.Fluid Pressure and Its Measurements:
Definition of pressure, Pressure at a point, Pascal's law, Variation of pressure with depth. Types of pressure. Measurement of pressure using simple, differential & inclined manometers (theory & problems). Introduction to Mechanical and electronic pressure measuring devices.

2.0 INTRODUCTION: Fluid is a state of matter which exhibits the property of flow. When a certain mass of fluids is held in static equilibrium by confining it within solid boundaries (Fig.1), it exerts force along direction perpendicular to the boundary in contact. This force is called fluid pressure (compression).

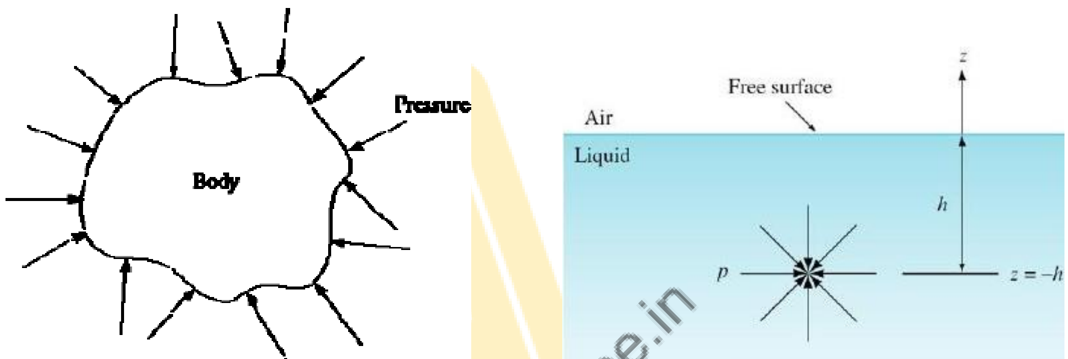


Fig.1 Definition of Pressure

In fluids, gases and liquids, we speak of pressure; in solids this is normal stress. For a fluid at rest, the pressure at a given point is the same in all directions. Differences or gradients in pressure drive a fluid flow, especially in ducts and pipes.

2.1 Definition of Pressure: Pressure is one of the basic properties of all fluids. Pressure (p) is the force (F) exerted on or by the fluid on a unit of surface area (A). Mathematically expressed:

$$p = \frac{F}{A} \quad \left(\frac{N}{m^2}\right)$$

The basic unit of pressure is Pascal (Pa). When a fluid exerts a force of 1 N over an area of 1m^2, the pressure equals one Pascal, i.e., 1 Pa = 1 N/m^2.Pascal is a very small unit, so that for typical power plant application, we use larger units:

Units: 1 kilopascal (kPa) = 10^3 Pa, and

1 megapascal (MPa) = 10^6 Pa = 10^3 kPa.

22 <u>Pressure at a Point and Pascal's Law:</u>

Pascal's Principle: Pressure extends uniformly in all directions in a fluid.

By considering the equilibrium of a small triangular wedge of fluid extracted from a static fluid body, one can show (Fig.2) that for *any* wedge angle θ, the pressures on the three faces of the wedge are equal in magnitude:

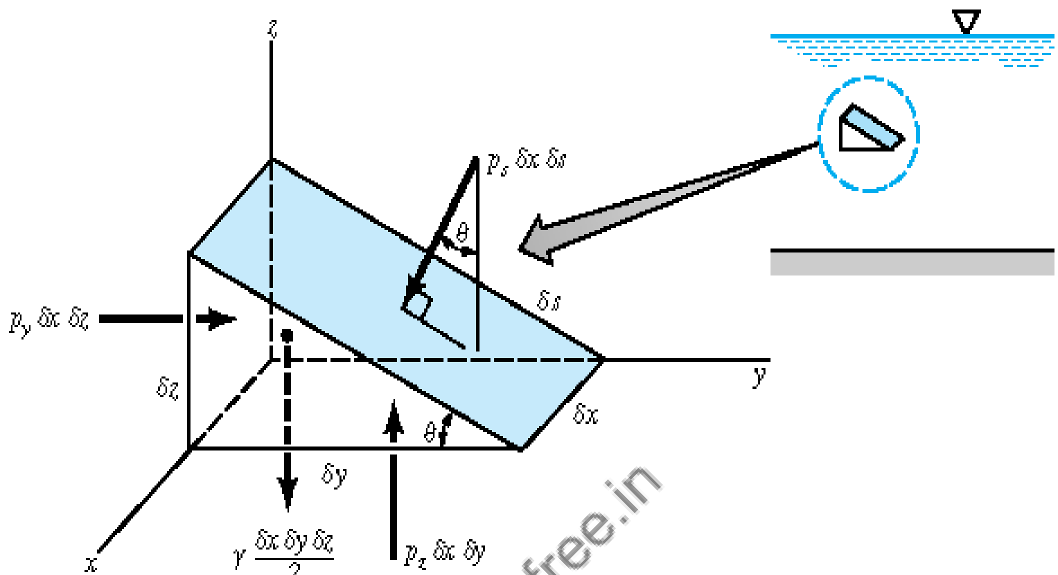


Fig.2 Pascal's Law

Independent of $p_x = p_y = p_z$ independent of 'θ'

Pressure at a point has the same magnitude in all directions, and is called **isotropic.**

This result is known as **Pascal's law**.

23 Pascal's Law: In any closed, static fluid system, a pressure change at any one point is transmitted undiminished throughout the system.

2.3.1 Application of Pascal's Law:

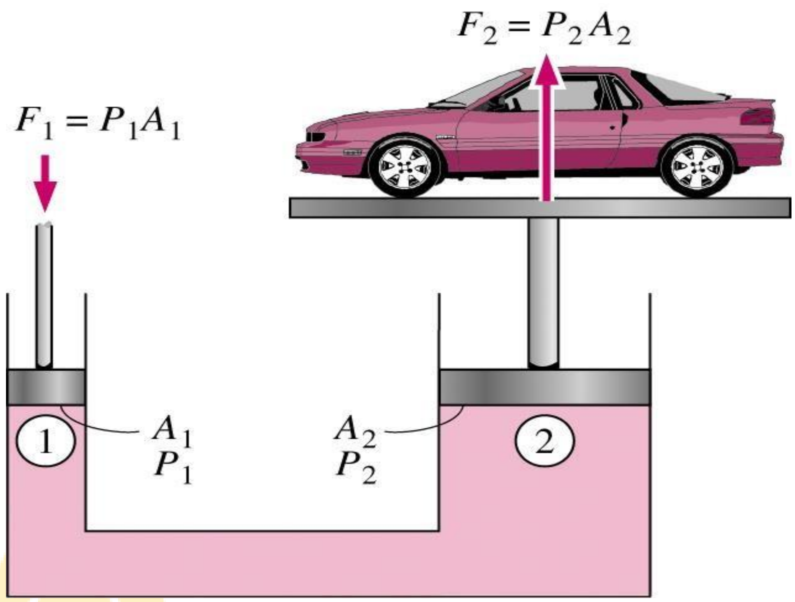


Fig.3 Application of Pascal's Law

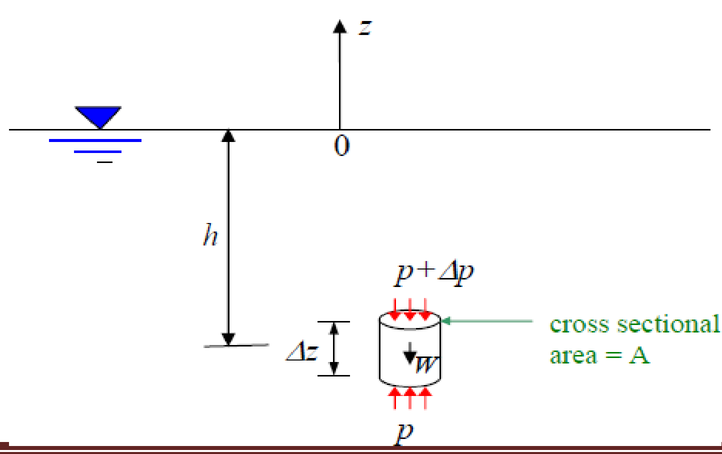
- Pressure applied to a confined fluid increases the pressure throughout by the same amount.
- In picture, pistons are at same height:

$$P_1 = P_2 \rightarrow \frac{F_1}{A_1} = \frac{F_2}{A_2} \rightarrow \frac{F_2}{F_1} = \frac{A_2}{A_1}$$

- Ratio A_2/A_1 is called ideal mechanical advantage

24 <u>Pressure Variation with Depth:</u>

Consider a small vertical cylinder of fluid in equilibrium, where *positive z is pointing vertically upward*. Suppose the origin z = 0 is set at the free surface of the fluid. Then the pressure variation at a depth *z* = -*h* below the free surface is governed by



$$(p + \Delta p)A + W = pA$$
$$\Rightarrow \Delta pA + \rho gA\Delta z = 0$$
$$\Rightarrow \Delta p = -\rho g\Delta z$$
$$\Rightarrow \frac{dp}{dz} = -\rho g \quad \text{or} \quad \frac{dp}{dz} = -\gamma \quad Eq.(1) \quad (\text{as } \Delta z \rightarrow 0)$$

Therefore, the hydrostatic pressure increases linearly with depth at the rate of the specific weight $\gamma = \rho g$ of the fluid.

Homogeneous fluid: ρ **is constant**

By simply integrating the above equation-1:

$$\int dp = -\int \rho g\, dz \quad \Rightarrow \quad p = -\rho gz + C$$

Where *C* is constant of integration

When $z = 0$ (on the free surface), $p = C = p_0$ = (the atmospheric pressure).

Hence, $$p = -\rho gz + p_0$$

Pressure given by this equation is called ABSOLUTE PRESSURE, i.e., measured above perfect vacuum.

However, for engineering purposes, it is more convenient to measure the pressure above a datum pressure at atmospheric pressure. By setting $p_0 = 0$,

$$p = -\rho gz + 0 = -\rho gz = \rho gh$$

$$p = \gamma h$$

The equation derived above shows that when the density is constant, **the pressure in a liquid at rest increases linearly with depth from the free surface.**

For a given pressure intensity 'h' will be different for different liquids since, 'γ' will be different for different liquids.

$$\therefore h = \frac{P}{\gamma}$$

Hint-1: To convert head of 1 liquid to head of another liquid.

$$S = \frac{\gamma}{\gamma_{Staandard}}$$

$$S_1 = \frac{\gamma_1}{\gamma_{Staandard}}$$

$p = \gamma_1 h_1$

$$\therefore \gamma_1 = S_1 \gamma_{Staandard}$$

$p = \gamma_2 h_2$

$$\gamma_{21} = S_2 \gamma_{Staandard}$$

$$\boxed{\gamma_1 h_1 = \gamma_2 h_2}$$

$$\therefore S_1 \gamma_{S\ tandard} h_1 = S_2 \gamma_{S\ tandard} h_2$$

$$\boxed{S_1 h_1 = S_2 h_2}$$

Hint: 2 $S_{water} \text{ x } h_{water} = S_{liquid} \text{ x } h_{liquid}$

$1\text{x } h_{water} = S_{liquid} \text{ x } h_{liquid}$

$$\boxed{h_{water} = S_{liquid} \text{ x } h_{liquid}}$$

Pressure head in meters of water is given by the product of pressure head in meters of liquid and specific gravity of the liquid.

Eg: 10meters of oil of specific gravity 0.8 is equal to 10x0.8 = 8 meters of water.

Eg: Atm pressure is 760mm of Mercury.

NOTE: $P = \gamma \; h$

$P \rightarrow$ kPa, $\gamma \rightarrow \frac{kN}{m^3}$, $h \rightarrow$ m

Solved Examples:

Ex. 1. Calculate intensity of pressure due to a column of 0.3m of (a) water (b) Mercury (c) Oil of specific gravity-0.8.

Soln: (a) Given: h = 0.3m of water

$$\gamma_{water} = 9.81 \frac{kN}{m^3}$$

$$p = ?$$

$$p_{water} = \gamma_{water} h_{water}$$

$$p_{water} = 2.943\ kPa$$

(b) Given: h = 0.3m of Hg

$\gamma_{mercury}$ = Sp.Gr. of Mercury X γ_{water} = 13.6 x 9.81

$\gamma_{mercury} = 133.416\ kN/m^3$

$p_{mercury} = \gamma_{mercury}\ h_{mercury}$

= 133.416 x 0.3

p = 40.025 kPa or 40.025 kN/m² → **$p = 40.025\ kPa$ or $40.025\ kN/m^2$**

(c) Given: h = 0.3 of Oil Sp.Gr. = 0.8

γ_{oil} = Sp.Gr. of Oil X γ_{water} = 0.8 x 9.8

$\gamma_{oil} = 7.848\ kN/m^3$

$p_{oil} = \gamma_{oil}\ h_{oil}$

= 7.848 x 0.3

$p_{oil} = 2.3544\ kPa$ or $2.3544\ kN/m^2$

Ex.2. Intensity of pressure required at a point is 40kPa. Find corresponding head in (a) water (b) Mercury (c) oil of specific gravity-0.9.

Solution: Given Intensity of pressure at a point 40 kPa i.e. $p = 40\ kN/m^2$

(a) Head of water h_{water} =?

$$h_{water} = \frac{p}{\gamma_{water}} = \frac{40}{9.81}$$

h_{water} = 4.077m of *water*

(b) Head of mercury '$h_{mercury}$ =?

$\gamma_{mercury}$ = Sp.Gr. of Mercury X γ_{water} = 13.6 x 9.81

$\gamma_{mercury}$ = 133.416 kN/m^3

$$h_{mercury} = \frac{p}{\gamma_{mercury}} = \frac{40}{133.416}$$

h_{water} = 0.3**m of *mercury***

(c) Head of oil 'h_{oil} =?

γ_{oil} = Sp.Gr. of Oil X γ_{water} = 0.9 x 9.81

γ_{oil} = 8.829 kN/m^3

$$h_{oil} = \frac{p}{\gamma_{oil}} = \frac{40}{8.829}$$

h_{oil} = 4.53**m of *oil***

Ex.3 Standard atmospheric pressure is 101.3 kPa Find the pressure head in (i) Meters of water (ii) mm of mercury (iii) m of oil of specific gravity 0.6.

(i) Meters of water h_{water}

$p = \gamma_{water}\ h_{water}$

101.3 = 9.81 x h_{water}

h_{water} = 10.3 m of water

(ii) Meters of water h_{water}

$p = \gamma_{mercury} \times h_{mercury}$

101.3 = (13.6x9.81) x $h_{mercury}$

h = 0.76 m of mercury

(iii) $p = \gamma_{oil}\ h_{oil}$

101.3 = (0.6 x 9.81) x h

h = 17.21m of oil of S = 0.6

Ex.4 An open container has water to a depth of 2.5m and above this an oil of S = 0.85 for a depth of 1.2m. Find the intensity of pressure at the interface of two liquids and at the bottom of the tank.

(i) At the Oil - water interface

$p_A = \gamma_{oil}\ h_{oil}$ = (0.85 x 9.81) x 1.2

p_A = 10 kPa

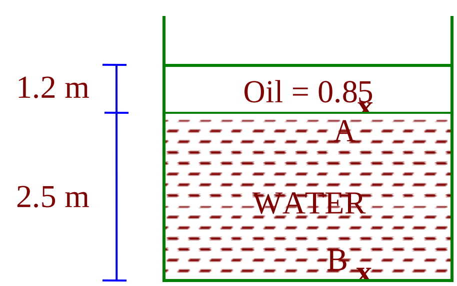
(ii) At the bottom of container

$p_B = \gamma_{oil}\ x h_{oil} + \gamma_{water} + h_{water}$

$p_B = p_A + \gamma_{water} h_{water}$

p_B = 10 kPa + 9.81 x 2.5

p_B = 34.525kPa

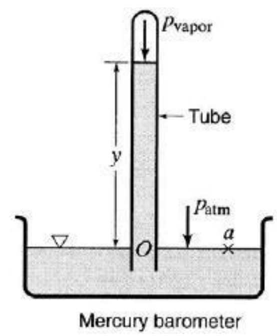


2.5 **Types of Pressure:** Air above the surface of liquids exerts pressure on the exposed surface of the liquid and normal to the surface.

- **Atmospheric pressure**

The pressure exerted by the atmosphere is called atmospheric pressure. Atmospheric pressure at a place depends on the elevation of the place and the temperature.

Atmospheric pressure is measured using an instrument called 'Barometer' and hence atmospheric pressure is also called Barometric pressure. *However, for engineering purposes, it is more convenient to measure the pressure above a datum pressure at atmospheric pressure.* By setting $p_{atmophere} = 0$,



$$p = -\rho gz = \rho gh$$

Unit: kPa . 'bar' is also a unit of atmospheric pressure 1-bar = 100 kPa.= 1 kg/cm^2

- **Absolute pressure:** **Absolute pressure** at a point is the intensity of pressure at that point measured with reference to absolute vacuum or absolute zero pressure. Absolute pressure at a point is the intensity of pressure at that point measured with reference to absolute vacuum or absolute zero pressure (Fig.4) .

  Absolute pressure at a point can never be negative since there can be no pressure less than absolute zero pressure.

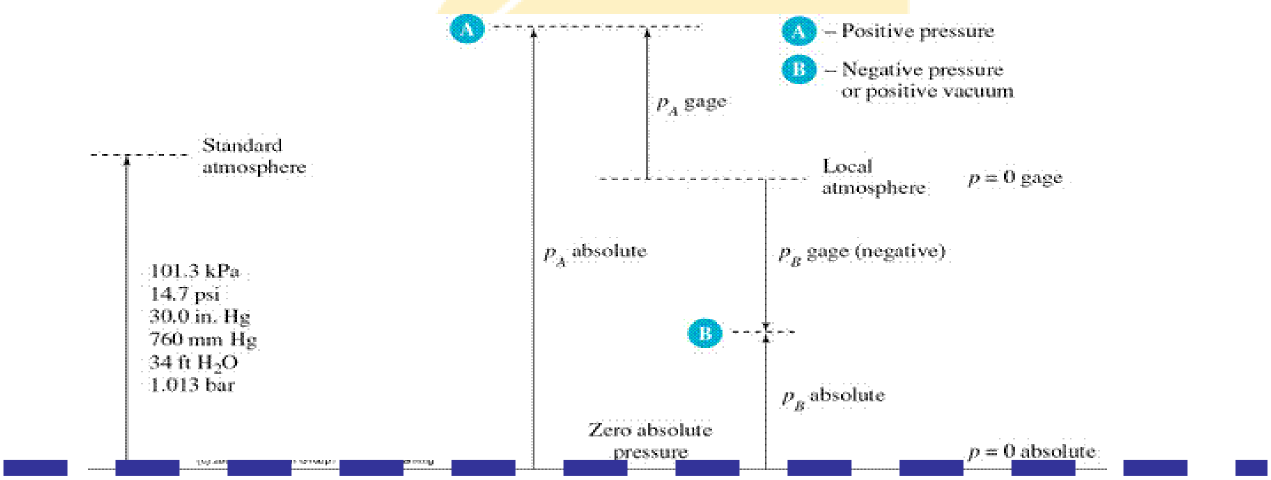


Fig.4 Definition of Absolute Pressure, Gauge Pressure and Vacuum Pressure

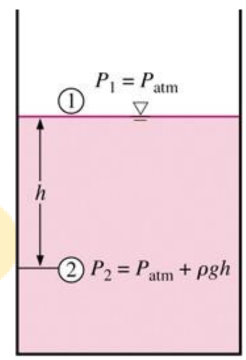
Gauge Pressure: If the intensity of pressure at a point is measurement with reference to atmosphere pressure, then it is called gauge pressure at that point.

Gauge pressure at a point may be more than the atmospheric pressure or less than the atmospheric pressure. Accordingly gauge pressure at the point may be positive or negative (Fig.4)

Negative gauge pressure: It is also called vacuum pressure. From the figure, It is the pressure measured below the gauge pressure (Fig.4).

Absolute pressure at a point = Atmospheric pressure ± Gauge pressure

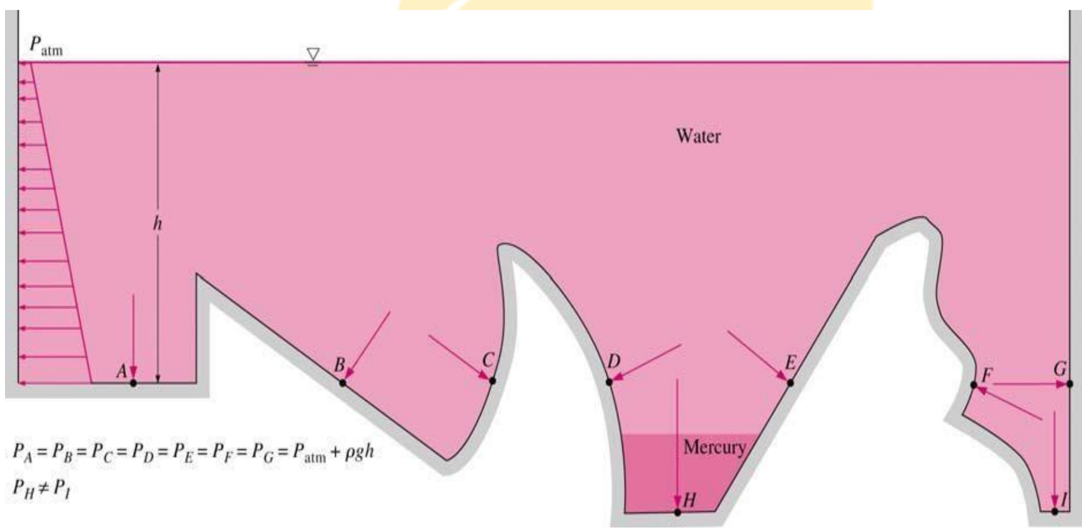
NOTE: If we measure absolute pressure at a Point below the free surface of the liquid, then, $\mathbf{p_2\ (absolute) = \gamma.\ h + p_{atm}}$ $\mathbf{p_1 = p_{atm}}$



If gauge pressure at a point is required, then atmospheric pressure is taken as zero, then,

$$\mathbf{p_2\ (gauge) = \gamma.\ h = \rho gh}$$

Also, the pressure is the same at all points with the same depth from the free surface regardless of geometry, provided that the points are interconnected by the same fluid. However, the thrust due to pressure is perpendicular to the surface on which the pressure acts, and hence its direction depends on the geometry.



Solved Example: Convert the following absolute pressure to gauge pressure:

(a) 120kPa (b) 3kPa (c) 15m of H_2O (d) 800mm of Hg.

Solution:

(a) $p_{abs} = p_{atm} + p_{gauge}$

$\therefore\ p_{gauge} = p_{abs} - p_{atm} = 120 - 101.3 = 18.7$ kPa

(b) $p_{gauge} = 3\text{-}101.3 = -98.3$ kPa

$p_{gauge} = 98.3$ kPa (vacuum)

(c) $h_{abs} = h_{atm} + h_{gauge}$

$15 = 10.3 + h_{gauge}$

$h_{gauge} = 4.7$m of water

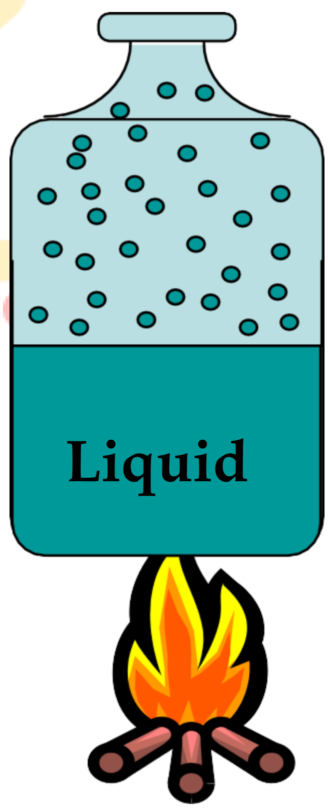
(d) $h_{abs} = h_{atm} + h_{gauge}$

$800 = 760 + h_{gauge}$

$h_{gauge} = 40$ mm of mercury

<u>2.6 Vpour Pressure:</u>

Vapor pressure is defined as the pressure at which a liquid will boil (vaporize) and is in equilibrium with its own vapor. Vapor pressure rises as temperature rises. For example, suppose you are camping on a high mountain (say 3,000 m in altitude); the atmospheric pressure at this elevation is about 70 kPa and the boiling temperature is around 90°C. This has consequences for cooking. For example, eggs have to be cooked longer at elevation to become hard-boiled since they cook at a lower temperature.



A pressure cooker has the opposite effect. Namely, the tight lid on a pressure cooker causes the pressure to increase above the normal atmospheric value. This causes water to boil at a temperature even greater than 100°C; eggs can be cooked a lot faster in a pressure cooker!

Fig.5

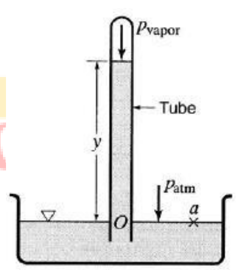
Vapor pressure is important to fluid flows because, in general, pressure in a flow decreases as velocity increases. This can lead to ***cavitation***, which is generally destructive and undesirable. In particular, at high speeds the local pressure of a liquid sometimes drops below the vapor pressure of the liquid. In such a case, ***cavitation*** occurs. In other words, a "cavity" or bubble of vapor appears because the liquid vaporizes or boils at the location where the pressure dips below the local vapor pressure.

Cavitation is not desirable for several reasons. First, it causes noise (as the cavitation bubbles collapse when they migrate into regions of higher pressure). Second, it can lead to inefficiencies and reduction of heat transfer in pumps and turbines (turbo machines). Finally, the collapse of these cavitation bubbles causes pitting and corrosion of blades and other surfaces nearby. The left figure below shows a cavitating propeller in a water tunnel, and the right figure shows cavitation damage on a blade.

2.7 Measurement of Pressure: Measurement of pressure

- Barometer
- Simple manometer
- Piezometer column
- Bourdon gage
- Pressure transducer

2.7.1 Barometer: A *barometer* is a device for measuring atmospheric pressure. A simple barometer consists of a tube more than 760 mm long inserted in an open container of mercury with a closed and evacuated end at the top and open tube end at the bottom and with mercury extending from the container up into the tube.



Mercury barometer

Strictly, the space above the liquid cannot be a true vacuum. It contains mercury vapor at its saturated vapor pressure, but this is extremely small at room temperatures (e.g. 0.173 Pa at 20°C). The atmospheric pressure is calculated from the relation $P_{atm} = \rho gh$ where ρ is the density of fluid in the barometer.

$$p_{at'o'} = \gamma_{mercury} \times y + p_{vapor} = p_{atm}$$

With negligible $p_{vapor} = 0$

$$\boxed{p_{atm} = \gamma_{mercury} \times y}$$

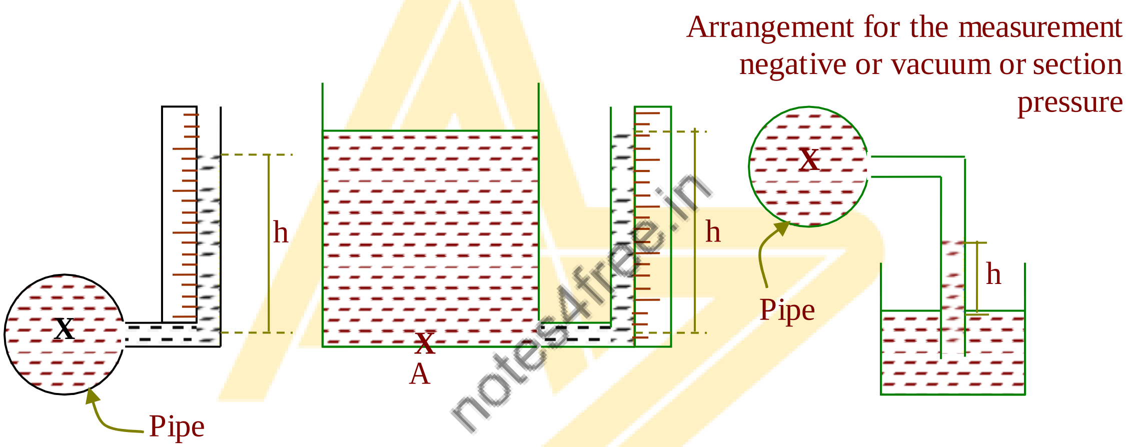
2.7.2 Simple Manometer: Simple monometers are used to measure intensity of pressure at a point. They are connected to the point at which the intensity of pressure is required. Such a point is called gauge point

♦ Types of Simple Manometers

Common types of simple manometers are

a) Piezometers
b) U-tube manometers
c) Single tube manometers
d) Inclined tube manometers

a) Piezometers



Piezometer consists of a glass tube inserted in the wall of the vessel or pipe at the level of point at which the intensity of pressure is to be measured. The other end of the piezometer is exposed to air. The height of the liquid in the piezometer gives the pressure head from which the intensity of pressure can be calculated.

To minimize capillary rise effects the diameters of the tube is kept more than 12mm.

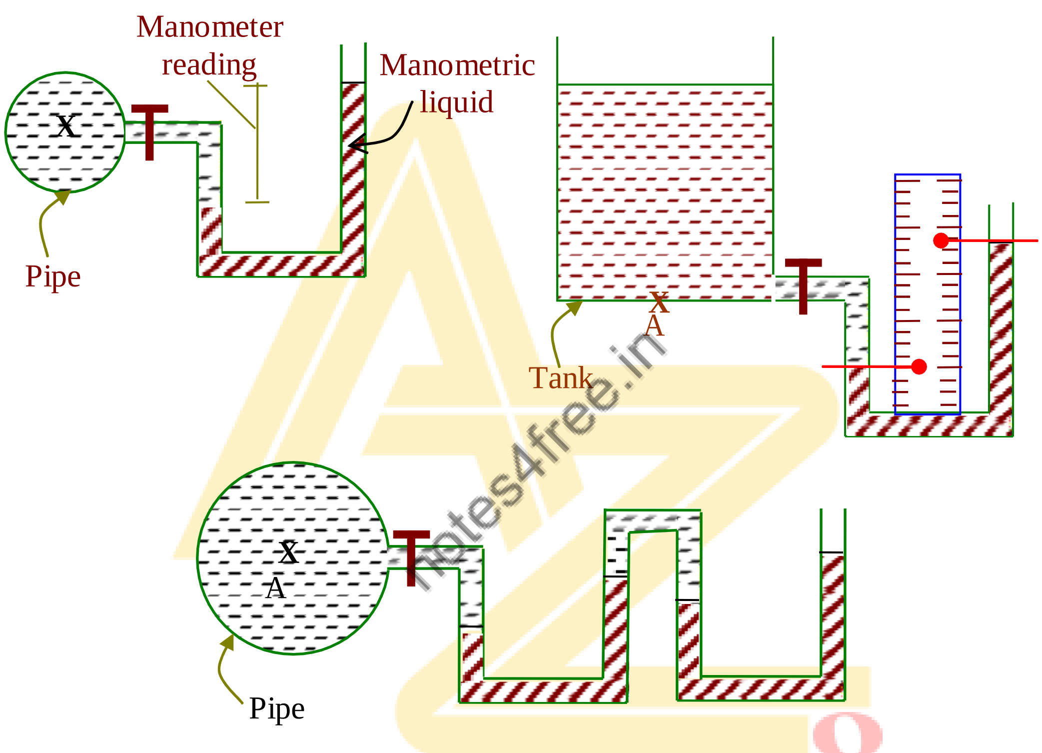
Merits

- Simple in construction
- Economical

Demerits

- Not suitable for high pressure intensity.
- Pressure of gases cannot be measured.

(b) U-tube Manometers:



A U-tube manometers consists of a glass tube bent in U-Shape, one end of which is connected to gauge point and the other end is exposed to atmosphere. U-tube consists of a liquid of specific of gravity other than that of fluid whose pressure intensity is to be measured and is called monometric liquid.

- **Manometric liquids**
  - Manometric liquids should neither mix nor have any chemical reaction with the fluid whose pressure intensity is to be measured.
  - It should not undergo any thermal variation.
  - Manometric liquid should have very low vapour pressure.
  - Manometric liquid should have pressure sensitivity depending upon the magnitude. Of pressure to be measured and accuracy requirement.

Gauge equations are written for the system to solve for unknown quantities.

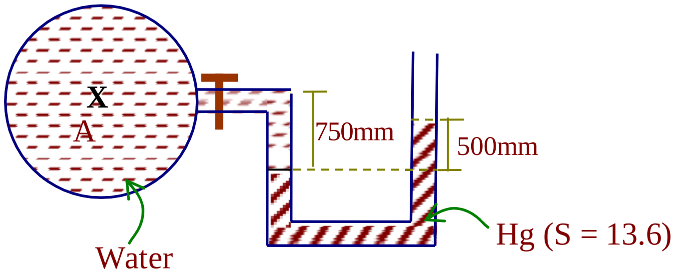
- **To write the gauge equation for manometers**

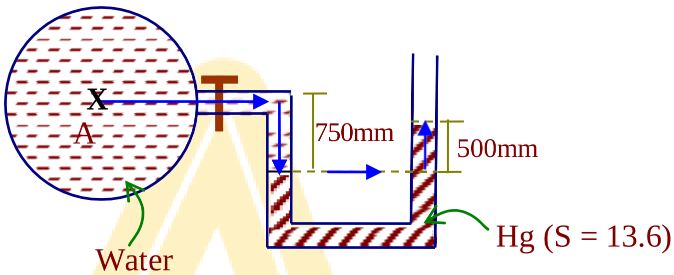
Steps:

1. Convert all given pressure to meters of water and assume unknown pressure in meters of waters.
2. Starting from one end move towards the other keeping the following points in mind.
   - Any horizontal movement inside the same liquid will not cause change in pressure.
   - Vertically downward movement causes increase in pressure and upward motion cause decrease in pressure.
   - Convert all vertical columns of liquids to meters of water by multiplying them by corresponding specify gravity.
   - Take atmospheric pressure as zero (gauge pressure computation).
3. Solve for the unknown quantity and convert it into the required unit.

Solved Problem:

1. Determine the pressure at A for the U- tube manometer shown in fig. Also calculate the absolute pressure at A in kPa.





Let ‘h_A’ be the pressure head at ‘A’ in ‘meters of water’.

$$h_A + 0.75 - 0.5\, x 13.6 = 0$$

$$h_A = 6.05\ m\ of\ \ water$$

$$p = \gamma h$$

$$= 9.81\ x\ 6.05$$

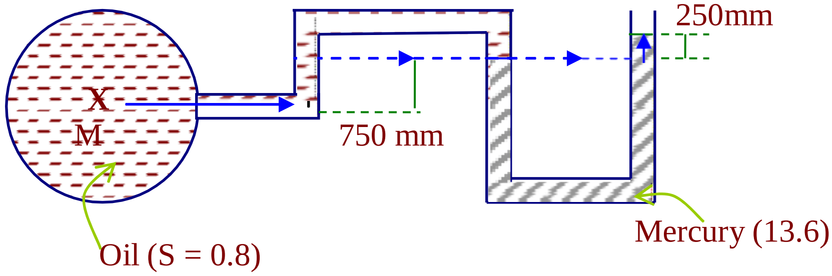
$$p = 59.35\ kPa (gauge\ \ pressure)$$

$$p_{abs} = p_{atm} + p_{gauge}$$

$$= 101.3 + 59.35$$

$$p_{abc} = 160.65\ kPa$$

2. For the arrangement shown in figure, determine gauge and absolute pressure at the point M.



Let 'h_M' be the pressure head at the point 'M' in m of water,

h_M - 0.75 x 0.8 – 0.25 x 13.6 = 0

h_M = 4 m of water

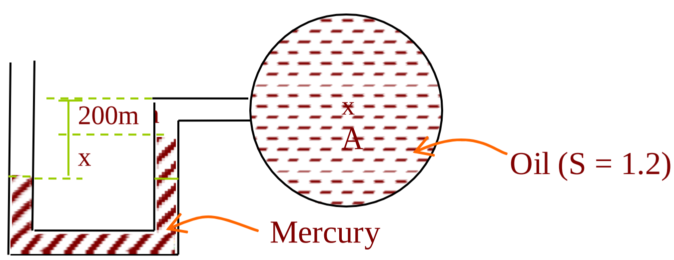
$p = \gamma h$

p = 39.24kPa

p_{abs} = 101.3 + 39.24

p_{abs} 140.54 kPa

3. If the pressure at 'At' is 10 kPa (Vacuum) what is the value of 'x'?



p_A = 10 kPa (Vacuum)

p_A = - 10 kPa

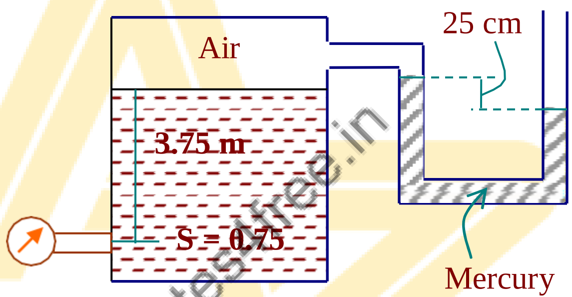
$$\frac{p_A}{\gamma} = \frac{-10}{9.81} = -1.019 \text{ m of water}$$

$$h_A = -1.019 \text{ m of water}$$

$$-1.019 + 0.2 \times 1.2 + x\,(13.6) = 0$$

$$x = 0.0572 \text{ m}$$

4. The tank in the accompanying figure consists of oil of S = 0.75. Determine the pressure gauge reading in $\frac{kN}{m^2}$.



Let the pressure gauge reading be 'h' m of water

$h - 3.75 \times 0.75 + 0.25 \times 13.6 = 0$

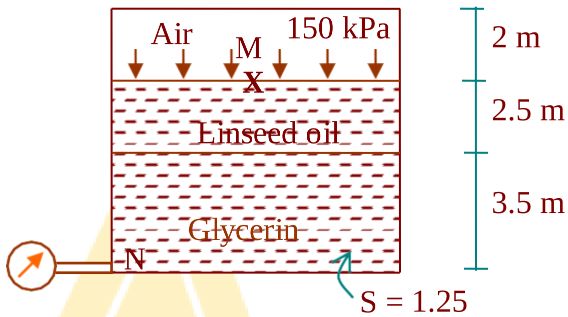
$h = -0.5875$ m of water

$p = \gamma h$

$p = -5.763$ kPa

$p = 5.763$ kPa (Vacuum)

5. A closed tank is 8m high. It is filled with Glycerine up to a depth of 3.5m and linseed oil to another 2.5m. The remaining space is filled with air under a pressure of 150 kPa. If a pressure gauge is fixed at the bottom of the tank what will be its reading. Also calculate absolute pressure. Take relative density of Glycerine and Linseed oil as 1.25 and 0.93 respectively.



$P_H = 150$ kPa

$h_M = \frac{150}{9.81}$

$h_M = 15.29$ m of water

Let 'h_N' be the pressure gauge reading in m of water.

$h_N - 3.5 \times 1.25 - 2.5 \times 0.93 = 15.29$

$h_N = 21.99$ m of water

$p = 9.81 \times 21.99$

$p = 215.72$ kPa (gauge)

$p_{abs} = 317.02$ kPa

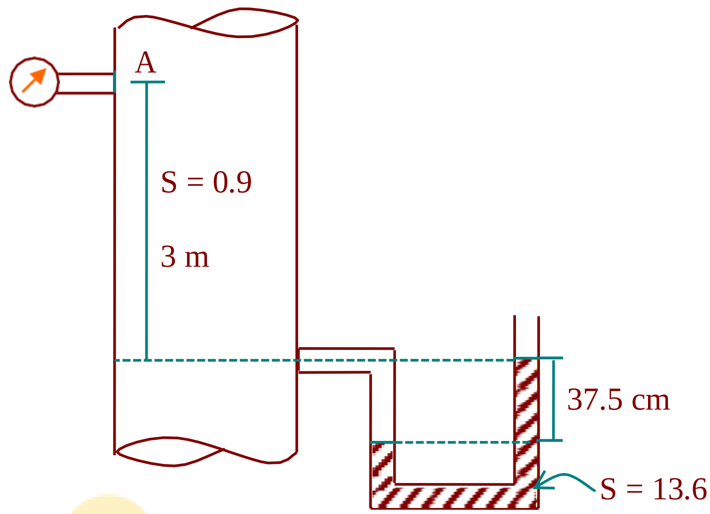
6. A vertical pipe line attached with a gauge and a manometer contains oil and Mercury as shown in figure. The manometer is opened to atmosphere. What is the gauge reading at 'A'? Assume no flow in the pipe.

$h_A - 3 \times 0.9 + 0.375 \times 0.9 - 0.375 \times 13.6 = 0$

$h_A = 2.0625$ m of water

$p = \gamma \times h$

$= 9.81 \times 21.99$



p = 20.23 kPa (gauge)

p_{abs} = 101.3 +20.23

p_{abs} = 121.53 kPa

- **DIFFERENTIAL MANOMETERS**

Differential manometers are used to measure pressure difference between any two points. Common varieties of differential manometers are:

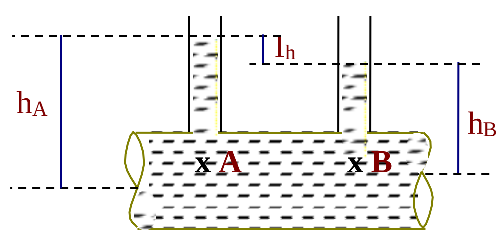
(a) Two piezometers.

(b) Inverted U-tube manometer.

(c) U-tube differential manometers.

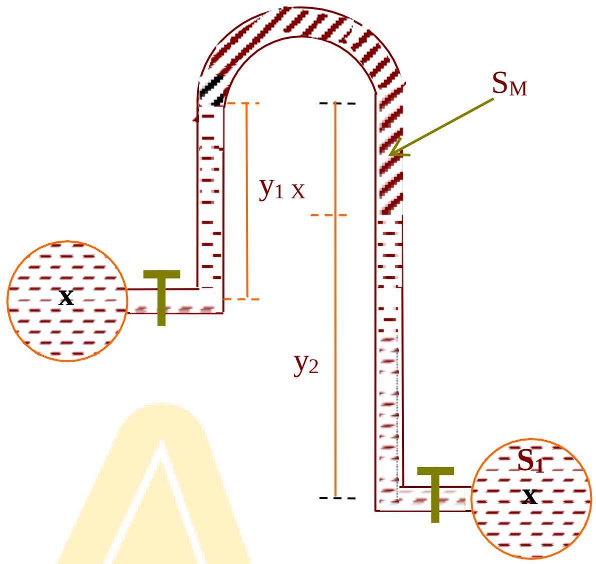
(d) Micro manometers.

(a) Two Pizometers



The arrangement consists of two pizometers at the two points between which the pressure difference is required. The liquid will rise in both the piezometers. The difference in elevation of liquid levels can be recorded and the pressure difference can be calculated. It has all the merits and demerits of piezometer.

(b) Inverted U-tube manometers:



Inverted U-tube manometer is used to measure small difference in pressure between any two points. It consists of an inverted U-tube connecting the two points between which the pressure difference is required. In between there will be a lighter sensitive manometric liquid. Pressure difference between the two points can be calculated by writing the gauge equations for the system.

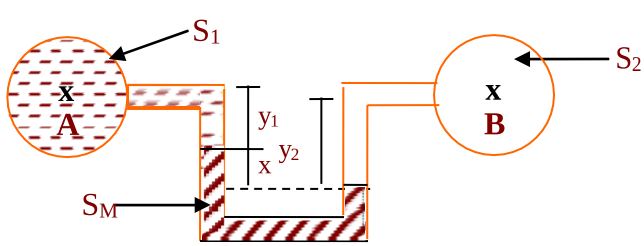
Let 'h_A' and 'h_B' be the pr head at 'A' and 'B' in meters of water

$h_A - (Y_1\ S_1) + (x\ S_M) + (y_2\ S_2) = h_B.$

$h_A - h_B = S_1\ y_1 - S_M\ x - S_2\ y_2,$

$p_A - p_B = \gamma\ (h_A - h_B)$

(c) U-tube Differential manometers



A differential U-tube manometer is used to measure pressure difference between any two points. It consists of a U-tube containing heavier manometric liquid, the two limbs of which are connected to the gauge points between which the pressure difference is required. U-tube differential manometers can also be used for gases. By writing the gauge equation for the system pressure difference can be determined.

Let 'h_A' and 'h_B' be the pressure head of 'A' and 'B' in meters of water

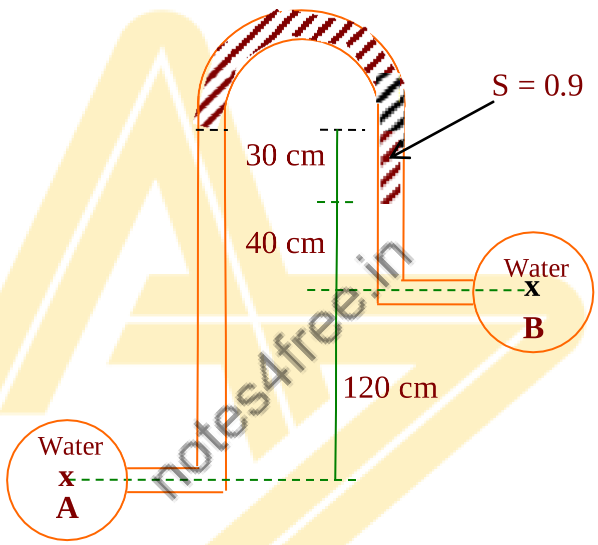
$h_A + S_1\ Y_1 + x\ S_M – Y_2\ S_2 = h_B$

$h_A – h_B = Y_2\ S_2 – Y_1\ S_1 – x\ S_M$

Solved Problems:

(1) An inverted U-tube manometer is shown in figure. Determine the pressure difference between A and B in N/M^2.

Let h_A and h_B be the pressure heads at A and B in meters of water.



