Module 1

Stress & Strain

Objectives:

Classify the stresses into various categories and define elastic properties of materials and compute stress and strain intensities caused by applied loads in simple and compound sections and temperature changes.

Learning Structure

- Classification Of Engineering Materials
- Choice Of Selection Of Engineering Materials
- Physical Properties Of Materials
- Mechanical Properties
- Stress, Strain And Hook's Law
- Stress – Strain Relation Or Diagram For Ductile Material
- Stress – Strain Relation Or Diagram For Brittle Material
- Problems
- Elongation Of Tapering Bars Of Circular Cross Section
- Elongation Of Tapering Bars Of Rectangular Cross Section
- Elongation In Bar Due To Self-Weight
- Compound Or Composite Bars
- Temperature Stresses In A Single Bar
- Temperature Stresses In A Composite Bar
- Simple Shear Stress And Shear Strain
- Complementary Shear Stresses
- Volumetric Strain
- Bulk Modulus
- Relation Between Elastic Constants
- Exercise Problems
- Outcomes

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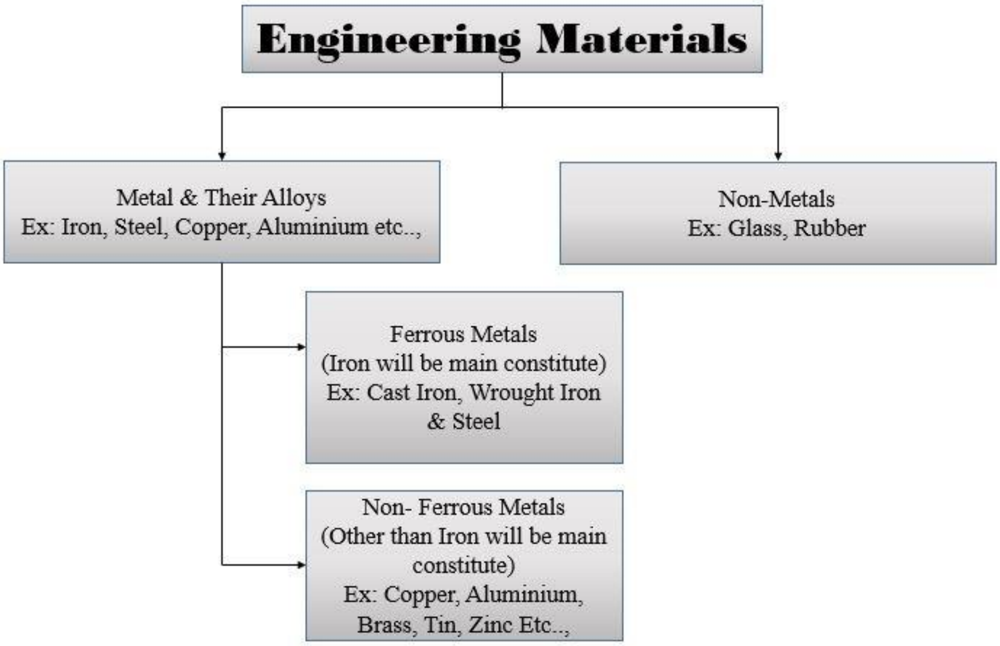
1.1 Classification of Engineering Materials



1.2 Choice of Selection of Engineering Materials

- Availability of materials.
- Sustainability of materials for the working conditions in service.
- Cost of materials.

1.3 Physical Properties of Materials

- Lustre
- Colour
- Size
- Density
- Shape

1.4 Mechanical Properties

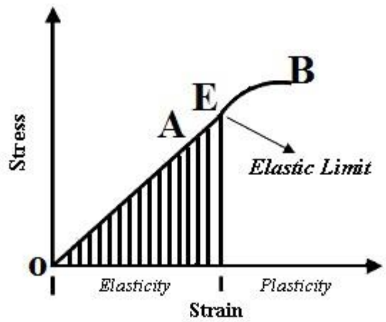
- **Load (F or P)**

  It is defined as any external force acting on a body

- **Elasticity**

  It is the property by virtue of which a material deformed under the load is enable to return to its original dimension when load is removed.

  If the body regains completely its original shape, it is said to be perfectly elastic.



In the above figure, the specimen is loaded upto point A, well within the elastic limit E. When load corresponding to point A is gradually removed the curve follows the same path AO and Strain completely disappears. Such a behaviour is known as Elastic behaviour. Steel is more elastic than rubber.

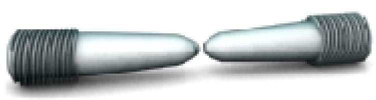
- **Plasticity**

  It is the converse of Elasticity. It is the property of a material which retains the deformation produced under the load permanently.

- **Ductility**

  It is the property of a material which exhibits large deformations in longitudinal direction under the application of tensile force before failure.

  A ductile material must be strong and plastic. The ductility is measured in terms of % elongation or % reduction in cross-sectional area of test specimen.



Ex: Mild steel, Brass, Aluminium, Nickel, Zinc, Tin, Lead etc..,

- **Brittleness**

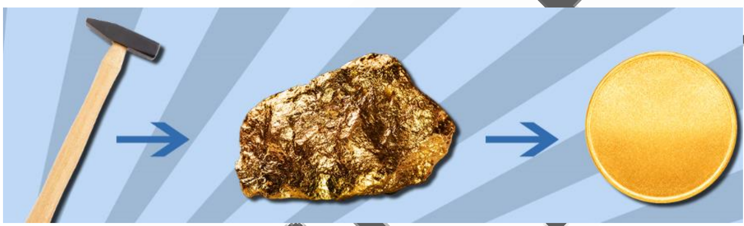
  It is the property of a material which exhibits little or no yielding before failure. Generally brittle materials are have higher strength in compression than in tension.

  Ex: Cast Iron, High carbon steel, Concrete, Stone, Glass, Ceramic materials etc..,

- **Malleability**

  It is the property of a material which permits the material to be extended in all directions without rupture.

  A malleable material possesses a high degree of plasticity but not necessarily great strength



  Ex: Gold, Lead, Soft steel, wrought iron, Copper, Aluminium, etc..,

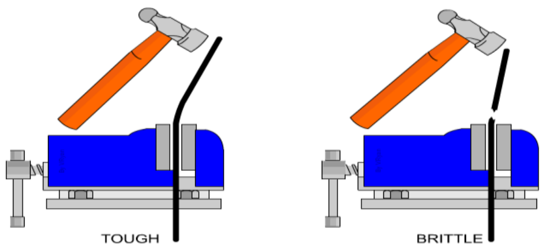
- **Strength**

  It is the ability of a material to resist the externally applied forces without breaking or yielding.

  The load required to cause fracture divided by the area of the test specimen is termed as ultimate strength of the material.

- **Toughness**

  It is the property of a material which enables it to absorb energy without fracture.

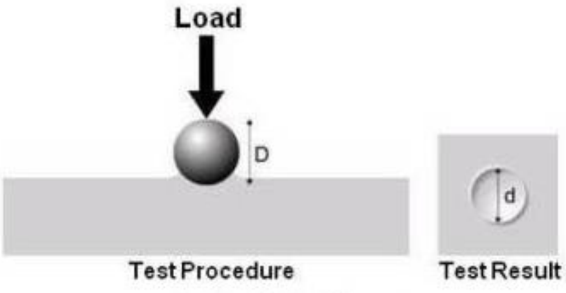


This property is desirable in parts subjected to impact and shock loads. Toughness is measured in terms of energy required per unit volume of the material to cause rupture under the action of gradually increasing tensile load.

- **Hardness**

  It is the ability of the material to resist indentation or surface abrasion.

  It embraces many different properties such as resistance to wear, scratching, deformation, machinability etc..,



- **Stiffness**

  It is the ability of a material to resist deformation under stress.

  The stiffness is measured by the modulus of elasticity in case of axially loaded members

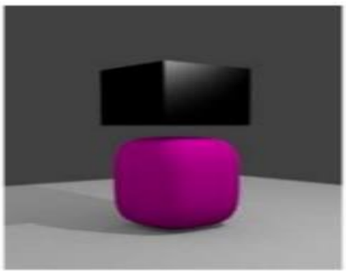
- **Creep**

  Whenever a member or part of a machine subjected to a constant stress at high temperature for a longer period of time, it will undergo a slow and permanent deformation called creep.

- **Resilience**

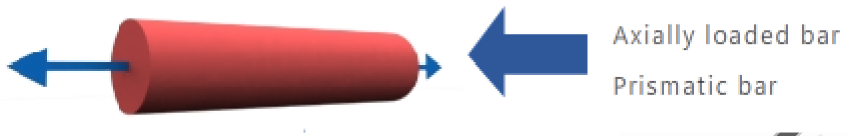
  It is the property of the material to absorb energy and to resist shock and Impact loads.

  It is measured by the amount of energy absorbed per unit volume within elastic limit,



1.5 Stress, Strain and Hook's law

The most fundamental concepts in mechanics of materials are **stress** and **strain**. These concepts can be illustrated in their most elementary form by considering a prismatic bar subjected to axial forces. A **prismatic bar** is a straight structural member having the same cross section throughout its length, and an **axial force** is a load directed along the axis of the member, resulting in either tension or compression in the bar.



1.5.1 Stress

When a body is acted upon by external force F, or Load P, internal resisting force is setup in the body such a body is said to be in state of stress, hence the resistance offered by the body against deformation due to the application of load is called as stress.

Or

The Internal resisting force per unit area at any section of the body is known as Stress

It is denoted by σ (Sigma),

$$\text{Stress } \sigma = \frac{\text{Applied Load or Force}}{\text{Cross-sectional Area}} = \frac{\text{F or P}}{\text{A}} \frac{\text{N}}{\text{mm}^2}$$

In general, the stresses s acting on a plane surface may be uniform throughout the area or may vary in intensity from one point to another.

1.5.1.1 Types of Stresses

1) Normal Stress
   a) Tensile Stress
   b) Compressive Stress
2) Shear Stress
3) Bearing Stress

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1. Normal Stress

A normal stress is a stress that occurs when a member is loaded by an axial force. (Axial force is the force acting along the axis of the specimen).

Normal stress can be either tensile or compressive in nature.

a) Tensile stress

When a load is acting in such a way that it tends to extend the material in the direction of application of load is called tensile load and the corresponding stress is called tensile stress.

$$\text{Tensile stress, } \sigma = \frac{\text{P or F}}{\text{A}} \frac{\text{N}}{\text{mm}^2}$$

b) Compressive stress

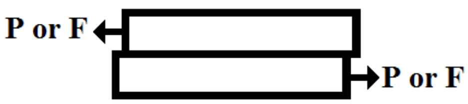
When a load is acting in such a way that it tends to shorten the material in the direction of application of load is called compressive load and the corresponding stress is called compressive stress.

$$\text{Compressive stress, } \sigma = \frac{\text{P or F}}{\text{A}} \frac{\text{N}}{\text{mm}^2}$$

When a **sign convention** for normal stresses is required, it is customary to define tensile stresses as positive and compressive stresses as negative.

2. Shear Stress

Shearing stress is a force that causes two contacting parts or layers to slide upon each other in opposite directions. The stress developed at the contacting surfaces is known as shear stress.



$$\text{Shear Stress, } \tau = \frac{\text{Shearing Force}}{\text{Shearing Area}} = \frac{\text{P or F}}{\text{A}} \frac{\text{N}}{\text{mm}^2}$$

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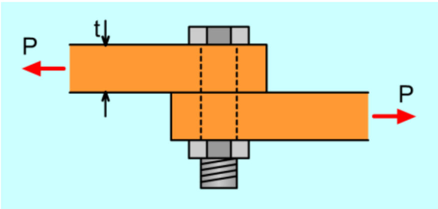
3. Bearing Stress

A Localised compressive stress at the surface of contact between two members of a machine part that are relatively at rest is known as Bearing stress or crushing stress.



$$\text{Bearing Stress} = \frac{P}{A} = \frac{P}{td}\,\frac{N}{mm^2}$$

Where,

t = Thickness of Plate

d = Diameter of the bolt

1.5.2 Strain

When a body is subjected to some external force there is some change in dimensions of the body.

The ratio of change in dimensions of the body to the original dimensions is known as Strain (ε)

$$\text{Strain } \varepsilon = \frac{\text{Change in Dimension}}{\text{Original Dimension}}$$

Strain is dimensionless

1.5.2.1 Types of Strain

1) Linear Strain
   a) Tensile Strain
   b) Compressive Strain
2) Lateral Strain
3) Shear Strain
4) Volumetric Strain

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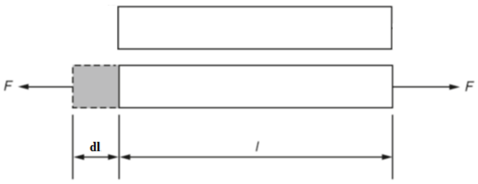
1. Linear Strain

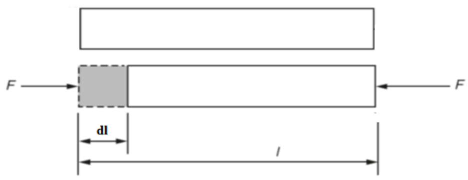
A straight bar will change in length when loaded axially, becoming longer when in tension and shorter when in compression. This change in dimensions in axial direction is known as Linear Strain.

Tensile Strain,



$$\text{Tensile Strain } \varepsilon = \frac{\text{Change in length (Extension)}}{\text{Original length}} = \frac{dl}{l} = \frac{\delta l}{l}$$

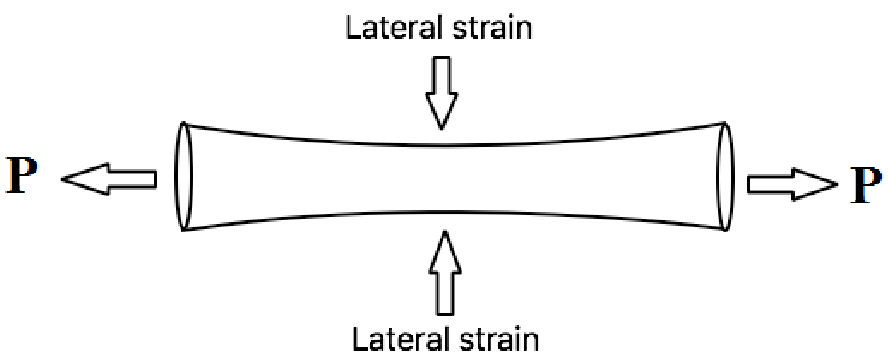
Compressive Strain,



$$\text{Compressive Strain } \varepsilon = \frac{\text{Change in Length (Reduction)}}{\text{Original Length}} = \frac{dl}{l} = \frac{\delta l}{l}$$

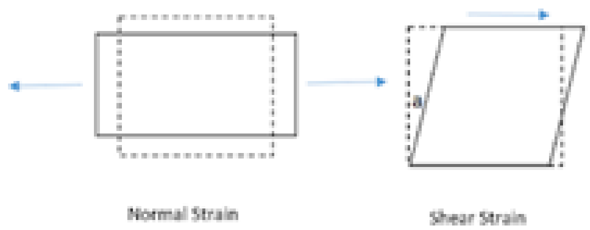
2. Lateral Strain

Lateral strain, also known as transverse strain, which takes place at right angles to the direction of applied load is known as lateral strain.



3. Shear Strain

Shear strain is the ratio of deformation to original dimensions. In the case of shear strain, it is the amount of deformation perpendicular to a given line rather than parallel to it.



4. Volumetric Strain

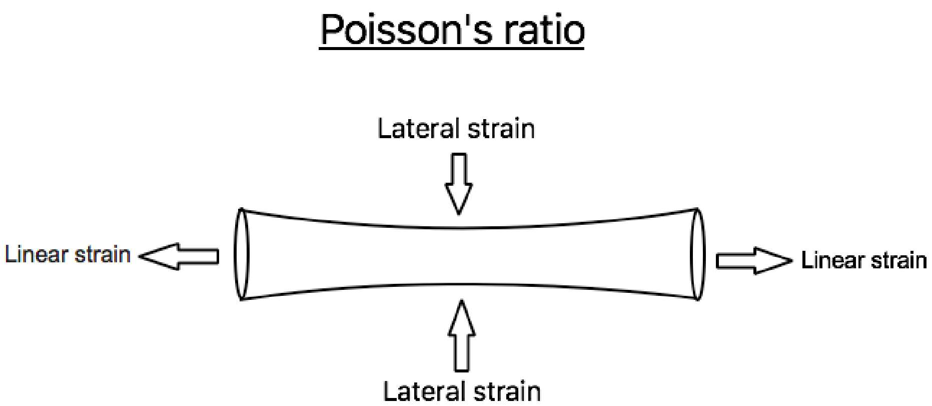
It is the ratio of change in volume to its original volume

$$\text{Volumetric Strain, } \varepsilon_v = \frac{\delta v}{v}$$

1.5.3 Poison's ratio

It is the ratio of lateral strain to linear strain

$$\text{Poison's ratio } \mu = \frac{\text{Lateral Strain}}{\text{Linear Strain}}$$



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1.5.4 Hook's Law

It states that **"When a material is loaded within its elastic limit, stress is directly proportional to the strain"**

$$\text{Stress } \alpha \text{ Strain}$$

$$\text{i.e } \frac{\text{Stress}}{\text{Strain}} = \text{Constant}$$

$$\text{i.e } \frac{\sigma}{\varepsilon} = E$$

Where,

E = A constant of proportionality known as Modulus of Elasticity E

σ = Stress & ε = Strain

Hook's law holds good for tension as well as compression.

1.5.5 Modulus of Elasticity or Young's Modulus (E)

Modulus of Elasticity or Young's Modulus (E) is the constant of proportionality and is defined as the ratio of linear stress to linear strain within elastic limit.

$$\text{Modulus of Elasticity, } E = \frac{\text{Linear stress (Tensile or Compressive)}}{\text{Linear Strain (Tensile or Compressive)}} = \frac{\sigma}{\varepsilon}$$

$$\therefore E = \frac{\sigma}{\varepsilon} \text{ MPa or GPa}$$

1.5.6 Factor of Safety (FOS)

It is defined as the ratio of ultimate stress or yield stress to the working or allowable or design stress

$$\text{FOS} = \frac{\text{Ultimate or Yield Stress}}{\text{Working or Allowable or Design Stress}}$$

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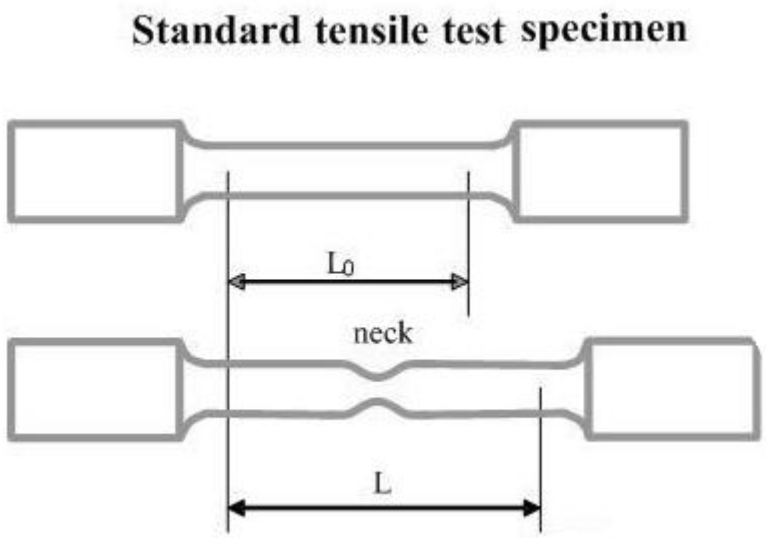
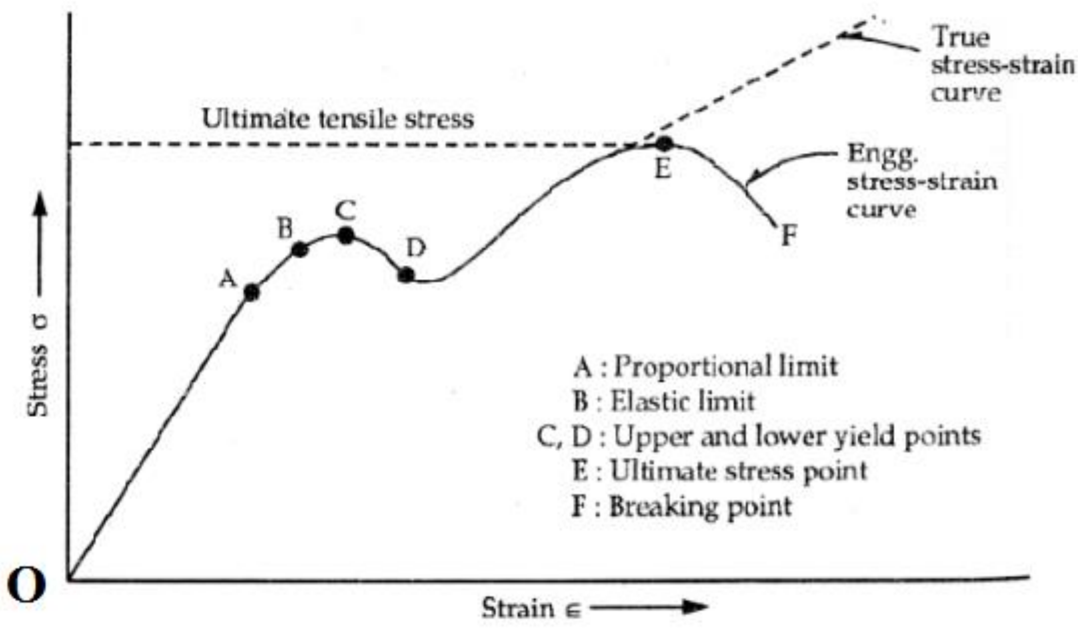
1.6 Stress – Strain Relation or Diagram for Ductile Material (Mild Steel or Low carbon steel)

A stress-strain diagram for a typical structural steel as a specimen in tension is shown in Figure. Strains are plotted on the horizontal axis and stresses on the vertical axis.



The load on the test specimen is increased gradually from zero in suitable increments till the specimen fails and the corresponding graph will be computed as shown in the figure below.



Proportional Limit (A)

From O to A the curve is straight and linear and hence proportional limit is the limiting value of stress upto which stress is directly proportional to strain and hence Hooke's law holds good upto point A.

Stress α Strain

Elastic Limit (B)

The point B is slightly beyond point A and is known as Elastic limit. Upto point B, the material will regain its original size and shape when load is removed. This indicates that the material has elastic properties upto point B.

Upper Yield Point (C)

If the material is stressed beyond point B, plastic deformation starts and the material does not regain its original size and shape upon unload and this phenomenon is called as **Yielding**.

A point at which Maximum load or stress required to initiate the plastic deformation or yielding of the material is called as **Upper yield point "C"**. At this point the dislocations or slip in the crystalline structure starts moving.

Lower Yield Point (D)

As the dislocations or slip is taking place in the material, it offers less resistance to the material and hence curve falls slightly.

A point at which minimum load or stress required to maintain the plastic deformation or yielding of the material is called as **Lower yield point "D"** and this point depicts the end of plastic deformation of the material.

Dislocations or slip become too much in number and they restrict each other's movement.

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Ultimate Stress point (E)

After Lower Yield point D, Strain Hardening in the materials takes place. **Strain hardening**, also known as **work hardening**, is the strengthening of a metal occurs because of dislocation movements within the crystal structure of the material and hence there is a positive rise in curve from D to E. In this region as stress increases strain also increases

At point E the specimen takes maximum load, and the corresponding stress at point E is called the **ultimate stress point "E"**.

Breaking Point (F)

Beyond the ultimate stress point is reached **Necking** takes place and the cross sectional area considerably decreases, the load carrying capacity of the specimen reduces and hence in the portion E to F the strain increases with decrease in stress. At point F the specimen breaks. The stress at this point is called breaking stress or fracture stress.

1.7 True Stress - Strain and Engineering Stress - Strain

Let P be the load, A_o be the original area of Cross-section, A be the area of cross-section at any instant.