$h_A – (190 \times 10^{-2}) + (0.3 \times 0.9) + (0.4)\ 0.9 = h_B$

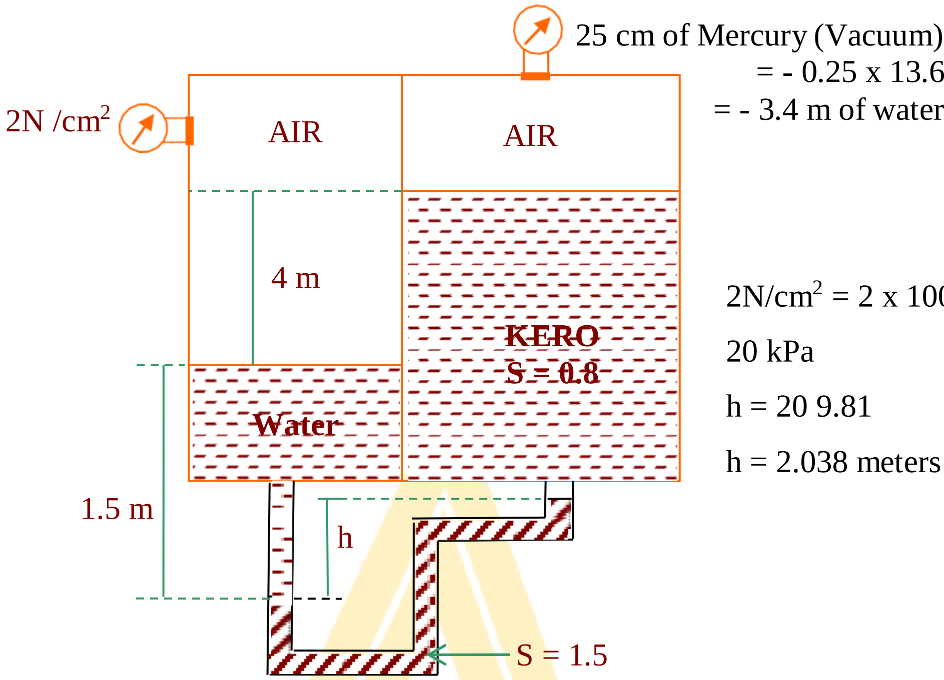
$h_A – h_B = 1.23$ meters of water

$p_A – p_B = \gamma\ (h_A – h_B) = 9.81 \times 1.23$

$p_A – p_B = 12.06$ kPa

$p_A – p_B = 12.06 \times 10^3\ N/m^2$

2. In the arrangements shown in figure. Determine the 'h'.



$2N/cm^2 = 2 \times 100^2\ N/m^2 =$

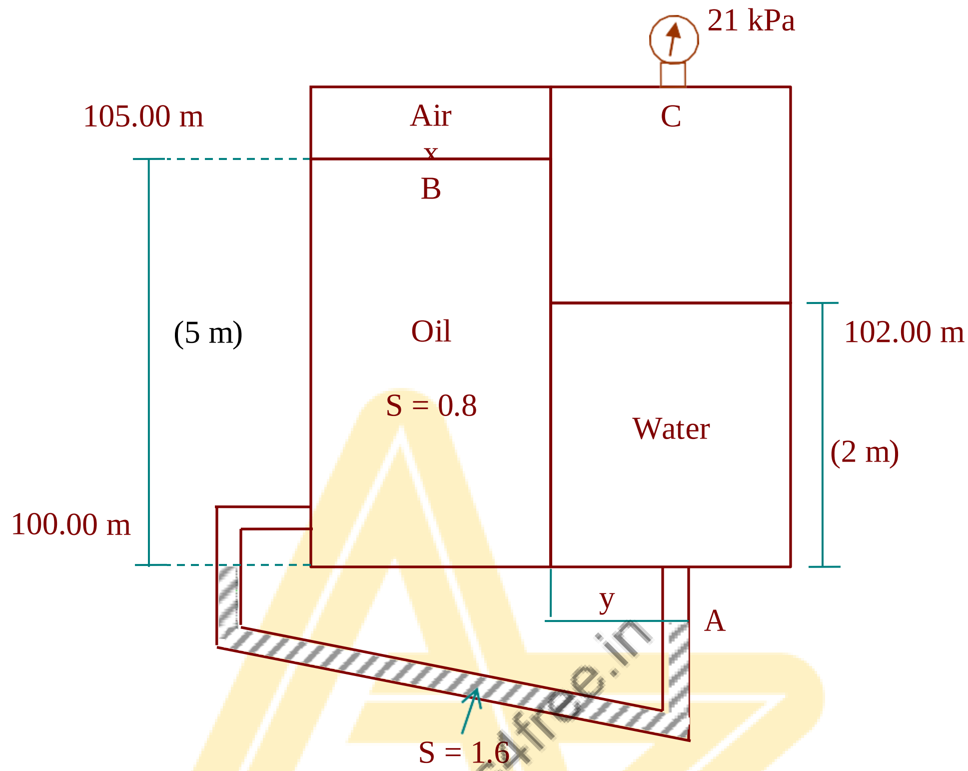
20 kPa

h = 20 9.81

h = 2.038 meters of water

2.038 + 1.5 – (4 + 1.5 – h) 0.8 = – 3.4, h = 3.6 m

3. In figure given, the air pressure in the left tank is 230 mm of Mercury (Vacuum). Determine the elevation of gauge liquid in the right limb at A. If liquid in the right tank is water.



$h_c = \frac{Pc}{\gamma}$

h_B = 230mm of Hg

$\frac{21}{9.81}$

$h_c = 2.14 m\ of\ water$

= 0.23 x 13.6

h_B = - 3.128 m of water

– 3.128 + 5 x 0.8 + y x 1.6 – (y + 2) = 2.14

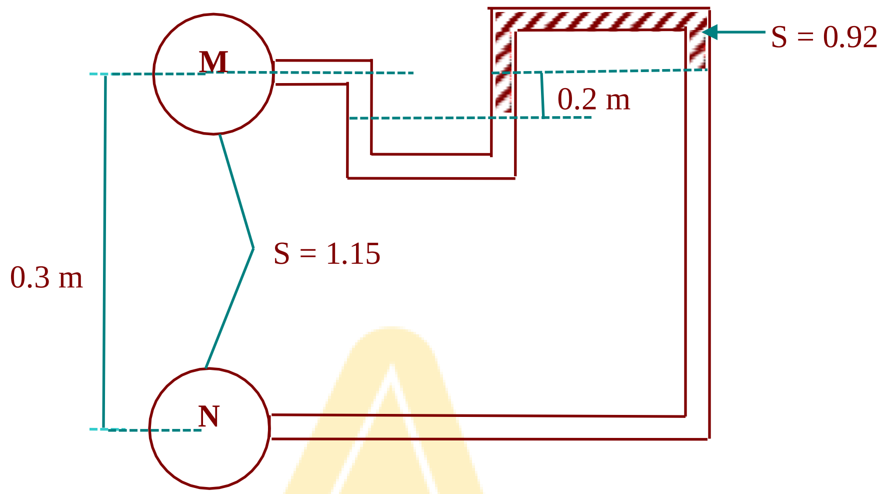
– 3.128 + 5 x 0.8 + y x 1.6 – y – 2 = 2.14

y = 5.446 m

∴ Elevation of A = 100 – 5.446

Elevation of A = 94.553m

4. Compute the pressure different between ‘M’ and ‘N’ for the system shown in figure.



Let ‘h_M’ and ‘ h_N’ be the pressure heads at M and N in m of water.

$hm + y \times 1.15 - 0.2 \times 0.92 + (0.3 - y + 0.2)\, 1.15 = hn$

$hm + 1.15\, y - 0.184 + 0.3 \times 1.15 - 1.15\, y + 0.2 \times 1.15 = hn$

$hm + 0.391 = hn$

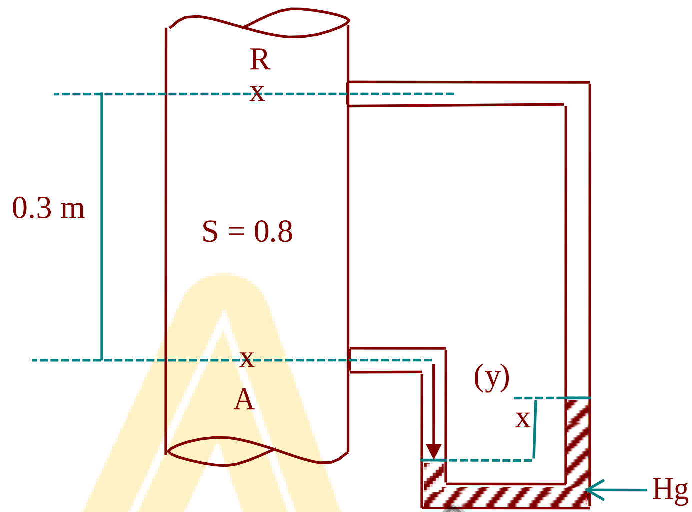
$hn - hm = 0.391$meters of water

$p_n - p_m = \gamma\, (h_N - h_m)$

$= 9.81 \times 0.391$

$p_n - p_m = 3.835$ kPa

5. Petrol of specify gravity 0.8 flows up through a vertical pipe. A and B are the two points in the pipe, B being 0.3 m higher than A. Connection are led from A and B to a U–tube containing Mercury. If the pressure difference between A and B is 18 kPa, find the reading of manometer.



$p_A - p_B = 18 kPa$

$$\frac{P_A - P_B}{\gamma}$$

$$h_A - h_B = \frac{18}{9.81}$$

$$h_A - h_B = 1.835 m \text{ of water}$$

$h_A + y \times 0.8 - x\ 13.6 - (0.3 + y - x)\ 0.8 = h_B$

$h_A - h_B = -0.8y + 13.66\ x + 0.24 + 0.8\ y - 0.8\ x$

$h_A - h_B = 12.8\ x + 0.24$

$1.835 = 12.8x + 0.24$

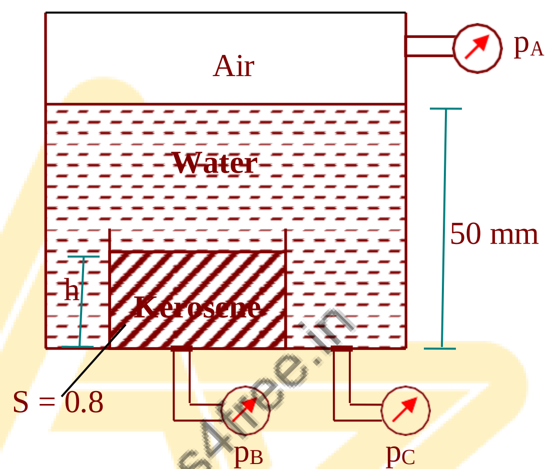
$x = 0.1246$ m

6. A cylindrical tank contains water to a height of 50mm. Inside is a small open cylindrical tank containing kerosene at a height specify gravity 0.8. The following pressures are known from indicated gauges.

p_B = 13.8 kPa (gauge)

p_C = 13.82 kPa (gauge)

Determine the gauge pressure p_A and height h. Assume that kerosene is prevented from moving to the top of the tank.



p_C = 13.82 kPa

h_C = 1.409 m of water

p_B = 13.8 kPa

h_B = 1.407 meters of water

$1.409 - 0.05 = h_A \therefore h_A = 1.359$ meters of water

$\therefore p_A = 1.359 \times 9.81$

$\therefore p_A = 13.33$ kPa

$h_B - h \times 0.8 - (0.05 - h) = h_A$

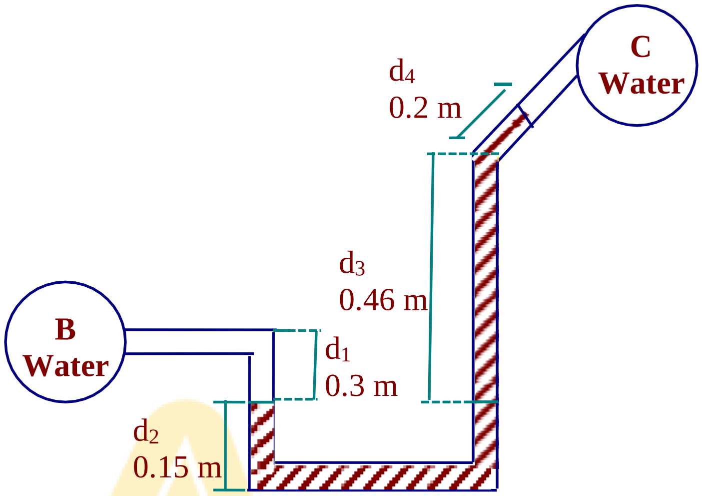
$1.407 - 0.8\,h - 0.05 + h = 1.359$

$0.2\,h = 1.359 - 1.407 + 0.05$

$0.2\,h = 0.002$

$h = 0.02$ m

7. Find the pressure different between A and B if d_1 = 300mm, d_2 = 150mm, d_3 = 460mm, d_4 = 200mm and 13.6.



Let h_A and h_B be the pressure head at A and B in m of water.

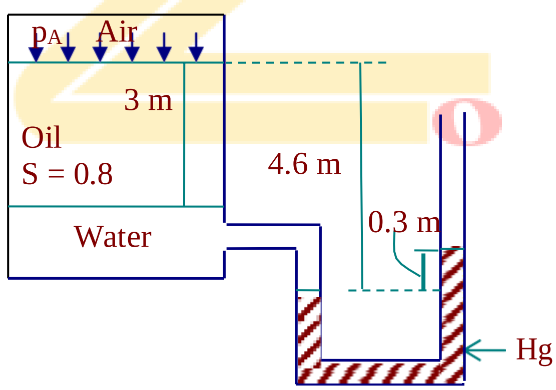
$h_A + 0.3 - (0.46 + 0.2 \sin 45)\, 13.6 = h_B$

$h_A - h_B = 7.88$m of water

$p_A - p_B = (7.88)(9.81)$

$p_A - p_B = 77.29$ kPa

8. What is the pressure p_A in the fig given below? Take specific gravity of oil as 0.8.



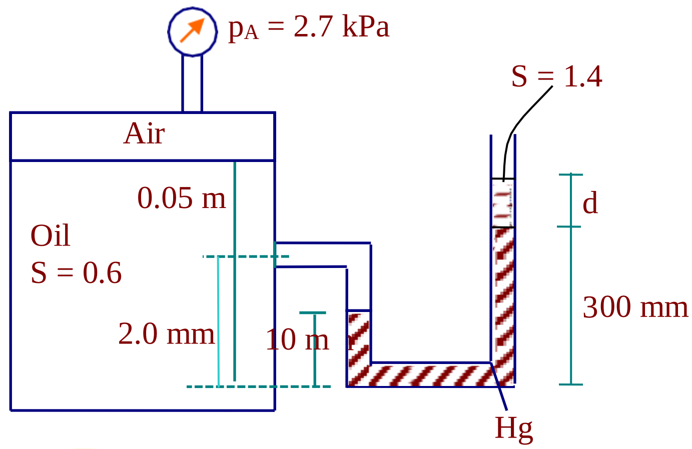
$h_A + (3 \times 0.8) + (4.6 - 0.3)(13.6) = 0$

$h_A = 2.24$ m of oil

$p_A = 9.81 \times 2.24$

$p_A = 21.97$ kPa

9. Find 'd' in the system shown in fig. If $p_A = 2.7\,kPa$



$$h_A = \frac{p_A}{\gamma} = \frac{2.7}{9.81}$$

$$h_A = 0.2752\ m\ of\ \ water$$

$$h_A + (0.05\,x\,0.6) + (0.05 + 0.02 - 0.01)0.6$$

$$+ (0.01x13.6) - (0.03\,x13.6) - d\,x1.4) = 0$$

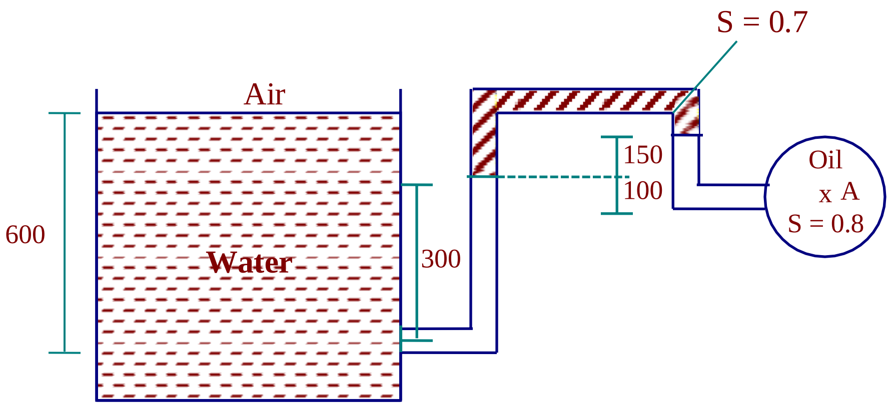
$$0.0692 - 1.4d = 0$$

$$d = 0.0494\ m$$

or

$$d = 49.4\ mm$$

10. Determine the absolute pressure at 'A' for the system shown in fig.



$h_A - (0.25 \times 0.8) + (0.15 \times 0.7) + (0.3 \times 0.8) - (0.6) = 0$

$h_A = 0.455$ m of water

$p_A = h_A \times 9.81$

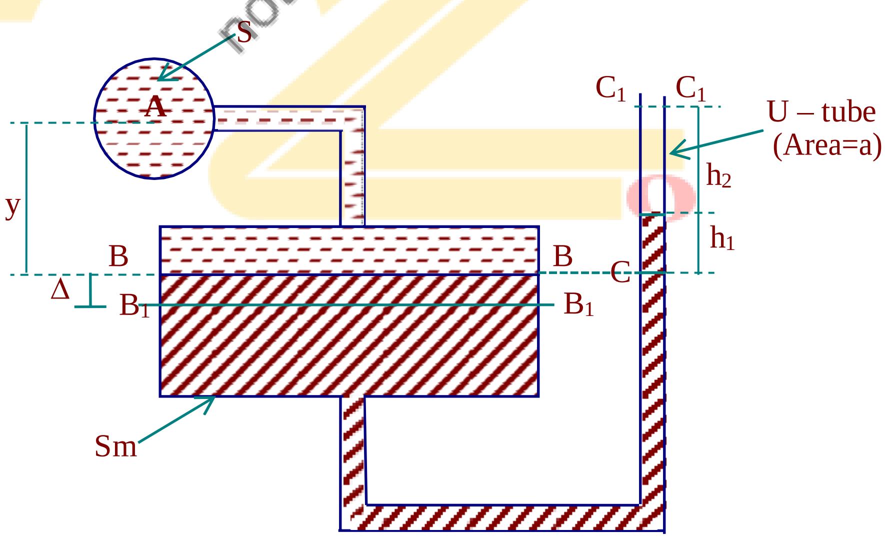
$p_A = 4.464$ kPa

$p_{abs} = 101.3 + 4.464$

$p_{abs} = 105.764$ kPa

SINGLE COLUMN MANOMETER:

Single column manometer is used to measure small pressure intensities.



A single column manometer consists of a shallow reservoir having large cross sectional area when compared to cross sectional area of U – tube connected to it. For any change in pressure, change in the level of manometeric liquid in the reservoir is small (Δ) and change in level of manometric liquid in the U- tube is large.

To derive expression for pressure head at A:

BB and CC are the levels of manometric liquid in the reservoir and U-tube before connecting the point A to the manometer, writing gauge equation for the system we have,

$+ y \times S - h_1 \times S_m = 0$

$\therefore Sy = S_m h_1$

Let the point A be connected to the manometer. B_1B_1 and C_1 C_1 are the levels of manometeric liquid. Volume of liquid between BBB_1B_1 = Volume of liquid between CCC_1C_1

$A\Delta = a\, h_2$

$\Delta = \dfrac{ah_2}{A}$

Let 'h_A' be the pressure head at A in m of water.

$h_A + (y + \Delta)\, S - (\Delta + h_1 + h_2)\, Sm = 0$

$h_A = (\Delta + h_1 + h_2)\, Sm - (y + \Delta)\, S$

$= \Delta\, Sm + \underline{h_1\, Sm} + h_2\, Sm - \underline{yS} - \Delta S$

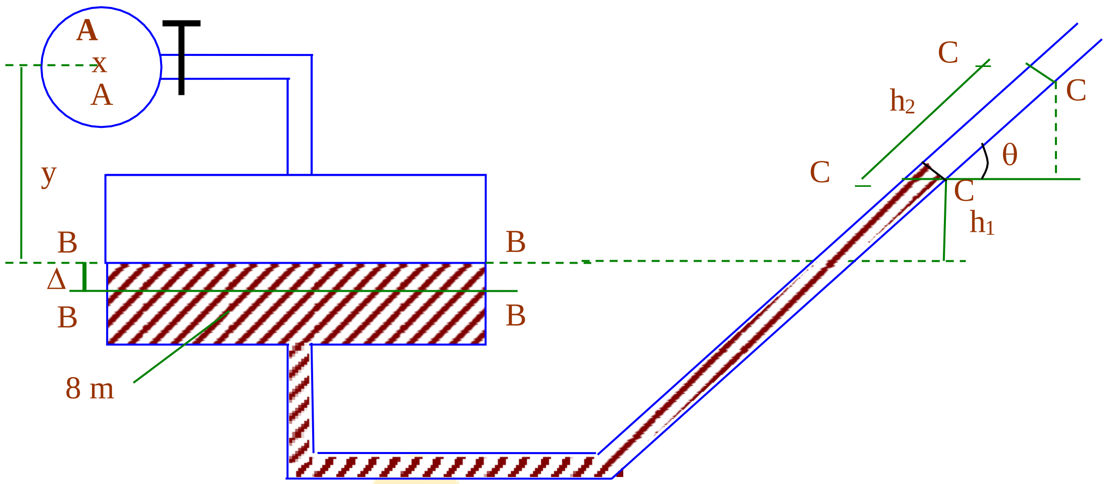
$h_A = \Delta\, (Sm - S) + h_2\, Sm$

$h_A = \dfrac{ah_2}{A}(Sm - S) + h_2\, Sm$

$\therefore$ It is enough if we take one reading to get 'h_2' If '$\dfrac{a}{A}$' is made very small (by increasing 'A') then the I term on the RHS will be negligible.

Then $h_A = h_2\, Sm$

INCLINED TUBE SINGLE COLUMN MANOMETER:



Inclined tube SCM is used to measure small intensity pressure. It consists of a large reservoir to which an inclined U – tube is connected as shown in fig. For small changes in pressure the reading 'h_2' in the inclined tube is more than that of SCM. Knowing the inclination of the tube the pressure intensity at the gauge point can be determined.

$$h_A = \frac{a}{A} h_2 \sin\theta (Sm - S) + h_2 \sin\theta . Sm$$

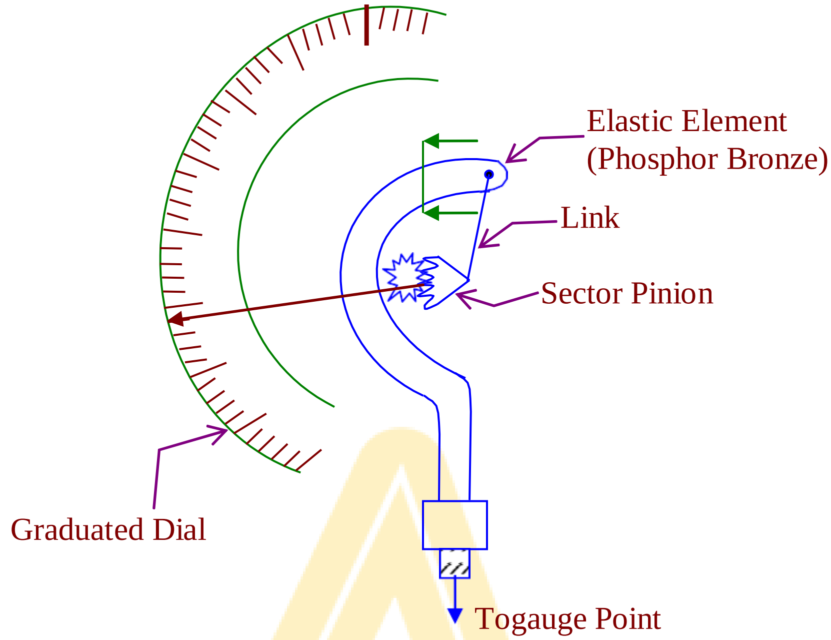
If '$\frac{a}{A}$' is very small then $h_A = (h_2 = \text{Sin}\theta)\, S_m$.

2.7.3 MECHANICAL GAUGES:

Pressure gauges are the devices used to measure pressure at a point. They are used to measure high intensity pressures where accuracy requirement is less. Pressure gauges are separate for positive pressure measurement and negative pressure measurement. Negative pressure gauges are called Vacuum gauges.

Mechanical gauge consists of an elastic element which deflects under the action of applied pressure and this deflection will move a pointer on a graduated dial leading to the measurement of pressure. Most popular pressure gauge used is Bordon pressure gauge.

BASIC PRINCIPLE:

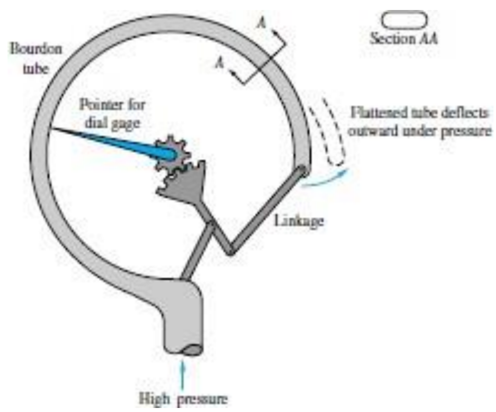


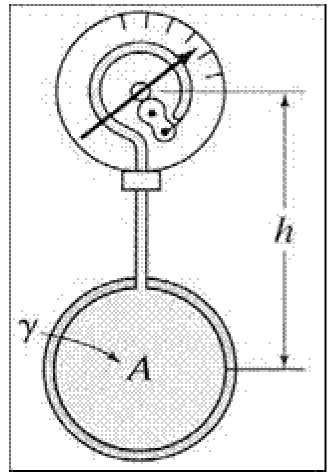
The arrangement consists of a pressure responsive element made up of phosphor bronze or special steel having elliptical cross section. The element is curved into a circular arc, one end of the tube is closed and free to move and the other end is connected to gauge point. The changes in pressure cause change in section leading to the movement. The movement is transferred to a needle using sector pinion mechanism. The needle moves over a graduated dial.

Bourdon gage:

Is a device used for measuring gauge pressures the pressure element is a hollow curved metallic tube closed at one end the other end is connected to the pressure to be measured. When the internal pressure is increased the tube tends to straighten pulling on a linkage to which is attached a pointer and causing the pointer to move. When the tube is connected the pointer shows zero. The *bourdon tube*, sketched in figure.

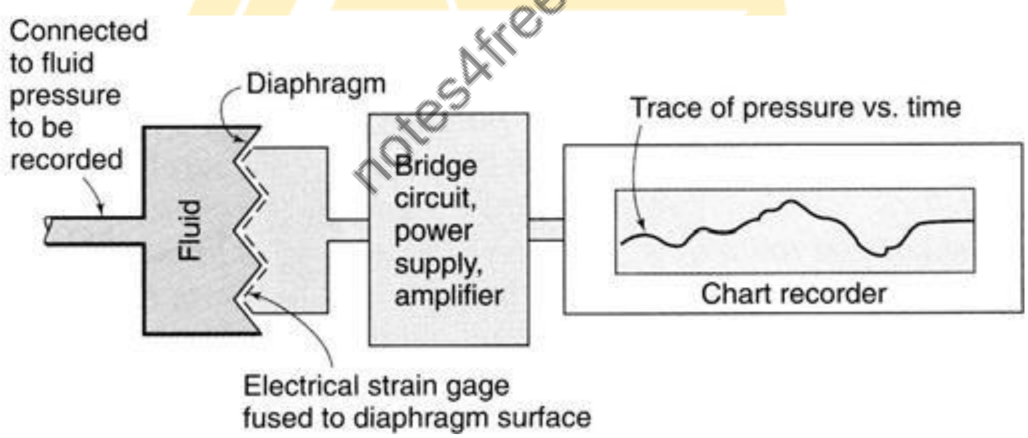
It can be used for the measurement of liquid and gas pressures up to 100s of MPa.





2.7.4 Electronic Pressure Measuring Devices:

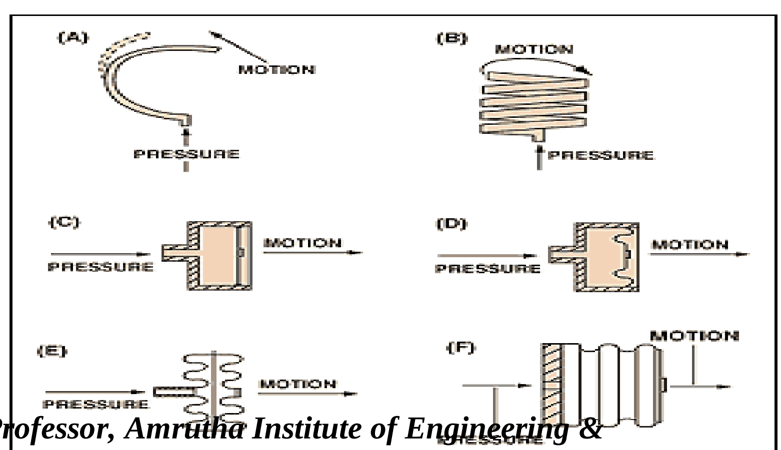
Electronic Pressure transducers convert pressure into an electrical output. These devices consist of a sensing element, transduction element and signal conditioning device to convert pressure readings to digital values on display panel.



Sensing Elements:

The main types of sensing elements are

- Bourdon tubes,
- Diaphragms,
- Capsules, and
- Bellows.



Pressure Transducers:

A transducer is a device that turns a mechanical signal into an electrical signal or an electrical signal into a mechanical response (e.g., Bourdon gage transfers pressure to displacement).

There are a number of ways to accomplish this kind of conversion

- Strain gage
- Capacitance
- Variable reluctance
- Optical

Normally Electronic Pressure transducers are costly compared to conventional mechanical gauges and need to be calibrated at National laboratories before put in to use.

FLUID DYNAMICS

Forces acting on the fluids

Following are the forces acting on the fluids

1. Self-Weight
2. Pressure Forces, F $_p$
3. Viscous Force, F $_v$
4. Turbulent Force, F $_t$
5. Surface Tension Force,
6. Compressibility Force,

Dynamics of fluid is governed by Newton's Second law of motion, which states that the resultant force on any fluid element must be equal to the product of the mass and the acceleration of the element.

$$\sum F = Ma$$

Surface tension forces and Compressibility forces are not significant and may be neglected. Hence

(1) becomes

$$\sum F = F_g + F_p + F_v + F_t$$

Reynold's Equation of motion and used in the analysis of Turbulent flows. For laminar flows, turbulent force becomes less significant and hence (1) becomes

$$\sum F = F_g + F_p + F_v$$

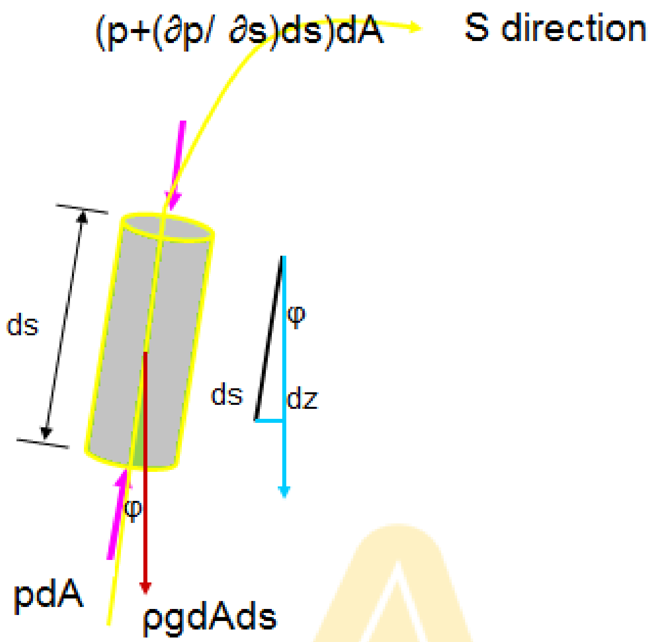
Navier - Stokes Equation. If viscous forces are neglected then the (1) reduces to

$$\sum F = F_g + F_p = M \times a$$

Euler's Equation of motion.

Euler equation of motion

Consider a stream lime in a flowing fluid in S direction as shown in the figure. On this stream line consider a cylindrical element having a cross sectional area *dA* and length *ds*.



Forces acting on the fluid element are: Pressure forces at both ends:

- Pressure force, pdA in the direction of flow
- Pressure force (p+(∂ p/∂ s)ds)dA in the direction opposite to the flow direction
- Weight of element *ρdads* acting vertically downwards

Let φ be the angle between the direction of flow and the line of action of the weight of the element. The resultant force on the fluid element in the direction of *s* must be equal to mass of fluid element× acceleration in direction *s (according to Newton's second law of motion)*

$$pda - (pda + (\partial p/\partial s)ds)dA) - \rho gd\cos\phi = \rho dadsa_s \qquad \text{(a)}$$

where $a_s$is the accelaration in direction of s now

$$a_s = \frac{dv}{dt}$$

where v is function of s and t

$$= \frac{\partial v}{\partial s}\frac{ds}{dt} + \frac{\partial v}{\partial t}$$
$$= v\frac{\partial v}{\partial s}\frac{ds}{dt} + \frac{\partial v}{\partial t}$$

since

$$\frac{ds}{dt} = v$$

If the flow is steady,

$$\frac{\partial v}{\partial t} = 0$$

hence,

$$a_s = v\frac{\partial v}{\partial s}$$

Substituting the valve of a_s in equation (a) and simplifying,

$$-\frac{\partial p}{\partial s}dsdA - \rho g dsdA\cos\phi = \rho dads \times v\frac{\partial v}{\partial s}$$

Dividing the whole equation by $\rho dsdA$,

$$-\frac{\partial p}{\rho\partial s} - g\cos\phi = v\frac{\partial v}{\partial s}$$

$$\Rightarrow \frac{\partial p}{\rho\partial s} + g\cos\phi + v\frac{\partial v}{\partial s} = 0$$

But from the figure we have

$$\cos\phi = \frac{dz}{ds}$$

Hence,

$$\frac{1}{\rho}\frac{\partial p}{\partial s} + g\frac{dz}{ds} + v\frac{\partial v}{\partial s} = 0$$

or

$$\frac{\partial p}{\rho} + gdz + vdv = 0 \qquad \text{(b)}$$

Equation (b)is known as Euler's equation of motion.

Bernoulli's Equation of motion from Euler's equation

Statement: *In a steady, incompressible fluid, the total energy remains same along a streamline throughout the reach.*

Bernoulli's equation may be obtained by integrating Euler's equation of motion i.e, equation (b) as

$$\int \frac{dp}{\rho} + \int gdz + \int vdv = constant$$

If the flow is in-compressible, ρ is constant and hence,

$$\frac{p}{\rho} + gz + \frac{v^2}{2} = constant$$

$$\Rightarrow \frac{p}{\rho g} + \frac{v^2}{2g} + z = constant \qquad \text{(c)}$$

Equation (c) is called as Bernoulli's equation, Where

$\frac{p}{\rho}$ = pressure energy per unit weight of the fluid or also called as pressure head

$\frac{v^2}{2g}$ = kinetic energy per unit weight of the fluid or kinetic head

z= potential energy per unit weight or potential head

Assumption made in deriving the Bernoulli's Equation

Following assumptions were made to derive the bernoulli's equation

•The flow is steady

•The flow is ideal (Viscosity of the fluid is zero)

•The flow is in-compressible

•The flow is irrotational.

Limitations on the use of the Bernoulli Equation

- **Steady flow**: The first limitation on the Bernoulli equation is that it is applicable to steady flow.
- **Friction-less flow**: Every flow involves some friction, no matter how small, and frictional effects may or may not be negligible.
- **In-compressible flow**: One of the assumptions used in the derivation of the Bernoulli equation is that ρ = constant and thus the flow is in-compressible. Strictly speaking, the Bernoulli equa- tion is applicable along a streamline, and the value of the constant *C*, in general, is different for different streamlines. But when a region of the flow is irrational, and thus there is no vorticity in the flow field, the value of the constant *C* remains the same for all streamlines, and, therefore, the Bernoulli equation becomes applicable across streamlines as well.

Kinetic Energy correction factor

In deriving the Bernoulli's Equation, the velocity head or the kinetic energy per unit weight of the fluid has been computed based on the assumption that the velocity is uniform over the entire cross section of the stream tube. But in real fluids, the velocity distribution is not uniform. Therefore, to obtain the kinetic energy possessed by the fluid at differently sections is obtained by integrating the kinetic energies possessed by different fluid particles.

It is more convenient to express the kinetic energy in terms of the mean velocity of flow. But the actual kinetic energy is greater than the computed using the mean velocity. Hence a correction factor called ' Kinetic Energy correction factor,α is introduced.

$$\frac{p_1}{\rho}+\alpha_1\left(\frac{v_1^2}{2g}\right)+z_1=\frac{p_2}{\rho}+\alpha_2\left(\frac{v_2^2}{2g}\right)+z_2+h_L=Constant$$

In most of the problems of turbulent flow, the value of $\alpha=1$.

Rotary or Vortex Motion

A mass of fluid in rotation about a fixed axis is called vortex. The rotary motion of fluid is also called vortex motion. In this case the rotating fluid particles have velocity in tangential direction. Thus the vortex motion is defined as motion in which the whole fluid mass rotates about an axis.

The vortex motion is of twotypes:

1. Free vortex.
2. Forced vortex.

Free vortex flow

Free vortex flow is that type of flow in which the fluid mass rotates without any external applied contact force. The whole mass rotates either due to fluid pressure itself or the gravity or due to rotation previously imparted. Energy is not expended to any outside source. The free vortex motion is also called Potential vortex or Ir-rotational vortex.

Relationship between velocity and radius in free vortex

It is obtained by putting the value of external torque equal to **Zero** or on other words the time rate of change of angular momentum, i.e., moment of the momentum must be Zero. Consider a fluid particle of mass 'M' at a radial distance 'r' from the axis of rotation, having a tangential velocity 'u'. Then,

$$Angular\ momentum = Mass \times velocity$$

$$Moment\ of\ the\ Momentum = Momentum \times radius = mur$$

$$Time\ rate\ of\ change\ of\ angular\ momentum = \frac{\partial (mur)}{\partial t}$$

But for free vortex,

$$\frac{\partial (mur)}{\partial t} = 0$$

Integrating, we get

$$\int \frac{\partial (mur)}{\partial t} = 0 \Rightarrow Mur = Constant = ur = constant$$

Forced vortex flow

Forced vortex motion is one in which the fluid mass is made to rotate by means of some external agencies. The external agency is generally the mechanical power which imparts the constant torque on the fluid mass. The forced vortex motion is also called flywheel vortex or rotational vortex. The fluid mass in this forced vortex flow rotates at constant angular velocity ω. The tangential velocity of any fluid particle is given by,

$$u = \omega \times r$$

where 'r' is the radius of the fluid particle from the axis of rotation. Hence angular velocity ω is given by,

$$\omega = \frac{u}{r} = constant$$

Variation of pressure of a rotating fluid in any plane is given by,

$$dp = \rho(\frac{\omega^2 r^2}{r})dr - \rho g dz$$

Integrating the above equation for points 1 and 2, we get

$$\int_1^2 dp = \int_1^2 \rho(\frac{\omega^2 r^2}{r})dr - \int_1^2 \rho g dz$$

$$\Rightarrow (p_2 - p_1) = [\rho\omega^2 \frac{r^2}{2}]_1^2 - \rho g[z]_1^2$$

$$\Rightarrow \quad = \frac{\rho}{2}[u_2^2 - u_1^2] - \rho g[z_2 - z_1]$$

if the point 1 and 2 lies on free surface of the liquid, then $p_1 = p_2$ and hence above equation reduces to

$$[z_2 - z_1] = \frac{1}{2g}[v_2^2 - v_1^2]$$

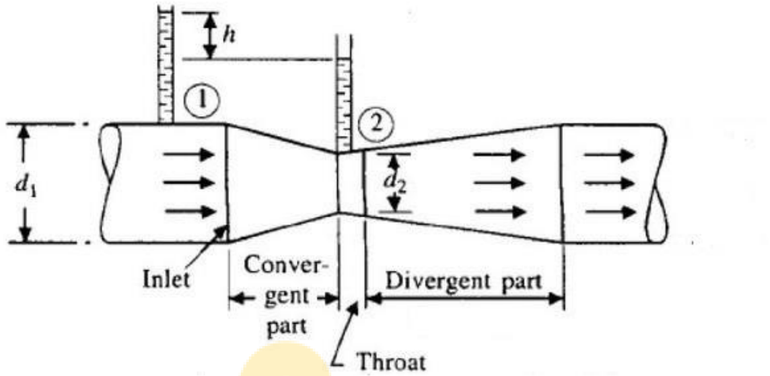
If the point 1 lies on the axis of rotaion, then $v_1 = \omega \times r_1 = \omega \times 0 = 0$,hence above equation reduces to,

$$Z = z_2 - z_1 = \frac{u_2^2}{2g} = \frac{\omega^2 r_2^2}{2g}$$

APPLICATIONS OF BERNOULLI'S EQUATION

Venturi Meter

Venturimeter is a device for measuring discharge in a pipe.



A Venturi meter consists of:

- Inlet/ Convergent cone
- Throat
- Outlet/ Divergent cone

The inlet section Venturi meter is same diameter as that type of the pipe to which it is connected, followed by the short convergent section with a converging cone angle of 21 ± 1^o and its length parallel to the axis is approximately equal to $2.7(D–d)$, where 'D' is the pipe diameter and 'd' is the throat diameter.

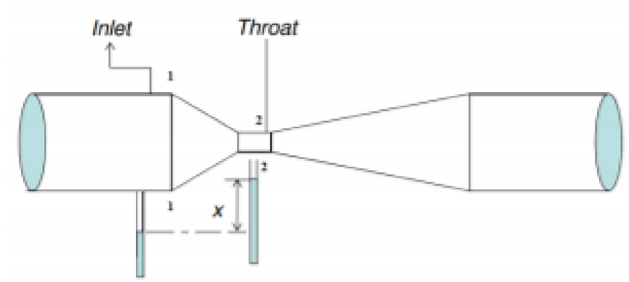
The cylindrical throat is a section of constant cross-section with its length equal to diameter. The flow is minimum at the throat. Usually, diameter of throat is $\frac{1}{2}$ the pipe diameter.

A long diverging section with a cone angle of about 5-7^o where in the fluid is retarded and a large portion of the kinetic energy is converted back into the pressure energy.

Principle of Venturi Meter:

The basic principle on which a Venturi meter works is that by reducing the cross-sectional area of the flow passage, a pressure difference is created between the two sections, this pressure difference enables the estimation of the flow rate through the pipe.

Expression for Discharge through Venturi meter



Let, d_1=diameter at section 1-1
p_1= pressure at section at 1-1
v_1= velocity at section at 1-1
a_1= area of cross-section at 1-1
d_2, p_2, v_2, a_2 be corresponding values at section 2-2.

Applying Bernoulli equation between 1-1 and 2-2 we have,

$$\frac{p_1}{\rho g}+\frac{v_1^2}{2g}+z_1=\frac{p_2}{\rho g}+\frac{v_2^2}{2g}+z_2$$

Since pipe is horizontal, $z_1=z_2$,
Hence,

$$\frac{p_1}{\rho g}+\frac{v_1^2}{2g}=\frac{p_2}{\rho g}+\frac{v_2^2}{2g}$$
$$\Rightarrow \frac{p_1-p_2}{\rho g}=\frac{v_2^2-v_1^2}{2g}$$
$$\Rightarrow h=\frac{v_2^2-v_1^2}{2g}$$

where $h=\frac{p_1-p_2}{\rho g}$, is the pressure difference between section 1-1 and 2-2.
from continuity equation, we have

$$a_1v_1=a_2v_2$$
$$\Rightarrow v_1=\frac{a_2v_2}{a_1}$$

Hence

$$h=\frac{v_2^2}{2g}\left[\frac{a_2^2-a_1^2}{a_1^2}\right]$$
$$\Rightarrow v_2=\frac{a_1}{\sqrt{a_1^2-a_2^2}}\sqrt{2gh} \qquad (1)$$

substituting the value of v_2 in equation $Q = a_2 v_2$ we have,

$$Q_{th} = \frac{a_1 a_2}{\sqrt{a_1^2 - a_2^2}} \sqrt{2gh}$$

Above equations is for ideal fluids and is called as the theoretical discharge equation of a venturi meter. For real fluids the equation changes to,

$$Q_{act} = C_d \frac{a_1 a_2}{\sqrt{a_1^2 - a_2^2}} \sqrt{2gh}$$

Expression for 'h' given by the differential manometer

- **Case 1:** when liquid in the manometer is heavier than the liquid flowing through the pipe.

$$h = x \left[\frac{S_H}{S_O} - 1 \right]$$

where: S_H is the specific gravity of heavier liquid

S_O is the specific gravity of liquid flowing through pipe.

x difference in liquid columns in U-tube.

- **Case 2:** when liquid in the manometer is lighter than the liquid flowing through the pipe.

$$h = x \left[1 - \frac{S_L}{S_O} \right]$$

where: S_L is the specific gravity of heavier liquid

S_O is the specific gravity of liquid flowing through pipe.

x difference in liquid columns in U-tube.

Orifice Meter

An orifice is a small aperture through which the fluid passes. The thickness of an orifice in the direction of flow is very small in comparison to its other dimensions.

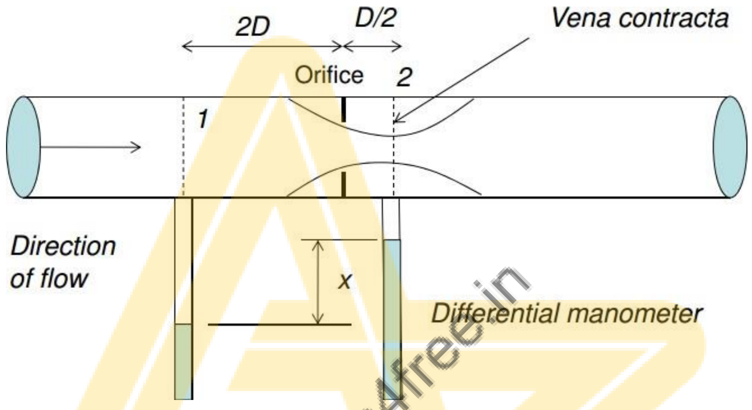
If a tank containing a liquid has a hole made on the side or base through which liquid flows, then such a hole may be termed as an orifice. The rate of flow of the liquid through such an orifice at a given time will depend partly on the shape, size and form of the orifice.

An orifice usually has a sharp edge so that there is minimum contact with the fluid and conse- quently minimum frictional resistance at the sides of the orifice. If a sharp edge is not provided, the flow depends on the thickness of the orifice and the roughness of its boundary surface too.

Orifice Meter

- It is a device used for measuring the rate of flow through a pipe.
- It is a cheaper device as compared to venturi meter. The basic principle on which the Orifice meter works is same as that of Venturi meter.
- It consists of a circular plate with a circular opening at the center. This circular opening is called an Orifice.
- The diameter of the orifice is generally varies from 0.4 to 0.8 times the pipe diameter.

Expression for Discharge through Orifice meter



Let, d_1=diameter at section 1-1

p_1= pressure at section at 1-1

v_1= velocity at section at 1-1

a_1= area of cross-section at 1-1

d_2, p_2, v_2, a_2 be corresponding values at section 2-2.