Engineering stress is the applied load divided by the original cross-sectional area of a material. Also known as nominal stress.

$$\text{Engineering Stress } \sigma = \frac{\text{Load}}{\text{Original Area of Cross−section}} = \frac{P}{A_o}\ \frac{N}{mm^2}$$

True stress is the applied load divided by the actual cross-sectional area (the changing area with respect to time) of the specimen at that load

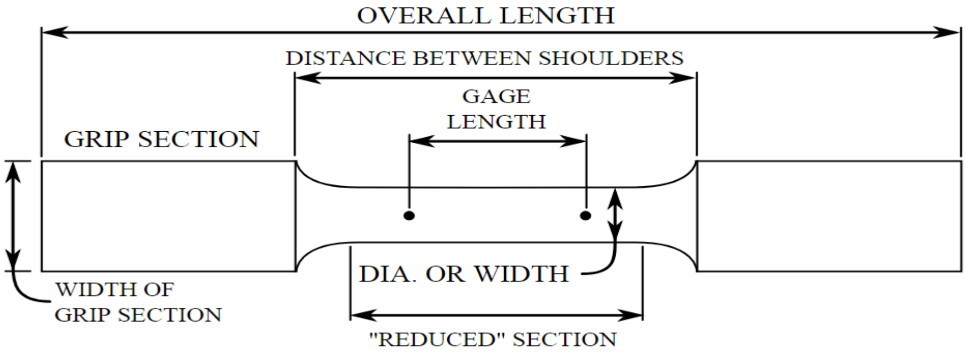
$$\text{True Stress } \sigma = \frac{\text{Load}}{\text{Actual Area of Cross−section at any instant}} = \frac{P}{A}\ \frac{N}{mm^2}$$

Engineering strain is the change in length to its original length in a tensile test. Also known as nominal strain.

$$\text{Engineering Strain } \varepsilon = \frac{\delta l}{l}$$

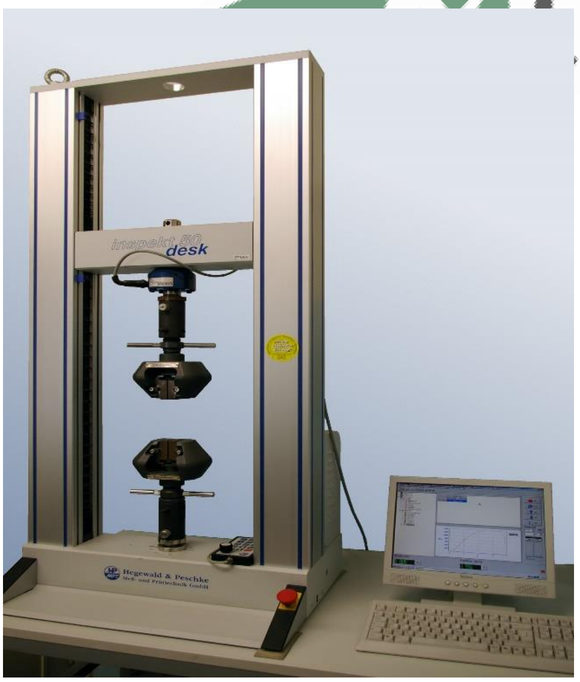
True strain is the sum of all the strains over the original length. True Strain $\varepsilon = \sum \frac{\delta l}{l}$

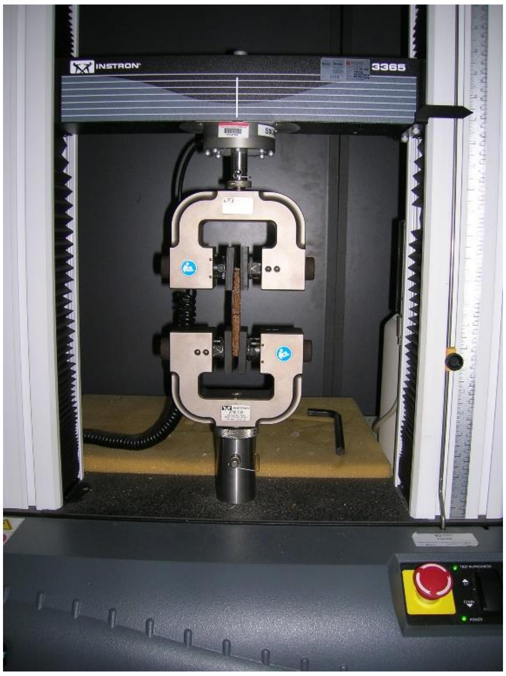
Additional Information – Photo Gallery



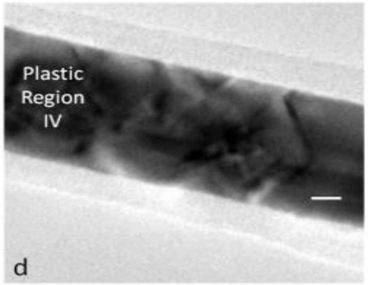
Detailed information for specimen preparation

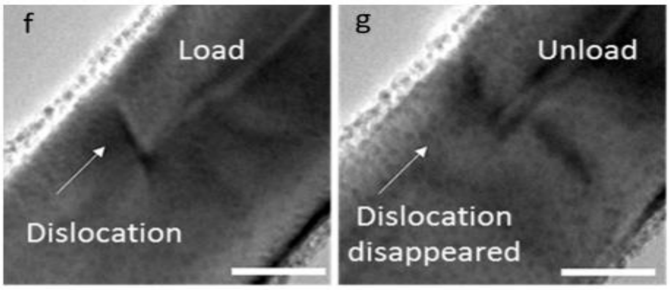
Various Specimens





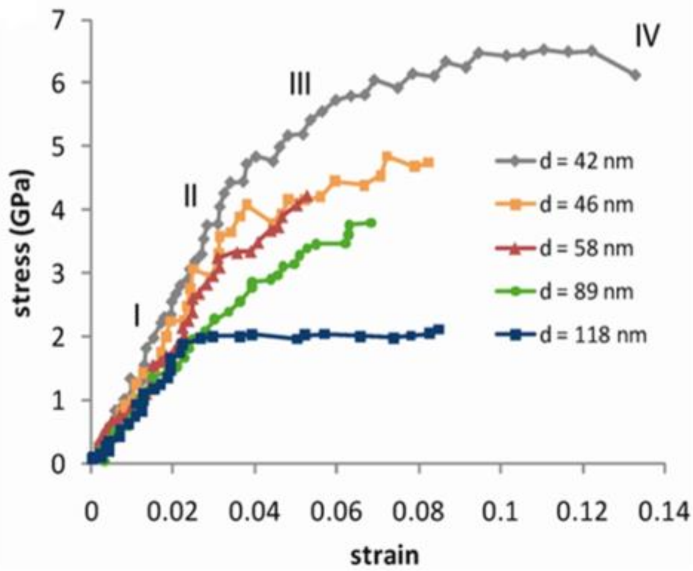
UTM for testing and Computer to plot Stress-Strain curve.





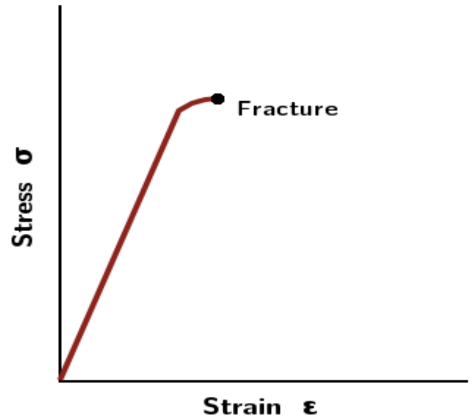
Location of plastic region Location of Dislocation upon Loading and Unloading

Necking in Tensile specimen



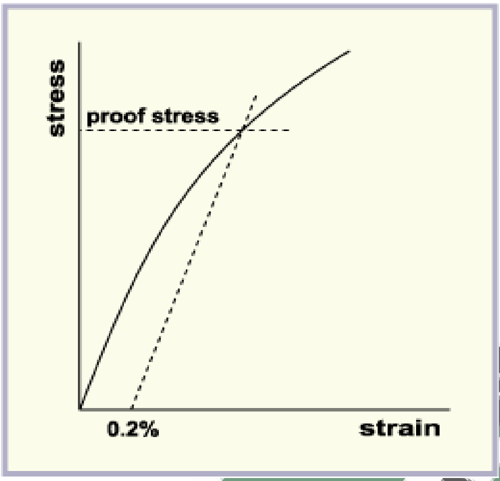
Specimens undergoing test Actual Stress-Strain curve

1.7 Stress – Strain Relation or Diagram for Brittle Material



Brittle materials, which includes cast iron, glass, and stone do not have a yield point, and do not strain-harden. Therefore, the ultimate strength and breaking strength are the same. A typical stress–strain curve is shown in the figure.

Proof Stress:



For materials which do not have clearly defined yield point, an arbitrary yield point is defined by drawing a line which is offset by a certain strain value and is parallel to the original stress-strain line (within proportional limit). The strain by which line is offset can be 0.1% or 0.2% and the corresponding stress is the Proof Stress at 0.1% or 0.2% strain respectively.

Additional formulas to remember:

Percentage Elongation in Length,

$$\% \text{ Elongation} = \frac{\text{Final Length} - \text{Initial Length}}{\text{Initial Length}} \text{ x } 100$$

$$\% \text{ Elongation} = \frac{L_f - L_i}{L_i} \text{ x } 100$$

Percentage Reduction in Area,

$$\% \text{ Reduction} = \frac{\text{Initial Area} - \text{Final Area}}{\text{Initial Area}} \text{ x } 100$$

$$\% \text{ Reduction} = \frac{A_i - A_f}{A_i} \text{ x } 100$$

1.8 Problems

1. The following data refer to a mild steel specimen tested in a laboratory

- **Diameter of the specimen - 25mm**
- **Length of the specimen – 300 mm**
- **Extension under a load of 15kN – 0.045 mm**
- **Load at yield point – 127.65kN**
- **Maximum load – 208.60kN**
- **Length of the specimen after failure – 375mm**
- **Neck diameter – 17.75mm**

Determine Young's modulus, Yield stress, Ultimate stress, % Elongation, %Reduction in area, safe or permissible stress adopting a factor of safety 2

Given Data:

d_o = 25mm, d_f = 17.75mm, L_o = 300mm, L_f = 375mm, δl = 0.045mm, F_{max} = 208.60kN

Area of the specimen, $A = \frac{\pi {d_o}^2}{4} = \frac{\pi x 25^2}{4} = \mathbf{490.87\ mm^2}$

$\sigma = \frac{F}{A} = \frac{15x10^3}{490.87} = \mathbf{30.55\ \frac{N}{mm^2}}$

$\varepsilon = \frac{\delta l}{l} = \frac{0.045}{300} = \mathbf{1.5x10^{-4}}$

→Young's Modulus $E = \frac{\sigma}{\varepsilon} = \frac{30.55}{1.5x10^{-4}} = \mathbf{203.66x10^3\ \frac{N}{mm^2}}$

⟶ Yield stress $\sigma_y = \frac{F}{A} = \frac{127.65x10^3}{490.87} = \mathbf{260.04\ \frac{N}{mm^2}}$

⟶ Ultimate stress $\sigma_u = \frac{F}{A} = \frac{208.60x10^3}{490.87} = \mathbf{424.95\ \frac{N}{mm^2}}$

⟶ % Elongation = $\frac{L_f - L_i}{L_i}$ x 100 = $\frac{375-300}{300}$ x100 = **25%**

→% Reduction = $\frac{A_i - A_f}{A_i}$ x 100

$A_f = \frac{\pi {d_f}^2}{4} = \frac{\pi x 17.75^2}{4} = \mathbf{247.44\ mm^2}$

∴ $\frac{490.87-247.44}{490.87}$ x100 = **49.59 %**

→ FOS = $\frac{\text{Ultimate or Yield Stress}}{\text{Working or Allowable or Design Stress}}$

Note:

In question they have asked for safe permissible stress hence take yield stress for calculation

For Maximum permissible stress take ultimate stress for calculation

∴ FOS = $\frac{\text{Yield Stress}}{\text{Working or Allowable or Design Stress}}$

Wkt, FOS = 2

$2 = \frac{260.04}{\text{Working or Allowable or Design Stress}}$

Working stress or Design stress $= \frac{260.04}{2} = \mathbf{130.02\ \frac{N}{mm^2}}$

2. A rod 150 cm long and a diameter 2 cm is subjected to an axial pull of 20kN. If the modulus of elasticity of material is 200 GPa. Determine stress, strain and Elongation of rod.