Applying Bernoulli equation between 1-1 and 2-2 we have,

$$\frac{p_1}{\rho g} + \frac{v_1^2}{2g} + z_1 = \frac{p_2}{\rho g} + \frac{v_2^2}{2g} + z_2$$

Since pipe is horizontal, $z_1=z_2$,
Hence,

$$\frac{p_1}{\rho g}+\frac{v_1^2}{2g}=\frac{p_2}{\rho g}+\frac{v_2^2}{2g}$$

$$\Rightarrow \frac{p_1-p_2}{\rho g}=\frac{v_2^2-v_1^2}{2g}$$

or

$$h=\frac{v_2^2-v_1^2}{2g}$$

or

$$2gh=v_2^2-v_1^2$$

$$v_2=\sqrt{2gh+v_1^2} \qquad \text{(i)}$$

Now section (2) is at the vena-contracta and a_2 represents the area at the vena-contracta. If the area a_o is the area of the orifice, then we have

$$C_c=\frac{a_2}{a_o}$$

where C_c is the co-efficient of contraction.
$\therefore$

$$a_2=a_o\times C_c$$

From continuity equation, we have

$$a_1v_1=a_2v_2 \qquad or \qquad \text{(ii)}$$

$$v_1=\frac{a_2v_2}{a_1}v_2=\frac{a_oC_c}{a_1}v_2 \qquad \text{(ii)}$$

Substituting the value of v_1 in equation (i), we get

$$v_2=\sqrt{2gh+\frac{a_o^2C_c^2v_2^2}{a_1^2}}$$

$$\Rightarrow v_2=\frac{\sqrt{2gh}}{\sqrt{1-\left(\frac{a_o}{a_1}\right)^2C_c^2}}$$

substituting the value of v_2 in equation $Q=a_2v_2$ we have,

$$Q=\frac{a_oC_c\sqrt{2gh}}{\sqrt{1-\left(\frac{a_o^2}{a_1^2}\right)C_c^2}} \qquad \text{(iv)}$$

Above equation can be simplified by using

$$C_c = C_d \frac{\sqrt{1-\left(\frac{a_o}{a_1}\right)^2}}{\sqrt{1-\left(\frac{a_o}{a_1}\right)^2 C_c^2}}$$

or

$$C_d = C_c \frac{\sqrt{1-\left(\frac{a_o}{a_1}\right)^2 C_c^2}}{\sqrt{1-\left(\frac{a_o}{a_1}\right)^2}}$$

Substituting the this value of C_c in(iv),

$$Q_{act} = a_o \times C_d \frac{\sqrt{1-\left(\frac{a_o}{a_1}\right)^2 C_c^2}}{\sqrt{1-\left(\frac{a_o}{a_1}\right)^2}} \times \frac{\sqrt{2gh}}{\sqrt{1-\left(\frac{a_o^2}{a_1^2}\right) C_c^2}}$$

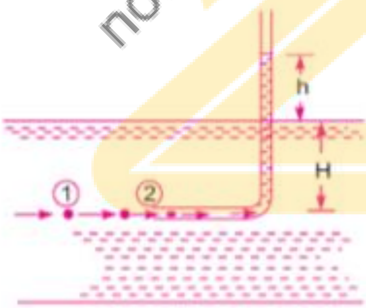
or

$$Q_{act} = \frac{C_d a_o a_1 \sqrt{2gh}}{\sqrt{a_1^2 - a_o^2}}$$

where C_d is the co-efficient of discharge for orifice meter.

Pitot tube

Pitot tube is a device used to measure the velocity of flow at any point in a pipe or a channel.
Principle: If the velocity at any point decreases, the pressure at that point increases due to the conversion of the Kinetic energy into pressure energy. In simplest form, the pitot tube consists of a glass tube, bent at right angles.



Let, p_1= pressure at section at 1-1
v_1= velocity at section at 1-1
p_2= pressure at section at 1-1
v_2= velocity at section at 1-1
H= depth of tube in the liquid
h= rise of liquid in the tube above free surface

Applying Bernoulli equation between 1-1 and 2-2 we have,

$$\frac{p_1}{\rho g}+\frac{v_1^2}{2g}+z_1=\frac{p_2}{\rho g}+\frac{v_2^2}{2g}+z_2$$

But $z_1=z_2$ as points(1)and (2) are on the same line and $v_2=0$

$\frac{p_1}{\rho g}$= pressure head at (1)=H

$\frac{p_2}{\rho g}$= pressure head at (2)=(h+H)

Substituting these values we get,

$$H+\frac{v_1^2}{2g}=(h+H)\therefore h=\frac{v_2}{2g} \quad or \quad v_1=\sqrt{2gh}$$

this is the theoretical velocity. Actual velocity is given by

$$(v_1)_{act}=C_v\sqrt{2gh}$$

there fore velocity at any point is,

$$v_{act}=C_v\sqrt{2gh}$$

P1. An oil of sp.gr. 0.8 is flowing through a venturimeter having inlet diameter 20cm and throat 10cm. The oil mercury differential manometer shows a reading of 25cm. Calculate the discharge of oil through the horizontal venturimeter. Take Cd= 0.98.

Solution. Given :

Sp. gr. of oil, $S_o = 0.8$

Sp. gr. of mercury, $S_h = 13.6$

Reading of differential manometer, $x = 25$ cm

$\therefore$ Difference of pressure head, $h = x\left[\frac{S_h}{S_o} - 1\right]$

$$= 25\left[\frac{13.6}{0.8} - 1\right] \text{ cm of oil} = 25\,[17 - 1] = 400 \text{ cm of oil}$$

Dia. at inlet, $d_1 = 20$ cm

$\therefore$ $a_1 = \frac{\pi}{4} d_1^2 = \frac{\pi}{4} \times 20^2 = 314.16 \text{ cm}^2$

$d_2 = 10$ cm

$\therefore$ $a_2 = \frac{\pi}{4} \times 10^2 = 78.54 \text{ cm}^2$

$C_d = 0.98$

$\therefore$ The discharge Q is given by equation (6.8)

or $$Q = C_d \frac{a_1 a_2}{\sqrt{a_1^2 - a_2^2}} \times \sqrt{2gh}$$

$$= 0.98 \times \frac{314.16 \times 78.54}{\sqrt{(314.16)^2 - (78.54)^2}} \times \sqrt{2 \times 981 \times 400}$$

$$= \frac{21421375.68}{\sqrt{98696 - 6168}} = \frac{21421375.68}{304} \text{ cm}^3\text{/s}$$

P2. A horizontal venturimeter with inlet diameter 30cm and throat diameter 15cm is used to measure the flow of water. The differential manometer connected to the inlet and throat is 20cm. Calculate the discharge. Take Cd= 0.98.

Solution. Given :

Dia. at inlet, $d_1 = 30$ cm

$\therefore$ Area at inlet, $a_1 = \frac{\pi}{4} d_1^2 = \frac{\pi}{4} (30)^2 = 706.85 \text{ cm}^2$

Dia. at throat, $d_2 = 15$ cm

$\therefore$ $a_2 = \frac{\pi}{4} \times 15^2 = 176.7 \text{ cm}^2$

$C_d = 0.98$

Reading of differential manometer $= x = 20$ cm of mercury.

$\therefore$ Difference of pressure head is given by (6.9)

or
$$h = x\left[\frac{S_h}{S_o} - 1\right]$$

where S_h = Sp. gravity of mercury = 13.6, S_0 = Sp. gravity of water = 1

$$= 20\left[\frac{13.6}{1} - 1\right] = 20 \times 12.6 \text{ cm} = 252.0 \text{ cm of water.}$$

The discharge through venturimeter is given by eqn. (6.8)

$$Q = C_d \frac{a_1 a_2}{\sqrt{a_1^2 - a_2^2}} \times \sqrt{2gh}$$

$$= 0.98 \times \frac{706.85 \times 176.7}{\sqrt{(706.85)^2 - (176.7)^2}} \times \sqrt{2 \times 9.81 \times 252}$$

$$= \frac{86067593.36}{\sqrt{499636.9 - 31222.9}} = \frac{86067593.36}{684.4}$$

$$= 125756 \text{ cm}^3/\text{s} = \frac{125756}{1000} \text{ lit/s} = \mathbf{125.756 \text{ lit/s.}}$$ **Ans.**

P3. A horizontal venturimeter with inlet diameter 20cm and throat diamete

0.8. The discharge of oil through venturimeter is 60li/s.
Find the reading of the oil-mercury manometer. Take
Cd= 0.98

Solution. Given : $d_1 = 20\text{ cm}$

$\therefore \quad a_1 = \frac{\pi}{4} 20^2 = 314.16\text{ cm}^2$

$d_2 = 10\text{ cm}$

$\therefore \quad a_2 = \frac{\pi}{4} \times 10^2 = 78.54\text{ cm}^2$

$C_d = 0.98$

$Q = 60\text{ litres/s} = 60 \times 1000\text{ cm}^3/\text{s}$

Using the equation (6.8), $Q = C_d \frac{a_1 a_2}{\sqrt{a_1^2 - a_2^2}} \times \sqrt{2gh}$

or $$60 \times 1000 = 9.81 \times \frac{314.16 \times 78.54}{\sqrt{(314.16)^2 - (78.54)^2}} \times \sqrt{2 \times 981 \times h} = \frac{1071068.78\sqrt{h}}{304}$$

or $$\sqrt{h} = \frac{304 \times 60000}{1071068.78} = 17.029$$

$\therefore \quad h = (17.029)^2 = 289.98\text{ cm of oil}$

But $$h = x\left[\frac{S_h}{S_o} - 1\right]$$

where S_h = Sp. gr. of mercury = 13.6

S_o = Sp. gr. of oil = 0.8

x = Reading of manometer

$$\therefore \quad 289.98 = x\left[\frac{13.6}{0.8} - 1\right] = 16x$$

$$\therefore \quad x = \frac{289.98}{16} = 18.12\text{ cm.}$$

$\therefore$ Reading of oil-mercury differential manometer = **18.12 cm. Ans.**

P5. The inlet and throat diameters of a horizontal venturimeter are 30cm and 10cm respectively. The liquid flowing through the venturimeter is water. The pressure intensity at inlet is 13.734N/cm^2 while the vacuum pressure head at the throat is 37 cm of mercury. Find the rate of flow. Assume that 4% of the differential head is lost between the inlet and the throat. Find also the values of C_d for the Venturimeter.

Dia. at inlet, $d_1 = 30$ cm

$\therefore$ $a_1 = \frac{\pi}{4}(30)^2 = 706.85 \text{ cm}^2$

Dia. at throat, $d_2 = 10$ cm

$\therefore$ $a_2 = \frac{\pi}{4}(10)^2 = 78.54 \text{ cm}^2$

Pressure, $p_1 = 13.734 \text{ N/cm}^2 = 13.734 \times 10^4 \text{ N/m}^2$

$\therefore$ Pressure head, $\frac{p_1}{\rho g} = \frac{13.734 \times 10^4}{1000 \times 9.81} = 14$ m of water

$\frac{p_2}{\rho g} = -37$ cm of mercury

$= \frac{-37 \times 13.6}{100}$ m of water $= -5.032$ m of water

Differential head, $h = p_1/\rho g - p_2/\rho g$

$= 14.0 - (-5.032) = 14.0 + 5.032$

$= 19.032$ m of water $= 1903.2$ cm

Head lost, $h_f = 4\%$ of $h = \frac{4}{100} \times 19.032 = 0.7613$ m

$\therefore$ $C_d = \sqrt{\frac{h - h_f}{h}} = \sqrt{\frac{19.032 - .7613}{19.032}} = 0.98$

$\therefore$ Discharge $= C_d \frac{a_1 a_2 \sqrt{2gh}}{\sqrt{a_1^2 - a_2^2}}$

$= \frac{0.98 \times 706.85 \times 78.54 \times \sqrt{2 \times 981 \times 1903.2}}{\sqrt{(706.85)^2 - (78.54)^2}}$

$= \frac{105132247.8}{\sqrt{499636.9 - 6168}} = 149692.8 \text{ cm}^3\text{/s} = \mathbf{0.14969 \text{ m}^3\text{/s. Ans.}}$

A 30cmX15cm Venturimeter is inserted in a vertical pipe carrying water flowing in the upward direction. A differential mercury manometer connected to the inlet and throat gives a reading of 20cm. Find the discharge. Take Cd = 0.98

Solution. Given :

Dia. at inlet, $d_1 = 30$ cm

$\therefore$ $a_1 = \frac{\pi}{4}(30)^2 = 706.85 \text{ cm}^2$

Dia. at throat, $d_2 = 15$ cm

$\therefore$ $a_2 = \frac{\pi}{4}(15)^2 = 176.7 \text{ cm}^2$

$$h = x\left[\frac{S_l}{S_o} - 1\right] = 20\left[\frac{13.6}{1.0} - 1.0\right] = 20 \times 12.6 = 252.0 \text{ cm of water}$$

$C_d = 0.98$

Discharge, $Q = C_d \frac{a_1 a_2}{\sqrt{a_1^2 - a_2^2}} \times \sqrt{2gh}$

A 20cmX10cm venturimeter is inserted in a vertical pipe carrying oil of sp.gr 0.8, the flow of oil is in the upward direction. The difference of levels between the throat and inlet section is 50cm. The oil mercury differential manometer gives a reading of 30cm of Mercury. Find the discharge of oil. Neglect the losses.

Solution. Dia. at inlet, $d_1 = 20$ cm

$\therefore$ $a_1 = \frac{\pi}{4}(20)^2 = 314.16 \text{ cm}^2$

Dia. at throat, $d_2 = 10$ cm

$\therefore$ $$a_2 = \frac{\pi}{4}(10)^2 = 78.54 \text{ cm}^2$$

Sp. gr. of oil, $S_o = 0.8$

Sp. gr. of mercury, $S_g = 13.6$

Differential manometer reading, $x = 30$ cm

$\therefore$ $$h = \left(\frac{p_1}{\rho g} + z_1\right) - \left(\frac{p_2}{\rho g} + z_2\right) = x\left[\frac{S_g}{S_o} - 1\right]$$

$$= 30\left[\frac{13.6}{0.8} - 1\right] = 30\,[17 - 1] = 30 \times 16 = 480 \text{ cm of oil}$$

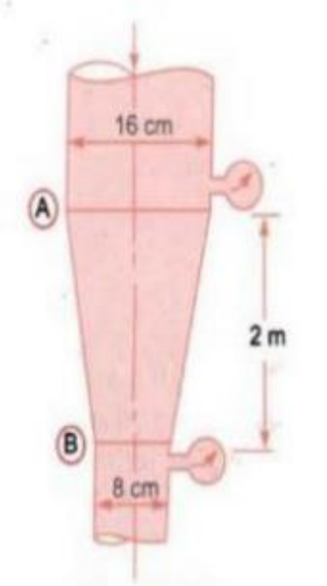
$$C_d = 1.0$$

The discharge, $$Q = C_d \frac{a_1 a_2}{\sqrt{a_1^2 - a_2^2}} \times \sqrt{2gh}$$

$$= \frac{1.0 \times 314.16 \times 78.54}{\sqrt{(314.16)^2 - (78.54)^2}} \times \sqrt{2 \times 981 \times 480} \text{ cm}^3/s$$

$$= \frac{23932630.7}{304} = 78725.75 \text{ cm}^3/s = \textbf{78.725 litres/s. Ans.}$$

In a vertical pipe conveying oil of specific gravity 0.8, two pressure gauges have been installed at A and B where the diameters are 16cm and 8cm respectively. A is 2 meters above B. The pressure gauge readings have shown that the pressure at B is greater than at A by 0.981N/cm2. Neglecting all losses, calculate the flow rate. If the gauges at A and B are replaced by tubes filled with the same fluid and connected to a U tube containing Mercury, Calculate the difference of level of Mercury in the two limbs of the U tube.



Solution. Given :

Sp. gr. of oil, $S_o = 0.8$

∴ Density, $\rho = 0.8 \times 1000 = 800 \frac{kg}{m^3}$

Dia. at A, $D_A = 16$ cm $= 0.16$ m

∴ Area at A, $A_1 = \frac{\pi}{4}(.16)^2 = 0.0201$ m^2

Dia. at B, $D_B = 8$ cm $= 0.08$ m

∴ Area at B, $A_2 = \frac{\pi}{4}(.08)^2 = 0.005026$ m^2

(i) Difference of pressures, $p_B - p_A = 0.981$ N/cm^2

$= 0.981 \times 10^4$ N/m$^2 = \frac{9810 N}{m^2}$

Difference of pressure head

∴ $$\frac{p_B - p_A}{\rho g} = \frac{9810}{800 \times 9.81} = 1.25$$

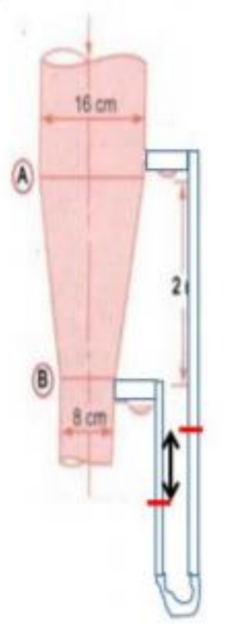
Applying Bernoulli's equation between A and B, taking the reference line passing through B, we have,

$p_1/\gamma + v_1^2/2g + z_1 = p_2/\gamma + v_2^2/2g + z_2 + h_L$

$(p_A/\gamma - p_B/\gamma) + z_A - z_B = (v_B^2/2g - v_A^2/2g)$

$(p_A/\gamma - p_B/\gamma) + 2.0 - 0.0 = (v_B^2/2g - v_A^2/2g)$

$-1.25+2.0 = (v_B^2/2g - v_A^2/2g)$



$0.75 = (v_B^2/ 2g - v_A^2/ 2g)$ ----------------- (1)

Now applying Continuity equation at A and B, we get,

$A_A V_A = A_B V_B$

$V_B = A_A V_A / A_B = 4V_A$

Substituting the value of V_B in equation (1), we get

$0.75 = 16\, v_A^2/ 2g - v_A^2/ 2g = 15\, v_A^2/ 2g$; Va= 0.99m/s

Rate of flow, Q= $A_A V_A$

Q= 0.99* 0.01989 Cum/s

Difference of level of mercury in the u – Tube:

Let x = Difference of Mercury level

Then h= x[(sm/so)- 1]

$h = (p_A/ \gamma + z_A) - (p_B/ \gamma + z_B)$

$= (p_A/ \gamma - p_B/ \gamma) + (z_A + z_B) = -1.25+2.00 = 0.75m$

0.75 = x[(13.6/0.8)-1] = 16x

X= 4.687 cm

P.9: Find the discharge of water flowing through a pipe 30cm diameter placed in an inclined position where a venturimeter is inserted, having a throat diameter of 15 cm. The difference of pressure between the main and the throat is measured by a liquid of sp.gr. 0.6 in an inverted U tube which gives a reading of 30cm. The loss of head between the main and the throat is 0.2 times the kinetic head of the pipe.

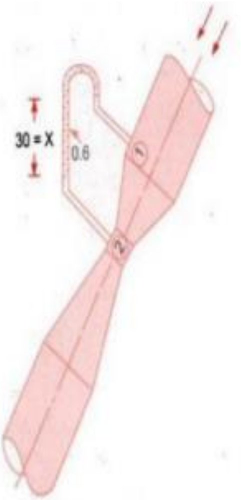
Solution. Dia. at inlet $= 30$ cm

$\therefore \quad d_1 = 30$ cm

$$a_1 = \frac{\pi}{4}(30)^2 = 706.85 \text{ cm}^2$$

Dia. at throat, $d_2 = 15$ cm

$$\therefore \quad a_2 = \frac{\pi}{4}(15)^2 = 176.7 \text{ cm}^2$$



Reading of differential manometer, $x = 30$ cm

Difference of pressure head, h is given by

$$\left(\frac{p_1}{\rho g} + z_1\right) - \left(\frac{p_2}{\rho g} + z_2\right) = h$$

Also $\quad h = x\left[1 - \frac{S_l}{S_o}\right]$

where $S_l = 0.6$ and $S_o = 1.0$

$$= 30\left[1 - \frac{0.6}{1.0}\right] = 30 \times .4 = 12.0 \text{ cm of water}$$

Loss of head, $h_L = 0.2 \times$ kinetic head of pipe $= 0.2 \times \frac{v_1^2}{2g}$

Now applying Bernoulli's equation at sections (1) and (2), we get

$$\frac{p_1}{\rho g} + z_1 + \frac{v_1^2}{2g} = \frac{p_2}{\rho g} + z_2 + \frac{v_2^2}{2g} + h_L$$

or $\quad \left(\frac{p_1}{\rho g} + z_1\right) - \left(\frac{p_2}{\rho g} + z_2\right) + \frac{v_1^2}{2g} - \frac{v_2^2}{2g} = h_L$

But $\quad \left(\frac{p_1}{\rho g} + z_1\right) - \left(\frac{p_2}{\rho g} + z_2\right) = h = 12.0$ cm of water

and $\quad h_L = 0.2 \times v_1^2/2g$

$$\therefore \quad 12.0 + \frac{v_1^2}{2g} - \frac{v_2^2}{2g} = 0.2 \times \frac{v_1^2}{2g}$$

$$\therefore \quad 12.0 + 0.8\frac{v_1^2}{2g} - \frac{v_2^2}{2g} = 0$$

Applying continuity equation at (1) and (2), we get

$$a_1 v_1 = a_2 v_2$$

$$\therefore \quad v_1 = \frac{a_2}{a_1} v_2 = \frac{\frac{\pi}{4}(15)^2 v_2}{\frac{\pi}{4}(30)^2} = \frac{v_2}{4}$$

Substituting this value of v_1 in equation (1), we get

$$12.0 + \frac{0.9}{2g}\left(\frac{v_2}{4}\right)^2 - \frac{v_2^2}{2g} = 0 \text{ or } 12.0 + \frac{v_2^2}{2g}\left[\frac{0.8}{16} - 1\right] = 0$$

$$\text{or} \quad \frac{v_2^2}{2g}[.05 - 1] = -12.0 \text{ or } \frac{0.95 v_2^2}{2g} = 12.0$$

$$\therefore \quad v_2 = \sqrt{\frac{2 \times 981 \times 12.0}{0.95}} = 157.4 \text{ cm/s}$$

$\therefore$ Discharge $= a_2 v_2$

$= 176.7 \times 157.4 \text{ cm}^3\text{/s} = 27800 \text{ cm}^3\text{/s} =$ **27.8 litres/s. Ans.**

MODULE 3
DIMENTIONAL ANALYSIS

1. WHAT IS DIMENSIONAL ANALYSIS?

Dimensional analysis is a means of simplifying a physical problem by appealing to dimensional homogeneity to reduce the number of relevant variables.

It is particularly useful for:
- presenting and interpreting experimental data;
- attacking problems not amenable to a direct theoretical solution;
- checking equations;
- establishing the relative importance of particular physical phenomena;
- physical modelling.

Example.
The drag force F per unit length on a long smooth cylinder is a function of air speed U, density ρ, diameter D and viscosity μ. However, instead of having to draw hundreds of graphs portraying its variation with all combinations of these parameters, dimensional analysis tells us that the problem can be reduced to a **single** dimensionless relationship

$$c_D = f(\mathrm{Re})$$

where c_D is the drag coefficient and Re is the Reynolds number.

In this instance dimensional analysis has reduced the number of relevant variables from 5 to 2 and the experimental data to a single graph of c_D against Re.

2. DIMENSIONS

2.1 Dimensions and Units

A *dimension* is the **type** of physical quantity.
A *unit* is a means of assigning a **numerical value** to that quantity.

SI units are preferred in scientific work.

2.2 Primary Dimensions

In fluid mechanics the *primary* or *fundamental* dimensions, together with their SI units are:

mass	M	(kilogram, kg)
length	L	(metre, m)
time	T	(second, s)
temperature	Θ	(kelvin, K)

In other areas of physics additional dimensions may be necessary. The complete set specified by the SI system consists of the above plus

electric current	I	(ampere, A)
luminous intensity	C	(candela, cd)
amount of substance	n	(mole, mol)

2.3 Dimensions of Derived Quantities

Dimensions of common derived mechanical quantities are given in the following table.

	Quantity	Common Symbol(s)	Dimensions
Geometry	Area	A	L^2
	Volume	V	L^3
	Second moment of area	I	L^4
Kinematics	Velocity	U	LT^{-1}
	Acceleration	a	LT^{-2}
	Angle	θ	1 (i.e. dimensionless)
	Angular velocity	ω	T^{-1}
	Quantity of flow	Q	L^3T^{-1}
	Mass flow rate	$\dot{m}$	MT^{-1}
Dynamics	Force	F	MLT^{-2}
	Moment, torque	T	ML^2T^{-2}
	Energy, work, heat	E, W	ML^2T^{-2}
	Power	P	ML^2T^{-3}
	Pressure, stress	p, τ	$ML^{-1}T^{-2}$
Fluid properties	Density	ρ	ML^{-3}
	Viscosity	μ	$ML^{-1}T^{-1}$
	Kinematic viscosity	ν	L^2T^{-1}
	Surface tension	σ	MT^{-2}
	Thermal conductivity	k	$MLT^{-3}\Theta^{-1}$
	Specific heat	c_p, c_v	$L^2T^{-2}\Theta^{-1}$
	Bulk modulus	K	$ML^{-1}T^{-2}$

2.4 Working Out Dimensions

In the following, [ ] means "dimensions of".

Example.

Use the definition $\tau = \mu \dfrac{dU}{dy}$ to determine the dimensions of viscosity.

Solution.

From the definition,

$$\mu = \frac{\tau}{dU/dy} = \frac{\textit{force / area}}{\textit{velocity / length}}$$

Hence,

$$[\mu] = \frac{MLT^{-2}/L^2}{LT^{-1}/L} = ML^{-1}T^{-1}$$

Alternatively, dimensions may be deduced indirectly from any known formula involving that quantity.

3. FORMAL PROCEDURE FOR DIMENSIONAL ANALYSIS

3.1 Dimensional Homogeneity

The Principle of Dimensional Homogeneity

*All **additive** terms in a physical equation must have the same dimensions.*

Examples:

$s = ut + \frac{1}{2}at^2$ – all terms have the dimensions of length (L)

$\frac{p}{\rho g} + \frac{V^2}{2g} + z = H$ – all terms have the dimensions of length (L)

Dimensional homogeneity is a useful tool for **checking** formulae. For this reason it is useful when analysing a physical problem to retain algebraic symbols for as long as possible, only substituting numbers right at the end. However, dimensional analysis cannot determine numerical factors; e.g. it cannot distinguish between $\frac{1}{2}at^2$ and at^2 in the first formula above.

Dimensional homogeneity is the basis of the formal dimensional analysis that follows.

3.2 Buckingham's Pi Theorem

Experienced practitioners can do dimensional analysis by inspection. However, the formal tool which they are unconsciously using is *Buckingham's Pi Theorem*[1]:

Buckingham's Pi Theorem

(1) If a problem involves
n relevant variables
m **independent** dimensions
then it can be reduced to a relationship between
$n - m$ non-dimensional parameters $\Pi_1, \ldots, \Pi_{n-m}$.

(2) To construct these non-dimensional Π groups:
(i) Choose *m* dimensionally-distinct *scaling variables* (aka *repeating variables*).
(ii) For each of the $n - m$ remaining variables construct a non-dimensional Π of the form

$$\Pi = (variable)(scale_1)^a (scale_2)^b (scale_3)^c \cdots$$

where *a*, *b*, *c*, ... are chosen so as to make each Π non-dimensional.

Note. In order to ensure dimensional independence in {MLT} systems it is common – but not obligatory – to choose the scaling variables as: a purely geometric quantity (e.g. a length), a kinematic (time- but not mass-containing) quantity (e.g. velocity or acceleration) and a dynamic (mass- or force-containing) quantity (e.g. density).

3.3 Applications

Example.
Obtain an expression in non-dimensional form for the pressure gradient in a horizontal pipe of circular cross-section. Show how this relates to the familiar expression for frictional head loss.

Step 1. Identify the relevant variables.

dp/dx, ρ, V, D, k_s, μ

Step 2. Write down dimensions.

$$\frac{dp}{dx} \quad \frac{[force/area]}{length} = \frac{MLT^{-2} \times L^{-2}}{L} = ML^{-2}T^{-2}$$

ρ ML^{-3}

V LT^{-1}

D L

k_s L

μ $ML^{-1}T^{-1}$

Step 3. Establish the number of independent dimensions and non-dimensional groups.

Number of relevant variables: $n = 6$

Number of independent dimensions: $m = 3$ (M, L and T)

Number of non-dimensional groups (Πs): $n - m = 3$

Step 4. Choose m (= 3) dimensionally-independent scaling variables.

e.g. geometric (D), kinematic/time-dependent (V), dynamic/mass-dependent (ρ).

Step 5. Create the Πs by non-dimensionalising the remaining variables: dp/dx, k_s and μ.

$$\Pi_1 = \frac{dp}{dx} D^a V^b \rho^c$$

Considering the dimensions of both sides:

$$M^0L^0T^0 = (ML^{-2}T^{-2})(L)^a(LT^{-1})^b(ML^{-3})^c$$

$$= M^{1+c}L^{-2+a+b-3c}T^{-2-b}$$

Equate powers of primary dimensions. Since M only appears in [ρ] and T only appears in [V] it is sensible to deal with these first.

M: $0 = 1 + c \Rightarrow c = -1$

T: $0 = -2 - b \Rightarrow b = -2$

L: $0 = -2 + a + b - 3c \Rightarrow a = 2 - b + 3c = 1$

Hence,

$$\Pi_1 = \frac{dp}{dx} D V^{-2}\rho^{-1} = \frac{D\frac{dp}{dx}}{\rho V^2}$$ (Check: OK – ratio of two pressures)

$$\Pi_2 = \frac{k_s}{D}$$ (by inspection, since k_s is a length)

$$\Pi_3 = \mu D^a V^b \rho^c$$

In terms of dimensions:

$$M^0L^0T^0 = (ML^{-1}T^{-1})(L)^a(LT^{-1})^b(ML^{-3})^c$$
$$= M^{1+c}L^{-1+a+b-3c}T^{-1-b}$$

Equating exponents:

M: $0 = 1 + c \Rightarrow c = -1$

T: $0 = -1 - b \Rightarrow b = -1$

L: $0 = -1 + a + b - 3c \Rightarrow a = 1 - b + 3c = -1$

Hence,

$$\Pi_3 = \frac{\mu}{\rho VD}$$ (Check: OK – this is the reciprocal of the Reynolds number)

Step 6. Set out the non-dimensional relationship.

$$\Pi_1 = f(\Pi_2, \Pi_3)$$

or

$$\frac{D\frac{dp}{dx}}{\rho V^2} = f\left(\frac{k_s}{D}, \frac{\mu}{\rho VD}\right) \qquad (*)$$

Step 7. Rearrange (if required) for convenience.

We are free to replace any of the Πs by a power of that Π, or by a product with the other Πs, provided we retain the same number of independent dimensionless groups. In this case we recognise that Π_3 is the reciprocal of the Reynolds number, so it looks better to use $\Pi_3' = (\Pi_3)^{-1} = \mathrm{Re}$ as the third non-dimensional group. We can also write the pressure gradient in terms of head loss: $\frac{dp}{dx} = \rho g \frac{h_f}{L}$. With these two modifications the non-dimensional relationship (*) then becomes

$$\frac{gh_f D}{LV^2} = f\left(\frac{k_s}{D}, \mathrm{Re}\right)$$

or

$$h_f = \frac{L}{D} \times \frac{V^2}{g} \times f\left(\frac{k_s}{D}, \mathrm{Re}\right)$$

Since numerical factors can be absorbed into the non-specified function, this can easily be identified with the Darcy-Weisbach equation

$$h_f = \lambda \frac{L}{D}\frac{V^2}{2g}$$

where λ is a function of relative roughness k_s/D and Reynolds number Re, a function given (Topic 2) by the Colebrook-White equation.

Example.
The drag force on a body in a fluid flow is a function of the body size (expressed via a characteristic length L) and the fluid velocity V, density ρ and viscosity μ. Perform a dimensional analysis to reduce this to a single functional dependence

$$c_D = f(\text{Re})$$

where c_D is a drag coefficient and Re is the Reynolds number.

What additional non-dimensional groups might appear in practice?

Notes.

(1) Dimensional analysis simply says that there **is** a relationship; it doesn't (except in the case of a single Π, which must, therefore, be constant) say what the relationship is. For the specific relationship one must appeal to theory or, more commonly, experimental data.

(2) If Π_1, Π_2, Π_3, ... are suitable non-dimensional groups then we are liberty to replace some or all of them by any powers or products with the other Πs, provided that we retain the same number of independent non-dimensional groups; e.g. $(\Pi_1)^{-1}$, $(\Pi_2)^2$, $\Pi_1/(\Pi_3)^2$.

(3) It is extremely common in fluid mechanics to find (often after the rearrangement mentioned in (2)) certain combinations which can be recognised as key parameters such as the Reynolds number ($\text{Re} = \rho UL/\mu$) or Froude number ($\text{Fr} = U/\sqrt{gL}$).

(4) Often the hardest part of the dimensional analysis is determining which <u>are</u> the relevant variables. For example, surface tension is always present in free-surface flows, but can be neglected if the Weber number $\text{We} = \rho U^2 L/\sigma$ is large. Similarly, all fluids are compressible, but compressibility effects on the flow can be ignored if the Mach number ($\text{Ma} = U/c$) is small; i.e. velocity is much less than the speed of sound.

(5) Although three primary dimensions (M,L,T) may appear when the variables are listed, they do not do so independently. The following example illustrates a case where M and T always appear in the combination MT^{-2}, hence giving only one independent dimension.

Example.
The tip deflection δ of a cantilever beam is a function of tip load *W*, beam length *l*, second moment of area *I* and Young's modulus *E*. Perform a dimensional analysis of this problem.

Step 1. Identify the relevant variables.

$\delta, W, l, I, E.$

Step 2. Write down dimensions.

δ	L
W	MLT^{-2}
l	L
I	L^4
E	$ML^{-1}T^{-2}$

Step 3. Establish the number of independent dimensions and non-dimensional groups.

Number of relevant variables: $n = 5$

Number of independent dimensions: $m = 2$ (L and MT^{-2} - *note*)

Number of non-dimensional groups (Πs): $n - m = 3$

Step 4. Choose m (= 2) dimensionally-independent scaling variables.
e.g. geometric (l), mass- or time-dependent (E).

Step 5. Create the Πs by non-dimensionalising the remaining variables: δ, I and W.
These give (after some algebra, not reproduced here):

$$\Pi_1 = \frac{\delta}{l}$$

$$\Pi_2 = \frac{I}{l^4}$$

$$\Pi_3 = \frac{W}{El^2}$$

Step 6. Set out the non-dimensional relationship.

$$\Pi_1 = f(\Pi_2, \Pi_3)$$

or

$$\frac{\delta}{l} = f\left(\frac{I}{l^4}, \frac{W}{El^2}\right)$$

This is as far as dimensional analysis will get us. Detailed theory shows that, for small elastic deflections,

$$\delta = \frac{1}{3}\frac{Wl^3}{EI}$$

or

$$\frac{\delta}{l} = \frac{1}{3}\left(\frac{W}{El^2}\right) \times \left(\frac{I}{l^4}\right)^{-1}$$

4. PHYSICAL MODELLING

4.1 Method

If a dimensional analysis indicates that a problem is described by a functional relationship between non-dimensional parameters Π_1, Π_2, Π_3, ... then full *similarity* requires that these parameters be the same at both full ("*prototype*") scale and model scale. i.e.

$$(\Pi_1)_m = (\Pi_1)_p$$

$$(\Pi_2)_m = (\Pi_2)_p$$

etc.

Example.
A prototype gate valve which will control the flow in a pipe system conveying paraffin is to be studied in a model. List the significant variables on which the pressure drop across the valve would depend. Perform dimensional analysis to obtain the relevant non-dimensional groups.

A 1/5 scale model is built to determine the pressure drop across the valve with water as the working fluid.

(a) For a particular opening, when the velocity of paraffin in the prototype is 3.0 m s^{-1} what should be the velocity of water in the model for dynamic similarity?

(b) What is the ratio of the quantities of flow in prototype and model?

(c) Find the pressure drop in the prototype if it is 60 kPa in the model.

(The density and viscosity of paraffin are 800 kg m^{-3} and 0.002 kg m^{-1} s^{-1} respectively. Take the kinematic viscosity of water as 1.0×10^{-6} m^2 s^{-1}).

Solution.
The pressure drop Δp is expected to depend upon the gate opening h, the overall depth d, the velocity V, density ρ and viscosity μ.

List the relevant variables:

Δp, h, d, V, ρ, μ

Write down dimensions:

Δp	$ML^{-1}T^{-2}$
h	L
d	L
V	LT^{-1}
ρ	ML^{-3}
μ	$ML^{-1}T^{-1}$

Number of variables: $n = 6$
Number of independent dimensions: $m = 3$ (M, L and T)
Number of non-dimensional groups: $n - m = 3$

Choose m (= 3) scaling variables:
geometric (d); kinematic/time-dependent (V); dynamic/mass-dependent (ρ).

Form dimensionless groups by non-dimensionalising the remaining variables: Δp, h and μ.

$$\Pi_1 = \Delta p d^a V^b \rho^c$$
$$M^0L^0T^0 = (ML^{-1}T^{-2})(L)^a(LT^{-1})^b(ML^{-3})^c$$
$$= M^{1+c}L^{-1+a+b-3c}T^{-2-b}$$

M: $0 = 1 + c \quad \Rightarrow \quad c = -1$

T: $0 = -2 - b \quad \Rightarrow \quad b = -2$

L: $0 = -1 + a + b - 3c \quad \Rightarrow \quad a = 1 + 3c - b \quad = 0$

$$\Rightarrow \quad \Pi_1 = \Delta p V^{-2}\rho^{-1} \quad = \frac{\Delta p}{\rho V^2}$$

$$\Pi_2 = \frac{h}{d}$$ (by inspection, since h is a length)

$$\Pi_3 = \mu d^a V^b \rho^c$$ (probably obvious by now, but here goes anyway ...)
$$M^0L^0T^0 = (ML^{-1}T^{-1})(L)^a(LT^{-1})^b(ML^{-3})^c$$
$$= M^{1+c}L^{-1+a+b-3c}T^{-1-b}$$

M: $0 = 1 + c \quad \Rightarrow \quad c = -1$

T: $0 = -1 - b + 0 \quad \Rightarrow \quad b = -1$

L: $0 = -1 + a + b - 3c \quad \Rightarrow \quad a = 1 + 3c - b \quad = -1$

$$\Rightarrow \quad \Pi_3 = \mu d^{-1}V^{-1}\rho^{-1} \quad = \frac{\mu}{\rho V d}$$

Recognition of the Reynolds number suggests that we replace Π_3 by

$$\Pi_3' = (\Pi_3)^{-1} = \frac{\rho V d}{\mu}$$

Hence, dimensional analysis yields

$$\Pi_1 = f(\Pi_2, \Pi_3')$$

i.e.

$$\frac{\Delta p}{\rho V^2} = f\left(\frac{h}{d}, \frac{\rho V d}{\mu}\right)$$

(a) Dynamic similarity requires that all non-dimensional groups be the same in model and prototype; i.e.

$$\Pi_1 = \left(\frac{\Delta p}{\rho V^2}\right)_p = \left(\frac{\Delta p}{\rho V^2}\right)_m$$

$$\Pi_2 = \left(\frac{h}{d}\right)_p = \left(\frac{h}{d}\right)_m$$ (automatic if similar shape; i.e. "geometric similarity")

$$\Pi_3' = \left(\frac{\rho V d}{\mu}\right)_p = \left(\frac{\rho V d}{\mu}\right)_m$$

From the last, we have a velocity ratio

$$\frac{V_p}{V_m} = \frac{(\mu/\rho)_p}{(\mu/\rho)_m}\frac{d_m}{d_p} = \frac{0.002/800}{1.0\times10^{-6}}\times\frac{1}{5} = 0.5$$

Hence,

$$V_m = \frac{V_p}{0.5} = \frac{3.0}{0.5} = 6.0\,\mathrm{m\,s^{-1}}$$

(b) The ratio of the quantities of flow is

$$\frac{Q_p}{Q_m} = \frac{(velocity\times area)_p}{(velocity\times area)_m} = \frac{V_p}{V_m}\left(\frac{d_p}{d_m}\right)^2 = 0.5\times5^2 = 12.5$$

(c) Finally, for the pressure drop,

$$\Pi_1 = \left(\frac{\Delta p}{\rho V^2}\right)_p = \left(\frac{\Delta p}{\rho V^2}\right)_m \Rightarrow \frac{(\Delta p)_p}{(\Delta p)_m} = \frac{\rho_p}{\rho_m}\left(\frac{V_p}{V_m}\right)^2 = \frac{800}{1000}\times0.5^2 = 0.2$$

Hence,

$$\Delta p_p = 0.2\times\Delta p_m = 0.2\times60 = 12.0\,\mathrm{kPa}$$

4.2 Incomplete Similarity ("Scale Effects")

For a multi-parameter problem it is often not possible to achieve full similarity. In particular, it is rare to be able to achieve full Reynolds-number scaling when other dimensionless parameters are also involved. For hydraulic modelling of flows with a free surface the most important requirement is *Froude-number scaling* (Section 4.3)

It is common to distinguish three levels of similarity.

Geometric similarity – the ratio of all corresponding lengths in model and prototype are the same (i.e. they have the same shape).

Kinematic similarity – the ratio of all corresponding lengths and times (and hence the ratios of all corresponding velocities) in model and prototype are the same.

Dynamic similarity – the ratio of all forces in model and prototype are the same; e.g. Re = (inertial force) / (viscous force) is the same in both.

Geometric similarity is almost always assumed. However, in some applications – notably river modelling – it is necessary to distort vertical scales to prevent undue influence of, for example, surface tension or bed roughness.

Achieving full similarity is particularly a problem with the Reynolds number Re = UL/ν.

- Using the same working fluid would require a velocity ratio inversely proportional to the length-scale ratio and hence impractically large velocities in the scale model.
- A velocity scale fixed by, for example, the Froude number (see Section 4.3) means that the only way to maintain the same Reynolds number is to adjust the kinematic viscosity (substantially).

In practice, Reynolds-number similarity is unimportant if flows in **both** model and prototype are fully turbulent; then momentum transport by viscous stresses is much less than that by turbulent eddies and so the precise value of molecular viscosity μ is unimportant. In some cases this may mean deliberately triggering transition to turbulence in boundary layers (for example by the use of tripping wires or roughness strips).

<u>Surface effects</u>

Full geometric similarity requires that not only the main dimensions of objects but also the surface roughness and, for mobile beds, the sediment size be in proportion. This would put impossible requirements on surface finish or grain size. In practice, it is sufficient that the surface be aerodynamically rough: $u_\tau k_s/\nu \ge 5$, where $u_\tau = \sqrt{\tau_w/\rho}$ is the friction velocity and k_s a typical height of surface irregularities. This imposes a minimum velocity in model tests.

<u>Other Fluid Phenomena</u>

When scaled down in size, fluid phenomena which were negligible at full scale may become important in laboratory models. A common example is surface tension.

4.3 Froude-Number Scaling

The most important parameter to preserve in hydraulic modelling of free-surface flows driven by gravity is the Froude number, $\text{Fr} = U/\sqrt{gL}$. Preserving this parameter between model (m) and prototype (p) dictates the scaling of other variables in terms of the length scale ratio.

<u>Velocity</u>

$$(\text{Fr})_m = (\text{Fr})_p$$

$$\left(\frac{U}{\sqrt{gL}}\right)_m = \left(\frac{U}{\sqrt{gL}}\right)_p \quad \Rightarrow \quad \frac{U_m}{U_p} = \left(\frac{L_m}{L_p}\right)^{1/2}$$

i.e. the velocity ratio is the square root of the length-scale ratio.

<u>Quantity of flow</u>

$$Q \sim \text{velocity} \times \text{area} \quad \Rightarrow \quad \frac{Q_m}{Q_p} = \left(\frac{L_m}{L_p}\right)^{5/2}$$

<u>Force</u>

$$F \sim \text{pressure} \times \text{area} \quad \Rightarrow \quad \frac{F_m}{F_p} = \left(\frac{L_m}{L_p}\right)^3$$

This arises since the pressure ratio is equal to the length-scale ratio – this can be seen from either hydrostatics (pressure $\propto$ height) or from the dynamic pressure (proportional to $(\text{velocity})^2$ which, from the Froude number, is proportional to length).

<u>Time</u>

$$t \sim \text{length} \div \text{velocity} \quad \Rightarrow \quad \frac{t_m}{t_p} = \left(\frac{L_m}{L_p}\right)^{1/2}$$

Hence the quantity of flow scales as the length-scale ratio to the 5/2 power, whilst the time-scale ratio is the square root of the length-scale ratio. For example, at 1:100 geometric scale, a full-scale tidal period of 12.4 hours becomes 1.24 hours.

Example.
The force exerted on a bridge pier in a river is to be tested in a 1:10 scale model using water as the working fluid. In the prototype the depth of water is 2.0 m, the velocity of flow is 1.5 m s^{-1} and the width of the river is 20 m.

(a) List the variables affecting the force on the pier and perform dimensional analysis. Can you satisfy all the conditions for complete similarity? What is the most important parameter to choose for dynamic similarity?

(b) What are the depth, velocity and quantity of flow in the model?

(c) If the hydrodynamic force on the model bridge pier is 5 N, what would it be on the prototype?

5. NON-DIMENSIONAL GROUPS IN FLUID MECHANICS

Dynamic similarity requires that the ratio of all forces be the same. The ratio of different forces produces many of the key non-dimensional parameters in fluid mechanics.

(Note that "inertial force" means "mass × acceleration" – since it is equal to the total applied force it is often one of the two "forces" in the ratio.)

Reynolds number	$\mathrm{Re} = \frac{\rho U L}{\mu}$	$= \frac{\textit{inertial force}}{\textit{viscous force}}$	(viscous flows)
Froude number	$\mathrm{Fr} = \frac{U}{\sqrt{gL}}$	$= \left(\frac{\textit{inertial force}}{\textit{gravitational force}}\right)^{1/2}$	(free-surface flows)
Weber number	$\mathrm{We} = \frac{\rho U^2 L}{\sigma}$	$= \frac{\textit{inertial force}}{\textit{surface tension}}$	(surface tension)
Rossby number	$\mathrm{Ro} = \frac{U}{\Omega L}$	$= \frac{\textit{inertial force}}{\textit{Coriolis force}}$	(rotating flows)
Mach number	$\mathrm{Ma} = \frac{U}{c}$	$= \left(\frac{\textit{inertial force}}{\textit{compressibility force}}\right)^{1/2}$	(compressible flows)

These groups occur regularly when dimensional analysis is applied to fluid-dynamical problems. They can be derived by considering forces on a small volume of fluid. They can also be derived by non-dimensionalising the differential equations of fluid flow (see White, Chapter 5), or the online notes for the 4th-year Computational Hydraulics unit.

MODULE-2

Fluid Kinematics

2.0 Definitions

Pressure or Pressure intensity (p): It is the Fluid pressure force per unit area of application. Mathematically, $P = \frac{p}{A}$. Units are Pascal or N/ m ^{2}.

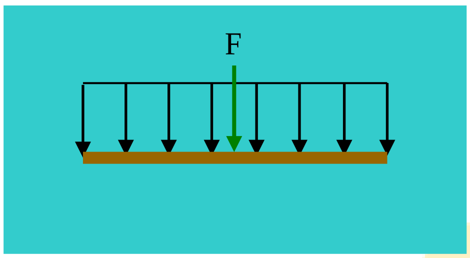
Total Pressure (P): This is that force exerted by the fluid on the contact surface (of the submerged surfaces), when the fluid comes in contact with the surface always acting normal to the contact surface. Units are N.