Given data:

l = 150 cm = 1500 mm

d = 2 cm = 20 mm

P = 20kN = $20\text{x}10^3$ N

E = 200 GPa = $200\text{x}10^3 \frac{N}{mm^2}$

Area = $\frac{\pi d^2}{4} = \frac{\pi 20^2}{4}$ = **314.15 mm²**

→ $\sigma = \frac{F}{A} = \frac{20x10^3}{314.15} = \mathbf{63.66\ \frac{N}{mm^2}}$

→ $E = \frac{\sigma}{\varepsilon}$ ⟶ $\varepsilon = \frac{\sigma}{E} = \frac{63.66}{200\text{x}10^3}$ = **3.183x10⁻⁴**

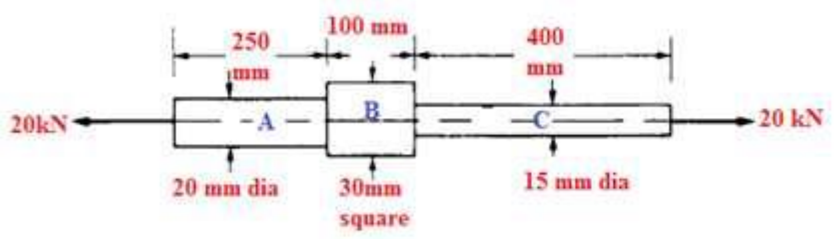
→ $\varepsilon = \frac{\delta l}{l}$ ⟶ $\delta l = \varepsilon \text{ x } l = 3.183\text{x}10^{-4} \text{ x } 1500$

δl = 0.477 mm

3. A bar of a rectangular section of 20 mm × 30 mm and a length of 500 mm is subjected to an axial compressive load of 60 kN. If E = 102 kN/mm2 and v = 0.34, determine the changes in the length and the sides of the bar.

- Since the bar is subjected to compression, there will be decrease in length, increase in breadth and depth. These are computed as shown below
- L = 500 mm, b = 20 mm, d = 30 mm, P = 60 x1000 = 60000 N, E = 102000 N/mm2
- Cross-sectional area A = 20 × 30 = 600 mm2
- Compressive stress σ = P/A = 60000/600 = 100 N/mm2
- Longitudinal strain $\varepsilon_L = \sigma/E$ = 100/102000 = 0.00098
- Lateral strain $\varepsilon_{lat} = \nu\ \varepsilon_L$ = 0.34 x 0.00098 = 0.00033
- Decrease in length $\delta L = \varepsilon_L L$ = 0.00098 x 500 = 0.49 mm
- Increase in breadth $\delta b = \varepsilon_{lat}\, b$ = 0.00033 x 20 = 0.0066 mm
- Increase in depth $\delta d = \varepsilon_{lat}\, d$ = 0.00033 x 30 = 0.0099 mm

4. Determine the stress in each section of the bar shown in the following figure when subjected to an axial tensile load of 20 kN. The central section is of square cross-section; the other portions are of circular section. What will be the total extension of the bar? For the bar material E = 210000MPa.



The bar consists of three sections with change in diameter. Loads are applied only at the ends. The stress and deformation in each section of the bar are computed separately. The total extension of the bar is then obtained as the sum of extensions of all the three sections. These are illustrated in the following steps.

The bar is in equilibrium under the action of applied forces

Stress in each section of bar = P/A and P = 20000N

NOTES4FREE in one Solution

i. Area of Bar A = π x 202/4 = 314.16 mm2

ii. Stress in Bar A : σA = 20000/ 314.16 = 63.66MPa

iii. Area of Bar B = 30 x30 = 900 mm2

iv. Stress in Bar B : σ_B = 20000/ 900 = 22.22MPa

v. Area of Bar C = π x 15_2/4 = 176.715 mm_2

vi. Stress in Bar C : σ_C = 20000/ 176.715 = 113.18MPa

Extension of each section of bar = σL/E and E = 210000 MPa

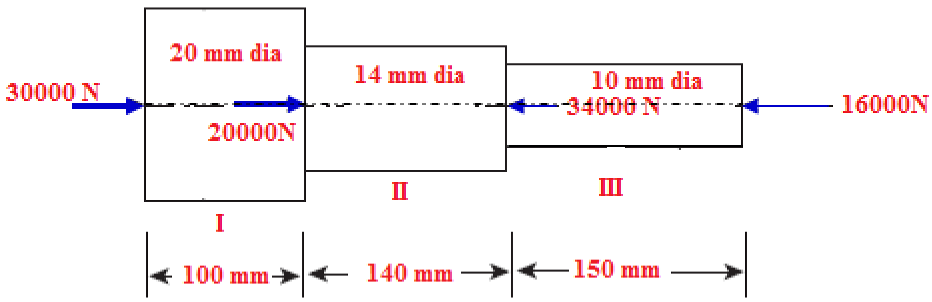
i. Extension of Bar A = 63.66 x 250 / 210000= 0.0758 mm

ii. Extension of Bar B = 22.22 x 100 / 210000= 0.0106 mm

iii. Extension of Bar C = 113.18 x 400 / 210000= 0.2155 mm

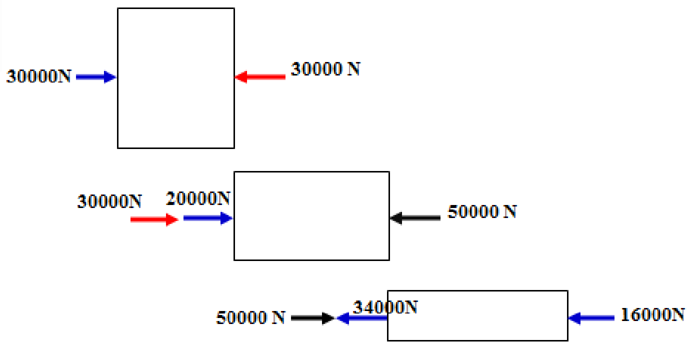
Total extension of the bar = **0. 302mm**

5. Determine the overall change in length of the bar shown in the figure below with following data: E = 100000 N/mm_2



The bar is with varying cross-sections and subjected to forces at ends as well as at other interior locations. It is necessary to study the equilibrium of each portion separately and compute the change in length in each portion. The total change in length of the bar is then obtained as the sum of extensions of all the three sections as shown below.

Forces acting on each portion of the bar for equilibrium



Sectional Areas

$A_I = \frac{\pi \times 20^2}{4} = 314.16\ mm^2$; $A_{II} = \frac{\pi \times 14^2}{4} = 153.94\ mm^2$; $A_{III} = \frac{\pi \times 10^2}{4} = 78.54\ mm^2$

Change in length in Portion I

Portion I of the bar is subjected to an axial compression of 30000N. This results in *decrease* in length which can be computed as

$$\delta L_I = \frac{P_I L_I}{A_I E} = \frac{30000 \times 100}{314.16 \times 100000} = 0.096\ mm$$

Change in length in Portion II

Portion II of the bar is subjected to an axial compression of 50000N (30000 + 20000). This results in *decrease* in length which can be computed as

$$\delta L_I = \frac{P_{II} L_{II}}{A_{II} E} = \frac{50000 \times 140}{153.94 \times 100000} = 0.455mm$$

Change in length in Portion III

Portion III of the bar is subjected to an axial compression of (50000 – 34000) = 16000N. This results in *decrease* in length which can be computed as

$$\delta L_I = \frac{P_{III} L_{III}}{A_{III} E} = \frac{16000 \times 150}{78.54 \times 100000} = 0.306mm$$

Since each portion of the bar results in decrease in length, they can be added without any algebraic signs.

Hence Total decrease in length = 0.096 + 0.455 + 0.306 = **0.857mm**

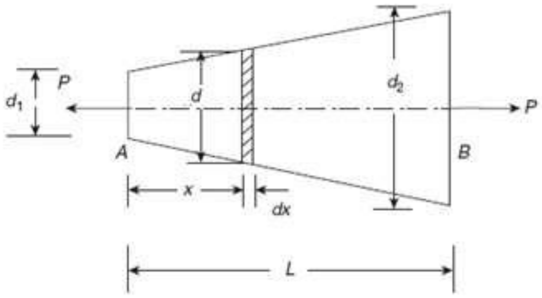
Note:

For equilibrium, if some portion of the bar may be subjected to tension and some other portion to compression resulting in increase or decrease in length in different portions of the bar. In such cases, the total change in length is computed as the sum of change in length of each portion of the bar with proper algebraic signs. Generally positive sign (+) is used for increase in length and negative sign (-) for decrease in length.

NOTES AK in One

1.9 Elongation of tapering bars of circular cross section

Consider a circular bar uniformly tapered from diameter **d1** at one end and gradually increasing to diameter **d2** at the other end over an axial length **L** as shown in the figure below.



Since the diameter of the bar is continuously changing, the elongation is first computed over an elementary length and then integrated over the entire length. Consider an elementary strip of diameter ***d*** and length ***dx*** at a distance of ***x*** from end ***A***.

Using the principle of similar triangles the following equation for d can be obtained

$$d = d_1 + \frac{d_2 - d_1}{L} x = d_1 + kx \text{ , } where\ k = \frac{d_2 - d_1}{L}$$

Cross–sectional area of the bar at x : $A_x = \frac{\pi\,(d_1+kx)^2}{4}$

Axial stress at x: $\sigma_x = \frac{P}{A_x} = \frac{4P}{\pi\,(d_1+kx)^2}$

Change in length over dx : $\delta dx = \frac{\sigma_x\,dx}{E} = \frac{4P\,dx}{\pi E\,(d_1+kx)^2}$

Total change in length: $\delta L = \int_0^L \frac{4P\,dx}{\pi E\,(d_1+kx)^2} = \frac{4P}{\pi E}\left[\frac{(d_1+kx)^{-1}}{-k}\right]_0^L$

After rearranging the terms: $\delta L = -\frac{4P}{\pi E k}\left[\frac{1}{(d_1+kx)}\right]_0^L$

Upon substituting the limits : $\delta L = -\frac{4P}{\pi E k}\left[\frac{1}{(d_1+kL)} - \frac{1}{d_1}\right]$

After rearranging the terms: $\delta L = \frac{4P}{\pi E k}\left[\frac{1}{d_1} - \frac{1}{(d_1+kL)}\right]$

But $(d_1 + kL) = d_1 + \frac{d_2 - d_1}{L} L = d_2$

With the above substitution: $\delta L = \frac{4P}{\pi E k}\left[\frac{1}{d_1} - \frac{1}{d_2}\right] = \frac{4P}{\pi E k}\left[\frac{d_2 - d_1}{d_1 d_2}\right]$

Substituting for $k = \frac{d_2 - d_1}{L}$ in the above expression, following equation for elongation of tapering bar of circular section can be obtained

Total change in length: $\delta L = \frac{4P\,L}{\pi E\,d_1 d_2}$

Problem:

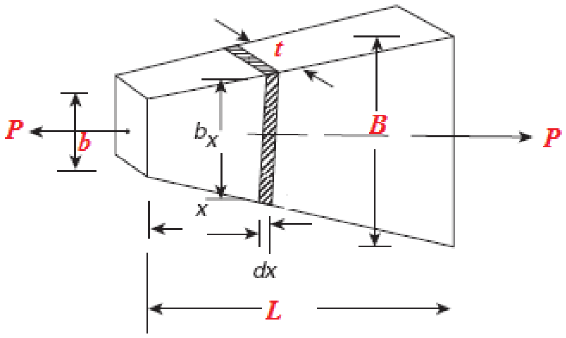
A bar uniformly tapers from diameter 20 mm at one end to diameter 10 mm at the other end over an axial length 300 mm. This is subjected to an axial compressive load of 7.5 kN. If *E* = 100 kN/mm2, determine the maximum and minimum axial stresses in bar and the total change in length of the bar.

P = 7500 N, E = 100000 N/mm^2, d_1 = 10mm, d_2 = 20mm, L = 300mm

- Minimum compressive stress occurs at d_2 = 20mm as the sectional area is maximum.
- Area at $d_2 = \frac{\pi \times 20^2}{4} = 314.16mm^2$
- $\sigma_{min} = \frac{7500}{314.16} = 23.87MPa$
- Maximum compressive stress occurs at d_1 = 10mm as the sectional area is minimum.
- Area at $d_1 = \frac{\pi \times 10^2}{4} = 78.54mm^2$
- $\sigma_{min} = \frac{7500}{78.54} = 95.5MPa$
- Total decrease in length: $\delta L = \frac{4P\,L}{\pi E\, d_1 d_2} = \frac{4 \times 7500 \times 300}{\pi \times 100000 \times 10 \times 20} = 0.143\text{mm}$

1.10 Elongation of tapering bars of rectangular cross section

Consider a bar of same thickness **t** throughout its length, tapering uniformly from a breadth **B** at one end to a breadth **b** at the other end over an axial length **L**. The flat is subjected to an axial force **P** as shown in the figure below.



Since the breadth of the bar is continuously changing, the elongation is first computed over an elementary length and then integrated over the entire length. Consider an elementary strip of breadth ***bx*** and length ***dx*** at a distance of ***x*** from left end.

Using the principle of similar triangles the following equation for *bx* can be obtained

S4FREE Solution

$$b_x = b + \frac{B-b}{L}x = b + kx \text{ , where } k = \frac{B-b}{L}$$

Cross–sectional area of the bar at x : $A_x = b_x t = (b+kx)t$

Axial stress at x: $\sigma_x = \frac{P}{A_x} = \frac{P}{(b+kx)t}$

Change in length over dx : $\delta dx = \frac{\sigma_x\, dx}{E} = \frac{P\, dx}{Et(b+kx)}$

Total change in length: $\delta L = \int_0^L \frac{P\, dx}{Et(b+kx)} = \frac{P}{Etk}[ln(b+kx)]_0^L$

Upon substituting the limits : $\delta L = \frac{P}{Etk}[ln(b+kL) - \ln(b)]$

$But\ (b+kL) = b + \frac{B-b}{L}L = B$

With the above substitution: $\delta L = \frac{P}{Etk}[ln(B) - \ln(b)] = \frac{P}{Etk}\ln(B/b)$

Substituting for $k = \frac{B-b}{L}$ in the above expression, following equation for elongation of tapering bar of rectangular section can be obtained

$$\delta L = \frac{P\,L}{Et(B-b)}\ln(B/b)$$

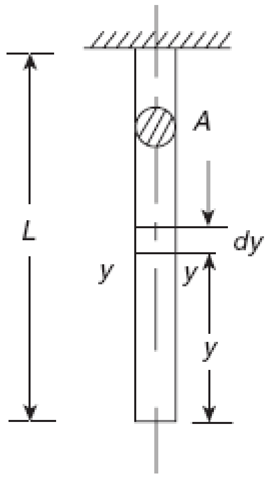
Problem

An aluminium flat of a thickness of 8 mm and an axial length of 500 mm has a width of 15 mm tapering to 25 mm over the total length. It is subjected to an axial compressive force *P*, so that the total change in the length of flat does not exceed 0.25 mm. What is the magnitude of *P*, if *E* = 67,000 N/mm2 for aluminium?

t = 8mm, B = 25mm,b = 15mm, L = 500 mm, δL = 0.25 mm, E = 67000MPa, P =?

$$P = \frac{Et(B-b)\delta L}{ln(B/b)\,L} = \frac{67000\ \times\ 8\ \times(25-15)\times 0.25}{ln(25/15)\times 500} = 5.246kN$$

1.11 Elongation in Bar Due to Self-Weight



Consider a bar of a cross-sectional area of **A** and a length **L** is suspended vertically with its upper end rigidly fixed as shown in the adjoining figure. Let the weight density of the bar is ρ. Consider a section y- y at a distance y from the lower end.

Weight of the portion of the bar below y-y = ρ A y

Stress at y-y : σy = ρ A y /A = ρ y

Strain at y-y : εy = ρ y / E

Change in length over dy: δdy = ρ y dy / E

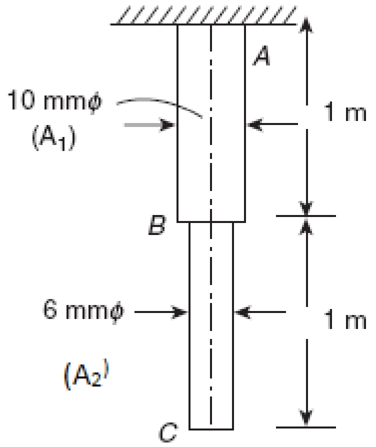
$$\text{Total change in length : } \delta L = \int_0^L \frac{\rho\, y\, dy}{E} = [\frac{\rho y^2}{2E}]_0^L = \frac{\rho L^2}{2E}$$

$$\text{This can also be written as : } \delta L = \frac{(\rho AL)L}{2AE} = \frac{WL}{2AE}$$

W = ρ A L represents the total weight of the bar

Problem:

A stepped steel bar is suspended vertically. The diameter in the upper half portion is 10 mm, while the diameter in the lower half portion is 6 mm. What are the stresses due to self-weight in sections B and A as shown in the figure. E = 200 kN/mm^2. Weight density, ρ = 0.7644x10^{-3} N/mm^3. What is the change in its length if E = 200000 MPa?



Stress at B will be due to weight of portion of the bar BC

Sectional area of BC: $A_2 = \pi \times 6^2/4 = 28.27$ mm^2

Weight of portion BC: $W_2 = \rho A_2 L_2 = 0.7644 \times 10^{-3} \times 28.27 \times 1000 = 21.61$N

Stress at B: $\sigma_B = W_2/A_2 = 21.61/28.27 =$ **0.764 MPa**

Stress at A will be due to weight of portion of the bar BC + AB

Sectional area of AB: $A_1 = \pi \times 10^2/4 = 78.54$ mm^2

Weight of portion AB: $W_1 = \rho A_1 L_1 = 0.7644 \times 10^{-3} \times 78.54 \times 1000 = 60.04$N

Stress at A: $\sigma_c = (W_1+W_2)/A_1 = (60.04+ 21.61) / 78.54 =$ **1.04 MPa**

Change in Length in portion BC

This is caused due to weight of BC and is computed as:

$$\delta L_{BC} = \frac{W_2 L_2}{2A_2 E} = \frac{21.61\times1000}{2\times28.27\times200000} = 0.00191\text{mm}$$

Change in Length in portion AB

This is caused due to weight of AB and due to weight of BC acting as a concentrated load at B and is computed as:

$$\delta L_{AB} = \frac{W_1 L_1}{2A_1 E} + \frac{W_2 L_1}{E\,A_1} = \frac{60.04\times1000}{2\times78.54\times200000} + \frac{21.61\times1000}{200000\times78.54} = 0.0033\text{mm}$$

Total change in length = 0.00191+ 0.0033 = **0.00521mm**

NOTES FREE
AK in One Solution

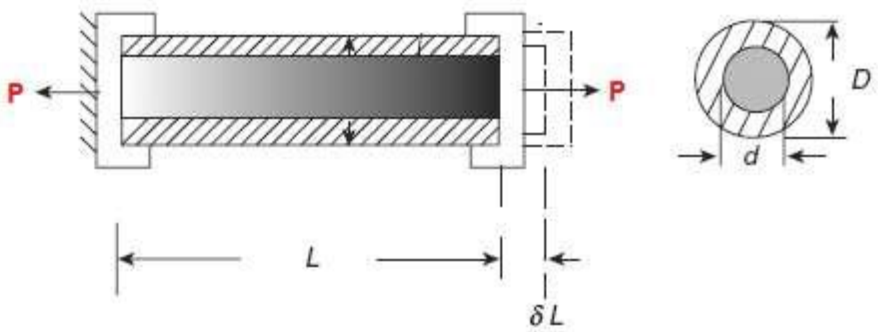
1.12 Compound or composite bars

A composite bar can be made of two bars of different materials rigidly fixed together so that both bars strain together under external load. As the strains in the two bars are same, the stresses in the two bars will be different and depend on their respective modulus of elasticity. A stiffer bar will share major part of external load.

In a composite system the two bars of different materials may act as suspenders to a third rigid bar subjected to loading. As the change in length of both bars is the same, different stresses are produced in two bars.

1.10.1 Stresses in a Composite Bar

Let us consider a composite bar consisting of a solid bar, of diameter ***d*** completely encased in a hollow tube of outer diameter ***D*** and inner diameter ***d,*** subjected to a tensile force **P** as shown in the following figure.



Let the extension of composite bar of length ***L*** be ***δL***. Let E_S and E_H be the modulus of elasticity of solid bar and hollow tube respectively. Let σ_S and σ_H be the stresses developed in the solid bar and hollow tube respectively.

Since change in length of solid bar is equal to the change in length of hollow tube, we can establish the relation between the stresses in solid bar and hollow tube as shown below :

$$\frac{\sigma_S L}{E_S} = \frac{\sigma_H L}{E_H} \text{ or } \sigma_S = \sigma_H \frac{E_S}{E_H}$$

$$\frac{\sigma_S L}{E_S} = \frac{\sigma_H L}{E_H} \text{ or } \sigma_S = \sigma_H \frac{E_S}{E_H}$$

Area of cross section of the hollow tube : $A_H = \frac{\pi(D^2 - d^2)}{4}$

Area of cross section of the solid bar : $A_S = \frac{\pi d^2}{4}$

Load carried by the hollow tube : $P_H = \sigma_H A_H$ and Load carried by the solid bar : $P_S = \sigma_S A_S$

But $P = P_S + P_H = \sigma_S A_S + \sigma_H A_H$

With $\sigma_S = \sigma_H \frac{E_S}{E_H}$, the following equation can be written

$$P = \sigma_H \frac{E_S}{E_H} A_s + \sigma_H A_H = \sigma_H \left(A_H + \frac{E_S}{E_H} A_s\right)$$

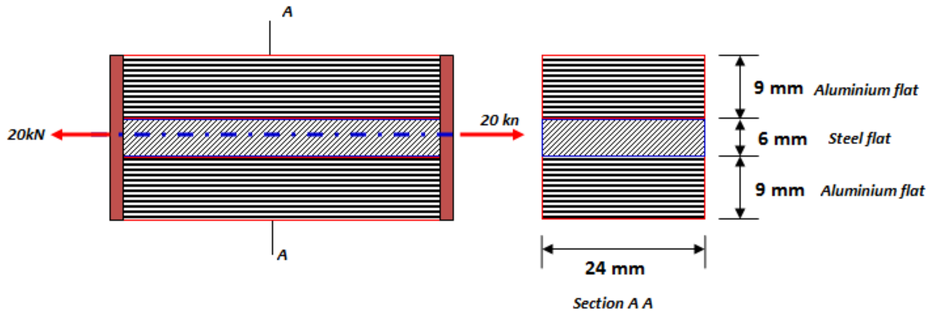
E_S/E_H is called *modular ratio*. Using the above equation stress in the hollow tube can be calculated. Next, the stress in the solid bar can be calculated using the equation

$$P = \sigma_S A_S + \sigma_H A_H$$

NOTES4FREE
All in One Solution

Problems.

A flat bar of steel of 24 mm wide and 6 mm thick is placed between two aluminium alloy flats 24 mm × 9 mm each. The three flats are fastened together at their ends. An axial tensile load of 20 kN is applied to the composite bar. What are the stresses developed in steel and aluminium alloy? Assume E_S = 210000 MPa and E_A = 70000MPa.



Area of Steel flat: $A_S = 24 \times 6 = 144\ mm^2$

Area of Aluminium alloy flats: $A_A = 2 \times 24 \times 9 = 432\ mm^2$

Since all the flats elongate by the same extent, we have the condition that $\frac{\sigma_S L}{E_S} = \frac{\sigma_A L}{E_A}$.

The relationship between the stresses in steel and aluminum flats can be established as:

$$\sigma_S = \sigma_A \frac{E_S}{E_A} = 3\,\sigma_A$$

Since $P = P_S + P_A = \sigma_S A_S + \sigma_A A_A$. This can be written as

$$P = 3\sigma_A A_s + \sigma_A A_A = \sigma_A (3A_s + A_A)$$

From which stress in aluminium alloy flat can be computed as:

$$\sigma_A = \frac{P}{(3A_s + A_A)} = \frac{20 \times 1000}{(3 \times 144 + 432)} = \mathbf{23.15}\boldsymbol{MPa}$$

Stress in steel flat can be computed as:

$$\sigma_S = 3 \times 23.15 = \mathbf{69.45}\boldsymbol{MPa}$$

2. A short post is made by welding steel plates into a square section and then filling inside with concrete. The side of square is 200 mm and the thickness t = 10 mm as shown in the figure. The steel has an allowable stress of 140 N/mm2 and the concrete has an allowable stress of 12 N/mm2. Determine the allowable safe compressive load on the post. E_C = 20 GPa, E_S = 200 GPa.