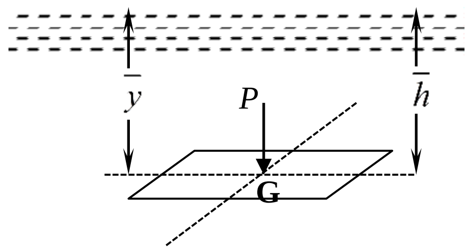
Centre of Pressure: It is defined as the point of application of the total pressure on the contact surface.

The submerged surface may be either plane or curved. In case of plane surface, it may be vertical, horizontal or inclined. Hence, the above four cases may be studied for obtaining the total pressure and centre of pressure.

2.1 Hydrostatic Forces on Plane Horizontal Surfaces:

- If a plane surface immersed in a fluid is horizontal, then
  - Hydrostatic pressure is uniform over the entire surface.
  - The resultant force acts at the centroid of the plane.





Consider a horizontal surface immersed in a static fluid as shown in Fig. As all the points on the plane are at equal depth from the free surface of the liquid, the pressure intensity will be equal on the entire surface and given by $p = \rho g \bar{y}$,

where $\bar{y}$ is the depth of the fluid surface Let A =

Area of the immersed surface

The total pressure force acting on the immersed surface is P

$$P = p \times \text{Area of the surface} = \rho g \bar{y} A$$

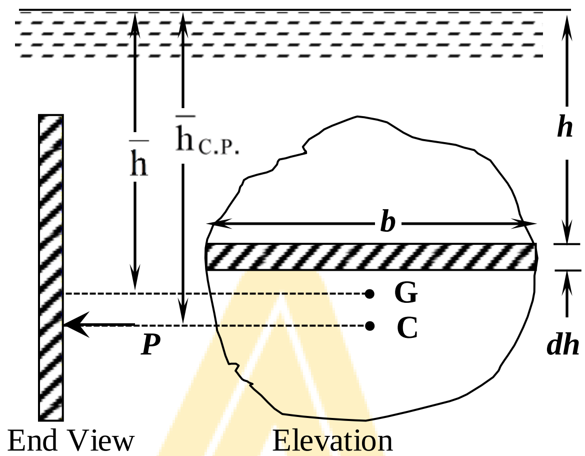
$$P = \rho g A \bar{y}$$

Where $\bar{y}$ is the centroidal distance immersed surface from the free surface of the liquid and $\bar{h}$ is the centre of pressure.

2.2 Hydrostatic Forces on Vertical Plane Surface:

Vertical Plane surface submerged in liquid

Consider a vertical plane surface of some arbitrary shape immersed in a liquid of mass density ρ as shown in Figure below:



Let, A = Total area of the surface

$\bar{h}$ = Depth of Centroid of the surface from the free surface

G = Centroid of the immersed surface

C = Centre of pressure

$\bar{h}_{C.P.}$ = Depth of centre of pressure

Consider a rectangular strip of breadth b and depth dy at a depth y from the free surface.

Total Pressure:

The pressure intensity at a depth y acting on the strip is $p = \rho g h$

Total pressure force on the strip = $dP = (\rho g h)dA$

$\therefore$ The Total pressure force on the entire area is given by integrating the above expression over the entire area $P = \int dP = \int (\rho g h)dA = \rho g \int h \, dA$ Eq.(1)

But $\int y \, dA$ is the Moment of the entire area about the free surface of the liquid given by

$$\int h \, dA = A\bar{h}$$

Substituting in Eq.(1), we get $P = \rho g A\bar{h} = \gamma A\bar{h}$ Eq.(2)

Where γ is the specific weight of Water

For water, $\rho = 1000 \text{ kg/m}^3$ and $g = 9.81 \text{ m/s}^2$. The force will be expressed in Newtons (N)

2.3 Hydrostatic Force on a Inclined submerged surface:

The other important utility of the hydrostatic equation is in the determination of force acting upon submerged bodies. Among the innumerable applications of this is the force calculation in storage tanks, ships, dams etc.

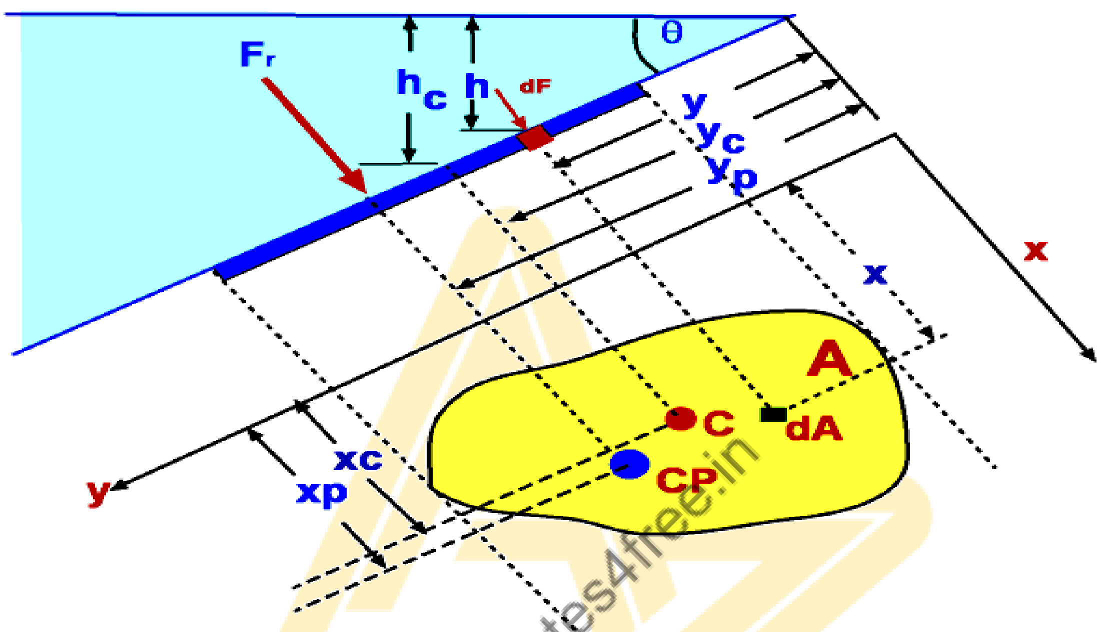


Figure 3.4 : Force upon a submerged object

First consider a planar arbitrary shape submerged in a liquid as shown in the figure. The plane makes an angle θ with the liquid surface, which is a free surface. The depth of water over the plane varies linearly. This configuration is efficiently handled by prescribing a coordinate frame such that the y-axis is aligned with the submerged plane. Consider an infinitesimally small area at a (x,y). Let this small area be located at a depth h from the free surface.dA = dx.dy

Differential Force acting on the differential area dA of plane,

$$dF = (\text{Pressure})\cdot(\text{Area}) = (\gamma h)\cdot(dA) \quad \text{(Perpendicular to plane)}$$

Then, Magnitude of total resultant force F_R

$$F_R = \int_A \gamma h dA = \int_A \gamma (y \sin\theta) dA$$ Where $h = y \sin\theta$

$$= \gamma \sin\theta \int_A y dA$$ ← 1^{st} moment of the area - Related with the center of area

※ $\int_A y dA = y_c A$ where yc: y coordinate of the center of area (Centroid)

c.f. Center or 1st moment

$\int_M x dm = M X_C$ & $\int_M y dm = M Y_C$ (XC & YC: Center of Mass)

$\int_A x dA$ = xc & $\int_A y dA$ = yc (xc&yc: Center of Area)

Moment of inertia or 2nd moment

$\int_M r^2 dm = I$ (2nd moment of Mass)

$\int_A y^2 dA = I_x$ &

Then,

$F_R = \gamma A y_c \sin\theta = (\gamma h_c) A$

Where γh_c : Pressure at the centroid = (Pressure at the centroid) × Area

- Magnitude of a force on an INCLINED plane
- Dependent on γ □, Area, and Depth of centroid
- Perpendicular to the surface (Direction)

i) Position of FR on y-axis 'yR' : y coordinate of the point of action of FR

Moment about x axis:

$$F_R y_R = (\gamma A y_c \sin\theta) y_R = \int_A y dF = \int_A \gamma \sin\theta y^2 dA = \gamma \sin\theta \int_A y^2 dA$$

$$\therefore h_R = \frac{\int_A h^2 dA}{h_c A} = \frac{I_x}{h_c A} \text{ where } I_x = \int_A y^2 dA$$:2nd moment of area

or, by using the parallel-axis theorem, $I_x = I_{xc} + Ay_c^2$

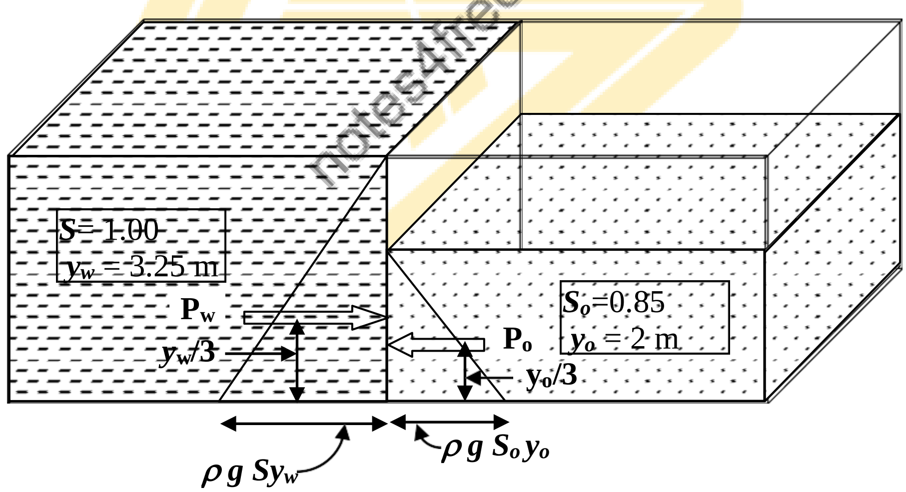
$$\therefore h_{C.P.} = \bar{h} + \frac{I_G \sin^2\theta°}{A\bar{h}}$$

(The centre of pressure below the centroid)

Solved Examples:

Q1. A rectangular tank 10 m x 5 m and 3.25 m deep is divided by a partition wall parallel to the shorter wall of the tank. One of the compartments contains water to a depth of 3.25 m and the other oil of specific gravity 0.85 to a depth of 2 m. Find the resultant pressure on the partition.

Solution:



The problem can be solved by considering hydrostatic pressure distribution diagram for both water and oil as shown in Fig.

From hydrostatic law, the pressure intensity ***p*** at any depth y_w is given by

$$p = S_o \rho g y_w$$

where ρ is the mass density of the liquid

Pressure force ***P = p x Area***

$$P_w = 1000 \times 10 \times 3.25 \times 5 \times 3.25 = 528.125 \text{ kN } (\rightarrow)$$

Acting at ***3.25/ 3 m*** from the base

P_o = 0.85 x 1000 x 10 x 2.0 x 5 x 2.0 = 200 kN (←)

Acting at ***2/ 3 m*** from the base.

Net Force ***P = P_w – P_o = 528.125 – 200.0 = 328.125 kN (→)***

Location:

Let ***P*** act at a distance ***y*** from the base. Taking moments of ***P_w, P_o*** and ***P*** about the base, we get

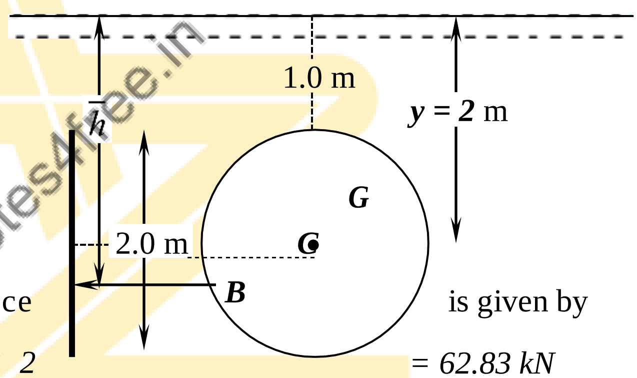
$P \times y = P_w \times y_w / 3 - P_o \times y_o / 3$

328.125 y = 528.125 x (3.25/ 3) – 200 x (2/ 3) or y = 1.337 m.

Q2. Determine the total force and location of centre of pressure for a circular plate of 2 m dia immersed vertically in water with its top edge 1.0 m below the water surface

Solution:

$$A = \frac{\pi \times D^2}{4} = \frac{\pi \times 2^2}{4} = 3.142 \text{m}$$



Assume

$\rho = 1000$ kg/ m^3 and g = 10 m/ s ^{2}

We know that the total pressure force is given by

$P = S_o \rho g A \bar{y} = 1000 \times 10 \times 3.142 \times 2 = 62.83$ kN

Centre of Pressure

The Centre of pressure is given by

$$\bar{h} = \bar{y} + \frac{I_g}{A\bar{y}}$$

$$I_g = \frac{\pi R^4}{4} = \frac{\pi \times 1^4}{4} = 0.785 \text{ m}^4$$

$$\bar{h} = 2 + \frac{0.785}{3.142 \times 2} = 2.125 \text{ m}$$

Q.3 *A large tank of sea water has a door in the side 1 m square. The top of the door is 5 m below the free surface. The door is hinged on the bottom edge. Calculate the force required at the top to keep it closed. The density of the sea water is 1033 kg/m*^{3}.

Solution: The total hydrostatic force $F = \gamma_{sea\ water} A\, h_c$

$$\gamma_{sea\ water} = 1033 \times 9.81 = 10133.73\, N/m^3$$

Given A = 1m X 1m = 1m^2

$$h_c = 5 + \frac{1}{2} = 5.5m$$

$$F = 10133.73 X 1 X 5.5 = 55735.5N$$

Acting at centre of pressure $(y_{c.p})$:

From the above $h_c = 5.5m,\ A = 1m^2$

$$(I_c)_{xx} = \frac{BD^3}{12} = \frac{1X1^3}{12} = 0.08333m^4$$

$$h_{C.P.} = h_c + \frac{(I_c)_{xx}}{Ah_c} = 5.5 + \frac{0.08333}{1X5.5} = 5.515m$$

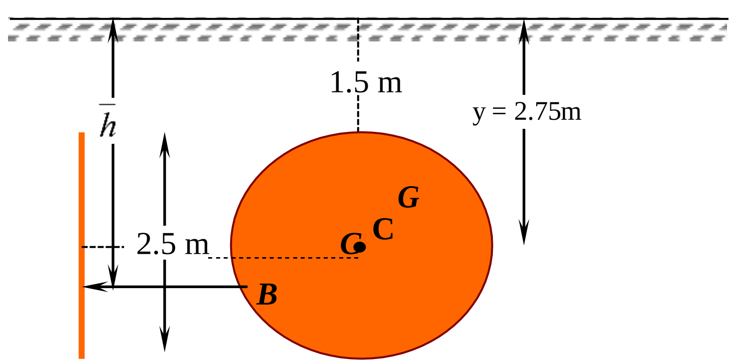
Distance of Hydrostatic force (F) from the bottom of the hinge = 6-5.515 = 0.48485m

The force 'P' required at the top of gate (1m from the hinge)

PX1 = FX0.48485 = 55735.5X0.48485

P = 27023.4 N = 27.023 kN

Q.4 Calculate the total hydrostatic force and location of centre of pressure for a circular plate of 2.5 m diameter immersed vertically in water with its top edge 1.5 m below the oil surface (Sp. Gr.=0.9)



Solution:

$$A = \frac{\pi \times D^2}{4} = \frac{\pi \times 2^2}{4} = 4.91\ m^2$$

Assume

ρ = 0.9X1000=900 kg/ m^3,g=9.8 m/ s^2

$$\gamma_{oil} = 900 \times 9.81 = 8829\ N/m^3$$

$h_c = 2.75m$

We know that the total pressure force is given by 'F'

$F = \gamma_{oil} A h_c$ = 8829 x 4.91 x 2 .75 = 238184 N = 238.184 kN

Centre of Pressure:

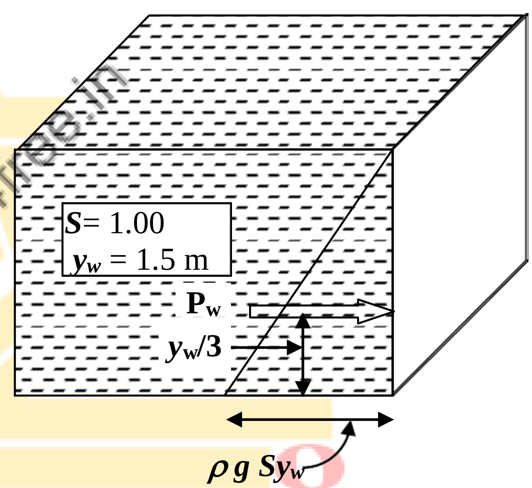
The Centre of pressure is given by

$$h_{C.P.} = h_c + \frac{(I_{cxx})}{A h_c}$$

$$I_g = \frac{\pi R^4}{4} = \frac{\pi \times 1.25^4}{4} = 1.9175 \text{ m}^4$$

$$h_{C.P.} = 2.75 + \frac{1.9175}{4.91 \times 2.75} = 2.892\text{m}$$

Q5 A square tank with 2 m sides and 1.5 m high contains water to a depth of 1 m and a liquid of specific gravity 0.8 on the water to a depth of 0.5 m. Find the magnitude and location of hydrostatic pressure on one face of tank.



Solution:

The problem can be solved by considering hydrostatic pressure distribution diagram for water as shown in Fig. From hydrostatic law, the pressure intensity ***p*** at any depth y_w is given by $p = S_o \rho g y_w$

where ρ is the mass density of the liquid

Pressure force ***P = p x Area***

$$P_w = 1000 \times 10 \times 2.0 \times 1.5 \times 1.5 = 45 \text{ kN } (\rightarrow)$$

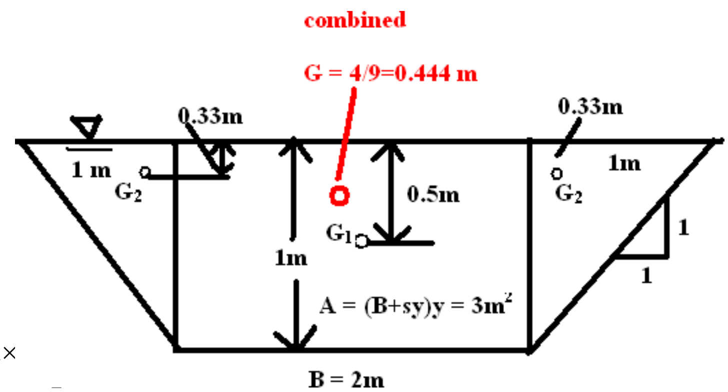
cting at ***1.5/ 3=0.5 m*** from the base

Q.6 **A** trapezoidal channel 2m wide at the bottom and 1m deep has side slopes 1:1. Determine: i) Total pressure ii) Centre of pressure, when it is full of water

Ans: Given B = 2m Area of flow A = (B+sy)y = 3m^2

The combined centroid will be located based on two triangular areas and one rectangle (shown as G_1, G_2, G_2)

$$\bar{y} = \frac{A_1 \times h_1 + A_2 \times h_2 + A_3 \times h_3}{A_1 + A_2 + A_3}$$



The total area A = 3m^2

Area of rectangle = 2 ×1 = 2m^2

Area of Triangle = $\frac{1}{2} \times 1 \times 1 = 0.5$m^2

$$\bar{y} = \frac{\left((2 \times 1) \times 0.5 + \left[\frac{1}{2} \times (1 \times 1) \right] \times 0.333 + \left[\frac{1}{2} \times (1 \times \right. \right.}{3}$$

i) The total pressure P = $\gamma_w \times A \times \bar{y}$ = 9810×3×0.444 = 13080N

ii) Centre of pressure

The centroidal moment of Inertia of Rectangle and Triangle is

$$I_{G1} = \frac{2 \times 1^3}{12} = 0.1667 m^4 \quad \textit{at} \quad 0.5m \ \textit{from water – surface}$$

$$I_{G1} = \frac{1 \times 1^3}{36} = 0.028 m^4 \quad \textit{at} \quad 0.333m \ \textit{from water – surface}$$

$$\bar{h} = \bar{y} + \frac{I_g}{A\bar{y}} \cdots Eq.1$$

The moment of Inertia about combined centroid can be obtained by using parallel axis theorem

$$I_G = \left((I_{G1}) + A_1 d_1^2 \right) + \left((I_{G2}) + A_2 d_2^2 \right) + \left((I_{G2}) + A_2 d_2^2 \right)$$ (as both the triangles are similar)

$$I_G = (0.1667 + 0.00618) + 2\left((I_{G2}) + A_2 d_2^2 \right)$$

$$I_G = (0.1667 + 0.00618) + 2(0.028 + 0.0062) = 0.2408 m^4$$

Substituting in Eq.1, The centre of pressure from free surface of water

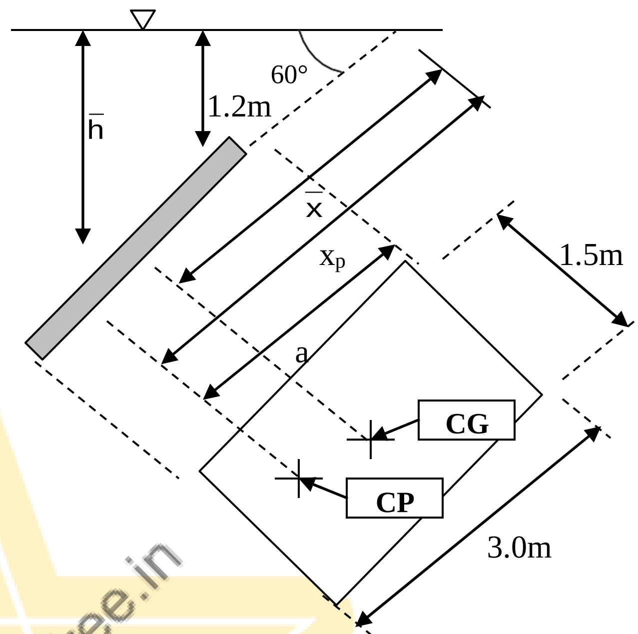
$$\bar{h} = \bar{y} + \frac{I_g}{A\bar{y}} \cdots Eq.1$$

$$\bar{h} = 0.444 + \frac{0.2408}{3 \times 0.444} = 0.6252 m$$

Q.7. A rectangular plate 1.5m x 3.0m is submerged in water and makes an angle of 60° with the horizontal, the 1.5m sides being horizontal. Calculate the magnitude of the force on the plate and the location of the point of application of the force, with reference to the top edge of the plate, when the top edge of the plate is 1.2m below the water surface.

Solution:

$$\bar{h} = \frac{1.2}{\sin 60°} + 1.5 = 1.386 + 1.5 = 2.886\text{m}$$



$$A = 3\text{m} \times 1.5\text{m} = 4.5\text{m}^2$$

$$\bar{h} = y \sin 60° = 2.886 \sin 60° = 2.499\text{m}$$

$$F = \rho g \bar{h} A = 1000 \times 9.81 \times 2.499 \times 3 \times 1.5$$

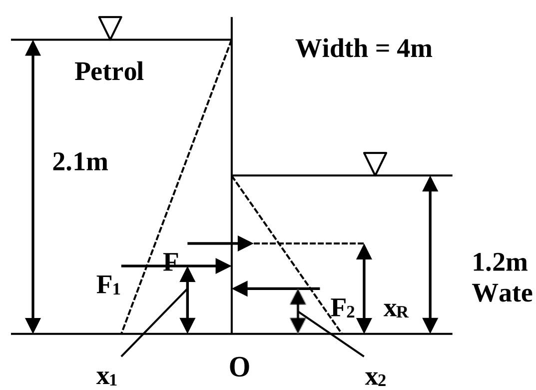
$$\therefore \quad F = 109.92 \times 10^3 \ \text{N} = 109.92 \ \text{kN}$$

$$h_{C.P.} = \bar{h} + \frac{I_G \sin^2 60°}{A\bar{h}}$$

$$\therefore h_{C.P.} = 2.886 + \frac{3^2}{12 \times 2.886} = 2.886 + 0.260 = 3.146m$$

From the top edge of the plate, a = 3.146 –1.386 = 1.760m

Q.8 A vertical bulkhead 4m wide divides a storage tank. On one side of the bulkhead petrol (S.G. = 0.78) is stored to a depth of 2.1m and on the other side water is stored to a depth of 1.2m. Determine the resultant force on the bulkhead and the position where it acts.



Solution:

$$F = \rho g \bar{h} A = \rho g \frac{h}{2} \cdot bh = \frac{1}{2} \rho g h^2 \cdot b$$

$$F_1 = \frac{1}{2} \times 780 \times 9.81 \times 2.1^2 \times 4 \ \text{N} = 67.5 \ \text{kN}$$

$$F_2 = \frac{1}{2} \times 1000 \times 9.81 \times 1.2^2 \times 4 \ \text{N} = 28.25 \ \text{kN}$$

Hence the resultant force

$$F_R = F_1 - F_2 = 67.5 - 28.25 = 39.25 \ \text{kN}$$

$$h_{C.P.} = \bar{h} + \frac{I_G}{A\bar{h}} = \frac{h}{2} + \frac{bh^3}{12}\,\frac{1}{bh(h/2)} = \frac{h}{2} + \frac{h}{6} = \tfrac{2}{3}h$$

From the diagram, $y = h - {}^{2}/_{3}h = {}^{1}/_{3}h$

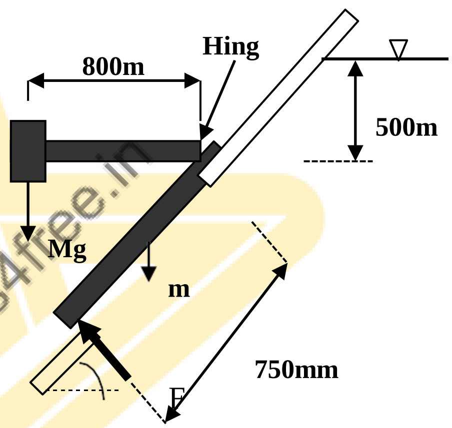
Hence, $y_1 = 2.1 / 3 = 0.7$m and $y_2 = 1.2 / 3 = 0.4$m

Taking moments about 'O', $\boxed{F_R \cdot y_R = F_1 \cdot y_1 - F_2 \cdot y_2}$

i.e. $39.25 \times y_R = 67.5 \times 0.7 - 28.25 \times 0.4$ and hence $y_R = 0.916$m

Q.9 *A* hinged, circular gate 750mm in diameter is used to close the opening in a sloping side of a tank, as shown in the diagram in **Error! Reference source not found.**. The gate is kept closed against water pressure partly by its own weight and partly by a weight on the lever arm. Find the mass M required to allow the gate to begin to open when the water level is 500mm above the top of the gate. The mass of the gate is 60 kg. (Neglect the weight of the lever arm.)

Solution:



$$a = \frac{500}{\sin 45°} = 707\text{mm}$$

$$\bar{x} = a + 375 = 1082\text{mm}$$

$$\bar{h} = \bar{x}\sin 45° = 765\text{mm}$$

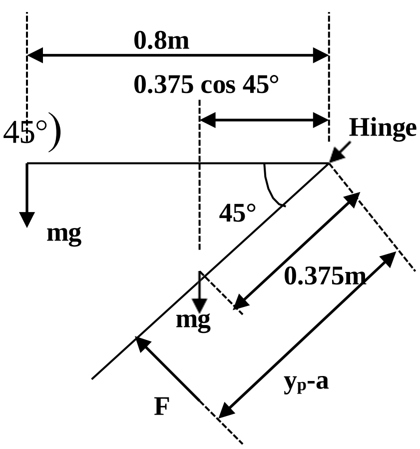
$$F = \rho g \bar{h} A = 1000 \times 9.81 \times 0.765 \times \left(\pi \times 0.75^2/4\right)$$

$$\therefore \quad F = 3.315 \times 10^3\ \text{N} = 3.315\,\text{kN}$$

$$x_P = \bar{x} + \frac{I_G}{A\bar{x}} = \bar{x} + \frac{\pi d^4}{64}\cdot\frac{4}{\pi d^2 \bar{x}} = \bar{x} + \frac{d^2}{16 \cdot \bar{x}}$$

$$x_P = 1.082 + \frac{0.75^2}{16 \times 1.082} = 1.082 + 0.032 = 1.114\text{m}$$

Taking moments about the hinge



$$F(x_P - a) = Mg \times 0.8 + mg \times 0.375\cos 45°$$

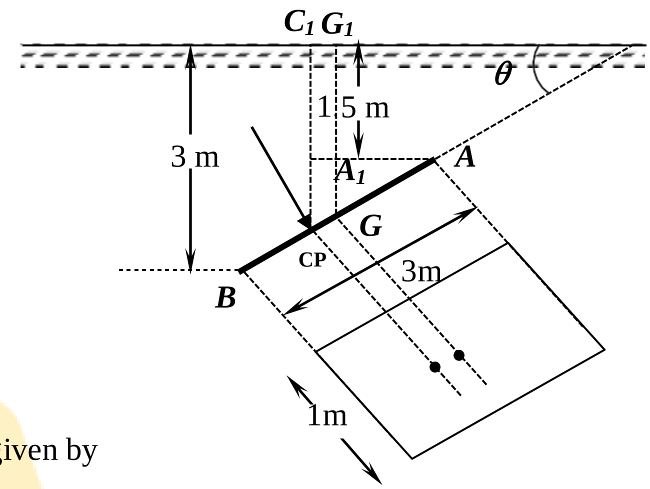
$$3315(1.114 - 0.707) = 9.81(M \times 0.8 + 60 \times 0.375\cos 45°)$$

$$M \times 0.8 = \frac{3315(1.114 - 0.707)}{9.81} - 60 \times 0.375\cos 45°$$

$$M \times 0.8 = 137.5 - 16 = 121.5$$

$$\therefore M = \frac{121.5}{0.8} = 152\ \text{kg}$$

Q.10. A rectangular plate 1 m x 3 m is immersed in water such that its upper and lower edge is at depths 1.5 m and 3 m respectively. Determine the total pressure acting on the plate and locate it.



Solution:

$A = 1 \times 3 = 3 \ m^2$

$\gamma_w = 9810 \ N/m^3$

$$h_c = \frac{3m + 1.5m}{2} = 2.25m$$

We know that the total pressure force is given by

$\boldsymbol{F = \gamma_{water} A h_c} = 9810 \times 3.0 \times 2.25 = 66217.5 \ N$

Sin θ = 1.5 / 3 = 0.5

θ = 30 º

Centre of Pressure; The Centre of pressure is given by

$$(I_C)_{xx} = \frac{bd^3}{12} = \frac{1 \times 3^3}{12} = 2.25 \ m^4$$

$h_c = 2.25m$

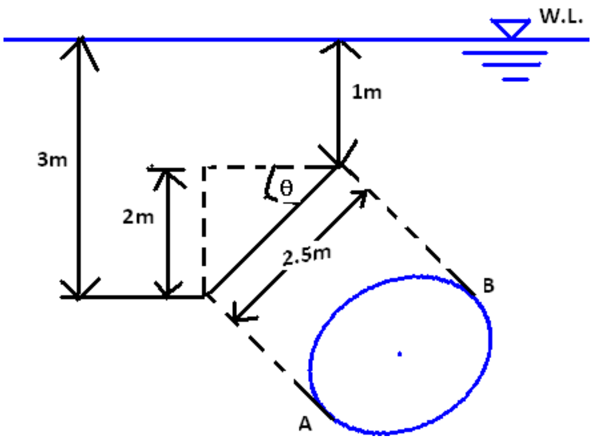
$$CC_1 = h_{C.P.} = h_c + \frac{(I_c)_{xx} \sin^2\theta}{A h_c}$$

$$CC_1 = h_{C.P.} = 2.25 + \frac{2.25 \ Sin^2 30^o}{3 \ X \ 2.25}$$

$$CC_1 = h_{C.P.} = 2.33333 \ m$$

Q 11. A circular plate 2.5m diameter is immersed in water, its greatest and least depth below the free surface being 3m and 1m respectively. Find
(i) The total pressure on one face of the plate and (ii) Position of centre of pressure

Ans: Given d = 2.5m,



$$\theta = Sin^{-1}\left(\frac{2}{2.5}\right)$$

$$\theta = 53.13^o$$

$$\bar{h} = 1+1 = 2m$$

$$A = \frac{\pi}{4} d^2 = \frac{\pi}{4} \times 2.5^2 = 4.909m^2$$

$$I_G = \frac{\pi}{64} d^4 = \frac{\pi}{64} \times (2.5)^4 = 1.917m^4$$

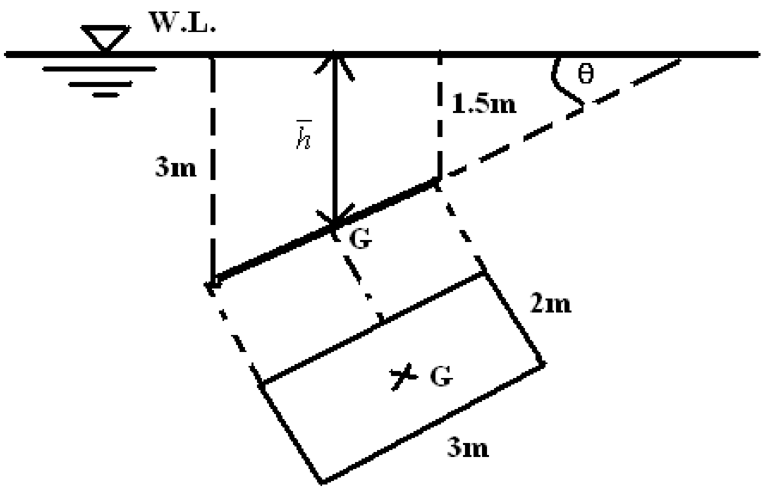
F = γ_wA $\bar{h}$ = 9.81×4.909×2 = 96.31 kN

$$\boldsymbol{h_{c.p.}} = \boldsymbol{\bar{h}} + \frac{\boldsymbol{I_G} \times \boldsymbol{Sin^2\theta}}{\boldsymbol{A \times \bar{h}}} = 2 + \frac{1.917 \times \boldsymbol{Sin}^2 53.13^o}{4.909 \times 2}$$

$$\boldsymbol{h_{c.p.}} = 2.125\boldsymbol{m}$$

Q.12. A 2m wide and 3m deep rectangular plane surface lies in water in such a way the top of and bottom edges are at a distance of 1.5m and 3m respectively from the surface. Determine the hydrostatic force and centre of pressure

Ans: Given A = 3m × 2m = 6m^2,



$$I_G = \frac{2 \times 3^3}{12} = 4.5m^4$$

Hydrostatic force

$$P = \gamma_w \times A \times \bar{h}$$

$$P = 9.81 \times 6 \times \left(\frac{3+1.5}{2}\right)$$

P = 132.435 kN

$$\sin\theta = \frac{(3.0 - 1.5)}{3} = 0.5$$

$$\theta = 30°$$

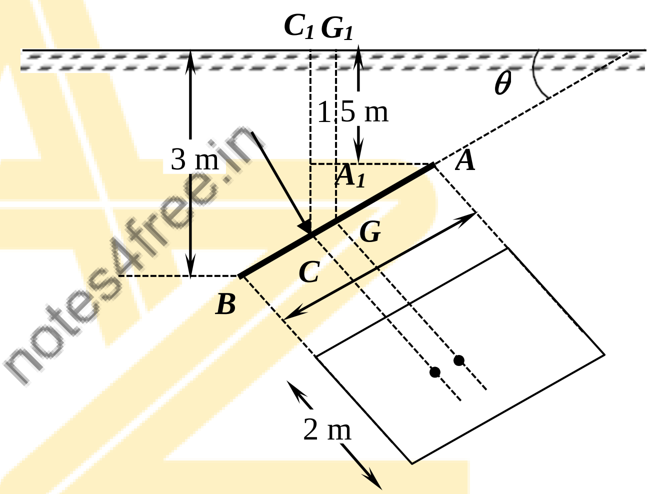
The centre of pressure

$$h_{C.P} = \bar{h} + \frac{I_G \times Sin^2\theta}{A\,\bar{h}}$$

$$h_{C.P} = 2.25 + \frac{4.5 \times \left(\frac{1}{4}\right)}{6 \times 2.25} = 2.33m$$

Q.13 A rectangular plate 2 m x 3 m is immersed in oil of specific gravity 0.85 such that its ends are at depths 1.5 m and 3 m respectively. Determine the total pressure acting on the plate and locate it.

Solution:



A = 2 x 3 = 6 m^2

$S_o = 0.85$

Assume

$\rho = 1000\ kg/m^3$

$g = 10\ m/s^2$

$\bar{y} = GG_1$

$\bar{h} = CC_1$

Sin $\boldsymbol{\theta = 1.5 / 3 = 0.5}$

$\boldsymbol{\theta = 30^o}$

$\boldsymbol{GG_1 = G_1A_1 + A_1G = G_1A_1 + AG\ Sin\ \theta}$

$\boldsymbol{GG_1 = 1.5 + (3/2)\ Sin\ 30 = 2.25\ m}$

We know that the total pressure force is given by

$\boldsymbol{P = S_o \rho g A}\ \bar{y} = 0.85 \times 1000 \times 10 \times 6 \times 2.25 = 114.75\ kN$

Centre of Pressure

The Centre of pressure is given by

$$\bar{h} = \bar{y} + \frac{I_g}{A\,\bar{y}} \sin^2\theta$$

$$I_g = \frac{bd^3}{12} = \frac{2\times 3^3}{12} = 4.5\ m^4$$

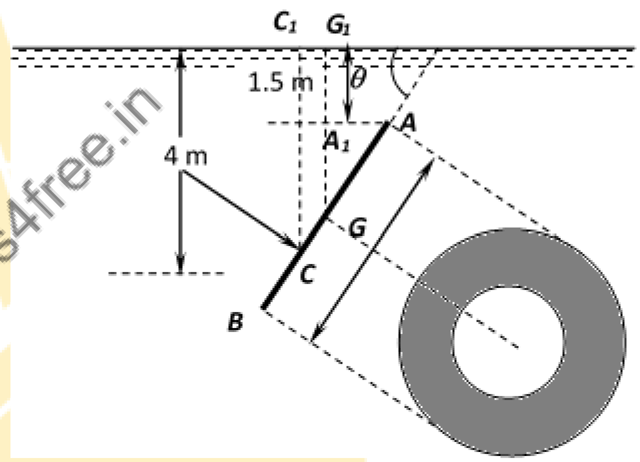
$$\bar{h} = 2.25 + \frac{4.5}{6\times 2.25}\sin^2 30 = 2.33\ m$$

Q.14. A Circular plate with a concentric hole is immersed in water in such a way that its greatest and least depth below water surface are 4 m and 1.5 m respectively. Determine the total pressure on the plate and locate it if the diameter of the plate and hole are 3 m and 1.5 m respectively.

Solution:

Assume

ρ = 1000 kg/ m^3 and g = 10 m/ s^2



$$A = \frac{\pi}{4}\left(D^2 - d^2\right) = \frac{\pi}{4}\left(3^2 - 1.5^2\right) = 5.3014\ \text{m}^2$$

$\bar{y} = GG_1$

$\bar{h} = CC_1$

Sin $\boldsymbol{\theta = 2.5 / 3 = 0.833}$ and $\boldsymbol{\theta = 30^o}$

$\boldsymbol{GG_1 = G_1A_1 + A_1G = G_1A_1 + AG\ Sin\ \theta}$

$\boldsymbol{GG_1 = 1.5 + (3/2)\ 0.833 = 2.75\ m}$

We know that the total pressure force is given by

$P = S_o\rho gA\,\bar{y} = 1000 \times 10 \times 5.3014 \times 2.75 = 144.7885\ kN$

Centre of Pressure

The Centre of pressure is given by

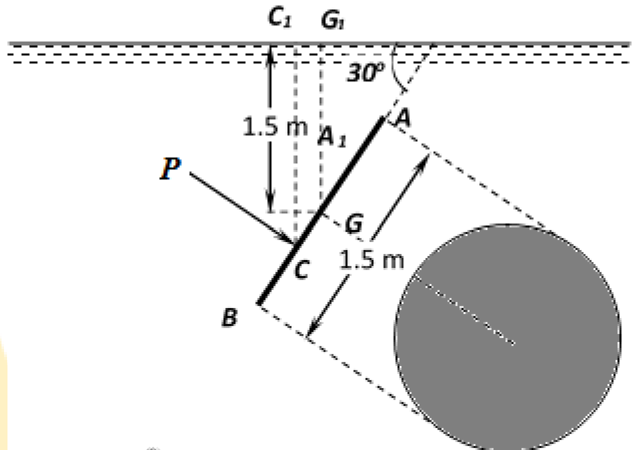
$$\bar{h} = \bar{y} + \frac{I_g}{A\bar{y}}\sin^2\theta$$

$$I_g = \frac{\pi}{4}\left(R^4 - r^4\right) = \frac{\pi}{4}\left(1.5^4 - 0.75^4\right) = 3.728\ \text{m}^4$$

$$\bar{h} = 2.75 + \frac{3.728}{5.3014 \times 2.75}\sin^2 30 = 2.814\ \text{m}$$

Q.15 . A circular plate of dia 1.5 m is immersed in a liquid of relative density of 0.8 with its plane making an angle of 30 ^{o} with the horizontal. The centre of the plate is at a depth of 1.5 m below the free surface. Calculate the total force on one side of the plate and location of centre of pressure.

Solution:



Assume

ρ = 1000 kg/ m^3 and g = 10 m/ s^2

$S_o = 0.80$

$$A = \frac{\pi D^2}{4} = \frac{\pi \times 1.5^2}{4} = 1.767 \text{ m}^2$$

$\bar{y} = GG_1$

$\bar{h} = CC_1$

$\theta = 30^o$

$GG_1 = G_1A_1 + A_1G = G_1A_1 + AG \sin\theta$

$GG_1 = 1.5 + (3/2)\, 0.833 = 2.75$ m

We know that the total pressure force is given by

$P = S_o\rho g A \bar{y} = 0.8 \times 1000 \times 10 \times 1.767 \times 2.75 = 38.874$ kN

Centre of Pressure

The Centre of pressure is given by

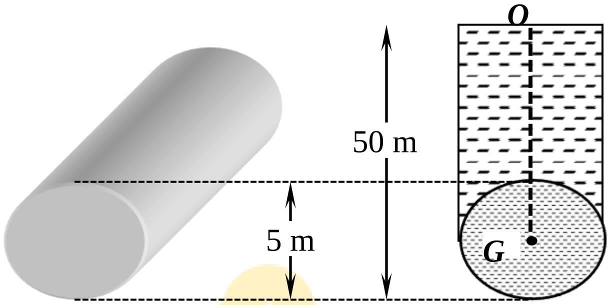
$$\bar{h} = \bar{y} + \frac{I_g}{A\bar{y}} \sin^2\theta$$

$$I_g = \frac{\pi R^4}{4} = \frac{\pi \times 0.75^4}{4} = 0.2485 \text{ m}^4$$

$$\bar{h} = 2.75 + \frac{0.2485}{1.767 \times 2.75} \sin^2 30 = 2.763 \text{ m}$$

Q.16 A vertical gate closes a circular tunnel of 5 m diameter running full of water, the pressure at the bottom of the gate is 0.5 MPa. Determine the hydrostatic force and the position of centre of pressure.

Solution: Assume ρ = 1000 kg/ m^3 and g = 10 m/ s ^{2}



Pressure intensity at the bottom of the gate is **= *p* =*S$_o$ρgy***

Where ***y*** is the depth of point from the free surface.

$0.5 \times 10^6 = 1000 \times 10 \times y$

$y = 50$ m

Hence the free surface of water is at 50 m from the bottom of the gate

$$A = \frac{\pi D^2}{4} = \frac{\pi \times 5^2}{4} = 19.635 \text{ m}^2$$

$$\bar{y} = OG = 50 - 2.5 = 47.5 \text{ m}$$

We know that the total pressure force is given by

$$P = S_o \rho g A \bar{y} = 1000 \times 10 \times 19.635 \times 47.5 = 9326.625 \text{ kN}$$

Centre of Pressure

The Centre of pressure is given by

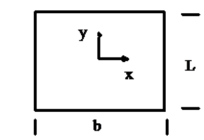
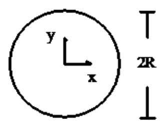
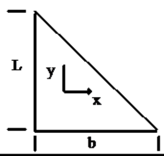
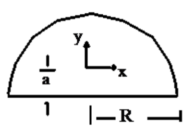
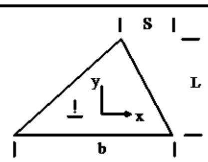
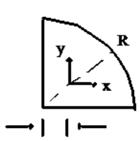
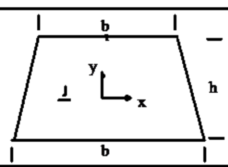
$$\bar{h} = \bar{y} + \frac{I_g}{A\bar{y}}$$

$$I_g = \frac{\pi R^4}{4} = \frac{\pi \times 2.5^4}{4} = 30.68 \text{ m}^4$$

$$\bar{h} = 47.5 + \frac{30.68}{19.635 \times 47.5} = 47.533 \text{ m}$$

i.e. 50.0 – 47.533 = 2.677 m from the bottom of the gate or tunnel.

PROPERTIES OF PLANE SECTIONS

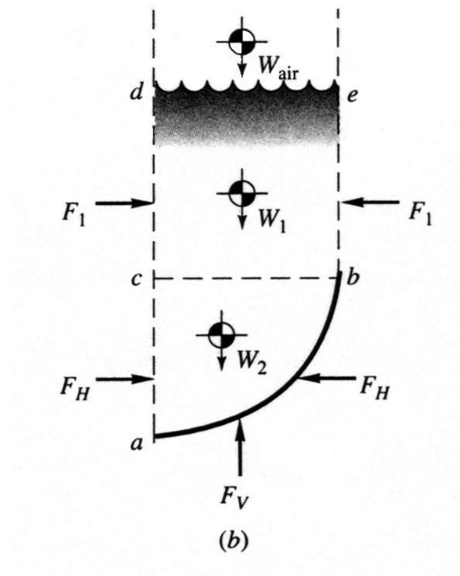
Geometry	Centroid	Moment of Inertia I_{xx}	Product of Inertia I_{xy}	Area
	$b/2$, $L/2$	$\frac{bL^3}{12}$	0	$b \cdot L$
	0 , 0	$\frac{\pi R^4}{4}$	0	πR^2
	$b/3$, $L/3$	$\frac{bL^3}{36}$	$-\frac{b^2L^2}{72}$	$\frac{b \cdot L}{2}$
	$0, a = \frac{4R}{3\pi}$	$R^4\left(\frac{\pi}{8} - \frac{8}{9\pi}\right)$	0	$\frac{\pi R^2}{2}$
	$a = \frac{L}{3}$	$\frac{bL^3}{36}$	$\frac{b(b-2s)L^2}{72}$	$\frac{1}{2} b \cdot L$
	$a = \frac{4R}{3\pi}$	$\left(\frac{\pi}{16} - \frac{4}{9\pi}\right)R^4$	$\left(\frac{1}{8} - \frac{4}{9\pi}\right)R^4$	$\frac{\pi R^2}{4}$
	$a = \frac{h(b+2b_1)}{3(b+b_1)}$	$\frac{h^3(b^2+4bb_1+b_1^2)}{36(b+b_1)}$	0	$(b+b_1)\frac{h}{2}$

Fluid Specific Weight

	lbf/ft^3	N/m^3		lbf/ft^3	N/m^3
Air	.0752	11.8	Seawater	64.0	10,050
Oil	57.3	8,996	Glycerin	78.7	12,360
Water	62.4	9,790	Mercury	846.	133,100
Ethyl	49.2	7,733	Carbon	99.1	15,570

2.4 Hydrostatic Forces on Curved Surfaces

Since this class of surface is curved, the direction of the force is different at each location on the surface. Therefore, we will evaluate the x and y components of net hydrostatic force separately.



Consider curved surface, a-b. Force balances in x & y directions yield

$F_h = F_H$

$Fv = W_{air} + W_1 + W_2$

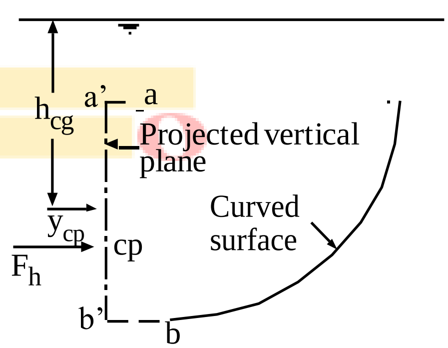
From this force balance, the basic rules for determining the horizontal and vertical component of forces on a curved surface in a static fluid can be summarized as follows:

Horizontal Component, F_h

The horizontal component of force on a curved surface equals the force on the plane area formed by the projection of the curved surface onto a vertical plane normal to the component.

The horizontal force will act through the c.p. (not the centroid) of the projected area.

from the Diagram:



All elements of the analysis are performed with the vertical plane. The original curved surface is important only as it is used to define the projected vertical plane.

Therefore, to determine the horizontal component of force on a curved surface in a hydrostatic fluid:

Vertical Component - Fv

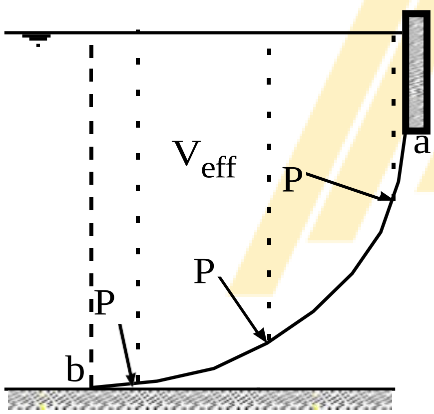
The vertical component of force on a curved surface equals the weight of the effective column of fluid necessary to cause **the pressure on the surface**.

The use of the words **effective column of fluid** is important in that there may not always actually be fluid directly above the surface. (See graphics below)

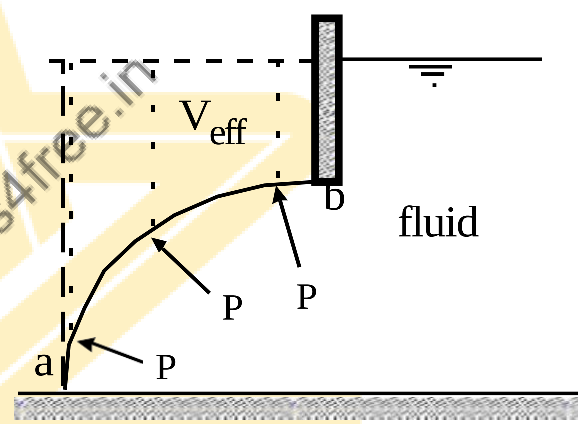
This effective column of fluid is specified by identifying the column of fluid that would be required to cause the pressure at each location on the surface.

Thus, to identify the Effective Volume - V_{eff}:

$$\mathbf{F_V = V_{eff}}$$



Fluid above the surface

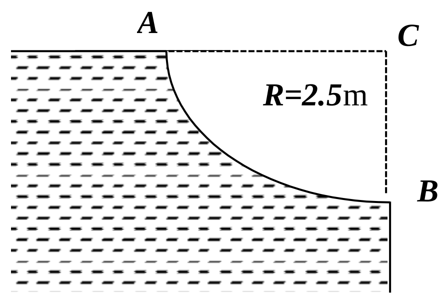


No fluid actually above surface

$$R = \sqrt{\left(\sum F_x^2\right) + \left(\sum F_y^2\right)} \qquad \theta = \tan^{-1}\left(\frac{\sum F_y}{\sum F_x}\right)$$

Q.17 Find the horizontal and vertical component of force and its point of application due to water per meter length of the gate AB having a quadrant shape of radius 2.5 m shown in Fig. Find also the resultant force in magnitude and direction.

Solution:



Assume

$\rho = 1000$ kg/ m^3 and g = 9.81 m/ s^2

R = 2.5 m, Width of gate = 1 m

Horizontal force F_x

F_h = Force on the projected area of the curved surface on the vertical plane

= Force on ***BC***

$A = 2.5 \times 1 = 2.5 \text{m}^2$

$$\bar{y} = \frac{2.5}{2} = 1.25\text{m}$$

$F = \gamma_{water} A h_c = 9810 \times 2.5 \times 1.25 = 30656 \text{ N} = 30.656\text{kN}$

This will act at a distance $\bar{h} = \frac{2}{3} \times 2.5 = \frac{5}{3}$ m from the free surface of liquid AC

Vertical Force F_y

F_y= Weight of water (imaginary) supported by ***AB***

= γ_{water} x Area of ***ACB*** x Length of gate

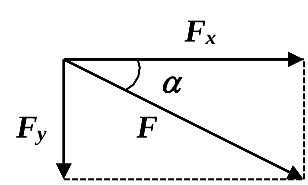
$$= 9810 \times \frac{\pi \times 2.5^2}{4} \times 1 = 48154\text{N} = 48.154\text{kN}$$

This will act at a distance The $\bar{x} = \frac{4 \times 2.5}{3\pi} = 1.061$ m from ***CB***

Resultant force

$$F = \sqrt{F_x^2 + F_y^2} = \sqrt{30.656^2 + 48.154^2} = 57.084\ kN$$ and its

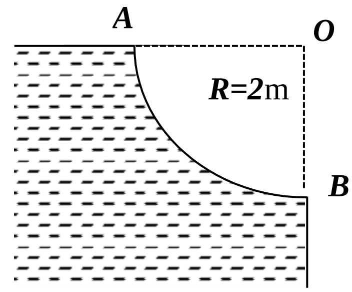
inclination is given by



$$\alpha = \tan^{-1}\frac{F_y}{F_x} = \tan^{-1}\frac{48.154}{30.656} = 57.51^o$$

Q.18 Find the horizontal and vertical component of force and its point of application due to water per meter length of the gate AB having a quadrant shape of radius 2 m shown in Fig. Find also the resultant force in magnitude and direction.

Solution:



Assume

$\rho = 1000$ kg/ m^3 and g = 10 m/ s ^{2}

R = 2 m, Width of gate = 1 m

Horizontal force F_x

F_x = Force on the projected area of the curved surface on the vertical plane

= Force on **BO** = $P = S_o \rho g A \bar{y}$

$A = 2 \times 1 = 2$ m^2

$\bar{y} = \frac{2}{2} = 1$ m

$F_x = 1000 \times 10 \times 2 \times 1 = 20$ kN

This will act at a distance $h = \frac{2}{3} \times 2 = \frac{4}{3}$ m from the free surface of liquid

Vertical Force F_y

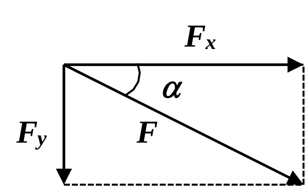
F_y= Weight of water (imaginary) supported by ***AB***

= $S_o \rho g$ x Area of ***AOB*** x Length of gate

= $1000 \times 10 \times \frac{\pi \times 2^2}{4} \times 1 = 31.416$ **kN**

This will act at a distance $\bar{x} = \frac{4 \times 2}{3\pi} = 0.848$ m from ***OB***

Resultant force $F = \sqrt{F_x^2 + F_y^2} = \sqrt{20^2 + 31.426^2} = 37.25$

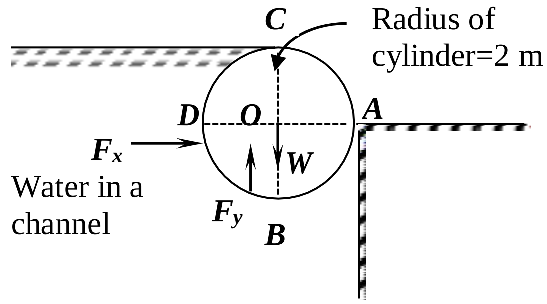


kN and its inclination is given by

$\alpha = \tan^{-1}\left[\frac{F_y}{F_x}\right] = \tan^{-1}\left[\frac{31.426}{20}\right] = 57.527^o$

Q.19. A cylinder holds water in a channel as shown in Fig. Determine the weight of 1 m length of the cylinder.

Solution:



Radius of Cylinder = R = 2m

Length of cylinder = 1 m

Weight of Cylinder = W

Horizontal force exerted by water= F_x

F_x = Force on vertical area ***BOC***

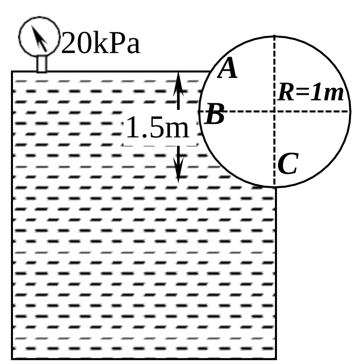
$= S_o \rho g A \bar{y} = 1000 \times 10 \times (4 \times 1) \times (2/2) = 40\ kN\ (\rightarrow)$

The vertical force exerted by water= F_y =Weight of water enclosed in ***BDCOB***

$$F_y = S_o \rho g \left(\frac{\pi \times 2^2}{4} \right) x L = 1000 x 10 x 3.142 = 31.416 kN\ (\uparrow)$$

For equilibrium of the cylinder the weight of the cylinder must be equal to the force exerted by the water on the cylinder. Hence, the weight of the cylinder is 31***.416 kN*** per meter length **.**

Q.20. Fig. shows the cross section of a tank full of water under pressure. The length of the tank is 2 m. An empty cylinder lies along the length of the tank on one of its corner as shown. Find the resultant force acting on the curved surface of the cylinder.



Solution:

R=1 m

L = 2 m

$p = \rho g h = 1000 \times 10 \times h = 20 \times 10^3$

$h = 2$ m

For this pressure, the free surface should be 2 m above A

Horizontal component of force F_x

$$F_x = S_o \rho g A \bar{y}$$

$A = 1.5x\ 2.0 = 3\ \text{m}^2$

$$\bar{y} = 2 + \frac{1.5}{2} = 2.75\ \text{m}$$

$F_x = 1000 \times 10 \times 3.0 \times 2.75 = 82.5\ kN\ (\rightarrow)$

The vertical force exerted by water = F_y

F_y = Weight of water enclosed in ***ABC***

= Weight of water enclosed in ***CODEABC***

= Weight of water enclosed in (***CODFBC – AEFB***)

But Weight of water enclosed in ***CODFBC***

= Weight of water enclosed in (***COB+ODFBO)***

$$= \rho g \left[\frac{\pi R^2}{4} + BO \times OD \right] \times 2 = 1000 \times 10 \left[\frac{\pi \times 1^2}{4} + 1 \times 2.5 \right] \times 2 = 65.708\ \text{kN}$$

Weight of water in ***AEFB = $S_o \rho g$[*** Area of ***AEFB] x 2.0***

= $S_o \rho g$[Area of (***AEFG+AGBH-AHB] x 2.0***

$\sin \theta = AH/ AO = 0.5/ 1.0 = 0.5\ \therefore \theta = 30^o$

$BH = BO - HO = 1.0 - AO \cos \theta = 1.0 - 1 \times \cos 30^o = 0.134$

Area ABH = Area ABO – Area AHO

$$= \pi R^2 \times \frac{30}{360} - \frac{AH \times HO}{2.0} = \pi \times 1^2 \times \frac{1}{12} - \frac{0.5 \times 0.866}{2.0} = 0.0453$$

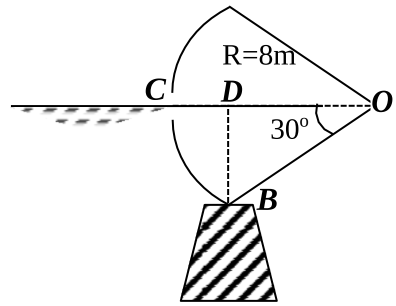
$\therefore$ Weight of water in AEFB = $1000 \times 10[AE \times AG + AG \times AH - 0.0453] \times 0.2$

$= 1000 \times 10[2.0 \times 0.134 + 0.134 \times 0.5 - .0453] \times 0.2$

= 5794 N

F_y =65708 – 5794 = 59914 N (Ans)

Q.21. Calculate the resultant water pressure on the Tainter gate of radius 8 m and width unity as shown in Fig.



Solution:

Horizontal component of force F_x

$F_x = S_o \rho g A \bar{y}$

$DB = OB \sin 30 = 8 \times 0.5 = 4.0$ m

$A = 4 \times 1.0 = 4$ m^2

$\bar{y} = \frac{4}{2} = 2$ m

The Horizontal force exerted by water = F_x

$F_x = 1000 \times 10 \times 4.0 \times 2.0 = 80.0$ kN ($\rightarrow$)

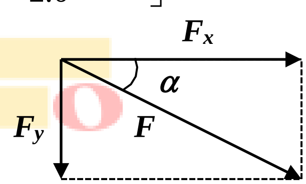
The vertical force exerted by water = F_y

F_y = Weight of water enclosed in ***CDBC***

= Weight of water enclosed in (***CDOBC*** – ***DOB***)

$$= S_o \rho g \left[\pi R^2 \times \frac{30}{360} - \frac{BD \times DO}{2.0}\right] = 1000 \times 10 \left[\pi \times 8^2 \times \frac{1}{12} - \frac{4.0 \times 8.8\cos 30}{2.0}\right] = 15.13 \text{ kN}$$

Resultant force $F = \sqrt{F_x^2 + F_y^2} = \sqrt{80^2 + 15.13^2} = 81.418$ kN



kN and its inclination is given by

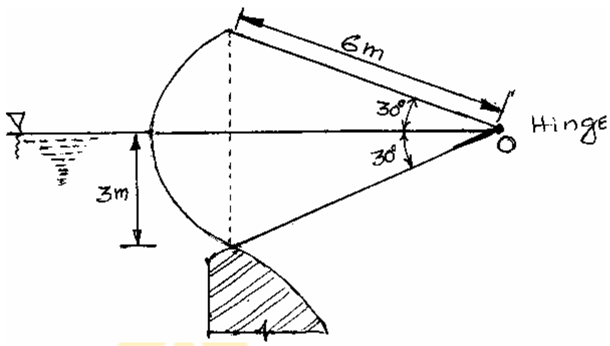
$$\alpha = \tan^{-1}\left[\frac{F_y}{F_x}\right] = \tan^{-1}\left[\frac{15.13}{80}\right] = 10.71^o$$

Q.22 Length of a Tainter gate perpendicular to paper is 0.50m. Find:

i) Total horizontal thrust of water on gate.

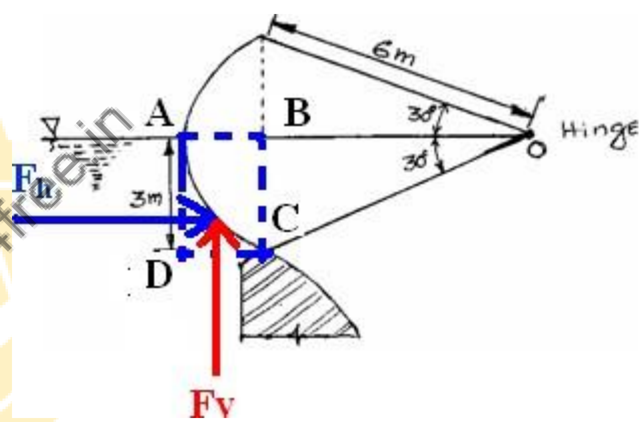
ii) Total vertical component of water pressure against gate.

iii) Resultant water pressure on gate and its inclination with horizontal.



Ans: Given L = 0.5m,

AD = BC = 3m, $\gamma W = 9.81\ kN/m^3$



(i) Total horizontal thrust of water on gate

$$Fh = \gamma W \times A \times \bar{h}$$

$$Fh = 9.81 \times (3.0 \times 0.5) \times \frac{3}{2}$$

Fh = 22.07 kN → Rightward

Acting at

$$\bar{h}_{c.p.} = \bar{h}_c + \frac{I_G \times Sin^2 90°}{A \times \bar{h}_c}$$

$$\bar{h}_{c.p.} = 1.5 + \frac{\left(\frac{0.5 \times 3^3}{12}\right) \times Sin^2 90°}{(3.0 \times 0.5) \times 1.5} = 1.5 + 0.5 = 2.0m$$

(ii) Total vertical component of water pressure against gate = upward thrust due area ABC

Upward thrust due area ABC = Area AOC - ΔOBC

Area ABC = $\frac{\pi \times \mathbf{R}^2}{12} - \frac{1}{2} \times \mathbf{OB} \times \mathbf{BC}$

Area ABC = $\frac{\pi \times 6^2}{12} - \frac{1}{2} \times 3\cos 30° \times 3$

Area ABC = 1.636 m2

Fv = γW × Area ABC × L

Fv = 9.81 × 1.636 × 0.5 = 8.024 kN ↑ upward

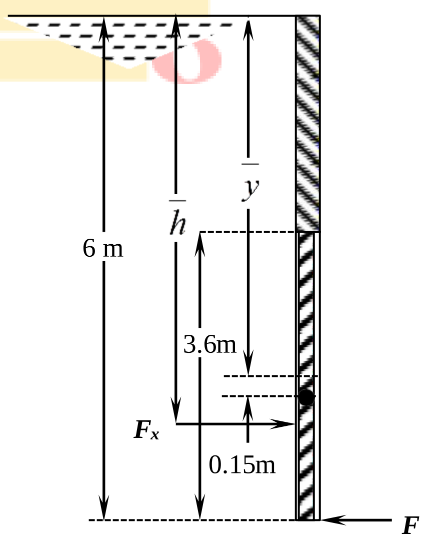
(iii) Resultant water pressure on gate and its inclination with horizontal

$$\mathbf{R} = \sqrt{\mathbf{F}_h^2 + \mathbf{F}_v^2} = \sqrt{(22.07)^2 + (8.024)^2} = 23.48 \quad \mathbf{kN}$$

$$\theta = \tan^{-1}\left(\frac{8.024}{22.07}\right) = 0.3637$$

Inclination $\theta = 20°$

Q23. A 3.6 m x 1.5 m wide rectangular gate MN is vertical and is hinged at point 150 mm below the centre of gravity of the gate. The total depth of water is 6 m. What horizontal force must be applied at the bottom of the gate to keep the gate closed?



Solution:

Total pressure acting on the gate is F_x

$F_x = S_o \rho g A \bar{y}$

= 1000 x 10 x (3.6 x 1.5) x (6-3.6/2)

= 226.8 kN

Acting at

$$\bar{h} = \bar{y} + \frac{I_g}{A\bar{y}}$$

$$I_g = \frac{bd^3}{12} = \frac{1.5 \times 3.6^3}{12} = 5.832 \text{ m}^4$$

$$\bar{h} = 4.2 + \frac{5.832}{5.4 \times 4.2} = 4.457 \text{ m}$$

Let ***F*** be the force applied at the bottom of the gate required to retain the gate in equilibrium.

From the conditions of equilibrium, taking moments about the hinge, we get

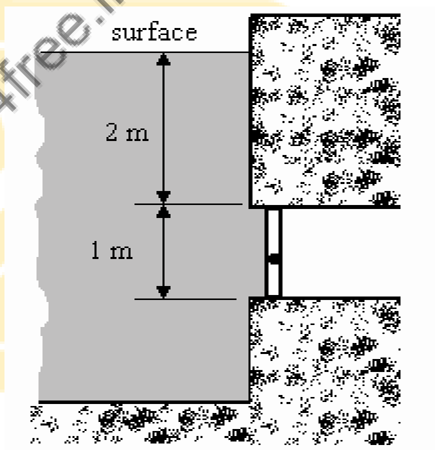
F (1.8 – 0.15) = F_x [4.457-(4.2+0.15)]

F = 14.707 kN (Ans).

Q.24 *A culvert in the side of a reservoir is closed by a vertical rectangular gate 2m wide and 1m deep as shown in figure. The gate is hinged about a horizontal axis which passes through the centre of the gate. The free surface of water in the reservoir is 2.5 m above the axis of the hinge. The density of water is 1000 kg/m^3. Assuming that the hinges are frictionless and that the culvert is open to atmosphere, determine*

(i) The force acting on the gate when closed due to the pressure of water.

(ii) The moment to be applied about the hinge axis to open the gate.



Solution: (i) The total hydrostatic force

$$F = \gamma A\, h_c$$

$$\gamma_{water} = 1000 \text{ x} 9.81 = 9810 \text{ N/m}^3$$

Given **A = 1m X 2m = 2m^2**

$$h_c = 2 + \frac{1}{2} = 2.5\text{m}$$

$$F = 9810 X 2 X 2.5 = 49050 N$$

(ii) The moment applied about hinge axis to open the gate is say 'M'

The centre of pressure ($h_{c.p}$):

From the above $h_c = 2.5m$, $A = 2m^2$

$$(I_c)_{xx} = \frac{BD^3}{12} = \frac{2X1^3}{12} = 0.167m^4$$

$$h_{C.P.} = h_c + \frac{(I_c)_{xx}}{Ah_c} = 2.5 + \frac{0.167}{2X2.5} = 2.53334\ m$$

Distance of Hydrostatic force (F) from the water surface = 2.5334m.

Distance of hinge from free surface = 2.5m

Distance between hinge and centre of pressure of force 'F' = 2.5334 m - 2.5m = 0.0334m

Taking moment about Hinge to open the gate 'M' = F X 0.0334 = 49050 N X 0.0334 m

The moment applied about hinge axis to open the gate 'M' = 1638.27 N-m

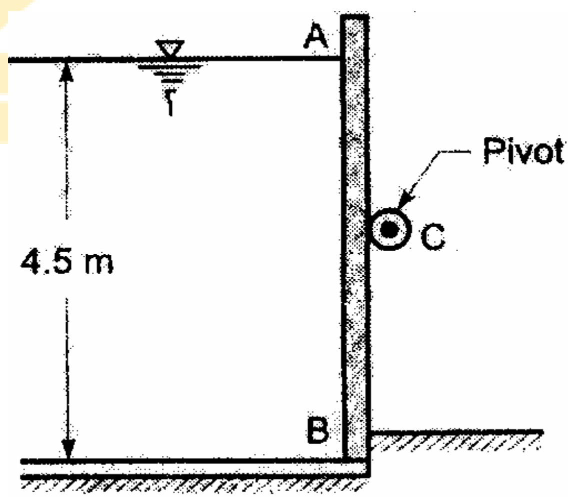
Q.25 Figure shows a rectangular flash board AB which is 4.5m high and is pivoted at C. What must be the maximum height of C above B so that the flash board will be on the verge of tipping when water surface is at A? Also determine if the pivot of the flash board is at a height h =1.5m, the reactions at B and C when the water surface is 4m above B.

Ans:

(i) The flash board would tip about the hinge point 'C' when the line of action of resultant 'R' pressure force 'F' lies from C to A anywhere on the board.

The limiting condition being the situation when the resultant force 'F' passes through 'C'

The resultant force 'F' also passes through the centroid of the pressure diagram and the centre lies at $\frac{1}{3} \times AB = \frac{4.5}{3} = 1.5m$



Hence the maximum height of 'C' from 'B' = (4.5m-3.0m) =1.5m (from bottom)

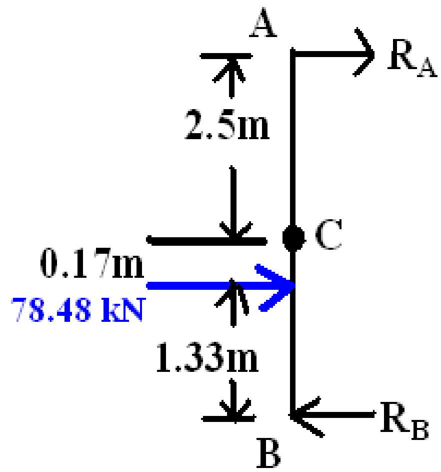
(ii) The pivot of the flash board is at a height h =1.5m from point B, the reactions at B and C when the water surface is 4m above B.

$$\bar{h} = \frac{4.0}{2} = 2.m$$

Hydrostatic force $P = \rho g A \bar{h}$ = 1000×9.81×(4.0×1.0)×2=78.48 kN acting at

$$h_{cp} = 2.0 + \frac{1 \times (4.0)^3 Sin^2 90°}{4.0 \times 2.0} = 2.67m \ \ \textit{from free water surfcae}$$

Or h = (4.0-2.67) = 1.33m from bottom



Let R_A and R_B be the reaction.

R_A + 78.48 = R_B
by taking moment about pivot ‘C’

R_A ×2.5 + 78.48×0.17 = R_B×1.5

On solving R_A =104.38kN R_B = 182.86 kN

2.5 Gravity Dam:

A **gravity dam** is a dam constructed from concrete or stone masonry and designed to hold back water by primarily utilizing the weight of the material alone to resist the horizontal pressure of water pushing against it. Gravity dams are designed so that each section of the dam is stable, independent of any other dam section

Gravity dams generally require stiff rock foundations of high bearing strength (slightly weathered to fresh); although they have been built on soil foundations in rare cases. The bearing strength of the foundation limits the allowable position of the resultant which influences the overall stability. Also, the stiff nature of the gravity dam structure is unforgiving to differential foundation settlement, which can induce cracking of the dam structure.

Gravity dams provide some advantages over embankment dams. The main advantage is that they can tolerate minor over-topping flows as the concrete is resistant to scouring. This reduces the requirements for a cofferdam during construction and the sizing of the spillway. Large overtopping flows are still a problem, as they can scour the foundations if not accounted for in the design. A disadvantage of gravity dams is that due to their large footprint, they are susceptible to uplift pressures which act as a de-stabilising force. Uplift pressures (buoyancy) can be reduced by internal and foundation drainage systems which reduces the pressures.

2.5.1 Forces Acting on Gravity Dams:

Forces that act on a gravity dam (Fig.1) are due to:

- Water Pressure(Hydrostatic)
- Uplift Pressure
- Earthquake Acceleration
- Silt Pressure
- Wave Pressure
- Ice Pressure

>> Self Weight (W) counters the forces listed above.

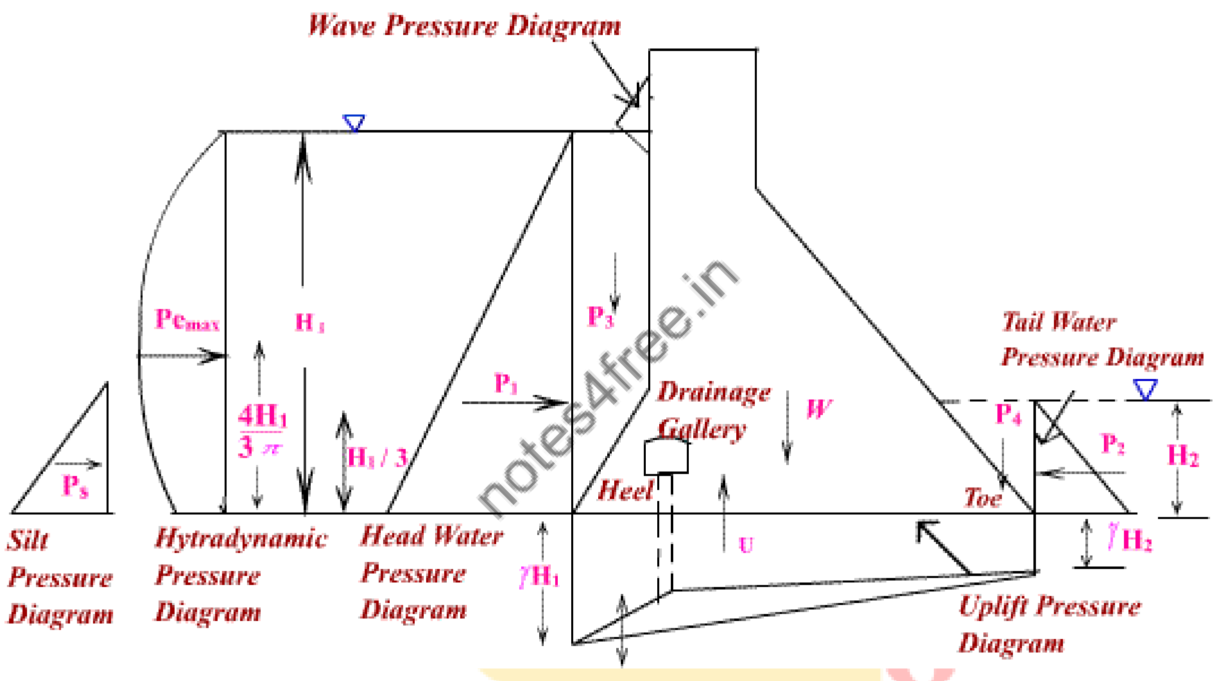


Fig. Forces on Gravity Dams

- **Force due to hydrostatic Pressure:**

Force due to hydrostatic Pressure is the major external force on a gravity dam. The intensity of pressure from zero at the water surface to the maximum (γH) at the base. The force due to this pressure is given by γH2, acting at H/3 from the base. In Fig.1, the forces P1 and P2 are due to hydrostatic pressure acting on the upstream and the downstream sides respectively. These are horizontal components of the hydrostatic force due to head water (upstream side) and tail water (downstream side) of the dam respectively.

The forces marked as P3 and P4 are the weight of water held over the inclined faces of the dam on the upstream slope and downstream slope respectively. These are the respective vertical components of the hydrostatic force on the two faces mentioned.

- **Force due to Uplift Pressure:**

Water that seeps through the pores, cracks and fissures of the foundation material and water that seeps through the body of the dam to the bottom through the joints between the body of the dam and the foundation at the base, exert an uplift pressure on the base of the dam. The force (U) due to this acts against the weight of the dam and thus contributes to destabilizing the dam.

According to the recommendation of the United States Bureau of Reclamation (USBR), the uplift pressure intensities at the heel (upstream end) and the toe (downstream end) are taken to be equal to the respective hydrostatic pressures. A linear variation of the uplift pressure is often assumed between the heel and the toe. Drainage galleries can be provided (Fig.) to relieve the uplift pressure. In such a case, the uplift pressure diagram gets modified as shown in Fig.

- **Earthquake Forces:**

The effect of an earthquake is perceived as imparting an acceleration to the foundations of the dam in the direction in which the wave travels at that moment. It can be viewed (resolved) as horizontal and vertical components of the random acceleration.

2.6 Lock Gates

Whenever a dam or a weir is constructed across a river or canal, the water levels on both the sides of the dam will be different. If it is desired to have navigation or boating in such a river or a canal, then a chamber, known as lock, is constructed between these two different water levels. Two sets of gates (one on the upstream side and the other on downstream side of the dam) are provided as shown in fig - 1.

Fig-1 : Lock Gate

(Source: http://www.codecogs.com/library/engineering/fluid_mechanics/water_pressure/lock-gate.php)

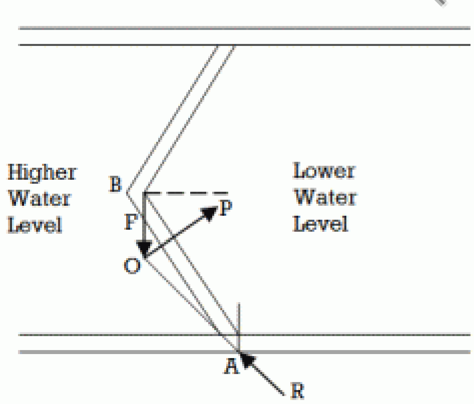


Fig-2(a) : Plan of lock gate

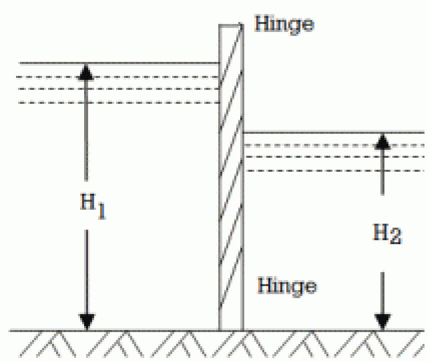


Fig-2(b) : Elevation of lock gate

Now consider a set of lock gates AB and BC hinged at the top and bottom at A and C respectively as shown in fig - 2(a). These gates will be held in contact at b by the water pressure, the water level being higher on the left hand side of the gates as shown in fig - 2(b).

Let,

- P = Water pressure on the gate AB or BC acting at right angles on it
- F = Force exerted by the gate BC acting normally to the contact surface of the two gates AB and BC (also known as reaction between the two gates), and
- R = Reaction at the upper and lower hinge

Since the gate AB is in equilibrium, under the action of the above three forces, therefore they will meet at one point. Let,P and F meet at O, then R must pass through this point.
Let, α = Inclination of the lock gate with the normal to the walls of the lock.
From the geometry of the figure ABO, we find that it is an isosceles triangle having its angles $\angle$ OBA and $\angle$ OAB both equal to α.

$$R\cos\alpha = F\cos\alpha$$
$$\therefore R = F \qquad (1)$$

and now resolving the force at right angles to AB

$$P = R\sin\alpha + F\sin\alpha = 2R\sin\alpha$$
$$\therefore R = \frac{P}{2\sin\alpha}$$
$$\therefore F = \frac{P}{2\sin\alpha} \qquad (2)$$

Now let us consider the water pressure on the top and bottom hinges of the gate, Let,

- H1 = Height of water to the left side of the gate.
- A1 = Wetted area (of one of the gates) on left side of the gate
- P1 = Total pressure of the water on the left side of the gate
- H2, A2, P2 = Corresponding values for right side on the gate
- RT = Reaction of the top hinge, and
- RB = Reaction of bottom hinge

Since the total reaction (R) will be shared by the two hinges (RT), therefore

$$R = R_T + R_B \qquad (3)$$

and total pressure on the lock gate,

$$P = wA\bar{x}$$
$$\Rightarrow P_1 = wA_1 \times \frac{H_1}{2} = \frac{wA_1H_1}{2}$$

Similarly, $P_1 = \frac{wA_2H_2}{2}$

Since the directions of P1 and P2 are in the opposite direction, therefore the resultant pressure,

$$P = P_1 - P_2$$

We know that the pressure P1 will act through its center of pressure, which is at a height of $\frac{H_1}{3}$ from the bottom of the gate. Similarly, the pressure P2 will also act through its center of pressure which is also at a height of $\frac{H_2}{3}$ from the bottom of the gate.
A little consideration will show, that half of the resultant pressure (i.e., P1 - P2 or P)will be resisted by the hinges of one lock gate (as the other half will be resisted by the other lock gates).

$$R_T \sin\alpha \times h = (\frac{P_1}{2} \times \frac{H_1}{3}) - (\frac{P_2}{2} \times \frac{H_2}{3}) \quad (4)$$

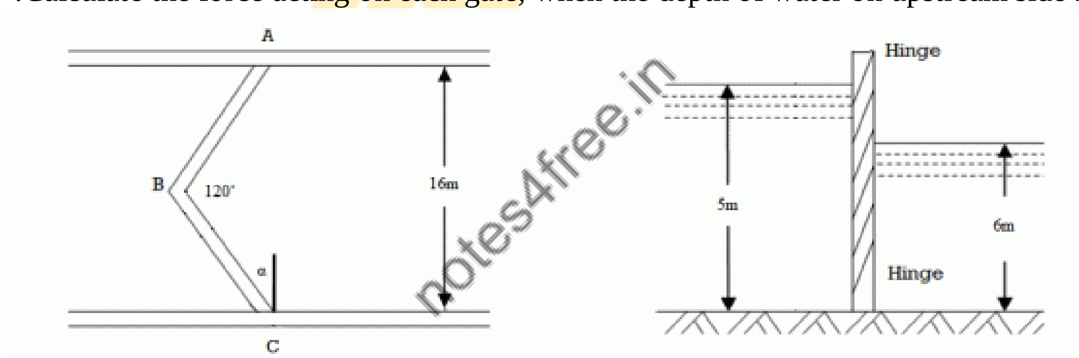
where h is the distance between the two hinges.

Also resolving the forces horizontally,

$$P_1 - P_2 = R_B \sin\alpha + R_T \sin\alpha \quad (5)$$

From equations (4) and (5) the values of RB and RT may be found out.

Q. 26 Two lock gates of 7.5m height are provided in a canal of 16m width meeting at an angle of 120°.Calculate the force acting on each gate, when the depth of water on upstream side is 5m.



Given,

- Height of lock gates = 7.5m
- Width of lock gates = 16m
- Inclination of gates = $120°$
- H = 5m

From the geometry of the lock gate, we find that inclination of the lock gates with the walls,

$$\alpha = \frac{180° - 120°}{2} = 30°$$

and

width of each gate = $\frac{16/2}{\cos\alpha} = \frac{8}{\cos 30°}$ = 9.24 m

∴ Wetted area of each gate, $A = 5 \times 9.24 = 46.2m^2$ and force acting on each gate,

$$P = wA \times \frac{H}{2} = 9.81 \times 46.2 \times \frac{5}{2} = 1133\ KN$$

15 CV 33 FLUID MECHANICS NOTES

MODULE-2

- **Module-2A : Hydrostatic forces on Surfaces**
- **Module-2B :Fundamentals of fluid flow (Kinematics)**

by

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Module-2B: Fundamentals of fluid flow (Kinematics)

Introduction. Methods of describing fluid motion. Velocity and Total acceleration of a fluid particle. Types of fluid flow, Description of flow pattern. Basic principles of fluid flow, three-dimensional continuity equation in Cartesian coordinate system. Derivation for Rotational and irroational motion. Potential function, stream function, orthogonality of streamlines and equipotential lines. Numerical problems on Stream function and velocity potential. Introduction to flow net.

2.7 Methods of Describing Fluid Motion:

Fluid kinematics refers to the features of a fluid in motion. It only deals with the motion of fluid particles without taking into account the forces causing the motion. Considerations of velocity, acceleration, flow rate, nature of flow and flow visualization are taken up under fluid kinematics.

A fluid motion can be analyzed by one of the two alternative approaches, called Lagrangian and Eulerian.

In Lagrangian approach, a particle or a fluid element is identified and followed during the course of its motion with time as demonstrated in

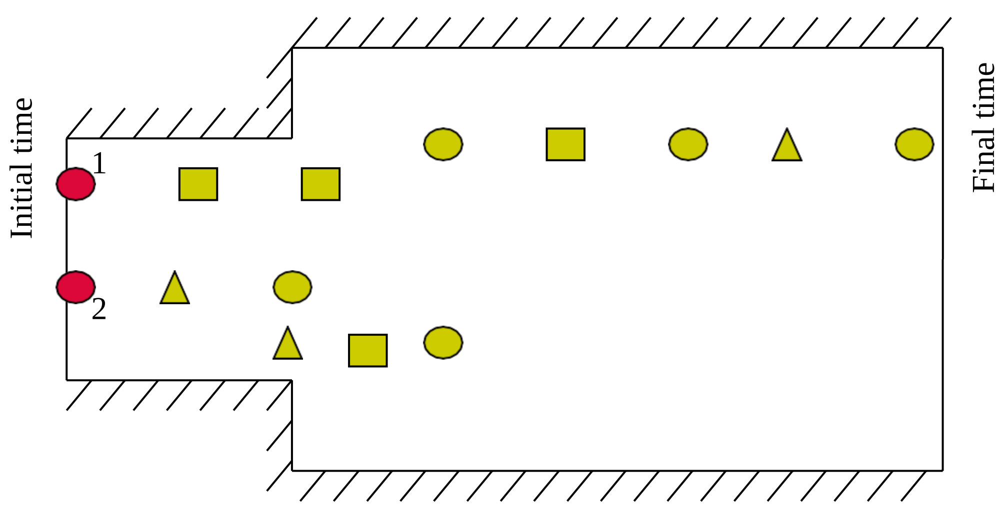


Fig. Lagrangian Approach (Study of each particle with time)

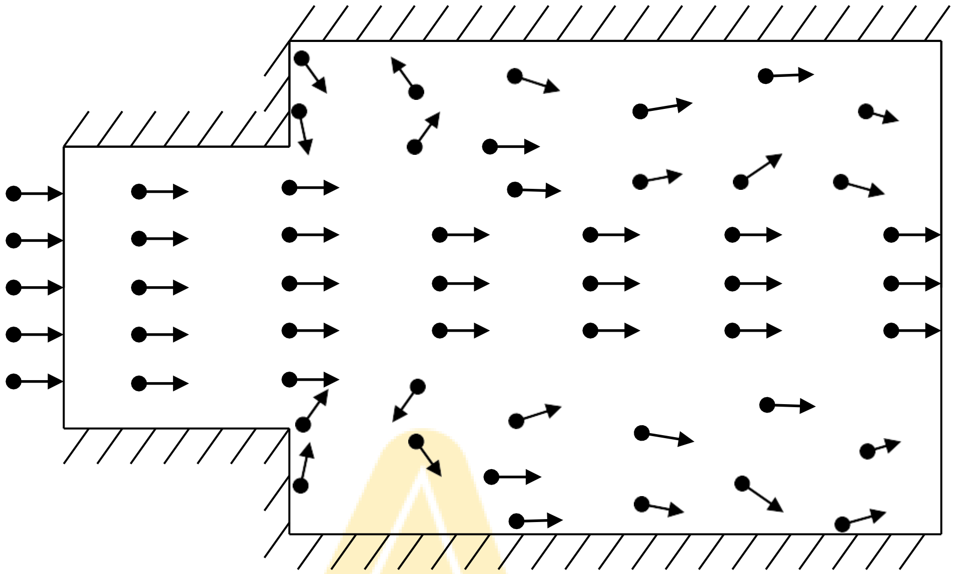


Fig. Eulerian Approach (Study at fixed station in space)

Example: To know the attributes of a vehicle to be purchased, you can follow the specific vehicle in the traffic flow all along its path over a period of time.

Difficulty in tracing a fluid particle (s) makes it nearly impossible to apply the Lagrangian approach. The alternative approach, called Eulerian approach consists of observing the fluid by setting up fixed stations (sections) in the flow field (Fig.).

Motion of the fluid is specified by velocity components as functions of space and time. This is considerably easier than the previous approach and is followed in Fluid Mechanics.

Example: Observing the variation of flow properties in a channel like velocity, depth etc, at a section.

2.8 *Velocity*

Velocity of a fluid along any direction can be defined as the rate of change of displacement of the fluid along that direction.

$$u = \frac{dx}{dt}$$

Where dx is the distance traveled by the fluid in time dt.

Velocity of a fluid element is a vector, which is a function of space and time.

Let V be the resultant velocity of a fluid along any direction and u, v and w be the velocity components in x, y and z-directions respectively.

Mathematically the velocity components can be written as

$u = f(x, y, z, t)$

$v = f(x, y, z, t)$

$w = f(x, y, z, t)$

and $V = ui + vj + wk = |V| = \sqrt{u^2 + v^2 + w^2}$

Where $u = \frac{dx}{dt}; v = \frac{dy}{dt}, w = \frac{dz}{dt}$

2.9 Acceleration

Acceleration of a fluid element along any direction can be defined as the rate of change of velocity of the fluid along that direction.

If a_x, a_y and a_z are the components of acceleration along-x, y and z- directions respectively, they can be mathematically written as

$$a_x = \frac{du}{dt}$$

But u = f (x, y, z, t) and hence by chain rule, we can write,

$$a_x = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} + \frac{\partial u}{\partial z}\frac{dz}{dt} + \frac{\partial u}{\partial t}$$

Similarly

$$a_y = \frac{\partial v}{\partial x}\frac{dx}{dt} + \frac{\partial v}{\partial y}\frac{dy}{dt} + \frac{\partial v}{\partial z}\frac{dz}{dt} + \frac{\partial v}{\partial t}$$

and

$$a_z = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt} + \frac{\partial w}{\partial t}$$

But $u = \frac{dx}{dt}; v = \frac{dy}{dt}, w = \frac{dz}{dt}$

Hence

Local accln

Convective accln

$$a_x = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} + \frac{\partial u}{\partial t}$$

$$a_y = u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} + \frac{\partial v}{\partial t}$$

Total Acceleration

If A is the resultant acceleration vector, it is given by

$$\boldsymbol{A} = \boldsymbol{a}_x \boldsymbol{i} + \boldsymbol{a}_y \boldsymbol{j} + \boldsymbol{a}_z \boldsymbol{k}$$

$$|\boldsymbol{a}| = \sqrt{\boldsymbol{a}_x^2 + \boldsymbol{a}_y^2 + \boldsymbol{a}_z^2}$$

For steady flow, the local acceleration will be zero

Problems

2.10 Types of fluid flow

2.10.1 Steady and unsteady flows:

A flow is said to be steady if the properties (P) of the fluid and flow do not change with time (t) at any section or point in a fluid flow.

$$\frac{\partial}{\partial t}(P) = 0$$

A flow is said to be unsteady if the properties (P) of the fluid and flow change with time (t) at any section or point in a fluid flow.

$$\frac{\partial}{\partial t}(P) \neq 0$$

Example: Flow observed at a dam section during rainy season, wherein, there will be lot of inflow with which the flow properties like depth, velocity etc.. will change at the dam section over a period of time representing it as unsteady flow.

2.10.2. Uniform and non- uniform flows:

A flow is said to be uniform if the properties (P) of the fluid and flow do not change (with direction) over a length of flow considered along the flow at any instant.

$$\frac{\partial}{\partial x}(P) = 0$$

A flow is said to be non-uniform if the properties (P) of the fluid and flow change (with direction) over a length of flow considered along the flow at any instant.

$$\frac{\partial}{\partial x}(P) \neq 0$$

Example Flow observed at any instant, at the dam section during rainy season, wherein, the flow varies from the top of the overflow section to the foot of the dam and the flow properties like depth, velocity etc., will change at the dam section at any instant between two sections, representing it as non-uniform flow.