Since the composite post is subjected to compressive load, both concrete and steel tube will shorten by the same extent. Using this condition following relation between stresses in concrete and steel can be established.

$$\frac{\sigma_C L}{E_C} = \frac{\sigma_S L}{E_S} \text{ or } \sigma_S = \sigma_C \frac{E_S}{E_C} = 10\,\sigma_C$$

Assume that load is such that $\sigma_s = 140$ N/mm^2. Using the above relationship, the stress in concrete corresponding to this load can be calculated as follows:

$$140 = 10\,\sigma_C \;\; or \;\; \sigma_C = 14\, N/mm^2 > 12 \text{ N/mm}^2$$

Hence the assumed load is not a safe load.

Instead assume that load is such that $\sigma_c = 12$ N/mm^2. The stress in steel corresponding to this load can be calculated as follows:

$$\sigma_s = 12 \times 10 \;\; or \;\; \sigma_s = 120\, N/mm^2 < 140 \text{ N/mm}^2$$

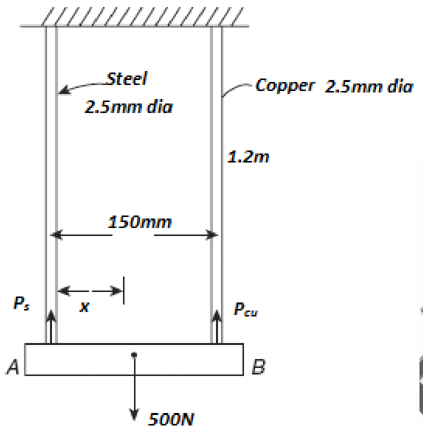
Hence the assumed load is a safe load which is calculated as shown below.

Area of concrete section Ac = 180 x180 = 32400mm^2.

Area of steel tube As = 200 x 200 – 32400 = 7600 mm^2.

$$P = \sigma_C A_C + \sigma_s A_s = 12 \times 32400 + 120 \times 7600 = \mathbf{1300.8}\boldsymbol{kN}$$

3. A rigid bar is suspended from two wires, one of steel and other of copper, length of the wire is 1.2 m and diameter of each is 2.5 mm. A load of 500 N is suspended on the rigid bar such that the rigid bar remains horizontal. If the distance between the wires is 150 mm, determine the location of line of application of load. What are the stresses in each wire and by how much distance the rigid bar comes down? Given $E_s = 3E_{cu} = 201000$ N/mm2.



i. Area of copper wire (Acu) = Area of steel wire(As) = π x 2.52/4 = 4.91 mm2

ii. For the rigid bar to be horizontal, elongation of both the wires must be same. This condition leads to the following relationship between stresses in steel and copper wires as:

$$\sigma_s = \frac{E_s}{E_{cu}}\sigma_{cu} = 3\sigma_{cu}$$

iii. Using force equilibrium, the stress in copper and steel wire can be calculated as:

$$P = P_s + P_{cu} = \sigma_s A_s + \sigma_{cu} A_{cu} = 3\,\sigma_{cu} A_s + \sigma_{cu} A_{cu} = \sigma_{cu}(3A_s + A_{cu})$$

$$\sigma_{cu} = \frac{P}{(A_{cu} + 3A_S)} = \frac{500}{(4.91 + 3 \times 4.91)} = 25.46\ MPa$$

$$\sigma_s = 3 \times 25.46 = 76.37\ MPa$$

iv. Downward movement of rigid bar = elongation of wires

$$\delta L_s = \frac{\sigma_s}{E_s}L = \frac{76.37}{201000} \times 1200 = 0.456\ mm$$

v. Position of load on the rigid bar is computed by equating moments of forces carried by steel and copper wires about the point of application of load on the rigid bar.

$$P_s\,x = P_c\,(150 - x)$$

$$(76.37 \times 4.91)x = (25.46 \times 4.91)\,(150 - x)$$

$$\frac{x}{150 - x} = 0.333$$

$$x = \mathbf{37.47mm}\ \textit{from steel wire}$$

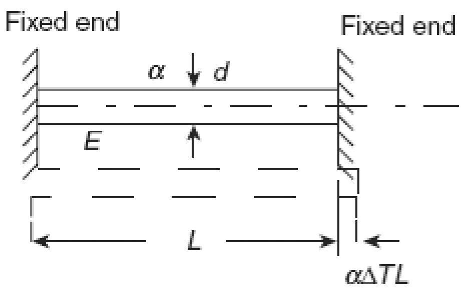
Note:

If the load is suspended at the centre of rigid bar, then both steel and copper wire carry the same load. Hence the stress in the wires is also same. As the moduli of elasticity of wires are different, strains in the wires will be different. This results in unequal elongation of wires causing the rigid bar to rotate by some magnitude. This can be prevented by offsetting the load or with wires having different length or with different diameter such that elongation of wires will be same.

NOTES4FREE
All in One Solution

1.13 Temperature stresses in a single bar

If a bar is held between two unyielding (rigid) supports and its temperature is raised, then a compressive stress is developed in the bar as its free thermal expansion is prevented by the rigid supports. Similarly, if its temperature is reduced, then a tensile stress is developed in the bar as its free thermal contraction is prevented by the rigid supports. Let us consider a bar of diameter ***d*** and length ***L*** rigidly held between two supports as shown in the following figure. Let α be the coefficient of linear expansion of the bar and its temperature is raised by ΔT (°C)



• Free thermal expansion in the bar = $\alpha \Delta T L$.

• Since the supports are rigid, the final length of the bar does not change. The fixed ends exert compressive force on the bar so as to cause shortening of the bar by $\alpha \Delta T L$.

• Hence the compressive strain in the bar = $\alpha \Delta T L / L = \alpha \Delta T$

• Compressive stress = $\alpha \Delta T E$

• Hence the thermal stresses introduced in the bar = $\alpha \Delta T E$

Note:

The bar can buckle due to large compressive forces generated in the bar due to temperature increase or may fracture due to large tensile forces generated due to temperature decrease.

Problem

A rail line is laid at an ambient temperature of 30°C. The rails are 30 m long and there is a clearance of 5 mm between the rails. If the temperature of the rail rises to 60°C, what is the stress developed in the rails?. Assume $\alpha = 11.5 \times 10^{-6}$/°C, E = *2,10,000 N/mm²*

• L = 30,000 mm, $\alpha = 11.5 \times 10^{-6}$/°C, Temperature rise ΔT = 60-30 = 30°C

• Free expansion of rails = $\alpha \Delta T L = 11.5 \times 10^{-6} \times 30 \times 30000 = 10.35$mm

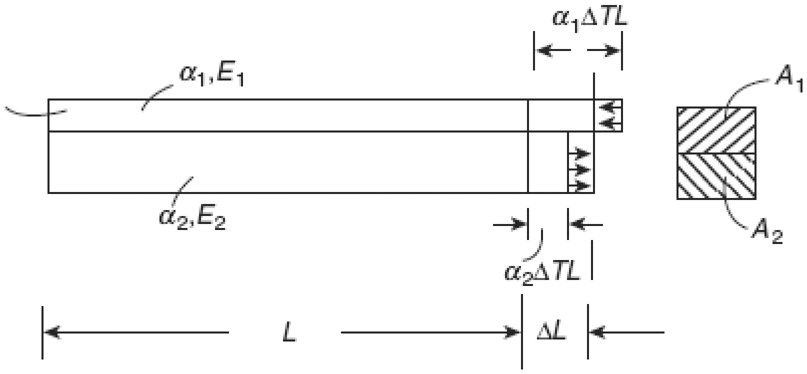
• Thermal expansion prevented by rails = Free expansion – clearance = 10.35 – 5 = 5.35mm

• Strain in the rails ε = 5.35/30000 = 0.000178

• Compressive stress in the rails = ε x E = 0.000178 x 210000 =**37.45N/mm²**.

1.14 Temperature Stresses in a Composite Bar

A composite bar is made up of two bars of different materials perfectly joined together so that during temperature change both the bars expand or contract by the same amount. Since the coefficient of expansion of the two bars is different thermal stresses are developed in both the bars. Consider a composite bar of different materials with coefficients of expansion and modulus of elasticity, as $\boldsymbol{\alpha_1}, \boldsymbol{E_1}$ and $\boldsymbol{\alpha_2}, \boldsymbol{E_2}$, respectively, as shown in the following figure. Let the temperature of the bar is raised by $\boldsymbol{\Delta T}$ *and* $\boldsymbol{\alpha_1} > \boldsymbol{\alpha_2}$



Free expansion in bar 1 = $\alpha_1 \Delta T L$ and Free expansion in bar 2 = $\alpha_2 \Delta T L$. Since both the bars expand by ΔL together we have the following conditions:

- Bar 1: $\Delta L < \alpha_1 \Delta T L$. The bar gets compressed resulting in compressive stress
- Bar 2: $\Delta L > \alpha_2 \Delta T L$. The bar gets stretched resulting in tensile stress.

$$\text{Compressive strain in Bar 1 : } \varepsilon_1 = \frac{\alpha_1 \Delta T L - \Delta L}{L}$$

$$\text{Tensile strain in Bar 2 : } \varepsilon_2 = \frac{\Delta L - \alpha_2 \Delta T L}{L}$$

$$\varepsilon_1 + \varepsilon_2 = \frac{\alpha_1 \Delta T L - \Delta L}{L} + \frac{\Delta L - \alpha_2 \Delta T L}{L} = (\alpha_1 - \alpha_2)\Delta T$$

Let σ_1 and σ_2 be the temperature stresses in bars. The above equation can be written as:

$$\frac{\sigma_1}{E_1} + \frac{\sigma_2}{E_2} = (\alpha_1 - \alpha_2)\Delta T$$

In the absence of external forces, for equilibrium, compressive force in Bar 1 = Tensile force in Bar 2. This condition leads to the following relation

$$\sigma_1 A_1 = \sigma_2 A_2$$

Using the above two equations, temperature stresses in both the bars can be computed. This is illustrated in the following example.

Note:

If the temperature of the composite bar is reduced, then a tensile stress will be developed in bar 1 and a compressive stress will be developed in bar 2 , since $\alpha_1 > \alpha_2$.

Problems

1 A steel flat of 20 mm × 10 mm is fixed with aluminium flat of 20 mm × 10 mm so as to make a square section of 20 mm × 20 mm. The two bars are fastened together at their ends at a temperature of 26°C. Now the temperature of whole assembly is raised to 55°C. Find the stress in each bar. E_s = 200 GPa, E_a = 70 GPa, $\alpha_s = 11.6 \times 10^{-6}$/°C, $\alpha_a = 23.2 \times 10^{-6}$/°C.

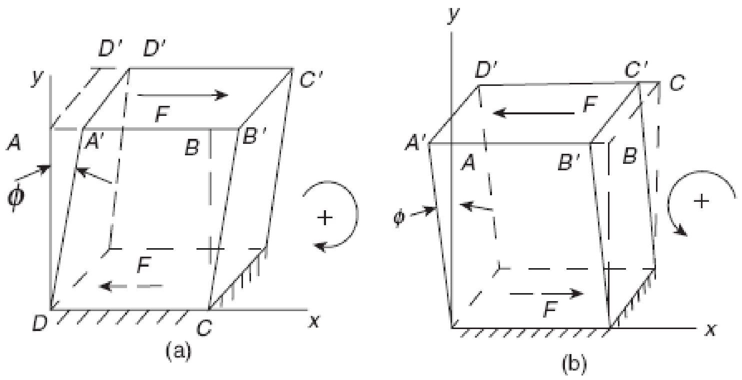
- Net temperature rise, ΔT = 55 − 26 = 29°C.
- Area of Steel flat (As) = Area of Aluminium flat (Aa) = 20 x10 =200 mm2
- For equilibrium, $\sigma_s A_s = \sigma_a A_a$; $\sigma_s = \sigma_a$ will be one of the conditions to be satisfied by the composite assembly.
- $But\ \frac{\sigma_a}{E_a} + \frac{\sigma_s}{E_s} = (\alpha_a - \alpha_s)\Delta T = (23.2 - 11.6) \times 29 \times 10^{-6} = 0.000336$
- $\frac{\sigma_s}{200000} + \frac{\sigma_a}{70000} = 0.000336$
- $270000\ \sigma_s = 4709600$;
- $\sigma_s(tensile) = \sigma_a(compressive) = 17.44MPa$ as $\alpha_a > \alpha_s$

2. A flat steel bar of 20 mm × 8 mm is placed between two copper bars of 20 mm × 6 mm each so as to form a composite bar of section of 20 mm × 20 mm. The three bars are fastened together at their ends when the temperature of each is 30°C. Now the temperature of the whole assembly is raised by 30°C. Determine the temperature stress in the steel and copper bars. $E_s = 2E_{cu}$= 210 kN/mm^2, $\alpha_s = 11 \times 10^{-6}$/°C, $\alpha_{cu} = 18 \times 10^{-6}$/°C.