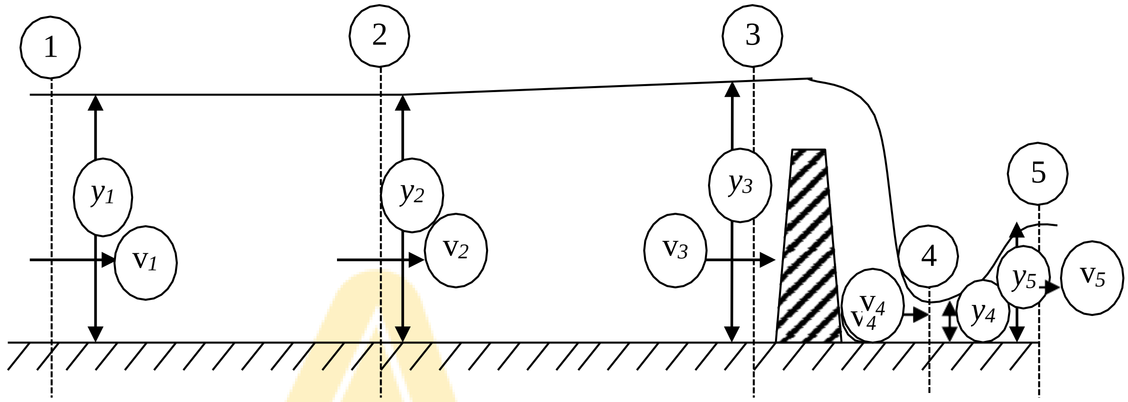


Fig. Different types of fluid flow

Consider a fluid flow as shown above in a channel. The flow is said to be steady at sections 1 and 2 as the flow does not change with respect to time at the respective sections (y_1=y_2 and v_1 = v_2)..

The flow between sections 1 and 2 is said to be uniform as the properties does not change between the sections at any instant (y_1=y_2 andv_1 = v_2).

The flow between sections 2 and 3 is said to be non-uniform flow as the properties vary over the length between the sections.

Non-uniform flow can be further classified as Gradually varied flow and Rapidly varied flow. As the name itself indicates, Gradually varied flow is a non-uniform flow wherein the flow/fluid properties vary gradually over a long length (Example between sections 2 and 3).

Rapidly varied flow is a non-uniform flow wherein the flow/fluid properties vary rapidly within a very short distance. (Example between sections 4 and 5).

Combination of steady and unsteady flows and uniform and non-uniform flows can be classified as steady-uniform flow (Sections 1 and 2), unsteady-uniform flow, steady-non-uniform flow (Sections 2 and 3) and unsteady-non-uniform flow (Sections 4 and 5).

2.10.3 One, Two and Three Dimensional flows

Flow is said to be one-dimensional if the properties vary only along one axis / direction and will be constant with respect to other two directions of a three-dimensional axis system.

Flow is said to be two-dimensional if the properties vary only along two axes / directions and will be constant with respect to other direction of a three-dimensional axis system.

Flow is said to be three-dimensional if the properties vary along all the axes / directions of a three-dimensional axis system.

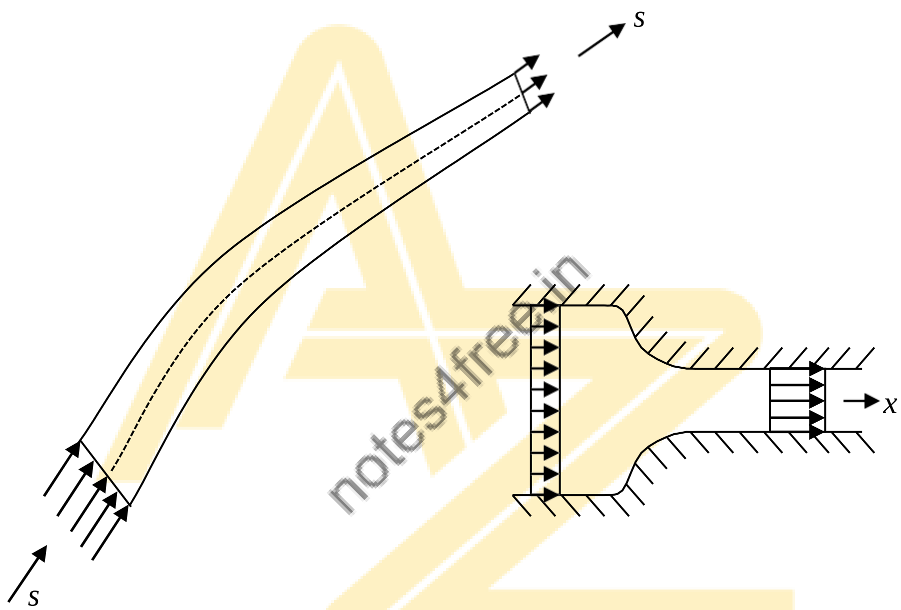


Fig. a) One- dimensional flow   Fig. b) Two-dimensional flow

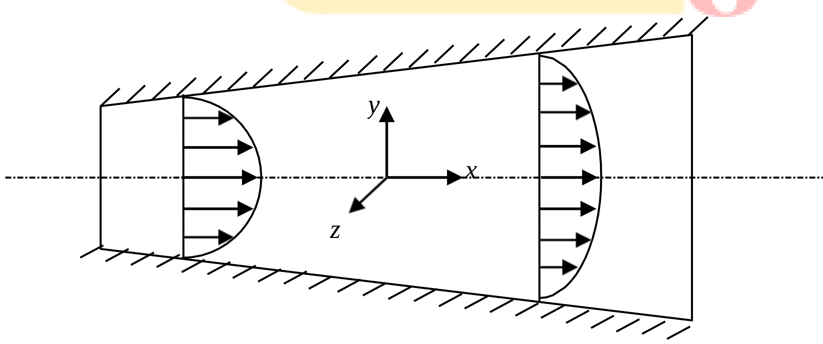
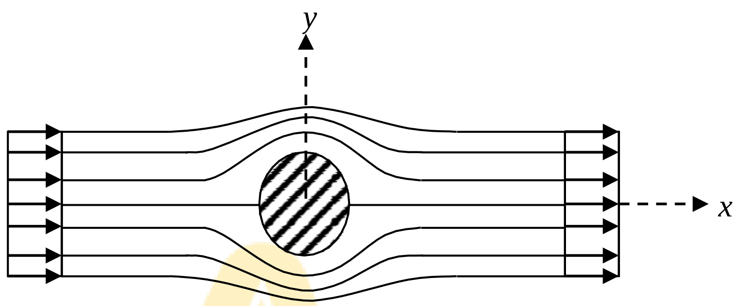


Fig. c) Three-dimensional flow

2.10.4. Description of flow pattern

Laminar and Turbulent flows:

When the flow occurs like sheets or laminates and the fluid elements flowing in a layer does not mix with other layers, then the flow is said to be laminar when the Reynolds number (Re) for the flow will be less than 2000.



$$R_e = \frac{\rho v D}{\mu}$$

Fig. 5Laminar flow

When the flow velocity increases, the sheet like flow gets mixes with other layer and the flow of fluid elements become random causing turbulence. There will be eddy currents generated and flow reversal takes place. This flow is said to be Turbulent when the Reynolds number for the flow will be greater than 4000. For flows with Reynolds number between 2000 to 4000 is said to be transition flow.

Fig. Compressible and Incompressible flows:

Flow is said to be Incompressible if the fluid density does not change (constant) along the flow direction and is Compressible if the fluid density varies along the flow direction

ρ= Constant (incompressible) and $\rho \neq$ Constant (compressible)

2.10.5 Path line, Streamline, Streak line and Stream tube:

Path Line: It is the path traced by a fluid particle over a period of time during its motion along the fluid flow.

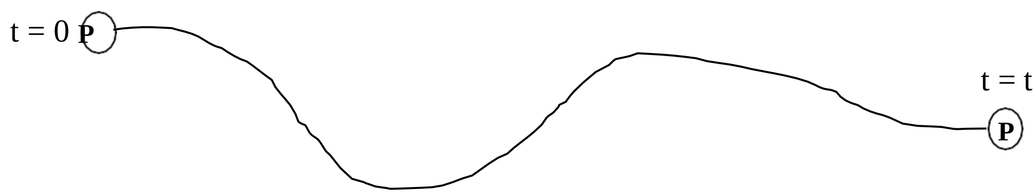


Fig. 7 Path line

Example Path traced by an ant coming out from its dwelling

Stream Lines

It is an imaginary line such that when a tangent is drawn at any point it gives the velocity of the fluid particle at that point and at that instant.

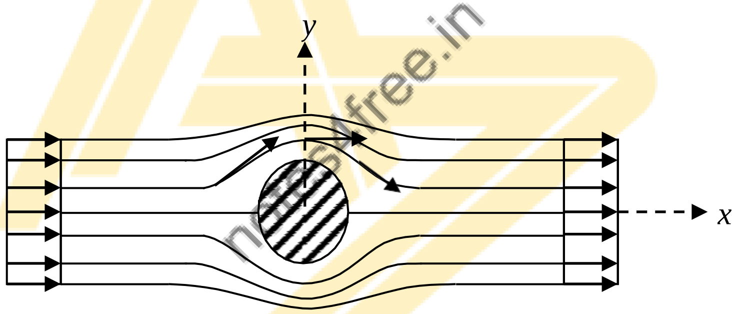


Fig. Stream lines

Example Path traced by the flow when an obstruction like a sphere or a stick is kept during its motion. The flow breaks up before the obstruction and joins after it crosses it.

Streak lines:

It is that imaginary line that connects all the fluid particles that has gone through a point/section over a period of time in a fluid motion.

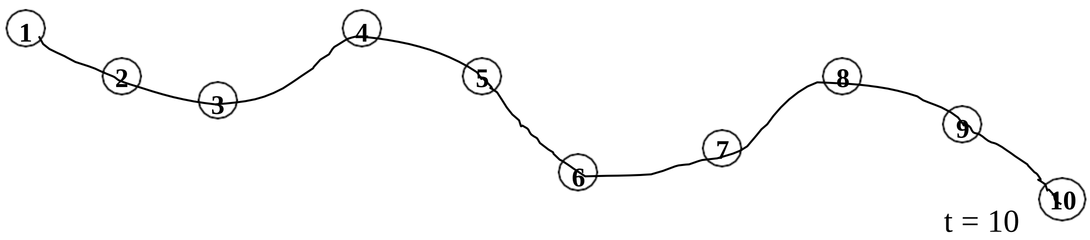


Fig. Streak lines

Stream tube:

It is an imaginary tube formed by stream line on its surface such that the flow only enters the tube from one side and leaves it on the other side only. No flow takes place across the stream tube. This concept will help in the analysis of fluid motion.

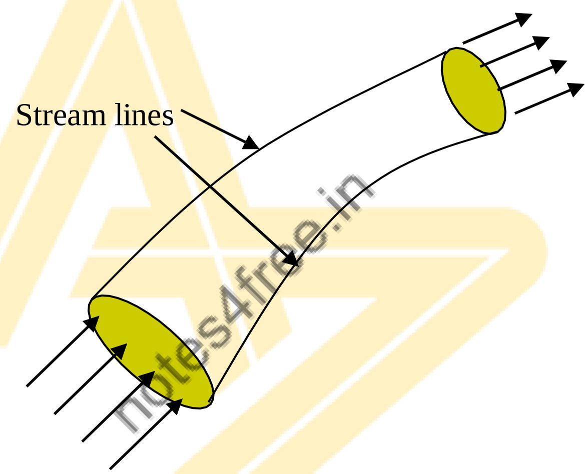


Fig. Stream tube

2.10.6. Rotational and Irrotational flows:

Flow is said to be Rotational if the fluid elements does not rotate about their own axis as they move along the flow and is Rotational if the fluid elements rotate along their axis as they move along the flow direction

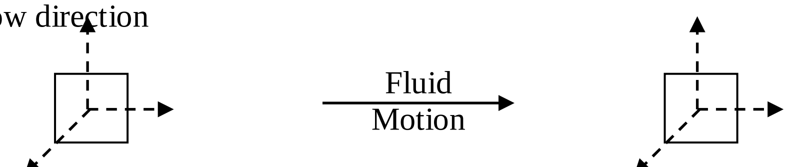
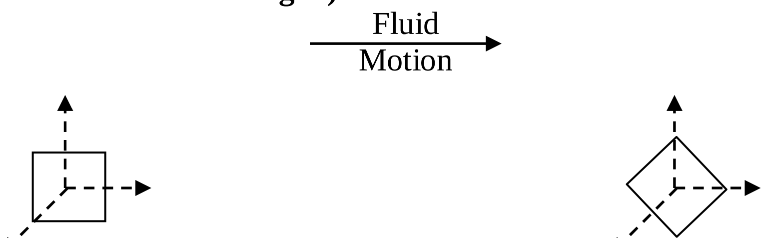


Fig. a) Irrotational flow



We know that for an irrotational two dimensional fluid flow, the rotational fluid elements about z axis must be zero.

$$w_z = \frac{1}{2}\left[\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right]$$

Substituting for *u* and v in terms of velocity potential-ϕ, we get

$$w_z = \frac{1}{2}\left[\frac{\partial}{\partial x}\left(-\frac{\partial \phi}{\partial y}\right) - \frac{\partial}{\partial y}\left(-\frac{\partial \phi}{\partial x}\right)\right] = \frac{1}{2}\left[\frac{\partial^2 \phi}{\partial x \partial y} - \frac{\partial^2 \phi}{\partial y \partial x}\right] = 0 \; Laplace \quad Eq.$$

Hence for the flow to be irrotational, the second partial derivative of Velocity potential -ϕ must be zero. This is true only when ϕ is a continuous function and exists.

Thus the properties of a velocity potential are:

1. If the velocity potential ϕ exists, then the flow should be irrotational
2. If the velocity potential ϕ satisfies the *Laplace Equation*, then it represents a possible case of a fluid flow.

Similarly for stream function ψ

$$w_z = \frac{1}{2}\left[\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right]$$

Substituting for *u* and v in terms of stream function-ψ, we get

$$w_z = \frac{1}{2}\left[\frac{\partial}{\partial x}\left(\frac{\partial \psi}{\partial x}\right) - \frac{\partial}{\partial y}\left(-\frac{\partial \psi}{\partial y}\right)\right] = \frac{1}{2}\left[\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2}\right] = 0 \; Laplace \quad Eq.$$

The above equation is known as ***Laplace equation*** in ψ

Thus the properties of a Stream function are:

1. If the Stream function ψ exists, then it represents a possible case of a fluid flow.
2. If the Stream function ψ satisfies the ***Laplace Equation***, then the flow should be irrotational.

2.10.7 Basic principles of fluid flow:

The derivation is based on the concept of Law of conservation of mass.

Continuity Equation

Statement: The flow of fluid in a continuous flow across a section is always a constant. Consider an enlarging section in a fluid flow of fluid density γ. Consider two sections 1 and 2 as shown in Fig. Let the sectional properties be as under

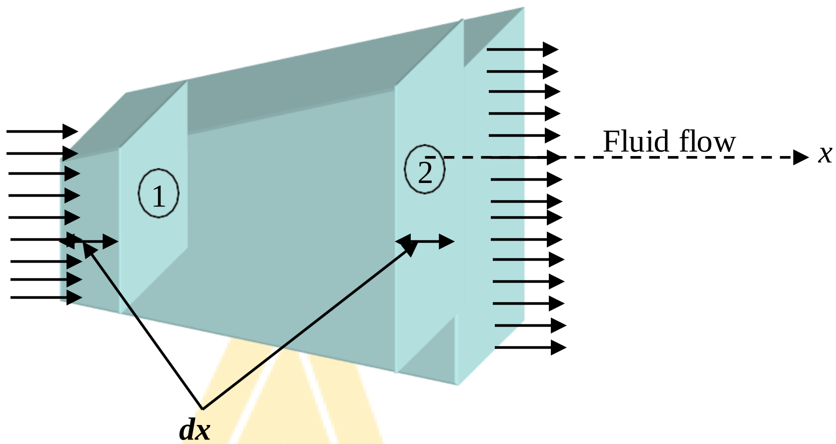


Fig. Fluid flow through a control volume

A_1 and A_2= Cross-sectional area, V_1 and V_2= Average flow velocity and

ρ_1 and ρ_2 = Fluid density at Section-1 and Section-2 respectively

dt is the time taken for the fluid to cover a distance dx

The mass of fluid flowing across section 1-1 is given by

m_1 = Density at section 1 x volume of fluid that has crossed section 1= $\rho_1 \times A_1 \times dx$

Mass rate of fluid flowing across section 1-1 is given by

$$\frac{m_1}{dt} = \frac{\textbf{(Density at section- 1} \times \textbf{volume of fluid that has crossed section- 1)}}{\textbf{dt}}$$

$$\rho_1 \times A_1 \times \frac{dx}{dt} = \rho_1 \times A_1 \times V_1 \cdots\cdots Eq.1$$

Similarly Mass rate of fluid flowing across section 2-2 is given by

$$\frac{m_2}{dt} = \frac{\textbf{(Density at section- 2} \times \textbf{volume of fluid that has crossed section- 2)}}{\textbf{dt}}$$

$$\rho_2 \times A_2 \times \frac{dx}{dt} = \rho_2 \times A_2 \times V_2 \cdots\cdots Eq.2$$

From law of conservation of mass, mass can neither be created nor destroyed. Hence, from Eqs. 1 and 2, we get

$$\rho_1 \times A_1 \times V_1 = \rho_2 \times A_2 \times V_2 \qquad \text{Eq.3}$$

If the density of the fluid is same on both side and flow is incompressible then $\rho_1 = \rho_2$ the equation 3 reduces to $A_1 \times V_1 = A_2 \times V_2$

The above equations discharge continuity equation in one dimensional form for a steady, incompressible fluid flow.

Rate of flow or Discharge (Q):

Rate of flow or discharge is said to be the quantity of fluid flowing per second across a section of a flow. Rate of flow can be expressed as mass rate of flow or volume rate of flow. Accordingly

Mass rate of flow = Mass of fluid flowing across a section / time

Rate of flow = Volume of fluid flowing across a section / time

2.10.7.1 Continuity Equation in three dimensional or differential form

Consider a parallelepiped ABCDEFGH in a fluid flow of density γ as shown in Fig. Let the dimensions of the parallelepiped be dx, dy and dz along x, y and z directions respectively. Let the velocity components along x, y and z be u, v and w respectively.

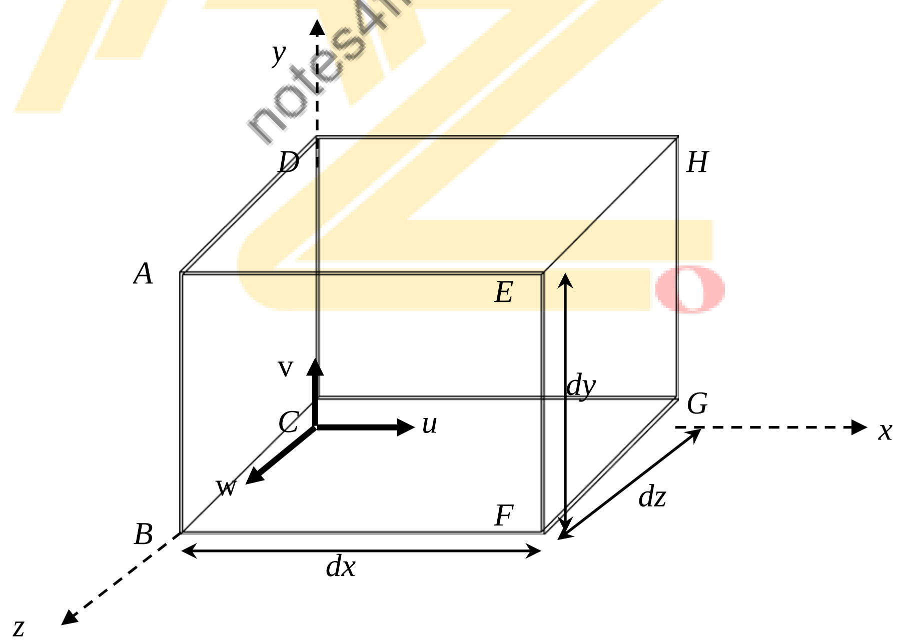


Fig. Parallelepiped in a fluid flow

Mass rate of fluid flow entering the section ABCD along x direction is given by ρ × Area × Vy

$M_{x1} = \rho\, u\, dy\, dz$ …(01)

Similarly mass rate of fluid flow leaving the section EFGH along × direction is given by,

$$M_{x2} = \left[\rho u + \frac{\partial}{\partial x}(\rho u)dx\right] dy\,dz \qquad \ldots(02)$$

Net gain in mass rate of the fluid along the x axis is given by the difference between the mass rate of flow entering and leaving the control volume. i.e. Eq. 1 – Eq. 2

$$dM_x = \rho u\,dy\,dz - \left[\rho u + \frac{\partial}{\partial x}(\rho u)dx\right] dy\,dz$$

$$dM_x = -\frac{\partial}{\partial x}(\rho u)dx\,dy\,dz \qquad \ldots(03)$$

Similarly net gain in mass rate of the fluid along the y and z axes are given by

$$dM_y = -\frac{\partial}{\partial y}(\rho\, \mathrm{v})dx\,dy\,dz \qquad \ldots(04)$$

$$dM_z = -\frac{\partial}{\partial z}(\rho\, \mathrm{w})dx\,dy\,dz \qquad \ldots(05)$$

Net gain in mass rate of the fluid from all the threeaxes are given by

$$dM = -\frac{\partial}{\partial x}(\rho u)dx\,dy\,dz - \frac{\partial}{\partial y}(\rho\, \mathrm{v})dx\,dy\,dz - \frac{\partial}{\partial z}(\rho\, \mathrm{w})dx\,dy\,dz$$

From law of conservation of Mass, the net gain in mass rate of flow should be zero and hence

$$\left[\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho\, \mathrm{v}) + \frac{\partial}{\partial z}(\rho\, \mathrm{w})\right] dx\,dy\,dz = 0$$

or

$$\left[\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho\, \mathrm{v}) + \frac{\partial}{\partial z}(\rho\, \mathrm{w})\right] = 0$$

This expression is known as the general Equation of Continuity in three dimensional form or differential form.

If the fluid is incompressible then the density is constant and hence

$$\left[\frac{\partial u}{\partial x} + \frac{\partial \mathrm{v}}{\partial y} + \frac{\partial \mathrm{w}}{\partial z}\right] = 0$$

The continuity equation in two-dimensional form for compressible and incompressible flows is respectively as below

$$\left[\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho\, \mathrm{v})\right] = 0$$

$$\left[\frac{\partial u}{\partial x} + \frac{\partial \mathrm{v}}{\partial y}\right] = 0$$

2.10.8 Velocity Potential Function (ϕ) and Stream Function (ψ):

2.10.8.1 Velocity Potential (ϕ):

Velocity Potential ϕ is a scalar function of space and time such that its negative derivative with respect to any direction gives the velocity component in that direction

Thus $\phi = \phi(x,y,z,t)$ and flow is steady then,

$u = -(\partial \phi / \partial x)$; $v = -(\partial \phi / \partial y)$; $w = -(\partial \phi / \partial z)$

Continuity equation for a three dimensional fluid flow is given by

$[(\partial u/\partial x)+(\partial v/\partial y) +(\partial w/\partial z)] = 0$

Substituting for u, v and w, we get

$[(\partial /\partial x)(-\partial \phi /\partial x)+(\partial /\partial y)(-\partial \phi /\partial y) +(\partial /\partial z) (-\partial \phi /\partial z)] = 0$

i.e. $[(\partial 2\phi /\partial x2)+(\partial 2\phi/\partial y2)+(\partial 2\phi /\partial z2)] = 0$

The above equation is known as Laplace equation in ϕ

For a 2 D flow the above equation reduces to

$$[(\partial^2\phi /\partial x^2)+(\partial^2\phi/\partial y^2)] = 0$$

We know that for an irrotational two dimensional fluid flow, the rotational fluid elements about z axis must be zero. i.e. $\omega z = ½ [(\partial v/\partial x) - (\partial u/\partial y)]$

Substituting for u and v, we get

$$w_z= ½ [(\partial /\partial x)(-\partial \phi /\partial y) - (\partial /\partial y)(-\partial \phi /\partial x)]$$

For the flow to be irrotational, the above component must be zero

$\omega z = ½ [(-\partial^2\phi /\partial x \partial y) - (-\partial^2\phi / \partial y \partial x)] = 0$

i.e. $(-\partial^2\phi /\partial x \partial y) = (-\partial^2\phi / \partial y \partial x)$

This is true only when ϕ is a continuous function and exists.

Thus the properties of a velocity potential are:

1. If the velocity potential ϕ exists, then the flow should be irrotational.
2. If the velocity potential ϕ satisfies the Laplace Equation, then it represents a possible case of a fluid flow.

Equi-potential lines:

It is an imaginary line along which the velocity potential ϕ is a constant

i.e. ϕ = Constant

$\therefore \quad d\phi = 0$

But $\phi = f(x,y)$ for a two dimensional steady flow

$\therefore \quad d\phi = (\partial \phi / \partial x)dx + (\partial \phi / \partial y)dy$

Substituting the values of u and v, we get

$d\phi = -u\,dx - v\,dy \Rightarrow 0$

or $u\,dx = -v\,dy$

or $(dy/dx) = -u/v$ … (01)

Where dy/dx is the slope of the equi-potential line.

2.10.8.2 Stream Function (ψ)

Stream Function ψ is a scalar function of space and time such that its partial derivative with respect to any direction gives the velocity component at right angles to that direction.

Thus $\psi = \psi(x,y,z,t)$ and flow is steady then,

$u = -(\partial \psi / \partial y)$; $v = (\partial \psi / \partial x)$

Continuity equation for a two dimensional fluid flow is given by

$[(\partial u/\partial x)+(\partial v/\partial y)] = 0$

Substituting for u and v, we get

$[(\partial /\partial x)(-\partial \psi /\partial y)+(\partial /\partial y)(\partial \psi /\partial x)] = 0$

i.e. $[(-\partial^2\psi /\partial x \partial y) + (\partial^2\psi / \partial y \partial x)] = 0$

or $(\partial^2\psi /\partial x \partial y) = (\partial^2\psi / \partial y \partial x)$

This is true only when ψ is a continuous function.

We know that for an irrotational two dimensional fluid flow, the rotational fluid elements about z axis must be zero.*i.e.* $\omega_z = \frac{1}{2}[(\partial v/\partial x) - (\partial u/\partial y)]$

Substituting for *u* and v, we get

$\omega_z = \frac{1}{2}[(\partial/\partial x)(\partial \psi/\partial x) - (\partial/\partial y)(-\partial \psi/\partial y)]$

For the flow to be irrotational, the above component must be zero

i.e. $[(\partial^2\phi/\partial x^2)+(\partial^2\phi/\partial y^2)] = 0$

The above equation is known as ***Laplace equation*** in ψ

Thus the properties of a Stream function are:

1. If the Stream function ψ exists, then it represents a possible case of a fluid flow.
2. If the Stream function ψ satisfies the Laplace Equation, then the flow should be irrotational.

Line of constant stream function or stream line

It is an imaginary line along which the stream function ψ is a constant

i.e. ψ = Constant

$d\psi = 0$

But $\psi = f(x,y)$ for a two dimensional steady flow

$d\psi = (\partial \psi/\partial x)dx + (\partial \psi/\partial y)dy$

Substituting the values of u and v, we get

$d\psi = v\,dx - u\,dy \Rightarrow 0$

or $v\,dx = u\,dy$

or $(dy/dx) = v/u$ … (02)

Where dy/dx is the slope of the Stream line.

From Eqs. 1 and 2, we get that the product of the slopes of equi-potential line and stream line is given by -1. Thus, the equi-potential lines and stream lines are orthogonal to each other at all the points of intersection.

2.10.8.3 Relationship between Stream function (ψ) and Velocity potential (ϕ)

We know that the velocity components are given by

$u = -(\partial\phi/\partial x)$ $v = -(\partial\phi/\partial y)$

and $u = -(\partial\psi/\partial y)$ $v = (\partial\psi/\partial x)$

Relation between (ϕ and ψ):

$$u = -\frac{\partial\phi}{\partial x} = -\frac{\partial\psi}{\partial y}$$

$$v = -\frac{\partial\phi}{\partial y} = \frac{\partial\psi}{\partial x}$$

Thus $u = -(\partial\phi/\partial x) = -(\partial\psi/\partial y)$ and $v = -(\partial\phi/\partial y) = (\partial\psi/\partial x)$

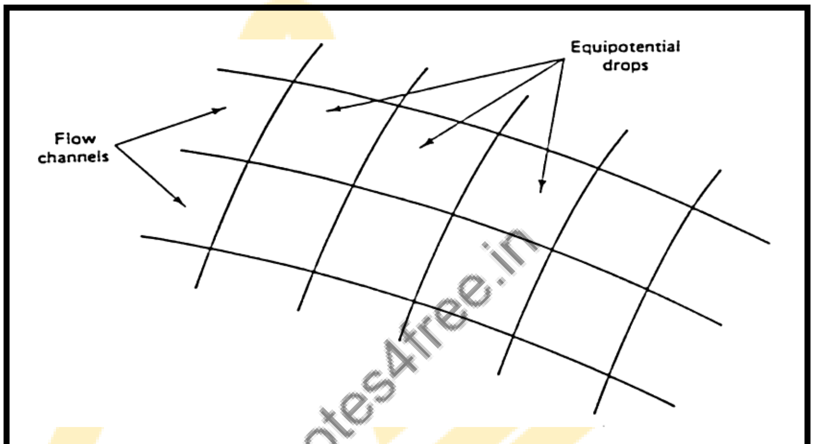
Hence $(\partial\phi/\partial x) = (\partial\psi/\partial y)$ and $(\partial\phi/\partial y) = -(\partial\psi/\partial x)$

ϕ-lines and ψ-lines intersect orthogonally

2.11 Flow net & its Applications:

A grid obtained by drawing a series of equi-potential lines and stream lines is called a Flow net. The flow net is an important tool in analysing two dimensional flow irrotational flow problems. A grid obtained by drawing a series of streamlines (ψ) and equipotential (ϕ) lines is known as flow net. The construction of flow net (ϕ-ψ lines) is restricted by certain conditions

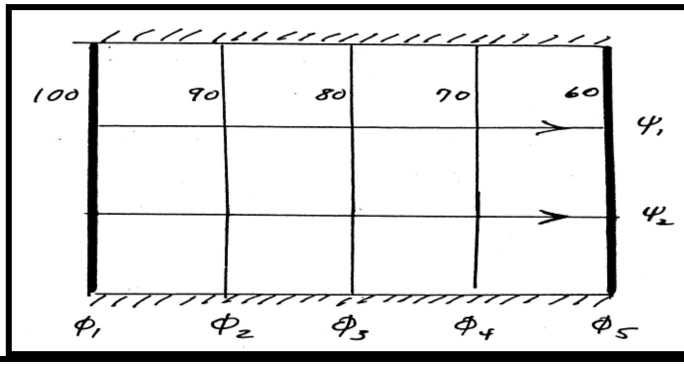
- ✓ The flow should be two dimensional
- ✓ The flow should be steady
- ✓ The flow should be Irrotational
- ✓ The flow is not governed by gravity force

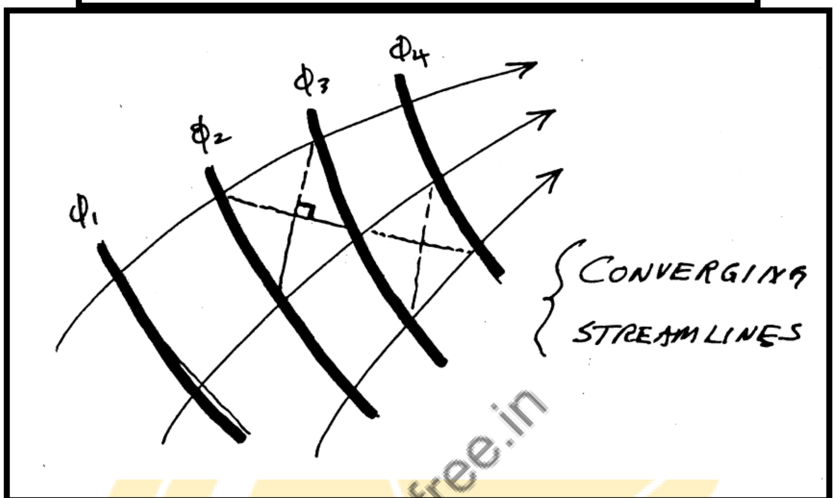


Uses of Flow net

To determine

- The streamlines and equipotential lines
- Quantity of seepage, upward lift pressure below the hydraulic structures (dam, gate, locks etc.)
- Velocity and pressure distribution, for given boundaries of flow
- To design streamlined structure
- Flow pattern near well





Methods of Drawing flow net

- Analytical Method
- Graphical Method
- Electrical Analogy Method
- Hydraulic Models
- Relaxation Method
- Hele Shaw or Viscous Analogy Method

The practical use of streamlines and velocity potential lines are:

(i) Quantity of seepage

(ii) Upward lift pressure below the hydraulic structures (dam, gate, locks etc.)

(iii) Velocity and pressure distribution, for given boundaries of flow

(iv) To design streamlined structure flow pattern near well

Solved Problems:

Q.1. The velocity field in a fluid is given by,

$$\boldsymbol{V_s} = (3\boldsymbol{x} + 2\,\boldsymbol{y})\boldsymbol{i} + (2\boldsymbol{z} + 3\boldsymbol{x}^2)\boldsymbol{j} + (2\boldsymbol{t} - 3\boldsymbol{z})\boldsymbol{k}$$

i. What are the velocity components *u*, v, and w?

ii. Determine the speed at the point (1,1,1).

iii. Determine the speed at time t=2 s at point (0,0,2)

$$u = (3x + 2\,y),\ \mathrm{v} = (2z + 3x^2),\ \mathrm{w} = (2t - 3z)k$$

Solution: The velocity components at any point *(x, y, z)* are

Substitute *x=1, y=1, z=1 in the above expression*

$u = (3*1+2*1) = 5$, v $= (2*1+3*1) = 5$, w $= (2t-3)$

$V^2 = u^2 + \mathrm{v}^2 + \mathrm{w}^2$

$= 5^2 + 5^2 + (2t-3)^2$

$$V_{(1,1,1)} = \sqrt{(4t^2 - 12t + 59)}$$

$= 4\,t^2 - 12\,t + 59$

Substitute *t = 2 s, x=0, y=0, z=2 in the above expression for u, v and w*

$u = 0$, v $= (4 + 0) = 4$, w $= (4 - 6) = -2$

$V^2_{(0,0,2,2)} = (0 + 15 + 4) = 20$

$V = 4.472$ *units*

Q. 2. The velocity distribution in a three-dimensional flow is given by:

$u = -x$, v $= 2y$ and w $= (3-z)$. Find the equation of the stream line that passes through point *(1,1,1).*

$$\frac{dx}{u} = \frac{dy}{\mathrm{v}} = \frac{dz}{\mathrm{w}} \text{ or } \frac{dx}{-x} = \frac{dy}{2y} = \frac{dz}{(3-z)}$$

$$\frac{dx}{-x} = \frac{dy}{2y}$$

Solution: The stream line equation is given by

Integrating we get

Where *A* is an integral constant. Substituting *x=1 & y=1, A = 0*

Considering the *x* and *z* components,

$$\log_e x = {}_2 \log_e y + A,$$

$$\frac{dx}{-x} = \frac{dy}{(3-z)}$$

$$-\log_e x = -\log_e (3-z) + B,$$

Integrating we get

Where *B* is an integral constant. Substituting *x=1 & z=1, B* = log *e 2*

$$\therefore -\log_e x = -\log_e (3-z) + \log_e 2 = -\log_e \left(\frac{3-z}{2}\right)$$

$$\text{or } x = \left(\frac{3-z}{2}\right)$$

From Eqs. 1 and 2, the final equation of the stream line that passes through the point (1,1,1) is

$$x = \frac{1}{\sqrt{y}} = \left(\frac{3-z}{2}\right)$$

Q3. A fluid particle moves in the following flow field starting from the points *(2,1,0)* at *t=0.* Determine the location of the fluid particle at *t = 3* s

$$u = \frac{t^2}{2x},\ v = \frac{ty^2}{18},\ w = \frac{z}{2t}$$

Solution

Integrating we get

$$u = \frac{dx}{dt} = \frac{t^2}{2x} \text{ or } 2xdx = t^2 dt$$

$$x^2 = \frac{t^3}{3} + A$$

$$x^2 = \frac{t^3}{3} + 4$$

$$x^2 = \frac{3^3}{3} + 4 = \sqrt{13}$$

Where *A* is an integral constant. Substituting *x=2, t=0, A = 4*

Integrating we get

$$v = \frac{dy}{dt} = \frac{ty^2}{18} \text{ or } \frac{dy}{y^2} = \frac{tdt}{18} \qquad -\frac{1}{y} = \frac{t^2}{36} + B$$

Where B is an integral constant.

$$\frac{1}{y} = 1 - \frac{t^2}{36} \qquad \frac{1}{y} = 1 - \frac{3^2}{36} = \frac{3}{4} \text{ or } y = \frac{4}{3}$$

$$w = \frac{dz}{dt} = \frac{z}{2t} \text{ or } \frac{2dz}{z} = \frac{dt}{t}$$

Substituting $y=1, t=0, B = -1$

At $t = 3$ s,

Integrating we get

$$2 \log_e z = \log_e t + C$$

Where C is an integral constant.

Substituting $z=0, t=0, C = 0$ $\quad 2 \log_e z = \log_e t \text{ or } z^2 = t$

At $t = 3$ s, $\quad z^2 = 3 \text{ or } z = \sqrt{3}$

From Eqs. 1, 2 and 3, at the end of 3 seconds the particle is at a point

$$\left(\sqrt{13}, \frac{4}{3}, \sqrt{3}\right)$$

Q.4. The following cases represent the two velocity components, determine the third component of velocity such that they satisfy the continuity equation:

(i) $u = x^2 + y^2 + z^2$; $v = xy^2 - yz^2 + xy$; **(ii)** $v = 2y^2$; $w = 2xyz$.

Solution:

The continuity equation for incompressible flow is given by

$[(\partial u/\partial x)+(\partial v/\partial y) +(\partial w/\partial z)] = 0$ ...(01)

$u = x^2 + y^2 + z^2$; $\quad (\partial u/\partial x) = 2x$

$v = xy^2 - yz^2 + xy$; $\quad (\partial v/\partial y) = 2xy - z2 + x$

Substituting in Eq. 1, we get

$2x + 2xy - z^2 + z + (\partial w/\partial z) = 0$

Rearranging and integrating the above expression, we get

$\mathbf{w = (-3xz - 2xyz + z^3/3) + f(x,y)}$

Similarly, solution of the second problem

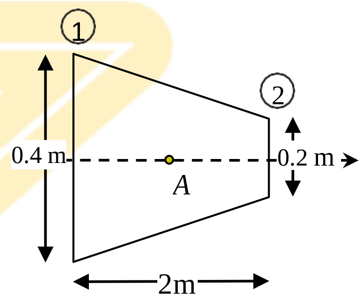
$\mathbf{u = -4xy - x^2y^2 + f(y,z).}$

Q.5. Find the convective acceleration at the middle of a pipe which converges uniformly from 0.4 m to 0.2 m diameter over a length of 2 m. The rate of flow is 20 lps. If the rate of flow changes uniformly from 20 lps to 40 lps in 30 seconds, find the total acceleration at the middle of the pipe at 15th second.

Solution: $D_1 = 0.4$ m, $D_2 = 0.2$ m, L = 2 m, Q = 20 lps = 0.02 m^3/s.

$Q_1 = 0.02\ m^3/s$ and $Q_2 = 0.04\ m^3/s$

Case (i): Flow is one dimensional and hence the velocity components v = w = *0*



∴Convective acceleration $= u(\partial u/\partial x)$

$A_1 = (\pi/4)(D_1^2) = 0.1257\ m^2$

$A_2 = (\pi/4)(D_2^2) = 0.0314\ m^2$

$u_1 = Q/A_1 = 0.02/0.1257 = 0.159\ m/s$

and $u_2 = Q/A_2 = 0.02/0.0314 = 0.637\ m/s$

As the diameter changes uniformly, the velocity will also Change uniformly. The velocity *u* at any distance *x* from inlet is given by

$u = u_1 + (u_2 - u_1)/(x/L) = 0.159 + 0.2388\,x$

$(\partial u/\partial x) = 0.2388$

∴Convective acceleration $= u(\partial u/\partial x) = (0.159 + 0.2388\,x)\,0.2388$

At A, $x = 1\ m$ and hence

(Convective accln) $_{x=1} = 94.99\ mm/s^2$

Case (ii): Total acceleration = (convective + local) acceleration at ***t* =*15* seconds**

Rate of flow $Q_{t=15} = Q_1 + (Q_2 - Q_1)(15/30) = 0.03\ m3/s.$

$u_1 = Q/A_1 = 0.03/0.1257 = 0.2386\ m/s$

and $u_2 = Q/A_2 = 0.03/0.0314 = 0.9554\ m/s$

The velocity u at any distance x from inlet is given by

$$u = u_1 + (u_2 - u_1)/(x/L) = 0.2386 + 0.3584\ x$$

$$(\partial u / \partial x) = 0.3584$$

$\therefore$ Convective acceleration $= u(\partial u / \partial x) = (0.2386 + 0.3584\ x)\ 0.3584$

At **A,** $x = 1\ m$ and hence

(Convective accln) $_{x = 1} = 0.2139\ m/s^2$

Local acceleration

Diameter at **A** is given by $D = D_1 + (D_1 - D_2)/(x/L) = 0.3\ m$

and $A = (\pi/4)(D^2) = 0.0707\ m^2$

When $Q_1 = 0.02\ m^3/s,\ u_1 = 0.02/0.0707 = 0.2829\ m/s$

When $Q_2 = 0.04\ m^3/s,\ u_2 = 0.02/0.0707 = 0.5659\ m/s$

Rate of change of velocity = Change in velocity/time

$= (0.5629 - 0.2829)/30 = 9.43 \times 10\ -3 m/s^2$

$\therefore$ **Total acceleration** $= 0.2139 + 9.43 \times 10^{-3} = 0.2233\ m/s^2$

Q.6. In a flow the velocity vector is given by V = 3xi + 4yj -7zk. Determine the equation of the stream line passing through a point M (1, 4, 5).

Ans: Given the Velocity vector V = 3xi+4yj -7zk

$\Rightarrow$ u = 3x ; v = 4y; w = -7z

To determine the equation of the stream line passing through a point M (1, 4, 5)

The 3-D equation of streamline is given by,

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w}$$

$$\frac{dx}{3x} = \frac{dy}{4y} = \frac{dz}{-7z} \cdots Eq.1$$

The streamline equation at point **M (1, 4, 5),** x =1, y = 4, z = 5

Substituting the values of x, y, and z in Eq.1

$$\frac{dx}{3} = \frac{dy}{16} = \frac{dz}{-35}$$

The equation of a streamline ds = 3i +16k – 35k

Q.7. A 250 mm diameter pipe carries oil of specific gravity 0.9 at a velocity of 3 m/s. At another section the diameter is 200 mm. Find the velocity at this section and the mass rate of flow of oil.

Solution:

$D_1 = 0.25\ m;\ D_2 = 0.2\ m;\ S_o = 0.9;\ V_1 = 3\ m/s;\ \rho = 1000\ kg/m^3$ ***(assumed);***

V2 = ?; Mass rate of flow = ***?***

From discharge continuity equation for steady incompressible flow, we have

$$Q = A_1V_1 = A_2V_2 \quad (01)$$

$$A_1 = (\pi/4)D_1^2 = (\pi/4)0.25^2 = 0.0499\ m^2$$

$$A_2 = (\pi/4)D_2^2 = (\pi/4)0.20^2 = 0.0314\ m^2$$

Substituting in Eq. 1, we get

$$Q = 0.0499 \text{ x } 3 = 0.1473\ m^3/s$$

Mass rate of flow = $\rho Q = 0.1479 \text{ x } 1000 = 147.9\ kg/m^3$ ***(Ans)***

$V_2 = (A_1 / A_2) \text{ x } V_1 = (D_1 / D_2)^2 \text{ x } V_1 = (0.25/0.2)^2 \text{ x } 3 = 4.6875\ m/s$ ***(Ans)***

Q.8. In a two dimensional incompressible flow the fluid velocity components are given by

$u = x - 4y$ and $v = -y - 4x$

Where u and v are x and y-components of velocity of flow. Show that the flow satisfies the continuity equation and obtain the expression for stream function. If the flow is potential, obtain also the expression for the velocity potential.

Solution:

$u = x - 4y$ and $v = -y - 4x$

$(\partial u / \partial x) = 1$ and $(\partial v / \partial y) = -1$

$(\partial u / \partial x) + (\partial v / \partial y) = 1 - 1 = 0.$

Hence it satisfies continuity equation and the flow is continuous and velocity potential exists.

Let ϕ be the velocity potential.

Then $(\partial \phi / \partial x) = -u = -(x - 4y) = -x + 4y$ (1)

and $(\partial \phi / \partial y) = -v = -(-y - 4x) = y + 4x$ (2)

Integrating Eq. 1, we get

$$\phi = (-x^2/2) + 4xy + C \quad (3)$$

Where C is an integral constant, which is independent of x and can be a function of y.

Differentiating Eq. 3 w.r.t. y, we get

$$(\partial \phi / \partial y) = 0 + 4x + (\partial C / \partial y) \Rightarrow y + 4x$$

Hence, we get $(\partial C / \partial y) = y$

Integrating the above expression, we get $C = y^2/2$

Substituting the value of C in Eq. 3, we get the general expression as

$$\phi = (-x^2/2) + 4xy + y^2/2$$

Stream Function

Let ψ be the velocity potential.

Then $(\partial \psi / \partial x) = \mathbf{v} = (-y - 4x) = -y - 4x$ (4)

and $(\partial \psi / \partial y) = u = -(x - 4y) = -x + 4y$ (5)

Integrating Eq. 4, we get

$$\psi = - y\, x - 4\, (x2/2) + K \qquad (6)$$

Where K is an integral constant, which is independent of x and can be a function of y.

Differentiating Eq. 6 w.r.t. y, we get

$$(\partial \psi / \partial y) = - x - 0 + (\partial K / \partial y) \Rightarrow -x + 4\, y$$

Hence, we get $(\partial K / \partial y) = 4\, y$

Integrating the above expression, we get $C = 4\, y^2/2 = 2\, y^2$

Substituting the value of K in Eq. 6, we get the general expression as

$$\psi = - y\, x - 2\, x^2 + 2\, y^2$$

Q.9. The components of velocity for a two dimensional flow are given by

$$u = x\, y; \qquad \mathbf{v} = x^2 - \frac{y^2}{2}$$

Check whether (i) they represent the possible case of flow and (ii) the flow is irrotational.