- Net temperature rise, $\Delta T = 30°C$.
- Area of Steel flat (A_s) = 20 x 8 = 160 mm^2
- Area of Copper flats (A_{cu}) = 2 x 20 x 6 =240 mm^2
- For equilibrium, $\sigma_s A_s = \sigma_{cu} A_{cu}$; $\sigma_s = 1.5\,\sigma_{cu}$ will be one of the conditions to be satisfied by the composite assembly.
- $But\ \frac{\sigma_{cu}}{E_{cu}} + \frac{\sigma_s}{E_s} = (\alpha_{cu} - \alpha_s)\Delta T = (18 - 11) \times 30\ \times 10^{-6} = 0.00021$
- $\frac{\sigma_{cu}}{105000} + \frac{1.5\sigma_{cu}}{210000} = 0.00021$
- σ_{cu} = 12.6MPa (compressive) and σ_s = 18.9MPa (tensile) as $\alpha_{cu} > \alpha_s$

1.15 Simple Shear stress and Shear Strain

Consider a rectangular block which is fixed at the bottom and a force *F* is applied on the top surface as shown in the figure (a) below.



Equal and opposite reaction *F* develops on the bottom plane and constitutes a couple, *tending to rotate the body in a clockwise direction*. This type of shear force is a *positive shear force* and the shear force per unit surface area on which it acts is called *positive shear stress* (τ). If force is applied in the opposite direction as shown in Figure (b), then they are termed as negative shear force and shear stress.

The *Shear Strain* (ϕ) = AA''/AD = tanϕ. Since ϕ is a very small quantity, tan$\phi \approx \phi$. Within the elastic limit, $\tau \propto \phi$ *or* $\tau = G\,\phi$

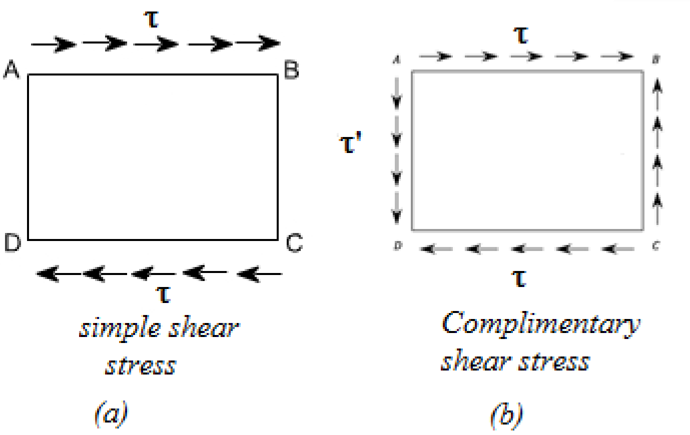
The constant of proportionality ***G*** is called *rigidity modulus or shear modulus.*

Note:

Normal stress is computed based on area perpendicular to the surface on which the force is acting, while, the shear stress is computed based on the surface area on which the force is acting. Hence shear stress is also called tangential stress.

1.16 Complementary Shear Stresses

Consider an element ABCD subjected to shear stress (τ) as shown in figure (a). We cannot have equilibrium with merely equal and opposite tangential forces on the faces AB and CD as these forces constitute a couple and induce a turning moment. The statical equilibrium demands that there must be tangential components (τ'') along AD and CB such that that can balance the turning moment. These tangential stresses (τ'') is termed as *complimentary shear stress*.



Let t be the thickness of the block. Turning moment due to τ will be (τ x t x L_{AB}) L_{BC} and Turning moment due to τ' will be (τ'' x t x L_{BC}) L_{AB}. Since these moments have to be equal for equilibrium we have:

$$(\tau \times t \times L_{AB})\, L_{BC} = (\tau'' \times t \times L_{BC})\, L_{AB}.$$

From which it follows that $\tau = \tau''$, that is, intensities of shearing stresses across two mutually perpendicular planes are equal.

1.17 Volumetric strain

This refers to the slight change in the volume of the body resulting from three mutually perpendicular and equal direct stresses as in the case of a body immersed in a liquid under pressure. This is defined as the *ratio of change in volume to the original volume* of the body. Consider a cube of side „**a**" strained so that each side becomes „**a** ± δ**a'.**

- Hence the linear strain = δa/a.
- Change in volume = $(a \pm \delta a)^3 - a^3 = \pm 3a^2\delta a$. (ignoring small higher order terms)
- Volumetric strain $\varepsilon_v = \pm 3a^2\, \delta a/a^3 = \pm 3\, \delta a/a$
- The volumetric strain is three times the linear strain

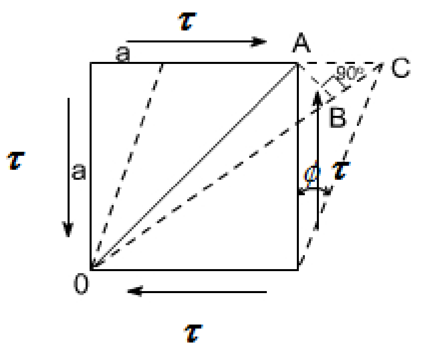
1.18 Bulk Modulus

This is defined as the ratio of the normal stresses (p) to the volumetric strain (ε_v) and denoted by **'K'.** Hence **K = p/**ε_v . This is also an elastic constant of the material in addition to E, G and ν.

1.19 Relation between elastic constants

1.19.1 Relation between E,G and ν

Consider a cube of material of side „a' subjected to the action of the shear and complementary shear stresses and producing the deformed shape as shown in the figure below.



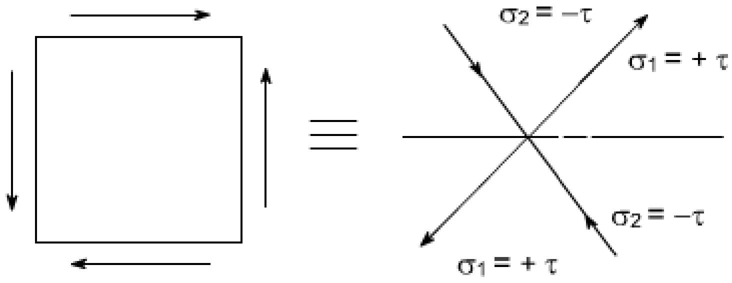
- Since, within elastic limits, the strains are small and the angle ACB may be taken as 45^0.
- Since angle between OA and OB is very small hence OA ≈ OB. BC, is the change in the length of the diagonal OA
- Strain on the diagonal OA = Change in length / original length = BC/OA

= AC cos45/ (a/sin45) = AC/ 2a = a φ / 2 a = φ / 2

- It is found that *strain along the diagonal is numerically half the amount of shear stain*.

- But from definition of rigidity modulus we have, $G = \tau / \phi$
- Hence, Strain on the diagonal OA = $\tau / 2G$

The shear stress system is equivalent or can be replaced by a system of direct stresses at 45^0 as shown below. One set will be compressive, the other tensile, and both will be equal in value to the applied shear stress.



Strain in diagonal OA due to direct stresses $= \frac{\sigma_1}{E} - \nu\frac{\sigma_2}{E} = \frac{\tau}{E} + \nu\frac{\tau}{E} = \frac{\tau}{E}(1+\nu)$

Equating the strain in diagonal OA we have $\frac{\tau}{2G} = \frac{\tau}{E}(1+\nu)$

Relation between E,G and ν can be expressed as : $\boldsymbol{E = 2G(1+\nu)}$

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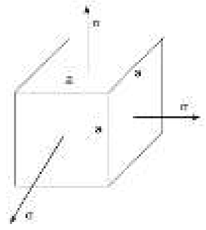
1.19.2 Relation between E,K and ν

Consider a cube subjected to three equal stresses a shown in the figure below.



Strain in any one direction $= \frac{\sigma}{E} - \nu\frac{\sigma}{E} - \nu\frac{\sigma}{E} = \frac{\sigma}{E}\,(1-2\nu)$

Since the volumetric strain is three times the linear strain: $\varepsilon_v = 3\frac{\sigma}{E}\,(1-2\nu)$

From definition of bulk modulus : $\varepsilon_v = \frac{\sigma}{K}$

$$3\frac{\sigma}{E}\,(1-2\nu) = \frac{\sigma}{K}$$

Relation between E,K and ν can be expressed as : $\boldsymbol{E = 3K(1-2\nu)}$

Note: Theoretically ν < 0.5 as E cannot be zero

1.19.3 Relation between E, G and K

We have E = 2G(1+ν) from which ν = (E - 2G) / 2G
We have E = 3K(1-2ν) from which ν = (3K -E) / 6K

(E - 2G) / 2G = (3K -E) / 6K or (6EK - 12GK) = (6GK - 2EG) or 6EK+2EG = (6GK +12GK)

Relation between E,G and K can be expressed as: $\mathbf{E} = \frac{\boldsymbol{9GK}}{\boldsymbol{(3K+G)}}$

1.20 Exercise problems

1. A steel bar of a diameter of 20 mm and a length of 400 mm is subjected to a tensile force of 40 kN. Determine (a) the tensile stress and (b) the axial strain developed in the bar if the Young‟s modulus of steel E = 200 kN/mm2

Answer: (a) Tensile stress = 127.23MPa, (b) Axial strain = 0.00064

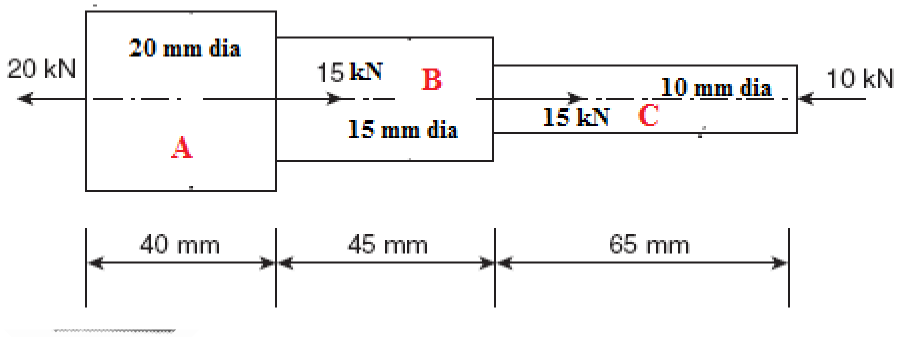
2. A 100 mm long bar is subjected to a compressive force such that the stress developed in the bar is 50 MPa. (a) If the diameter of the bar is 15 mm, what is the axial compressive force? (b) If E for bar is 105 kN/mm2, what is the axial strain in the bar?

Answer: (a) Compressive force = 8.835 kN, (b) Axial strain = 0.00048

3. A steel bar of square section 30 × 30 mm and a length of 600 mm is subjected to an axial tensile force of 135 kN. Determine the changes in dimensions of the bar. E = 200 kN/mm2, v = 0.3.

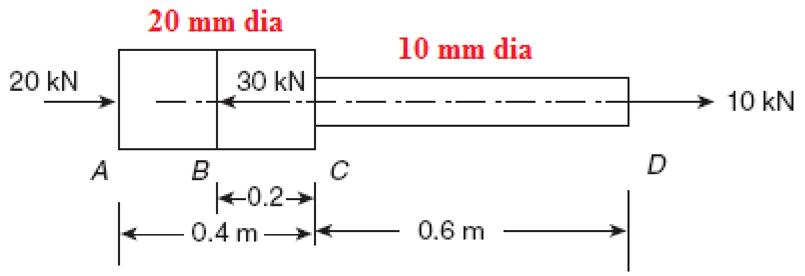
Answer: Increase in length δl = 0.45 mm, Decrease in breadth δb = 6.75 × 10–3 mm

4. A stepped circular steel bar of a length of 150 mm with diameters 20, 15 and 10 mm along lengths 40, 50 and 65 mm, respectively, subjected to various forces is shown in figure below. If E = 200 kN/mm2, determine the total change in its length.



Answer : Total decrease in length = 0.022mm

5. A stepped bar is subjected to axial loads as shown in the figure below. If *E* = 200 GPa, calculate the stresses in each portion *AB*, *BC* and *CD*. What is the total change in length of the bar?