Solution:

$$u = x\, y; \qquad \text{and} \qquad \mathbf{v} = x^2 - \frac{y^2}{2}$$

$(\partial u / \partial x) = y$ $\quad\quad\quad\quad$ $(\partial v / \partial y) = -y$

$(\partial u / \partial y) = x$ $\quad\quad\quad\quad$ $(\partial v / \partial x) = 2x$

For a possible case of flow the velocity components should satisfy the equation of continuity.

i.e. $$\left(\frac{\partial u}{\partial x}\right) + \left(\frac{\partial v}{\partial y}\right) = 0$$

Substituting, we get $y - y = 0.$

Hence it is a possible case of a fluid flow.

For flow to be irrotational in a two dimensional fluid flow, the rotational component in z direction (ωz) must be zero, where

$$w_z = \frac{1}{2}\left[\left(\frac{\partial v}{\partial x}\right) - \left(\frac{\partial u}{\partial y}\right)\right] = \frac{1}{2}[2x - x] \neq 0$$

Hence, the flow is not irrotational.

Q.10. Find the components of velocity along x and y for the velocity potential $\phi = a \text{ Cos } xy.$ Also calculate the corresponding stream function.

Solution:

$\phi = a \text{ Cos } xy.$

$$\left(\frac{\partial \phi}{\partial x}\right) = -u = -ay Sin(xy) \quad\quad (1)$$

and $$\left(\frac{\partial \phi}{\partial y}\right) = -v = -ax Sin(xy) \quad\quad (2)$$

Hence $u = ay \text{ Sin } xy$ and $v = ax \text{ Sin } xy.$

Q.11. The stream function and velocity potential for a flow are given by,

$\psi = 2xy$ and $\phi = x^2 - y^2$

Show that the conditions for continuity and irrotational flow are satisfied

Solution:

From the properties of Stream function, the existence of stream function shows the possible case of flow and if it satisfies Laplace equation, then the flow is irrotational.

(i) $\psi = 2xy$

$$(\partial \psi / \partial x) = 2y \quad \text{and} \quad (\partial \psi / \partial y) = 2x$$

$$(\partial^2 \psi / \partial x^2) = 0 \quad \text{and} \quad (\partial^2 \psi / \partial y^2) = 0$$

$$(\partial^2 \psi / \partial x \, \partial y) = 2 \quad \text{and} \quad (\partial^2 \psi / \partial y \, \partial x) = 2$$

$$(\partial^2 \psi / \partial x \, \partial y) = (\partial^2 \psi / \partial y \, \partial x)$$

Hence the flow is Continuous.

$$(\partial^2 \psi / \partial x^2) + (\partial^2 \psi / \partial y^2) = 0$$

As it satisfies the Laplace equation, the flow is irrotational.

From the properties of Velocity potential, the existence of Velocity potential shows the flow is irrotational and if it satisfies Laplace equation, then it is a possible case of flow

(ii) $\phi = x^2 - y^2$

$$(\partial \phi / \partial x) = 2x \quad \text{and} \quad (\partial \phi / \partial y) = -2y$$

$$(\partial^2 \phi / \partial x^2) = 2 \quad \text{and} \quad (\partial^2 \phi / \partial y^2) = -2$$

$$(\partial^2 \phi / \partial x \, \partial y) = 0 \quad \text{and} \quad (\partial^2 \phi / \partial y \, \partial x) = 0$$

$$\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$$

Hence the flow is irrotational

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$$

As it satisfies the Laplace equation, the flow is Continuous.

Q.12. In a 2-D flow, the velocity components are $u = 4y$ and $v = -4x$

i. Is the flow possible?
ii. if so, determine the stream function
iii. What is the pattern of stream lines?

Solution:

For a possible case of fluid flow, it has to satisfy continuity equation.

i.e. $$\left(\frac{\partial u}{\partial x}\right) + \left(\frac{\partial v}{\partial y}\right) = 0 \qquad (1)$$

$$u = 4y \quad \text{and} \quad v = -4x$$

$$(\partial u / \partial x) = 0 \qquad (\partial v / \partial y) = 0$$

Substituting in Eq. 1, we get ***0.***

Hence the flow is possible.

Stream function

We know that $(\partial \psi / \partial x) = v = -4x$ (2)

and $(\partial \psi / \partial y) = -u = -4y$ (3)

$\psi = -2x^2 + C(y)$ (4)

Where ***C*** is an integral constant and a function of ***y.***

Differentiating Eq. 4, w.r.t. ***y***, we get

$$(\partial \psi / \partial y) = 0 + \partial C(y) / \partial y = -u = -4y$$

Integrating the above expression w.r.t. ***y*** we get

$$C(y) = -2y^2.$$

Substituting the above value in Eq. 4, we get the general expression as

$$\psi = -2x^2 - 2y^2 = -2(x^2 + y^2)$$

The above equation is an expression of concentric circles and hence the stream function is concentric circles.

Q.13. A stream function in a two dimensional flow is $\psi = 2xy$. **D**etermine the corresponding velocity potential.

Solution:

Given $\psi = 2xy.$

$u = -(\partial \phi / \partial x) = -(\partial \psi / \partial y) = -2x$ (01)

$v = -(\partial \phi / \partial y) = (\partial \psi / \partial x) = 2y$ (02)

Integrating Eq. 1, w.r.t. x, we get

$\phi = 2x^2/2 + C = x^2 + C(y)$ (03)

Where ***C(y)*** is an integral constant independent of ***x***

Differentiating Eq. 3 w.r.t. y, we get

$$(\partial \phi / \partial y) = 0 + (\partial C(y) / \partial y) = -2y$$

Integrating the above expression w.r.t. y, we get

$$C(y) = -y^2$$

Substituting for ***C(y)*** in Eq. 3, we get the general expression for ϕ as

$\boldsymbol{\phi = x^2 + C = x^2 - y^2}$ ***(Ans)***

Q.14. The velocity potential for a flow is given by the function $\boldsymbol{\phi = x^2 - y^2}$**.** Verify that the flow is incompressible.

Solution:

From the properties of velocity potential, we have that if $\boldsymbol{\phi}$ satisfies Laplace equation, then the flow is steady incompressible continuous fluid flow.

Given $\boldsymbol{\phi = x^2 - y^2}$

$$(\partial \phi / \partial x) = 2x \qquad (\partial \phi / \partial y) = -2y$$

$$(\partial^2 \phi / \partial x^2) = 2 \qquad (\partial^2 \phi / \partial^2 y) = -2$$

From Laplace Equation, we have $\boldsymbol{(\partial^2 \phi / \partial x^2) + (\partial^2 \phi / \partial^2 y) = 2 - 2 = 0}$

Q.15. If for a two dimensional potential flow, the velocity potential is given by ϕ = x (2y-1). Determine the velocity at the point P (4, 5). Determine also the value of stream function ψ at the point 'P'.

Ans:

(i) The velocity at the point P (4, 5), x =4, y = 5

$$\phi = x\,(2y-1).$$

$$\frac{\partial \phi}{\partial x} = -u = (2y - 1), \qquad u = (1 - 2y)$$

$$\frac{\partial \phi}{\partial y} = -v = x \times 2, \qquad v = -2x$$

u at 'P'(4,5) = -9 Units/s

v(4,5) at 'P' = -8 Units/s

Velocity at P = -9i-8j, Velocity $\sqrt{(-9)^2 + (-8)^2} = 12.04 \quad \textit{Units}$

(ii) Stream function $\psi_{P\ (4,5)}$

Given $\phi = x(2y-1)$

$$\frac{\partial \phi}{\partial x} = -u = (2y-1) = \frac{\partial \psi}{\partial y}$$

$$\frac{\partial \phi}{\partial y} = -v = x \times 2 = -\frac{\partial \psi}{\partial x}$$

$$\frac{\partial \psi}{\partial y} = -u = (2y-1) \ldots Eq.1$$

$$\frac{\partial \psi}{\partial x} = u = -2x \ldots Eq.2$$

Integrating Eq.1 with respect 'y' we get

$$\int d\psi = \psi = \frac{2 \times y^2}{2} - y + C(f(x)) \cdots Eq.3$$

Differentiating Eq.3 with respect to 'x'

$$\frac{\partial \psi}{\partial x} = \frac{\partial C}{\partial x} \qquad from \quad Eq.2 \quad \frac{\partial \psi}{\partial x} = -2x$$

$$\frac{\partial C}{\partial x} = -2x \qquad Integrating \rightarrow C = -x^2$$

Substituting value of C in Eq.3

$$\psi = \left(\boldsymbol{y}^2 - \boldsymbol{y} - \boldsymbol{x}^2\right)$$

Q.16. A stream function is given by $\psi = 2x^2 - 2y^2$. Determine the velocity and velocity potential function at (1, 2)

Ans: Given: $\psi = 2x^2 - 2y^2$

$$\frac{\partial \boldsymbol{\psi}}{\partial \boldsymbol{x}} = 4\boldsymbol{x} = -\boldsymbol{v};\ \boldsymbol{v} = -4\boldsymbol{x} \Rightarrow \textbf{\textit{Velocity at}}\ (1,2), \quad \boldsymbol{v} = -4 \quad \textbf{\textit{Units}}$$

$$\frac{\partial \boldsymbol{\psi}}{\partial \boldsymbol{y}} = -4\boldsymbol{y} = \boldsymbol{u};\ \boldsymbol{u} = -4\boldsymbol{y} \Rightarrow \textbf{\textit{Velocity at}}\ (1,2), \quad \boldsymbol{u} = -8 \quad \textbf{\textit{Units}}$$

Resultant velocity $V_{(1,2)}$ = $\sqrt{(-4)^2 + (-8)^2} = 8.94\,\textbf{\textit{Units}}$

$$\frac{\partial \boldsymbol{\phi}}{\partial \boldsymbol{x}} = -\boldsymbol{u} \quad \Rightarrow \frac{\partial \boldsymbol{\phi}}{\partial \boldsymbol{x}} = -(-4\boldsymbol{y}) = 4\boldsymbol{y} \Rightarrow \boldsymbol{\phi} = 4 \times \boldsymbol{x} \times \boldsymbol{y} + \boldsymbol{C}(\boldsymbol{f}(\boldsymbol{y})\boldsymbol{only}) \cdots \boldsymbol{eq}1$$

$$\frac{\partial \boldsymbol{\phi}}{\partial \boldsymbol{y}} = -\boldsymbol{v} \quad \Rightarrow \frac{\partial \boldsymbol{\phi}}{\partial \boldsymbol{x}} = -(-4\boldsymbol{x}) = 4\boldsymbol{x} \Rightarrow \boldsymbol{\phi} = 4 \times \boldsymbol{x} \times \boldsymbol{y} + \boldsymbol{C}(\boldsymbol{f}(\boldsymbol{x})\boldsymbol{only}) \cdots \boldsymbol{eq}2$$

$$\frac{\partial \phi}{\partial y} = (4x + \frac{\partial C}{\partial y}) \Rightarrow \frac{\partial C}{\partial y} = 4x - \frac{\partial \phi}{\partial y} \Rightarrow \frac{\partial C}{\partial y} = 4x - \left(\frac{\partial \psi}{\partial x}\right) \Rightarrow \frac{\partial C}{\partial y} = 4x - 4x = 0$$

From Eq.1 $\frac{\partial C}{\partial y} = 0$ *Integrating* $C = 0$

$$\therefore \phi = 4 \times x \times y \qquad \Rightarrow \phi = 4 \times 1 \times 2 = 8 \; Units$$

Q.17. The velocity potential ϕ for a two dimensional flow is given by $(x^2 - y^2) + 3xy$. Calculate: (i) the stream function ψ and (ii) the flow rate passing between the stream lines through (1, 1) and (1, 2).

Ans: Given $\phi = (x^2 - y^2) + 3xy$

(i) To determine the ψ function

$$d\psi = \frac{\partial \psi}{\partial x} dx + \frac{\partial \psi}{\partial y} dy \cdots Eq.(1)$$

$$d\psi = -v\,dx + u\,dy \cdots Eq.(2)$$

As per definition of velocity potential (ϕ) and stream function (ψ);

$$\frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y} = u \text{ and } \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x} = v$$

$$u = \frac{\partial \phi}{\partial x} = (2x + 3y) = \frac{\partial \psi}{\partial y} \text{ and } \frac{\partial \phi}{\partial y} = (-2y + 3x) = \left(-\frac{\partial \psi}{\partial x}\right) = v$$

Substituting the value of u and v in terms of x and y in equation 2, we obtain

$$d\psi = -v\,dx + u\,dy = -(-2y + 3x)dx + (2x + 3y)dy$$
$$d\psi = (2y + 3x)dx + (2x + 3y)dy \cdots Eq.3$$

Integrating the equation-3 (partially w.r.t 'x' the 'dx-term' and w.r.t 'y' the 'dy-term')

$$\psi = \left(2xy + \frac{3}{2}x^2\right) + \left(2xy + \frac{3}{2}y^2\right) = 4xy + \frac{3}{2}\left(x^2 + y^2\right)$$

$$\boxed{\psi = 4xy + \frac{3}{2}\left(x^2 + y^2\right)}$$

(ii) The flow rate passing between the stream lines through (1, 1) and (1, 2).

The equation of stream function is given by $\boxed{\psi = 4xy + \frac{3}{2}\left(x^2 + y^2\right)}$

The value of Point streamline at (1, 1) is obtained by substituting x = 1, y = 1

$$\psi_{(1,1)} = 4xy + \frac{3}{2}\left(x^2 + y^2\right) = 4\times1\times1 + \frac{3}{2}(1^2 + 1^2) = 7\,Units$$

The value of Point streamline at (1, 2) is obtained by substituting x = 1, y = 2

$$\psi_{(1,2)} = 4xy + \frac{3}{2}\left(x^2 + y^2\right) = 4\times1\times2 + \frac{3}{2}(1^2 + 2^2) = 15.5\,Units$$

The flow rate passing between the stream lines through (1, 1) and (1, 2)

$$\mathbf{q = \psi_{(1,2)} - \psi_{(1,1)} = (15.5-7)}$$

$$\boxed{\mathbf{q = 8.5\ m^2/s/unit\ width}}$$

Q.18. The velocity components in a 2-dimensional incompressible flow field are expressed as

$$u = \left(\frac{y^3}{3} + 2x - x^2 \times y\right), \quad v = \left(x \times y^2 - 2y - \frac{x^3}{3}\right)$$

Is the flow irrotational? If so determine the corresponding stream function.

Ans: Given the components of velocity

$$u = \left(\frac{y^3}{3} + 2x - x^2 \times y\right), \quad v = \left(x \times y^2 - 2y - \frac{x^3}{3}\right)$$

The condition for Irrorational flow

$$\left(\frac{\partial v}{\partial x}\right) = \left(\frac{\partial u}{\partial y}\right)$$

$$LHS \quad \frac{\partial v}{\partial x} = \frac{\partial}{\partial x}\left(x \times y^2 - 2y - \frac{x^3}{3} \right) \text{ and } RHS \quad \frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left(\frac{y^3}{3} + 2x - x^2 \times y \right)$$

i.e. LHS = $(y^2 - x^2)$ and RHS = $(y^2 - x^2)$

Hence the flow is Irrorational

The corresponding stream function 'ψ'can be obtained by using following relationship

$$\frac{\partial \psi}{\partial x} = v = \left(x \times y^2 - 2y - \frac{x^3}{3} \right) \cdots Eq.1$$

$$\frac{\partial \psi}{\partial y} = -u = -\left(\frac{y^3}{3} + 2x - x^2 \times y \right) \cdots Eq.2$$

Integrating Eq.1 with respect to 'x'

$$\psi = \frac{x^2 \times y^2}{2} - 2 \times x \times y - \frac{x^4}{12} + C_1(f(y)) \cdots Eq.3$$

Differentiating Eq.3 with respect to 'y'

$$\frac{\partial \boldsymbol{\psi}}{\partial \boldsymbol{y}} = \boldsymbol{x}^2 \times \boldsymbol{y} - 2\boldsymbol{x} + \frac{\partial \boldsymbol{C}_1}{\partial \boldsymbol{y}}$$

$$\frac{\partial \boldsymbol{C}_1}{\partial \boldsymbol{y}} = -\frac{\boldsymbol{y}^3}{3}$$

$$\textbf{Integrating}, \boldsymbol{C}_1 = -\frac{\boldsymbol{y}^4}{12} + \boldsymbol{C}; \quad (\textbf{assuming } \boldsymbol{C} = 0)$$

$$\boldsymbol{C}_1 = -\frac{\boldsymbol{y}^4}{12}$$

The stream function 'ψ' is given by

$$\psi = \frac{x^2 \times y^2}{2} - 2 \times x \times y - \frac{x^4}{12} + -\frac{y^4}{12}$$

Module 5:
COMPRESSIBLE FLOWS

Overview

In general, the liquids and gases are the states of a matter that comes under the same category as "fluids". The incompressible flows are mainly deals with the cases of constant density. Also, when the variation of density in the flow domain is negligible, then the flow can be treated as incompressible. Invariably, it is true for liquids because the density of liquid decreases slightly with temperature and moderately with pressure over a broad range of operating conditions. Hence, the liquids are considered as incompressible. On the contrary, the compressible flows are routinely defined as "variable density flows". Thus, it is applicable only for gases where they may be considered as incompressible/compressible, depending on the conditions of operation. During the flow of gases under certain conditions, the density changes are so small that the assumption of constant density can be made with reasonable accuracy and in few other cases the density changes of the gases are very much significant (e.g. high speed flows). Due to the dual nature of gases, they need special attention and the broad area of in the study of motion of compressible flows is dealt separately in the subject of "gas dynamics". Many engineering tasks require the compressible flow applications typically in the design of a building/tower to withstand winds, high speed flow of air over cars/trains/airplanes etc. Thus, gas dynamics is the study of fluid flows where the compressibility and the temperature changes become important. Here, the entire flow field is dominated by Mach waves and shock waves when the flow speed becomes supersonic. Most of the flow properties change across these waves from one state to other. In addition to the basic fluid dynamics, the knowledge of thermodynamics and chemical kinetics is also essential to the study of gas dynamics.

Thermodynamic Aspects of Gases

In high speed flows, the kinetic energy per unit mass $\left(V^2/2\right)$ is very large which is substantial enough to strongly interact with the other properties of the flow. Since the science of energy and entropy is the thermodynamics, it is essential to study the thermodynamic aspects of gases under the conditions compressible high speed flows.

<u>Perfect gas</u>: A gas is considered as a collection of particles (molecules, atoms, ions, electrons etc.) that are in random motion under certain intermolecular forces. These forces vary with distances and thus influence the microscopic behavior of the gases. However, the thermodynamic aspect mainly deals with the global nature of the gases. Over wide ranges of pressures and temperatures in the compressible flow fields, it is seen that the average distance between the molecules is more than the molecular diameters (about 10-times). So, all the flow properties may be treated as macroscopic in nature. A perfect gas follows the relation of pressure, density and temperature in the form of the fundamental equation.

$$p = \rho RT\ ;\ R = \frac{\bar{R}}{M} \tag{4.1.1}$$

Here, M is the molecular weight of the gas, R is the gas constant that varies from gas to gas and $\bar{R}\left(=8314\,\text{J/kg.K}\right)$ is the universal gas constant. In a calorically perfect gas, the other important thermodynamic properties relations are written as follows;

$$c_p = \left(\frac{\partial h}{\partial}\right)_p\ ;\ c_v = \left(\frac{\partial e}{\partial}\right)_v\ ;\ c_p - c_v = R$$
$$c_p = \frac{\gamma R}{\gamma - 1}\ ;\ c_v = \frac{R}{\gamma - 1}\ ;\ \gamma = \frac{c_p}{c_v} \tag{4.1.2}$$

In Eq. (4.1.2), the parameters are specific heat at constant pressure $\left(c_p\right)$, specific heat at constant volume $\left(c_v\right)$, specific heat ratio $\left(\gamma\right)$, specific enthalpy $\left(h\right)$ and specific internal energy $\left(e\right)$.

First law of thermodynamics: A system is a fixed mass of gas separated from the surroundings by a flexible boundary. The heat added (q) and work done (w) on the system can cause change in energy. Since, the system is stationary, the change in internal energy. By definition of first law, we write,

$$\delta q + \delta w = de \tag{4.1.3}$$

For a given de, there are infinite number of different ways by which heat can be added and work done on the system. Primarily, the three common types of processes are, adiabatic (no addition of heat), reversible (no dissipative phenomena) and isentropic (i.e. reversible and adiabatic).

Second law of thermodynamics: In order to ascertain the direction of a thermodynamic process, a new state variable is defined as 'entropy (s)'. The change in entropy during any incremental process (ds) is equal to the actual heat added divided by the temperature (dq/T), plus a contribution from the irreversible dissipative phenomena (ds_{irrev}) occurring within the system.

$$ds = \frac{\delta q}{T} + ds_{irrev} \tag{4.1.4}$$

Since, the dissipative phenomena always increases the entropy, it follows that

$$ds \geq \frac{\delta q}{T}\ ; \quad ds \geq 0 \text{ (Adiabatic process)} \tag{4.1.5}$$

Eqs. (4.1.4 & 4.1.5) are the different forms of *second law of thermodynamics*. In order to calculate the change in entropy of a thermodynamic process, two fundamental relations are used for a calorically perfect gas by combining both the laws of thermodynamics;

$$s_2 - s_1 = c_p \ln\left(\frac{T_2}{T_1}\right) - R\ln\left(\frac{p_2}{p_1}\right)$$

$$s_2 - s_1 = c_v \ln\left(\frac{T_2}{T_1}\right) + R\ln\left(\frac{\rho_1}{\rho_2}\right)$$

An isentropic process is the one for which the entropy is constant and the process is reversible and adiabatic. The isentropic relation is given by the following relation;

$$\frac{p_2}{p_1}=\left(\frac{\rho_2}{\rho_1}\right)^{\gamma}=\left(\frac{T_2}{T_1}\right)^{\gamma/(\gamma-1)} \qquad (4.1.7)$$

Important Properties of Compressible Flows

The simple definition of compressible flow is the variable density flows. In general, the density of gases can vary either by changes in pressure and temperature. In fact, all the high speed flows are associated with significant pressure changes. So, let us recall the following fluid properties important for compressible flows;

Bulk modulus (E_v): It is the property of that fluid that represents the variation of density (ρ) with pressure (p) at constant temperature (T). Mathematically, it is represented as,

$$E_v=-\forall\left(\frac{\partial p}{\partial \forall}\right)_T=\rho\left(\frac{\partial \rho}{\partial T}\right)_T \qquad (4.1.8)$$

In terms of finite changes, it is approximated as,

$$E_v=\frac{(\Delta\forall/\forall)}{\Delta T}=-\frac{(\Delta\rho/\rho)}{\Delta T} \qquad (4.1.9)$$

Coefficient of volume expansion (β): It is the property of that fluid that represents the variation of density with temperature at constant pressure. Mathematically, it is represented as,

$$\beta=\frac{1}{\forall}\left(\frac{\partial \forall}{\partial T}\right)_p=-\frac{1}{\rho}\left(\frac{\partial \rho}{\partial T}\right)_p \qquad (4.1.10)$$

In terms of finite changes, it is approximated as,

$$\beta=\frac{(\Delta\forall/\forall)}{\Delta T}=-\frac{(\Delta\rho/\rho)}{\Delta T} \qquad (4.1.11)$$

<u>Compressibility</u>(κ) : It is defined as the fractional change in the density of the fluid element per unit change in pressure. One can write the expression for κ as follows;

$$\kappa = \frac{1}{\rho}\left(\frac{d\rho}{dp}\right) \quad \Rightarrow d\rho = \rho\kappa\, dp \tag{4.1.12}$$

In order to be more precise, the compression process for a gas involves increase in temperature depending on the amount of heat added or taken away from the gas. If the temperature of the gas remains constant, the definition is refined as *isothermal compressibility* (κ_T) . On the other hand, when no heat is added/taken away from the gases and in the absence of any dissipative mechanisms, the compression takes place isentropically. It is then, called as *isentropic compressibility* (κ_s) .

$$\kappa_T = \frac{1}{\rho}\left(\frac{\partial\rho}{\partial p}\right)_T ; \quad \kappa_s = \frac{1}{\rho}\left(\frac{\partial\rho}{\partial p}\right)_s \tag{4.1.13}$$

Being the property of a fluid, the gases have high values of compressibility $\left(\kappa_T = 10^{-5}\,\mathrm{m}^2/\mathrm{N} \text{ for air at 1atm}\right)$ while liquids have low values of compressibility much less than that of gases $\left(\kappa_T = 5\times10^{-10}\,\mathrm{m}^2/\mathrm{N} \text{ for water at 1atm}\right)$. From the basic definition (Eq. 4.1.12), it is seen that whenever a fluid experiences a change in pressure dp , there will be a corresponding change in $d\rho$. Normally, high speed flows involve large pressure gradient. For a given change in dp , the resulting change in density will be small for liquids (low values of κ) and more for gases (high values of κ). Therefore, for the flow of liquids, the relative large pressure gradients can create much high velocities without much change in densities. Thus, the liquids are treated to be incompressible. On the other hand, for the flow of gases, the moderate to strong pressure gradient leads to substantial changes in the density (Eq.4.1.12) and at the same time, it can create large velocity changes. Such flows are defined as compressible flows where the density is a variable property and the fractional change in density $(d\rho/\rho)$ is too large to be ignored.

Fundamental Equations for Compressible Flow

Consider a compressible flow passing through a rectangular control volume as shown in Fig. 4.1.1. The flow is one-dimensional and the properties change as a function of *x*, from the region '1' to '2' and they are velocity(u), pressure(p), temperature(T), density (ρ) and internal energy (e). The following assumptions are made to derive the fundamental equations;

- Flow is uniform over left and right side of control volume.
- Both sides have equal area (A), perpendicular to the flow.
- Flow is inviscid, steady and nobody forces are present.
- No heat and work interaction takes place to/from the control volume.

Let us apply mass, momentum and energy equations for the one dimensional flow as shown in Fig. 4.1.1.

<u>Conservation of Mass</u>:

$$-\rho_1 u_1 A + \rho_2 u_2 A = 0 \Rightarrow \rho_1 u_1 = \rho_2 u_2 \quad (4.1.14)$$

<u>Conservation of Momentum</u>:

$$\rho_1(-u_1 A)u_1 + \rho_2(u_2 A)u_2 = -(-p_1 A + p_2 A) \Rightarrow p_1 + \rho_1 u_1^2 = p_2 + \rho_2 u_2^2 \quad (4.1.15)$$

<u>Steady Flow Energy Conservation</u>:

$$\frac{p_1}{\rho_1} + e_1 + \frac{u_1^2}{2} = \frac{p_2}{\rho_2} + e_2 + \frac{u_2^2}{2} \Rightarrow h_1 + \frac{u_1^2}{2} = h_2 + \frac{u_2^2}{2}$$

Here, the enthalpy $h\left(=e+\frac{p}{\rho}\right)$ is defined as another thermodynamic property of the gas.

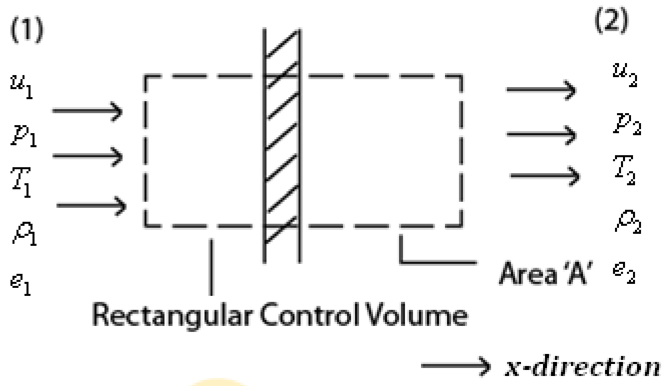


Fig. 4.1.1: Schematic representation of one-dimensional flow.

Wave Propagation in a Compressible Media

Consider a gas confined in a long tube with piston as shown in Fig. 4.2.1(a). The gas may be assumed to have infinite number of layers and initially, the system is in equilibrium. In other words, the last layer does not feel the presence of piston. Now, the piston is given a very small 'push' to the right. So, the layer of gas adjacent to the piston piles up and is compressed while the reminder of the gas remains unaffected. With due course of time, the compression wave moves downstream and the information is propagated. Eventually, all the gas layers feel the piston movement. If the pressure pulse applied to the gas is small, the wave is called as sound wave and the resultant compression wave moves at the "speed of sound". However, if the fluid is treated as incompressible, the change in density is not allowed. So, there will be no piling of fluid due to instantaneous change and the disturbance is felt at all other locations at the same time. So, the speed of sound depends on the fluid property i.e. 'compressibility'. The lower is its value; more will be the speed of sound. In an ideal incompressible medium of fluid, the speed of sound is infinite. For instance, sound travels about 4.3-times faster in water (1484 m/s) and 15-times as fast in iron (5120 m/s) than air at 20ºC.

Let us analyze the piston dynamics as shown in Fig. 4.2.1(a). If the piston moves at steady velocity dV , the compression wave moves at speed of sound a into the stationary gas. This infinitesimal disturbance creates increase in pressure and density next to the piston and in front of the wave. The same effect can be observed by keeping the wave stationary through dynamic transformation as shown in Fig. 4.2.1 (b). Now all basic one dimensional compressible flow equations can be applied for a very small control enclosing the stationary wave.

Continuity equation: Mass flow rate $(\dot{m})$ is conserved across the stationary wave.

$$\dot{m} = \rho a A = (\rho + d\rho)(a - dV) A \quad \Rightarrow dV = \left(\frac{a}{\rho}\right) d\rho \tag{4.2.1}$$

Momentum equation: As long as the compression wave is thin, the shear forces on the control volume are negligibly small compared to the pressure force. The momentum balance across the control volume leads to the following equation;

$$(p + dp) A - pA = \dot{m} a - \dot{m}(a - dV) \quad \Rightarrow dV = \left(\frac{1}{\rho a}\right) dp \tag{4.2.2}$$

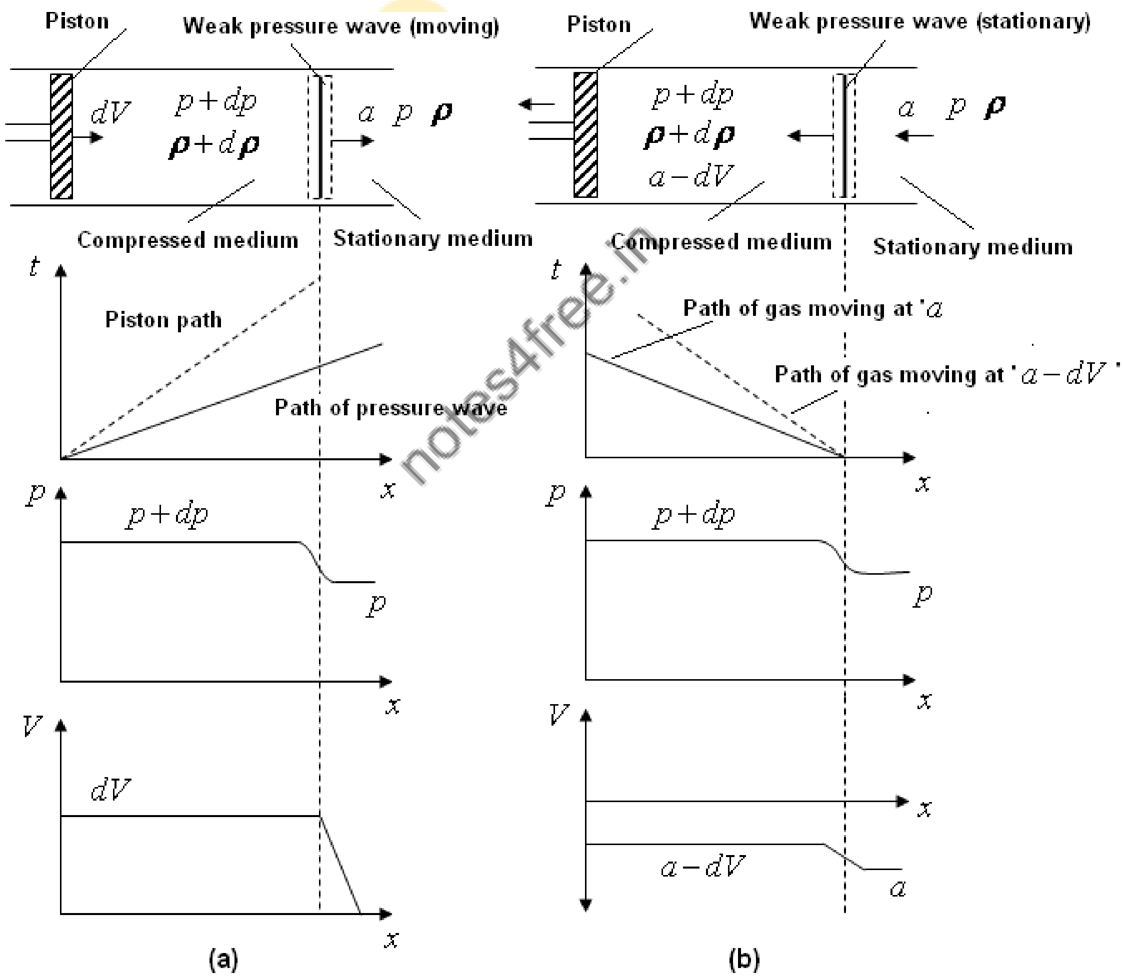


Fig. 4.2.1: Propagation of pressure wave in a compressible medium: (a) Moving wave; (b) Stationary wave.

Energy equation: Since the compression wave is thin, and the motion is very rapid, the heat transfer between the control volume and the surroundings may be neglected and the thermodynamic process can be treated as *adiabatic*. Steady flow energy equation can be used for energy balance across the wave.

$$h+\frac{a^2}{2}=(h+dh)+\frac{(a-dV)^2}{2} \quad \Rightarrow dV=\left(\frac{1}{a}\right)dh \tag{4.2.3}$$

Entropy equation: In order to decide the direction of thermodynamic process, one can apply $T-ds$ relation along with Eqs (4.2.2 & 4.2.3) across the compression wave.

$$T\,ds=dh-\frac{dp}{\rho}=0 \quad \Rightarrow ds=0 \tag{4.2.4}$$

Thus, the flow is isentropic across the compression wave and this compression wave can now be called as sound wave. The speed of the sound wave can be computed by equating Eqs.(4.2.1 & 4.2.2).

$$\left(\frac{a}{\rho}\right)=\left(\frac{1}{\rho a}\right) \quad \Rightarrow a^2=\frac{dp}{d\rho}=\left(\frac{\partial p}{\partial \rho}\right)_s$$

Further simplification of Eq. (4.2.5) is possible by evaluating the differential with the use of isenropic equation.

$$\frac{p}{\rho^\gamma}=\text{constant} \quad \Rightarrow \ln p-\gamma \ln \rho=\text{constant} \tag{4.2.6}$$

Differentiate Eq. (4.2.6) and apply perfect gas equation $\left(p=\rho RT\right)$ to obtain the expression for speed of sound. is obtained as below;

$$\left(\frac{\partial p}{\partial \rho}\right)_s=\frac{\gamma p}{\rho} \quad \Rightarrow a=\sqrt{\frac{\gamma p}{\rho}}=\sqrt{\gamma RT} \tag{4.2.7}$$

Mach number

It may be seen that the speed of sound is the thermodynamic property that varies from point to point. When there is a large relative speed between a body and the compressible fluid surrounds it, then the compressibility of the fluid greatly influences the flow properties. Ratio of the local speed (V) of the gas to the speed of sound (a) is called as local Mach number (M) .

$$M = \frac{V}{a} = \frac{V}{\sqrt{\gamma RT}} \tag{4.2.8}$$

There are few physical meanings for Mach number;

(a) It shows the compressibility effect for a fluid i.e. $M < 0.3$ implies that fluid is incompressible.

(b) It can be shown that Mach number is proportional to the ratio of kinetic to internal energy.

$$\frac{\left(V^2/2\right)}{e} = \frac{V^2/2}{c_v T} = \frac{V^2/2}{RT/(\gamma-1)} = \frac{(\gamma/2)V^2}{a^2/(\gamma-1)} = \frac{\gamma(\gamma-1)}{2}M^2 \tag{4.2.9}$$

(c) It is a measure of directed motion of a gas compared to the random thermal motion of the molecules.

$$M^2 = \frac{V^2}{a^2} = \frac{\text{directed kinetic energy}}{\text{random kinetic energy}} \tag{4.2.10}$$

Compressible Flow Regimes

In order to illustrate the flow regimes in a compressible medium, let us consider the flow over an aerodynamic body (Fig. 4.2.2). The flow is uniform far away from the body with free stream velocity (V_∞) while the speed of sound in the uniform stream is a_∞. Then, the free stream Mach number becomes $M_\infty(=V_\infty/a_\infty)$. The streamlines can be drawn as the flow passes over the body and the local Mach number can also vary along the streamlines. Let us consider the following distinct flow regimes commonly dealt with in compressible medium.

Subsonic flow: It is a case in which an airfoil is placed in a free stream flow and the local Mach number is less than unity everywhere in the flow field (Fig. 4.2.2-a). The flow is characterized by smooth streamlines with continuous varying properties. Initially, the streamlines are straight in the free stream, but begin to deflect as they approach the body. The flow expands as it passed over the airfoil and the local Mach number on the top surface of the body is more than the free stream value. Moreover, the local Mach number (M) in the surface of the airfoil remains always less than 1, when the free stream Mach number (M_∞) is sufficiently less than 1. This regime is defined as subsonic flow which falls in the range of free stream Mach number less than 0.8 i.e. $M_\infty \le 0.8$.

Transonic flow: If the free stream Mach number increases but remains in the subsonic range close to 1, then the flow expansion over the air foil leads to supersonic region locally on its surface. Thus, the entire regions on the surface are considered as mixed flow in which the local Mach number is either less or more than 1 and thus called as *sonic pockets* (Fig. 4.2.2-b). The phenomena of sonic pocket is initiated as soon as the local Mach number reaches 1 and subsequently terminates in the downstream with a shock wave across which there is discontinuous and sudden change in flow properties. If the free stream Mach number is slightly above unity (Fig. 4.2.2-c), the shock pattern will move towards the trailing edge and a second shock wave appears in the leading edge which is called as *bow shock*. In front of this bow shock, the streamlines are straight and parallel with a uniform supersonic free stream Mach number. After passing through the bow shock, the flow becomes subsonic close to the free stream value. Eventually, it further expands over the airfoil surface to supersonic values and finally terminates with trailing edge shock in the downstream. The mixed flow patterns sketched in Figs. 4.2.2 (b & c), is defined as the *transonic regime*.

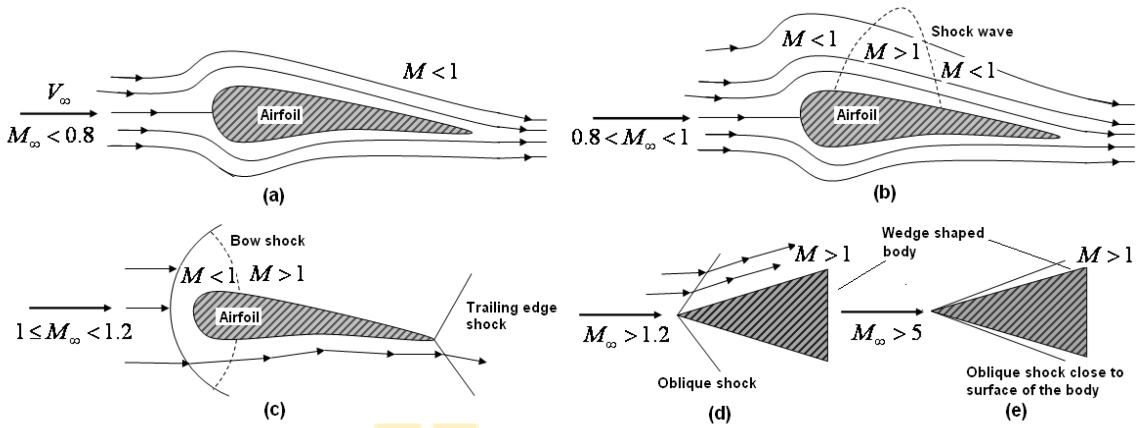


Fig. 4.2.2: Illustration of compressible flow regime: (a) subsonic flow; (b & c) transonic flow; (d) supersonic flow; (d) hypersonic flow.

<u>Supersonic flow</u>: In a flow field, if the Mach number is more than 1 everywhere in the domain, then it defined as supersonic flow. In order to minimize the drag, all aerodynamic bodies in a supersonic flow, are generally considered to be sharp edged tip. Here, the flow field is characterized by straight, oblique shock as shown in Fig. 4.2.2(d). The stream lines ahead of the shock the streamlines are straight, parallel and horizontal. Behind the oblique shock, the streamlines remain straight and parallel but take the direction of wedge surface. The flow is supersonic both upstream and downstream of the oblique shock. However, in some exceptional strong oblique shocks, the flow in the downstream may be subsonic.

<u>Hypersonic flow</u>: When the free stream Mach number is increased to higher supersonic speeds, the oblique shock moves closer to the body surface (Fig. 4.2.2-e). At the same time, the pressure, temperature and density across the shock increase explosively. So, the flow field between the shock and body becomes hot enough to ionize the gas. These effects of thin shock layer, hot and chemically reacting gases and many other complicated flow features are the characteristics of *hypersonic flow*. In reality, these special characteristics associated with hypersonic flows appear gradually as the free stream Mach numbers is increased beyond 5.

As a rule of thumb, the compressible flow regimes are classified as below;

$$M < 0.3 \text{ (incompressible flow)}$$
$$M < 1 \text{ (subsonic flow)}$$
$$0.8 < M < 1.2 \text{ (transonic flow)}$$
$$M > 1 \text{ (supersonic flow)}$$
$$M > 5 \text{ and above (hypersonic flow)}$$

Rarefied and Free Molecular Flow: In general, a gas is composed of large number of discrete atoms and molecules and all move in a random fashion with frequent collisions. However, all the fundamental equations are based on overall macroscopic behavior where the continuum assumption is valid. If the mean distance between atoms/molecules between the collisions is large enough to be comparable in same order of magnitude as that of characteristics dimension of the flow, then it is said to be low density/rarefied flow. Under extreme situations, the mean free path is much larger than the characteristic dimension of the flow. Such flows are defined as free molecular flows. These are the special cases occurring in flight at very high altitudes (beyond 100 km) and some laboratory devices such as electron beams.

Isentropic and Characteristics States

An isentropic process provides the useful standard for comparing various types of flow with that of an idealized one. Essentially, it is the process where all types of frictional effects are neglected and no heat addition takes place. Thus, the process is considered as reversible and adiabatic. With this useful assumption, many fundamental relations are obtained and some of them are discussed here.

Stagnation/Total Conditions

When a moving fluid is decelerated isentropically to reach zero speed, then the thermodynamic state is referred to as *stagnation/total* condition/state. For example, a gas contained in a high pressure cylinder has no velocity and the thermodynamic state is known as *stagnation/total* condition (Fig. 4.3.1-a). In a real flow field, if the actual conditions of pressure (p), temperature (T), density (ρ), enthalpy(h), internal energy (e), entropy (s) etc. are referred to as static conditions while the associated stagnation parameters are denoted as $p_0, T_0, \rho_0, h_0, e_0$ and s_0, respectively. The stagnation state is fixed by using second law of thermodynamics where $s = s_0$ as represented in *enthalpy-entropy diagram called as the Mollier diagram* (Fig. 4.3.1-b).

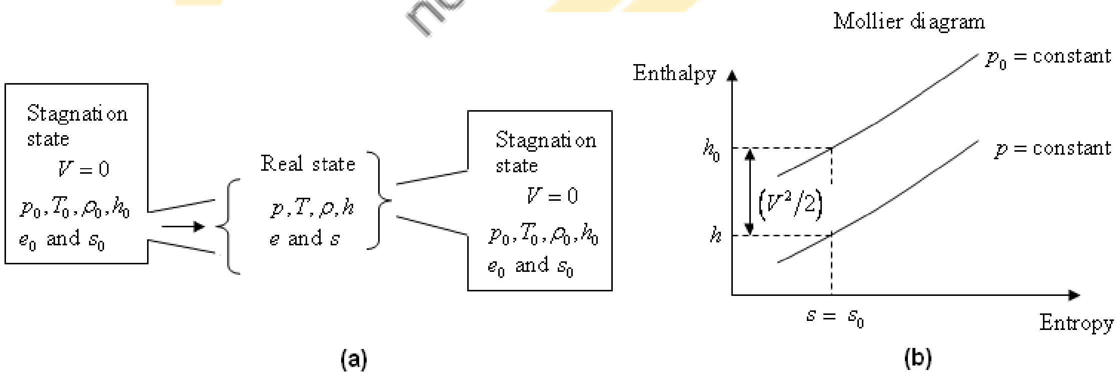


Fig 4.3.1: (a) Schematic representation of stagnation condition; (b) Mollier diagram.

The simplified form of energy equation for steady, one-dimensional flow with no heat addition, across two regions 1 and 2 of a control volume is given by,

$$h_1 + \frac{u_1^2}{2} = h_2 + \frac{u_2^2}{2} \tag{4.3.1}$$

For a calorically perfect gas, replacing, $h = c_p T$, so the Eq. (4.3.1) becomes,

$$c_p T_1 + \frac{u_1^2}{2} = c_p T_2 + \frac{u_2^2}{2} \tag{4.3.2}$$

If the region '1' refers to any arbitrary real state in the flow field and the region '2' refers to stagnation condition, then Eq. (4.3.2) becomes,

$$c_p T + \frac{u^2}{2} = c_p T_0 \tag{4.3.3}$$

It can be solved for (T_0/T) as,

$$\begin{aligned} \frac{T_0}{T} &= 1 + \frac{u^2}{2c_p T} = 1 + \frac{u^2}{2\gamma RT/(\gamma-1)} = 1 + \frac{u^2}{2a^2/(\gamma-1)} \\ \text{or, } \frac{T_0}{T} &= 1 + \left(\frac{\gamma-1}{2}\right)\left(\frac{u}{a}\right)^2 = 1 + \left(\frac{\gamma-1}{2}\right)M^2 \end{aligned} \tag{4.3.4}$$

For an isentropic process, the thermodynamic relation is given by,

$$\frac{p_0}{p} = \left(\frac{\rho_0}{\rho}\right)^{\gamma} = \left(\frac{T_0}{T}\right)^{\frac{\gamma}{\gamma-1}} \tag{4.3.5}$$

From, Eqs (4.3.4) and (4.3.5), the following relations may be obtained for stagnation pressure and density.

$$\begin{aligned} \frac{p_0}{p} &= \left(1 + \frac{\gamma-1}{2}M^2\right)^{\frac{\gamma}{\gamma-1}} \\ \frac{\rho_0}{\rho} &= \left(1 + \frac{\gamma-1}{2}M^2\right)^{\frac{1}{\gamma-1}} \end{aligned} \tag{4.3.6}$$

In general, if the flow field is isentropic throughout, the stagnation properties are constant at every point in the flow. However, if the flow in the regions '1' and '2' is non-adiabatic and irreversibile, then $T_{01} \neq T_{02}$; $p_{01} \neq p_{02}$; $\rho_{01} \neq \rho_{02}$

Characteristics Conditions

Consider an arbitrary flow field, in which a fluid element is travelling at some Mach number (M) and velocity (V) at a given point 'A'. The static pressure, temperature and density are p, T and ρ, respectively. Now, imagine that the fluid element is adiabatically slowed down (if $M > 1$) or speeded up (if $M < 1$) until the Mach number at 'A' reaches the sonic state as shown in Fig. 4.3.2. Thus, the temperature will change in this process. This imaginary situation of the flow field when a real state in the flow is brought to sonic state is known as the *characteristics conditions*. The associated parameters are denoted as p^*, T^*, ρ^*, a^* etc.

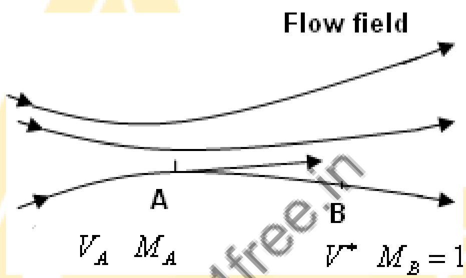


Fig. 4.3.2: Illustration of characteristics states of a gas.

Now, revisit Eq. (4.3.2) and use the relations for a calorically perfect gas, by replacing, $c_p = \frac{\gamma R}{\gamma - 1}$ and $a = \sqrt{\gamma RT}$. Another form of energy equation is obtained as below;

$$\frac{a_1^2}{\gamma - 1} + \frac{u_1^2}{2} = \frac{a_2^2}{\gamma - 1} + \frac{u_2^2}{2} \qquad (4.3.7)$$

At the imagined condition (point 2) of Mach 1, the flow velocity is sonic and $u_2 = a^*$. Then the Eq. (4.3.7) becomes,

$$\frac{a^2}{\gamma - 1} + \frac{u^2}{2} = \frac{a^{*2}}{\gamma - 1} + \frac{a^{*2}}{2}$$
$$\text{or, } \frac{a^2}{\gamma - 1} + \frac{u^2}{2} = \frac{\gamma + 1}{2(\gamma - 1)} a^{*2} \qquad (4.3.8)$$

Like stagnation properties, these imagined conditions are associated properties of any fluid element which is actually moving with velocity u . If an actual flow field is non-adiabatic from $A \rightarrow B$, then $a^*_A \neq a^*_B$ .On the other hand if the general flow field is adiabatic throughout, then a^* is a constant value at every point in the flow. Dividing u^2 both sides for Eq. (4.3.8) leads to,

$$\frac{(a/u)^2}{\gamma-1} + \frac{1}{2} = \frac{\gamma+1}{2(\gamma-1)}\left(\frac{a^*}{u}\right)^2$$

$$\text{or, } M^2 = \frac{2}{\left[(\gamma+1)/M^{*2}\right]-(\gamma-1)} \qquad (4.3.9)$$

This equation provides the relation between actual Mach number (M) and *characteristics Mach number* (M^*) . It may be shown that when M approaches infinity, M^* reaches a finite value. From Eq. (4.3.9), it may be seen that

$$\begin{aligned} &M = 1 \Rightarrow M^* = 1 \\ &M < 1 \Rightarrow M^* < 1 \\ &M > 1 \Rightarrow M^* > 1 \\ &M \rightarrow \infty \Rightarrow M^* \rightarrow \sqrt{\frac{\gamma+1}{\gamma-1}} \end{aligned} \qquad (4.3.10)$$

Relations between stagnation and characteristics state

The stagnation speed and characteristics speed of sound may be written as,

$$a_0 = \sqrt{\gamma R T_0}; \; a^* = \sqrt{\gamma R T^*} \qquad (4.3.11)$$

Rewrite Eq. (4.3.7) for stagnation conditions as given below;

$$\frac{a^2}{\gamma-1} + \frac{u^2}{2} = \frac{a_o^2}{\gamma-1} \qquad (4.3.12)$$

Equate Eqs. (4.3.8) and (4.3.12),

$$\frac{\gamma+1}{2(\gamma-1)} a^{*2} = \frac{a_0^2}{\gamma-1} \Rightarrow \left(\frac{a^*}{a_0}\right)^2 = \frac{T^*}{T_0} = \frac{2}{\gamma+1} \qquad (4.3.13)$$

More useful results may be obtained for Eqs. (4.3.4) & (4.3.6), when we define

$p = p^*;\ T = T^*;\ \rho = \rho^*; a = a^*$ for Mach 1

$$\frac{p^*}{p_0} = \left(\frac{2}{\gamma+1}\right)^{\frac{\gamma}{\gamma-1}};\ \frac{\rho^*}{\rho_0} = \left(\frac{2}{\gamma+1}\right)^{\frac{1}{\gamma-1}} \qquad (4.3.14)$$

With $\gamma = 1.4$ (for air), the Eqs (4.3.13) & (4.3.14) reduces to constant value.

$$\left(\frac{a^*}{a_0}\right)^2 = \frac{T^*}{T_0} = 0.833;\quad \frac{p^*}{p_0} = 0.528;\ \frac{\rho^*}{\rho_0} = 0.634 \qquad (4.3.15)$$

Critical speed and Maximum speed

The critical speed of the gas (u^*) is same as that speed of sound (a^*) at sonic state i.e. $u^* = a^*$ at $M = 1$. A gas can attain the maximum speed (u_{max}) when it is hypothetically expanded to zero pressure. The static temperature corresponding to this state is also zero. The maximum speed of the gas represents the speed corresponding to the complete transformation of kinetic energy associated with the random motion of gas molecules into the directed kinetic energy. Rearranging Eq. (4.3.3), one can obtain the following equation;

$$T_0 = T + \left(\frac{\gamma-1}{2\gamma}\right)u^2;\ \text{At } T = 0;\quad u = u_{max} = \sqrt{\frac{2\gamma R T_0}{\gamma-1}}$$

$$\text{or, } \left(\frac{u_{max}}{a_0}\right)^2 = \frac{2}{\gamma-1} \qquad (4.3.16)$$

Now, the Eqs (4.3.13) & (4.3.16) can be simplified to obtain the following relation;

$$\frac{u_{max}}{a^*} = \sqrt{\frac{\gamma+1}{\gamma-1}} \qquad (4.3.17)$$

Steady Flow Adiabatic Ellipse

It is an ellipse in which all the points have same total energies. Each point differs from the other owing to relative proportions of thermal and kinetic energies corresponding to different Mach numbers. Now, rewrite Eq. (4.3.3) by replacing $c_p = \frac{\gamma R}{\gamma - 1}$ and $a = \sqrt{\gamma RT}$;

$$\frac{u^2}{2} + \frac{\gamma R}{\gamma - 1} T = c_1 \quad \Rightarrow u^2 + \left(\frac{2}{\gamma - 1}\right) a^2 = c \qquad (4.3.18)$$

When, $T = 0$, $u = u_{max}$ so that the constant appearing in Eq. (4.3.18) can be considered as, $c = u_{max}^2$. Then, Eq. (4.3.18) is written as follows;

$$u^2 + \left(\frac{2}{\gamma - 1}\right) a^2 = u_{max}^2 \quad \Rightarrow \frac{u^2}{u_{max}^2} + \left(\frac{2}{\gamma - 1}\right) \frac{a^2}{u_{max}^2} = 1 \qquad (4.3.19)$$

Replacing the value of u_{max}^2 from Eq. (4.3.16) in Eq. (4.3.19), one can write the following expression;

$$\frac{u^2}{u_{max}^2} + \frac{a^2}{a_0^2} = 1 \qquad (4.3.20)$$

This is the equation of an ellipse with major axis as u_{max} and minor axis as a_0 as shown in Fig. 4.3.3. Now, rearrange Eq. (4.3.20) in the following form;

$$a^2 = a_0^2 - \left(\frac{u^2}{u_{max}^2}\right) a_0^2 \qquad (4.3.21)$$

Now, differentiate Eq. (4.3.21) with respect to u and simplify;

$$\frac{da}{du} = -\left(\frac{\gamma - 1}{2}\right)\left(\frac{u}{a}\right) = -\left(\frac{\gamma - 1}{2}\right) M \quad \Rightarrow M = -\left(\frac{2}{\gamma - 1}\right) \frac{da}{du} \qquad (4.3.22)$$

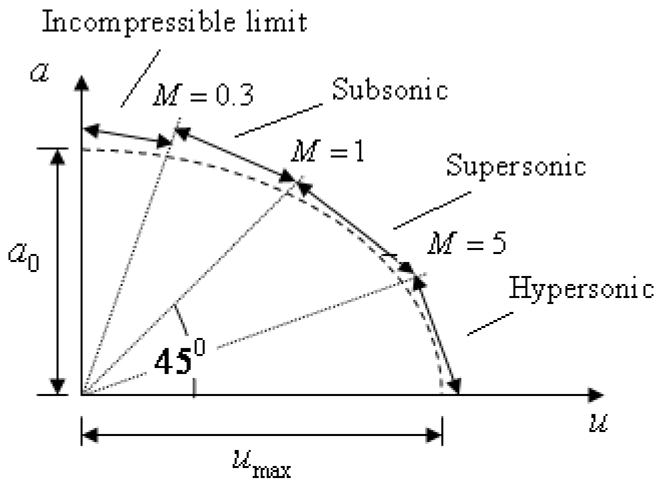


Fig. 4.3.3: Steady flow adiabatic ellipse.

Thus, the change of slope from point to point on the ellipse indicates the change in Mach number and hence the speed of sound and velocity. So, it gives the direct comparison of the relative magnitudes of thermal and kinetic energies. Different compressible flow regimes can be obtained with the knowledge of slope in Fig. 4.3.2. The following important inferences may be drawn;

- In high Mach numbers flows, the changes in Mach number are mainly due to the changes in speed of sound.
- At low Mach numbers flows, the changes in Mach number are mainly due to the changes in the velocity.
- When the flow Mach number is below 0.3, the changes in speed of sound is negligible small and the flow is treated as incompressible.

One-Dimensional Analysis

Mach Waves

Consider an aerodynamic body moving with certain velocity (V) in a still air. When the pressure at the surface of the body is greater than that of the surrounding air, it results an infinitesimal compression wave that moves at speed of sound (a). These disturbances in the medium spread out from the body and become progressively weaker away from the body. If the air has to pass smoothly over the surface of the body, the disturbances must 'warn' the still air, about the approach of the body. Now, let us analyze two situations: (a) the body is moving at subsonic speed $(V < a; M < 1)$; (b) the body is moving at supersonic speed $(V > a; M > 1)$.

Case I: During the motion of the body, the sound waves are generated at different time intervals (t) as shown in Fig. 4.4.1. The distance covered by the sound waves can be represented by the circle of radius $(at,\ 2at,\ 3at.......\text{so on})$. During same time intervals (t), the body will cover distances represented by, $Vt,\ 2Vt,\ 3Vt$ so on . At subsonic speeds $(V < a;\ M < 1)$, the body will always remains inside the family of circular sound waves. In other words, the information is propagated through the sound wave in all directions. Thus, the surrounding still air becomes aware of the presence of the body due to the disturbances induced in the medium. Hence, the flow adjusts itself very much before it approaches the body.

Case II: Consider the case, when the body is moving at supersonic speed $(V > a;\ M > 1)$. With a similar manner, the sound waves are represented by circle of radius $(at,\ 2at,\ 3at.......\text{so on})$ after different time (t) intervals. By this time, the body would have moved to a different location much faster from its initial position. At any point of time, the location of the body is always outside the family of circles of sound waves. The pressure disturbances created by the body always lags behind the body that created the disturbances. In other words, the information reaches the surrounding air much later because the disturbances cannot overtake the body. Hence, the flow cannot adjust itself when it approaches the body. The nature induces a wave across which the flow properties have to change and this line of disturbance is known as "Mach wave". These mach waves are initiated when the speed of the body approaches the speed of sound $(V = a; M = 1)$. They become progressively stronger with increase in the Mach number.

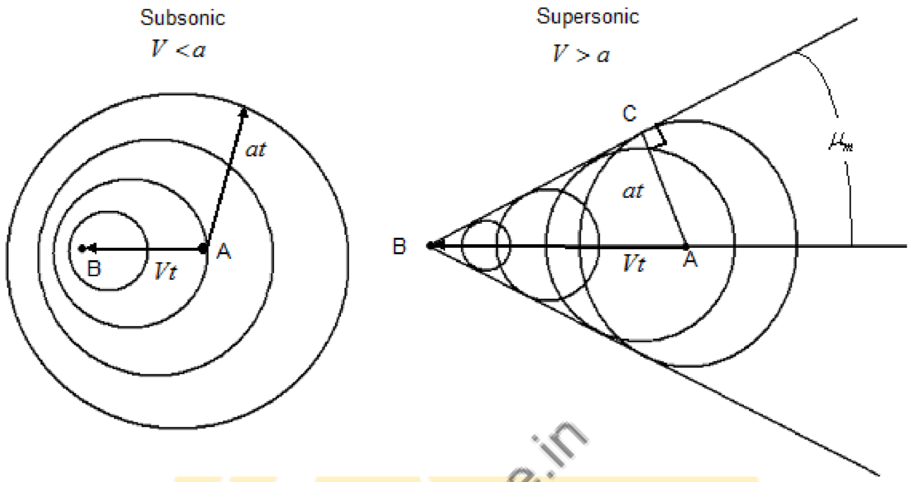


Fig. 4.4.1: Spread of disturbances at subsonic and supersonic speeds.

Some silent features of a *Mach wave* are listed below;

- The series of wave fronts form a disturbance envelope given by a straight line which is tangent to the family of circles. It will be seen that all the disturbance waves lie within a cone (Fig. 4.4.1), having a *vertex/apex* at the body at time considered. The locus of all the leading surfaces of the waves of this cone is known as *Mach cone*.
- All disturbances confine inside the Mach cone extending downstream of the moving body is called as *zone of action*. The region outside the Mach cone and extending upstream is known as *zone of silence*. The pressure disturbances are largely concentrated in the neighborhood of the Mach cone that forms the outer limit of the zone of action (Fig. 4.4.2).

- The half angle of the Mach cone is called as the Mach angle (μ_m) that can be easily calculated from the geometry of the Fig. 4.4.1.

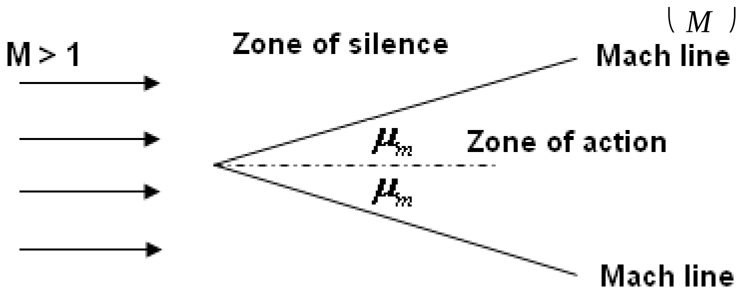


Fig. 4.4.2: Illustration of a Mach wave.

Shock Waves

Let us consider a subsonic and supersonic flow past a body as shown in Fig. 4.3.3. In both the cases, the body acts as an obstruction to the flow and thus there is a change in energy and momentum of the flow. The changes in flow properties are communicated through pressure waves moving at speed of sound everywhere in the flow field (i.e. both upstream and downstream). As shown in Fig. 4.3.3(a), if the incoming stream is subsonic i.e. $M_\infty < 1;\ V_\infty < a_\infty$, the sound waves propagate faster than the flow speed and warn the medium about the presence of the body. So, the streamlines approaching the body begin to adjust themselves far upstream and the flow properties change the pattern gradually in the vicinity of the body. In contrast, when the flow is supersonic, (Fig. 4.3.3-b) i.e. $M_\infty > 1;\ V_\infty > a_\infty$, the sound waves overtake the speed of the body and these weak pressure waves merge themselves ahead of the body leading to compression in the vicinity of the body. In other words, the flow medium gets compressed at a very short distance ahead of the body in a very thin region that may be comparable to the mean free path of the molecules in the medium. Since, these compression waves propagate upstream, so they tend to merge as *shock wave*. Ahead of the shock wave, the flow has no idea of presence of the body and immediately behind the shock; the flow is subsonic as shown in Fig. 4.3.3(b).

The thermodynamic definition of a shock wave may be written as "the instantaneous compression of the gas". The energy for compressing the medium, through a shock wave is obtained from the kinetic energy of the flow upstream the shock wave. The reduction in kinetic energy is accounted as heating of the gas to a static temperature above that corresponding to the isentropic compression value. Consequently, in flowing through the shock wave, the gas experiences a decrease in its available energy and accordingly, an increase in entropy. So, the compression through a shock wave is considered as an irreversible process.

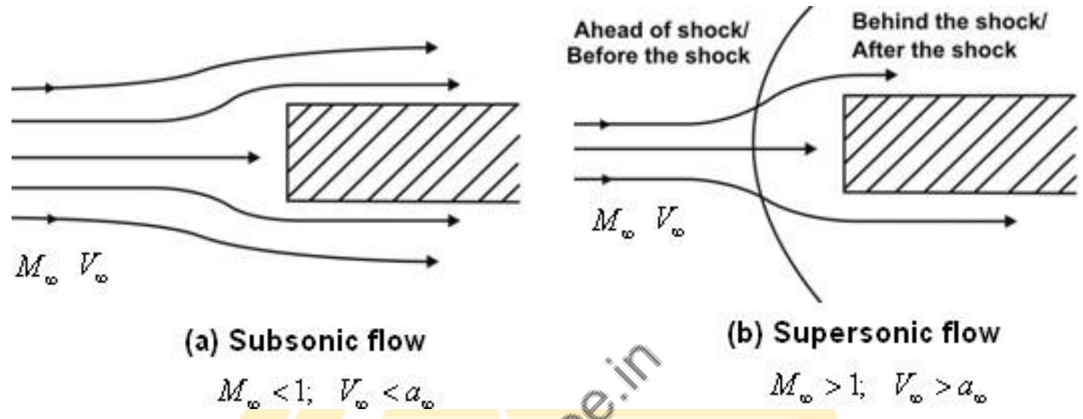


Fig. 4.4.3: Illustration of shock wave phenomena.

Normal Shock Waves

A normal shock wave is one of the situations where the flow properties change drastically in one direction. The shock wave stands perpendicular to the flow as shown in Fig. 4.4.4. The quantitative analysis of the changes across a normal shock wave involves the determination of flow properties. All conditions of are known ahead of the shock and the unknown flow properties are to be determined after the shock. There is no heat added or taken away as the flow traverses across the normal shock. Hence, the flow across the shock wave is adiabatic $(\dot{q} = 0)$.

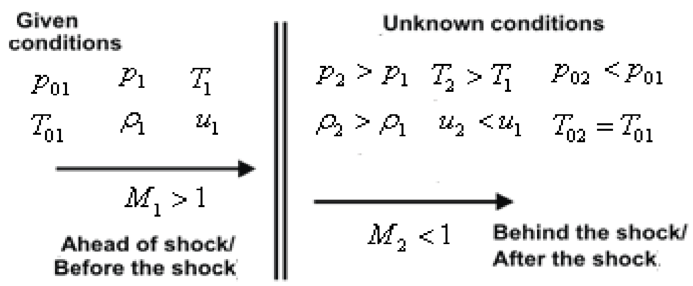


Fig. 4.4.4: Schematic diagram of a standing normal shock wave.

Two-Dimensional Analysis

Oblique Shock Wave

The normal shock waves are straight in which the flow before and after the wave is normal to the shock. It is considered as a special case in the general family of oblique shock waves that occur in supersonic flow. In general, oblique shock waves are straight but inclined at an angle to the upstream flow and produce a change in flow direction as shown in Fig. 4.5.1(a). An infinitely weak oblique shock may be defined as a *Mach wave* (Fig. 4.5.1-b). By definition, an oblique shock generally occurs, when a supersonic flow is 'turned into itself' as shown in Fig. 4.5.1(c). Here, a supersonic flow is allowed to pass over a surface, which is inclined at an angle (θ) to the horizontal. The flow streamlines are deflected upwards and aligned along the surface. Since, the upstream flow is supersonic; the streamlines are adjusted in the downstream an oblique shock wave angle (β) with the horizontal such that they are parallel to the surface in the downstream. All the streamlines experience same deflection angle across the oblique shock.

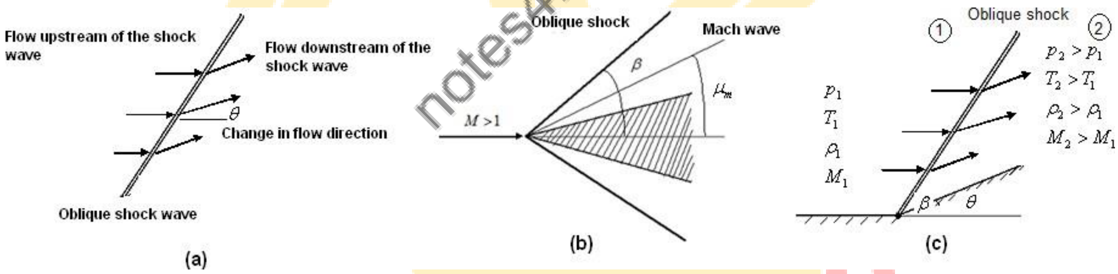


Fig. 4.5.1: Schematic representation of an oblique shock.
(a) Attached shock; (b) Detached shock; (c) Strong and weak shock.

Oblique Expansion Waves

Another class of two dimensional waves occurring in supersonic flow shows the opposite effects of oblique shock. Such types of waves are known as *expansion waves*. When the supersonic flow is "turned away from itself", an expansion wave is formed as shown in Fig. 4.5.5(a). Here, the flow is allowed to pass over a surface which is inclined at an angle (θ) to the horizontal and all the flow streamlines are deflected downwards. The change in flow direction takes place across an expansion fan centered at point 'A'. The flow streamlines are smoothly curved till the downstream flow becomes parallel to the wall surface behind the point 'A'. Here, the flow properties change smoothly through the expansion fan except at point 'A'. An infinitely strong oblique expansion wave may be called as a *Mach wave*. An expansion wave emanating from a sharp convex corner is known as a *centered expansion* which is commonly known as *Prandtl-Meyer expansion wave*. Few features of PM expansion waves are as follows;

- Streamlines through the expansion wave are smooth curved lines.
- The expansion of the flow takes place though an infinite number of Mach waves emitting from the center 'A'. It is bounded by forward and rearward Mach lines as shown in Fig. 4.5.5(b). These Mach lines are defined by Mach angles i.e.

$$\text{Forward Mach angle: } \mu_{m1} = \sin^{-1}\left(1/M_1\right)$$
$$\text{Rearward Mach angle: } \mu_{m2} = \sin^{-1}\left(1/M_2\right) \qquad (4.5.11)$$

- The expansion takes place through a continuous succession of Mach waves such that there is no change in entropy for each Mach wave. Thus, the expansion process is treated as isentropic.
- The Mach number increases while the static properties such as pressure, temperature and density decrease during the expansion process.

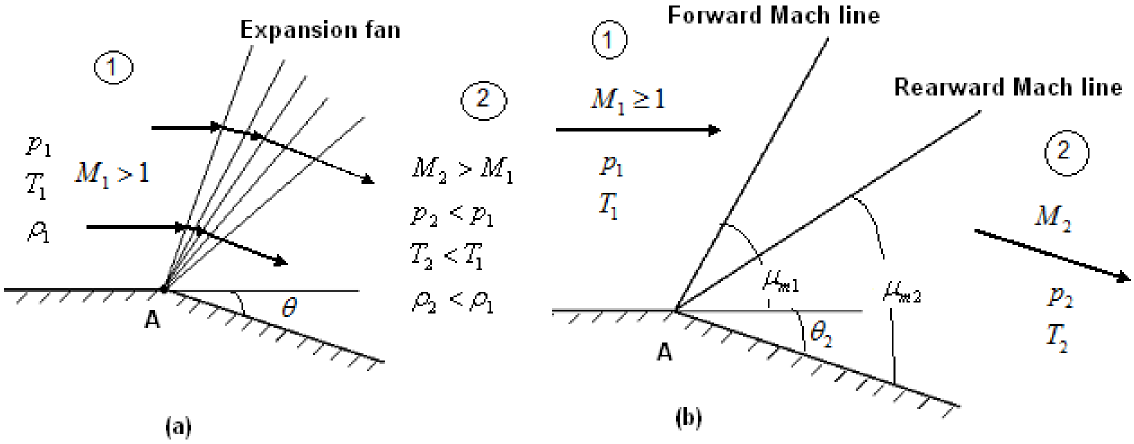


Fig. 4.5.5: Schematic representation of an expansion fan.

The quantitative analysis of expansion fan involves the determination of M_2, p_2, T_2 and ρ_2 for the given upstream conditions of M_1, p_1, T_1, ρ_1 and θ_2. Consider the infinitesimal changes across a very weak wave (Mach wave) as shown in Fig. 4.5.6.

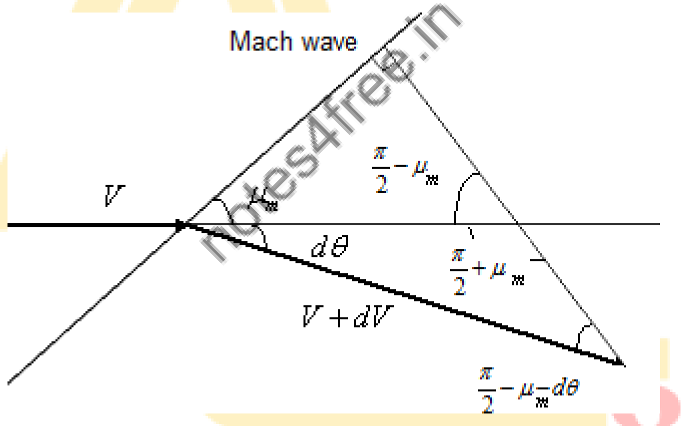


Fig. 4.5.6: Infinitesimal change across a Mach wave.

Use trigonometric identities and Taylor series expansion, Eq. (4.5.12) can be simplified as below;

$$d\theta = \frac{(dV/V)}{\tan \mu_m} \quad (4.5.13)$$

Since, $\sin \mu_m = \frac{1}{M} \Rightarrow \tan \mu_m = \frac{1}{\sqrt{M^2-1}}$, so the Eq. (4.5.13) can be simplified and integrated further from region '1' to '2',

$$d\theta = \sqrt{M^2-1}\frac{dV}{V} \quad \Rightarrow \int_{\theta_1}^{\theta_2} d\theta = \int_{M_1}^{M_2} \sqrt{M^2-1}\frac{dV}{V} \quad (4.5.14)$$

From the definition of Mach number,

$$V = Ma \quad \Rightarrow \frac{dV}{V} = \frac{dM}{M} + \frac{da}{a} \quad (4.5.15)$$

For a calorically perfect gas, the energy equation can be written as,

$$\left(\frac{a_o}{a}\right)^2 = 1 + \frac{\gamma-1}{2}M^2 \quad \Rightarrow \frac{da}{a} = -\left(\frac{\gamma-1}{2}\right)M\left(1+\frac{\gamma-1}{2}M^2\right)^{-1}dM \quad (4.5.16)$$

Use Eqs (4.5.15 & 4.5.16) in Eq. (4.5.14) and integrate from $\theta = 0$ to θ_2,

$$\int_{\theta_1}^{\theta_2} d\theta = \theta_2 - 0 = \int_{M_1}^{M_2} \frac{\sqrt{M^2-1}}{1+\frac{\gamma-1}{2}M^2}\frac{dM}{M} \quad (4.5.17)$$

The integral in the Eq. (4.5.18) is known as *Prandtl-Meyer function*, $\nu(M)$.

$$\nu(M) = \int \frac{\sqrt{M^2-1}}{1+\frac{\gamma-1}{2}M^2}\frac{dM}{M} = \sqrt{\frac{\gamma+1}{\gamma-1}}\left[\tan^{-1}\sqrt{\frac{\gamma-1}{\gamma+1}\left(M^2-1\right)}\right] - \tan^{-1}\sqrt{M^2-1} \quad (4.5.18)$$

Finally, Eq. (4.5.17) reduces to,

$$\theta_2 = \nu(M_2) - \nu(M_1) \quad (4.5.19)$$

Thus, for a given upstream Mach number M_1, one can obtain $\nu(M_1)$, subsequently calculate using given $\nu(M_2)$ and θ_2. Since, the expansion process is isentropic, the flow properties can be calculated from isentropic relations.

Hypersonic Flow

Introduction to Hypersonic Flow

The hypersonic flows are different from the conventional regimes of supersonic flows. As a rule of thumb, when the Mach number is greater than 5, the flow is classified as hypersonic. However, the flow does not change its feature all of a sudden during this transition process. So, the more appropriate definition of hypersonic flow would be regime of the flow where certain physical flow phenomena become more important with increase in the Mach number. One of the physical meanings may be given to the Mach number as the measure of the ordered motion of the gas to the random thermal motion of the molecules. In other words, it is the ratio of ordered energy to the random energy as given in Eq. (4.6.1).

$$M^2 = \frac{(1/2)V^2}{(1/2)a^2} = \frac{\text{Ordered kinetic energy}}{\text{Random kinetic energy}} \qquad (4.6.1)$$

In the case of hypersonic flows, it is the directed/ordered kinetic energy that dominates over the energy associated with random motion of the molecules. Now, recall the energy equation expressed in the form of flow velocity (V), speed of sound $\left(a = \sqrt{\gamma RT}\right)$ and stagnation speed of sound $\left(a_0 = \sqrt{\gamma RT_0}\right)$.

$$\frac{a_0^2}{\gamma-1} = \frac{a^2}{\gamma-1} + \frac{V^2}{2} \quad \Rightarrow \left(\frac{a}{a_0}\right)^2 + \left(\frac{\gamma-1}{2}\right)\left(\frac{V}{a_0}\right)^2 = 1 \qquad (4.6.2)$$

Eq. (4.6.2) forms an adiabatic ellipse which is obtained for steady flow energy equation. When the flow approaches the hypersonic limit, the ratio becomes $\frac{a}{a_0} \ll 1$.

Then, Eq. (4.6.2) simplifies to the following expression.

$$V^2 \approx \frac{2a_0^2}{\gamma-1} \approx \frac{2\gamma RT_0}{\gamma-1} \qquad (4.6.3)$$

In other words, the entire kinetic energy of the flow gets converted to internal energy of the flow which is a function total temperature (T_0) of the flow.

The study/research on hypersonic flows revels many exciting and unknown flow features of aerospace vehicles in the twenty-first century. The presence of special features in a hypersonic flow is highly dependent on type of trajectory, configuration of the vehicle design, mission requirement that are decided by the nature of hypersonic atmosphere encountered by the flight vehicle. Therefore, the hypersonic flight vehicles are classified in four different types, based on the design constraints imposed from mission specifications.

- Reentry vehicles (uses the rocket propulsion system)
- Cruise and acceleration vehicle (air-breathing propulsion such as ramjet/scramjet)
- Reentry vehicles (uses both air-breathing and rocket propulsion)
- Aero-assisted orbit transfer vehicle (presence of ions and plasma in the vicinity of spacecraft)

Characteristics Features of Hypersonic Flow

There are certain physical phenomena that essentially differentiate the hypersonic flows as compared to the supersonic flows. Even though, the flow is treated as supersonic, there are certain special features that appear when the speed of the flow is more than the speed of sound typically beyond the Mach number of 5. Some of these characteristics features are listed here;

Thin shock layer: It is known from oblique shock relation $(\theta-\beta-M)$ that the shock wave angle (β) decreases with increase in the Mach number (M) for weak shock solution. With progressive increase in the Mach number, the shock wave angle reaches closer to the flow deflection angle (θ) . Again, due to increase in temperature rise across the shock wave, if chemical reaction effects are included, the shock wave angle will still be smaller. Since, the distance between the body and the shock wave is small, the increase in the density across the shock wave results in very high mass fluxes squeezing through small areas. The flow region between the shock wave and the body is known as *thin shock layer* as shown in Fig. 4.6.1(a). It is the basic characteristics of hypersonic flows that shock waves lie closer to the body and shock layer is thin. Further, the shock wave merges with the thick viscous boundary layer growing from the body surface. The complexity of flow field increases due to thin shock layer where the boundary layer thickness and shock layer thickness become comparable.

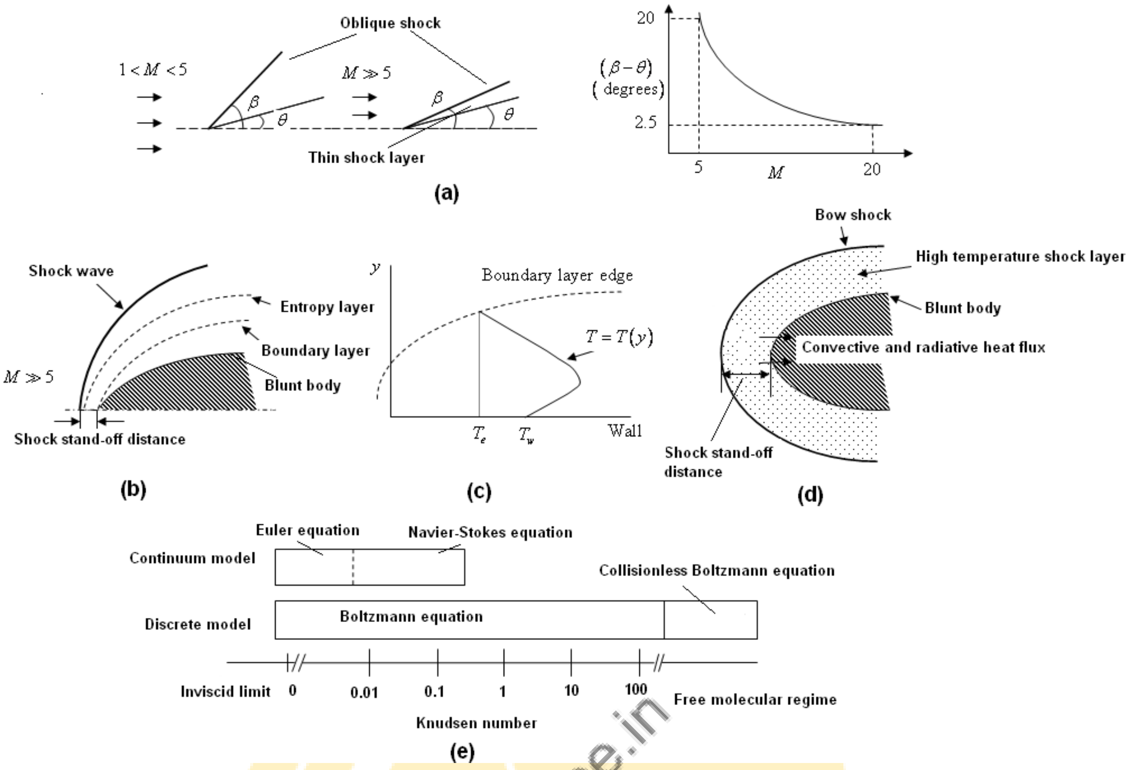


Fig. 4.6.1: Few important phenomena in a hypersonic flow: (a) Thin shock layer; (b) Entropy layer; (c) Temperature profile in a boundary layer; (d) High temperature shock layer; (e) Low density effects.

Entropy layer: The aerodynamic body configuration used in hypersonic flow environment is typically blunt to avoid thin shock layers to be closer to the body. So, there will be a detached bow shock standing at certain distance from the nose of the body and this shock wave is highly curved (Fig. 4.6.1-b). Since, the flow process across the shock is a non-isentropic phenomena, an entropy gradient is developed that varies along the distance of the body. At the nose portion of the blunt body, the bow shock resembles normal to the streamline and the centerline of the flow will experience a larger entropy gradient while all other neighboring streamlines undergo the entropy changes in the weaker portion of the shock. It results in an *entropy layer* that persists all along the body. Using the classical *Crocco's theorem*, the entropy layer may be related to vorticity. Hence, the entropy layer in high Mach number flows, exhibits strong gradient of entropy which leads to higher vorticity at higher magnitudes. Due to the presence of entropy layer, it becomes difficult to predict the boundary layer properties. This phenomenon in the hypersonic flow is called as *vortcity generation*. In addition to thin shock layer, the entropy layer also interacts with viscous boundary layer that leads to very complicated and unknown flow features.

Viscous-Inviscid interaction: When a high velocity, hypersonic flow is slowed down in the vicinity of the aerodynamic body due to viscous effects within the boundary layer, the major portion of the kinetic energy is transformed into the internal energy of the gas known as *viscous dissipation* leading to increase in temperature. For a cold wall, the typical temperature profile in a boundary layer is shown in Fig. 4.6.1(c). Since, the pressure is constant in the normal direction through the boundary layer, the increase in temperature results decrease in density. In order to pass through a given mass flux at reduced density, the thickness of the boundary layer must be larger. Thus, the displacement thickness increases, causing the body shape to appear much thicker and displacing outer inviscid flow. Hence, the free stream flow encounters an inflated object which changes the shock shape and in turn boundary layer parameters such as surface pressure, wall heat flux, skin friction etc. Again, when the boundary layer becomes thick, it essentially merges with the thin shock layer. Thus, there are major interactions of viscous boundary layer, thin shock layer and outer inviscid flows. This phenomenon is known as viscous-inviscid interaction and has important effect on the surface pressures and the stability of hypersonic vehicles.

High temperature effects: The kinetic energy of the high speed, hypersonic flow is dissipated by the effect of friction within the boundary layer (Fig. 4.6.1-d). The extreme viscous dissipation can result in substantial increase in temperature (~10000 K) exciting the vibration within the molecules and can cause dissociation, ionization in the gas. Typically, in the range of 2000K-4000K, the oxygen molecules start dissociating and with increase in temperature, dissociation of nitrogen molecules takes place. Further increase in temperature (> 9000 K), ionization of both oxygen and nitrogen can start. This leads to chemical reaction within the boundary layer. As a result, the gases within the boundary layer will have variable specific heat ratio and gas constant which are functions of both temperature and pressure. Therefore treatment of air or any fluid flowing with hypersonic speed over any configuration should be done properly by incorporating all the microscopic changes which essentially leads to change in thermodynamic properties with temperature. If the vibrational excitation and chemical reactions takes place very rapidly in comparison to time taken by the fluid element to move in the flow field, then it is called as *equilibrium flow*. When there is sufficient time lag, then it is treated as *non-*

equilibrium flow. All these phenomena are called as *high temperature real gas effects*. The presence of high temperature reacting plasma in the vicinity of the flight vehicle influence the aerodynamic parameters, aerodynamic heating and subsequently, communication is blocked. Flight parameters like pitch, roll, drag, lift, defection of control surfaces get largely deviated from their usual estimate of calorically perfect gas. The presence of hot fluid in the vicinity of vehicle surface induces heat transfer not only through convection but also through radiation. Communication waves which are necessarily radio waves get absorbed by free electrons formed from ionization of atmospheric fluid. This phenomenon is called as *communication blackout* where on board flight parameters and ground communication is lost.

Low density flow: At standard sea level conditions, all the fluids are treated as *continuum* so that the global behavior is same as that of average fluid properties. In these conditions, the fluid contains certain desired number of molecules and the average distance between two successive collisions of the molecules is specified by its *mean free path* $\left(\lambda \approx 7 \times 10^{-9}\,\mathrm{m}\right)$. Since, the hypersonic flows are encountered at very high altitude (~100 km), the density of the medium is very less and the mean free path may be in the order of 0.3m. So, the air is no longer a continuous substance, rather treated as individual and widely spaced particles in the matter. Under these conditions, all the fundamental equations based on continuum assumption break down and they are dealt with the concepts of kinetic theory. This regime of the aerodynamics is known as *low-density flows*. Further increase in altitude (~ 150 km), the air density becomes so low that only a few molecules impact on the surface per unit time. This regime of flow is known as *free molecular flow*. Thus, a hypersonic vehicle moves in different flow regimes during the course of its flight i.e. from a dense atmosphere to a rarefied atmosphere. The similarity parameter that governs different regimes of the flow for certain characteristic dimension L, is then defined as Knudsen number (Kn).

$$\mathrm{Kn} = \frac{\lambda}{L} \tag{4.6.4}$$

Large value of Kn implies free molecular flow $(\text{Kn} \to \infty)$ while small value of Kn is the regime of continuum flow $(\text{Kn} < 0.2)$ as shown in Fig. 4.6.1(e). In the inviscid limit, the value of Kn approaches to zero while the free molecular flow regime begins with $\text{Kn} = 1$. In the low density regimes, the *Boltzmann equation* is used to deal with the fundamental laws.

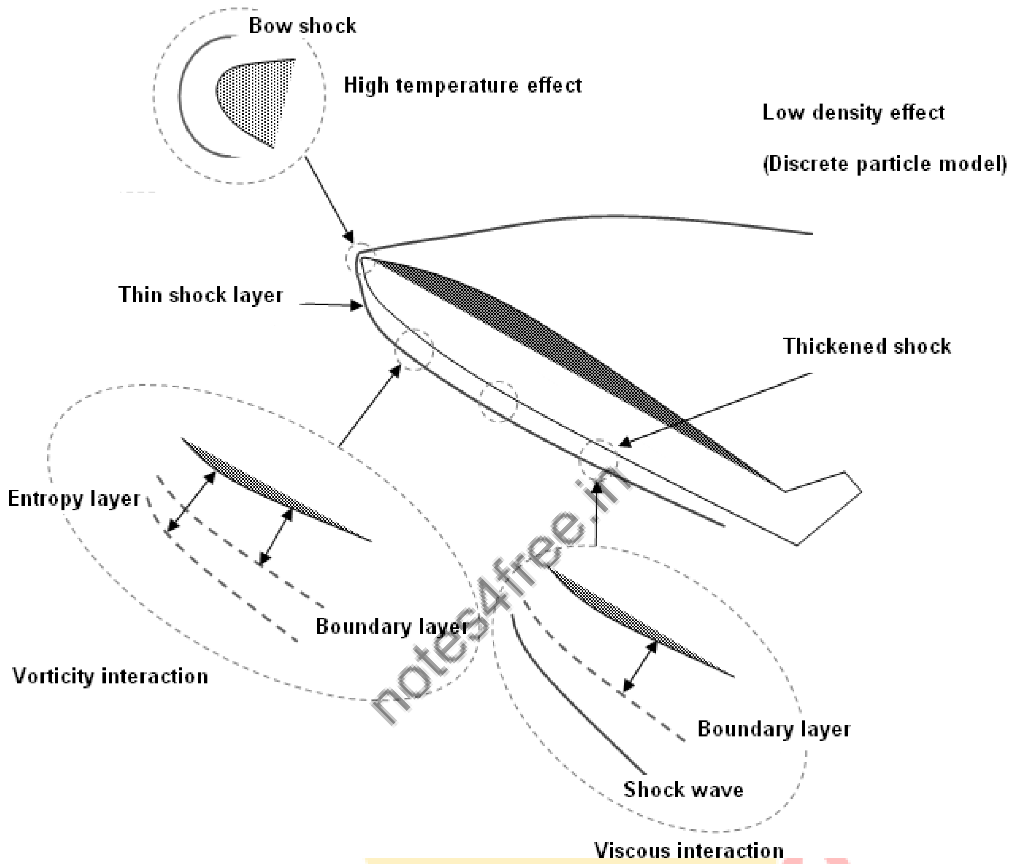


Fig. 4.6.2: Characteristics features of hypersonic flow.

From these characteristics of hypersonic flows, it is clear that Mach number to be greater than 5 is the most formal definition of hypersonic flow rather it is desired to have some of the characteristics features summarized in Fig. 4.6.2. It is more important that one of these characteristics features should appear in the flow phenomena so that the definition becomes more appropriate. There are many challenges for experimental simulation of hypersonic flow in the laboratory. Understanding the challenges faced by hypersonic flight and driving solutions these problems on case to case basic are the most research themes on hypersonic flows.

Hypersonic Flow

Inviscid Hypersonic Flow Relations

In general, the hypersonic flows are characterized with viscous boundary layers interacting the thin shock layers and entropy layers. The analysis of such flow fields is very complex flows and there are no standard solutions. In order to get some quantitative estimates, the flow field at very high Mach numbers is generally analyzed with inviscid assumption so that the mathematical complications are simplified. In conventional supersonic flows, the shock waves are usually treated as mathematical and physical discontinuities. At hypersonic speeds, some approximate forms of shock and expansion relations are obtained in the limit of high Mach numbers.

Hypersonic shock relations

Consider the flow through a straight oblique shock as shown in Fig. 4.7.1(a). The notations have their usual meaning and upstream and downstream conditions are denoted by subscripts '1' and '2', respectively. Let us revisit the exact oblique shock relations and simplify them in the limit of high Mach numbers.

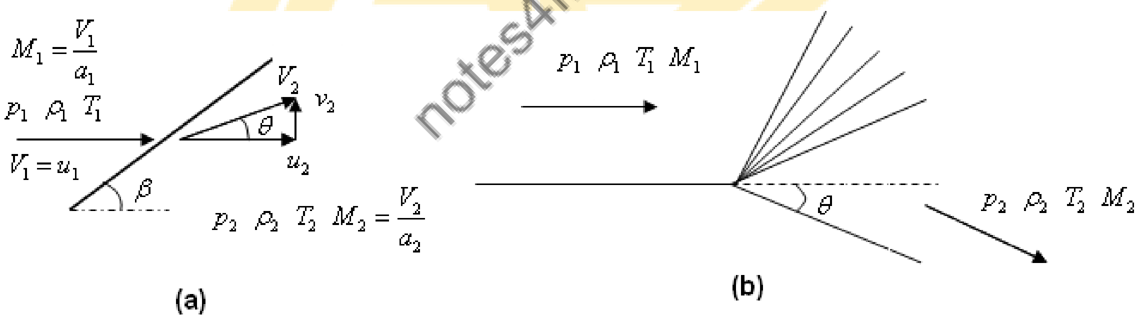


Fig. 4.7.1: Geometry of shock and expansion wave: (a) oblique shock; (b) centered expansion wave.

What is fluid flow?

Fluid flows encountered in everyday life include

meteorological phenomena (rain, wind, hurricanes, floods, fires)

environmental hazards (air pollution, transport of contaminants)

heating, ventilation and air conditioning of buildings, cars etc.

combustion in automobile engines and other propulsion systems

interaction of various objects with the surrounding air/water

complex flows in furnaces, heat exchangers, chemical reactors etc.

processes in human body (blood flow, breathing, drinking ...)

and so on and so forth

What is CFD?

Computational Fluid Dynamics (CFD) provides a qualitative (and sometimes even quantitative) prediction of fluid flows by means of

- mathematical modeling (partial differential equations)
- numerical methods (discretization and solution techniques)
- software tools (solvers, pre- and postprocessing utilities)

CFD enables scientists and engineers to perform 'numerical experiments' (i.e. computer simulations) in a 'virtual flow laboratory'

Why use CFD?

Numerical simulations of fluid flow (will) enable

Architects to design comfortable and safe living environments

designers of vehicles to improve the aerodynamic characteristics

chemical engineers to maximize the yield from their equipment

petroleum engineers to devise optimal oil recovery strategies

surgeons to cure arterial diseases (computational hemodynamics)

meteorologists to forecast the weather and warn of natural disasters

safety experts to reduce health risks from radiation and other hazards