Answer: Total increase in length = 0.35mm

6. A 400-mm-long aluminium bar uniformly tapers from a diameter of 25 mm to a diameter of 15 mm. It is subjected to an axial tensile load such that stress at middle section is 60 MPa. What is the load applied and what is the total change in the length of the bar if E = 67,000 MPa? (*Hint: At the middle diameter = (25+15)/2 = 20 mm*).

Answer: Load = 18.85kN, Increase in length = 0.382 mm

7. A short concrete column of 250 mm × 250 mm in section strengthened by four steel bars near the corners of the cross-section. The diameter of each steel bar is 30 mm. The column is subjected to an axial compressive load of 250 kN. Find the stresses in the steel and the concrete. Es = 15 Ec = 210 GPa. If the stress in the concrete is not to exceed 2.1 N/mm^2, what area of the steel bar is required in order that the column may support a load of 350 kN?

Answer: Stress in concrete = 2.45N/mm^2, Stress in steel = 36.75N/mm^2, Area of steel = 7440 mm^2

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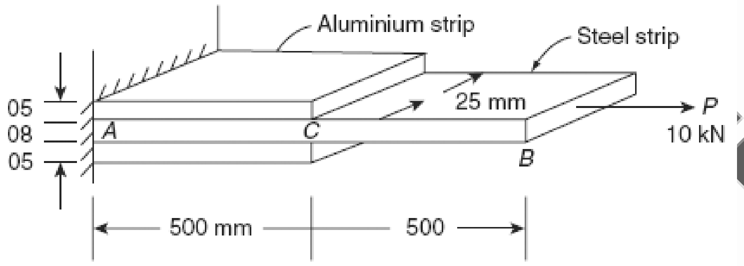
8. Two aluminium strips are rigidly fixed to a steel strip of section 25 mm × 8 mm and 1 m long. The aluminium strips are 0.5 m long each with section 25 mm × 5 mm. The composite bar is subjected to a tensile force of 10 kN as shown in the figure below. Determine the deformation of point B. Es = 3EA = 210 kN/mm2. *Answer: 0.203mm*

(*Hint: Portion CB is a single bar, Portion AC is a composite bar. Compute elongation separately for both the portions and add*)



Module 2

Compound Stresses

Objectives:

Derive the equations for principal stress and maximum in-plane shear stress and calculate their magnitude and direction. Draw Mohr circle for plane stress system and interpret this circle.

Learning Structure

- 2.1 Introduction
- 2.2 Plane Stress Or 2–D Stress System Or Biaxial Stress System
- 2.3 Expressions For Normal And Tangential Components Of Stress On A Given Plane
- 2.4 Mohr's Circle
- 2.5 Problems
- 2.6 Thick Cylinders
- 2.7 Thin Cylinders
- Outcomes
- Further Reading

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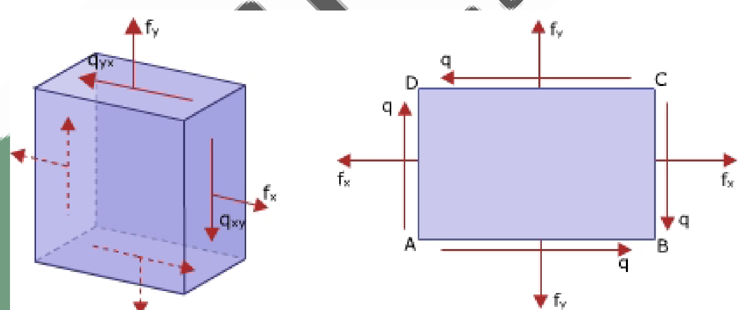
2.1 Introduction

Structural members are subjected to various kinds of loads. This results in combination of different stresses which changes from point to point. When an element (considered at any point) in a body is subjected to a combination of normal stresses (tensile and/or compressive) and shear stresses over its various planes, the stress system is known as compound stress system. In a compound stress system, the magnitude of normal stress may be maximum o n some plane and minimum on some plane, when compared with those acting on the element. Similarly, the magnitude of shear stresses may also be maximum on two planes when compared with those acting on the element. Hence, for the considered compound stress system it is important to find the magnitudes of maximum and minimum normal stresses, maximum shear stresses and the inclination of planes on which they act.

2.2 PLANE STRESS OR 2–D STRESS SYSTEM OR BIAXIAL STRESS SYSTEM

Generally a body is subjected to 3–D state of stress system with both normal and shear stresses acting in all the three directions. However, for convenience, in most problems, variation of stresses along a particular direction can be neglected and the remaining stresses are assumed to act in a plane. Such a system is called 2–D stress system and the body is called plane stress body.



In a general two dimensional stress system, a body consists of two normal stresses (fx and fy), which are mutually perpendicular to each other, with a state of shear (q) as shown in figure. Further, since planes AD and BC carry normal stress fx they are called planes of fx. These planes are parallel to Y–axis. Similarly, planes AB and CD represent planes of fy, which are parallel to X–axis.

2.2.1 PRINCIPAL STRESSES AND PRINCIPAL PLANES

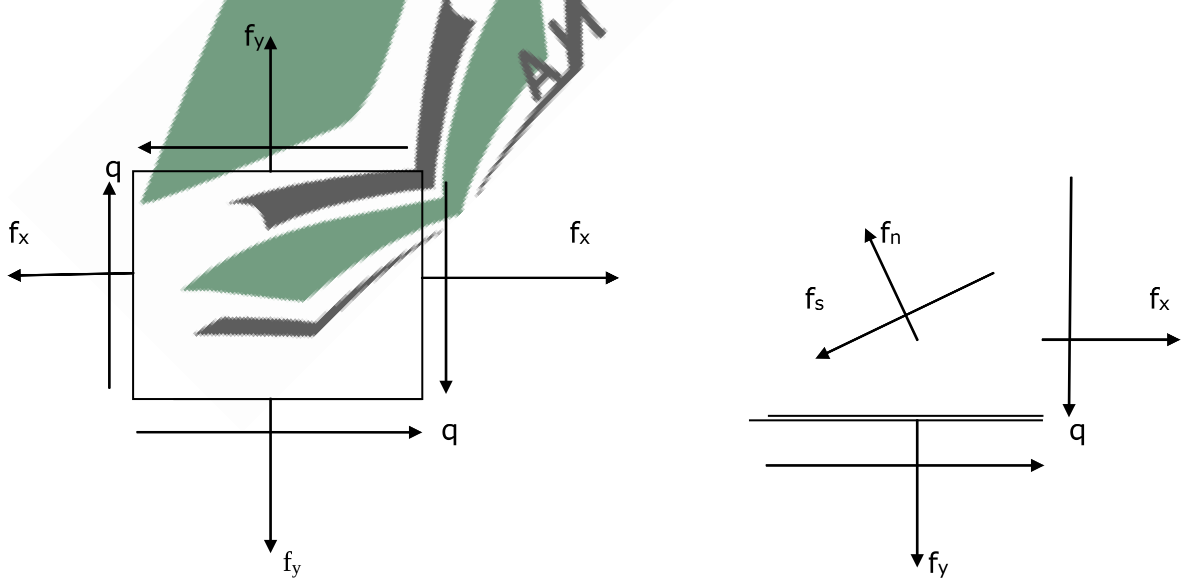
For a given compound stress system, there exists a maximum normal stress and a minimum normal stress which are called the Principal stresses. The planes on which these Principal stresses act are called Principal planes. In a general 2-D stress system, there are two Principal planes which are always mutually perpendicular to each other. Principal planes are free from shear stresses. In other words Principal planes carry only normal stresses.

2.2.2 MAXIMUM SHEAR STRESSES ANDITS PLANES

For a given 2–D stress system, there will be two maximum shear stresses (of equal magnitude) which act on two planes. These planes are called planes of maximum shear. These planes are mutually perpendicular. Further, these planes may or may not carry normal stress. The planes of maximum shear are always inclined at 450 with Principal planes.

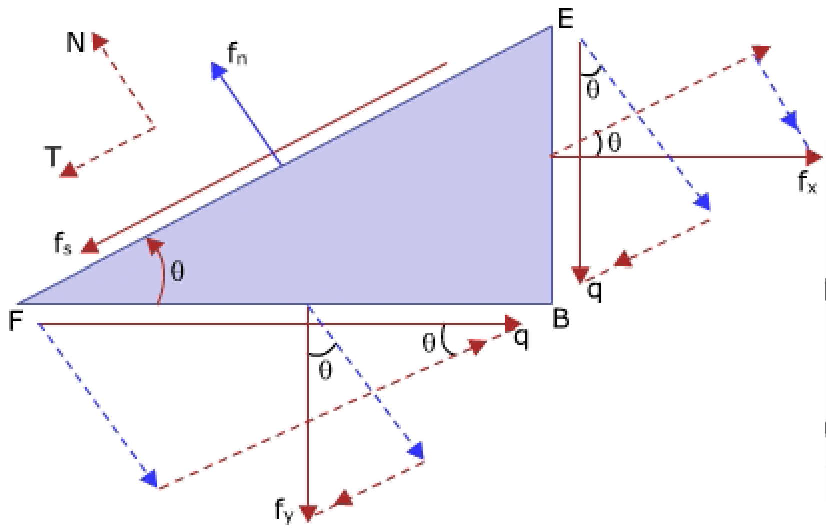
2.3 EXPRESSIONS FOR NORMAL AND TANGENTIAL COMPONENTS OF STRESS ON A GIVEN PLANE

Consider a rectangular element ABCD of unit thickness subjected to a general 2-D stress system as shown in figure. Let f n and f s represent the normal and tangential components of resultant stress ‘R' on any plane EF which is inclined at an angle ‘?' measured counter clockwise with respect to the plane of f y or X–axis.



To derive expression for f_n

Consider the Free Body Diagram of portion FBE as shown in figure.



Applying equilibrium along N-direction, we have

$$\Sigma F_N = 0 \quad [\ \nwarrow\ +ve]$$

$$f_n (EF.1) - f_x (BE.1) \sin\theta - q(BE.1)\cos\theta - f_y (BF.1)\cos\theta - q(BF.1)\sin\theta = 0$$

$$f_n = f_x \frac{BE}{EF} \sin\theta + q \frac{BE}{EF} \cos\theta + f_y \frac{BF}{EF} \cos\theta + q \frac{BF}{EF} \sin\theta$$

$$\text{Since} \quad \frac{BE}{EF} = \sin\theta \quad \text{and} \quad \frac{BF}{EF} = \cos\theta$$

$$\therefore f_n = f_x \sin^2\theta + 2q \sin\theta \cos\theta + f_y \cos^2\theta$$

$$\text{But} \quad \cos 2\theta = 2\cos^2\theta - 1 \qquad \text{Hence} \quad \cos^2\theta = \frac{1}{2}(1 + \cos 2\theta)$$

$$\text{Also} \quad \cos 2\theta = 1 - 2\sin^2\theta \qquad \text{Hence} \quad \sin^2\theta = \frac{1}{2}(1 - \cos 2\theta)$$

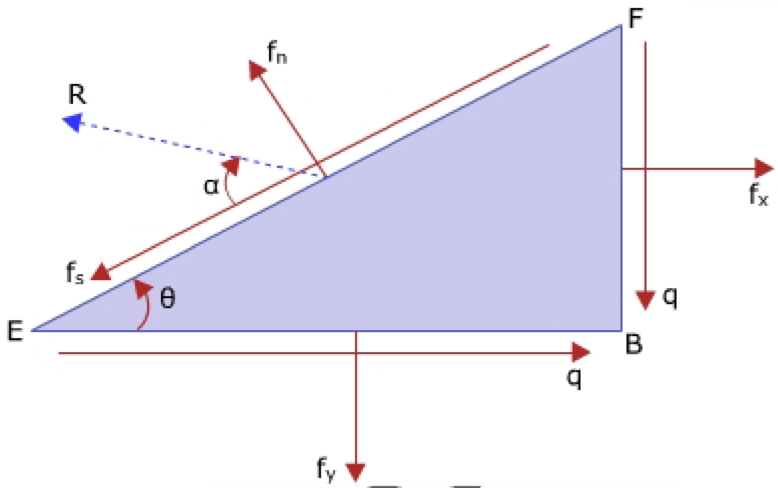
$$f_x = f_x \frac{1}{2}(1 - \cos 2\theta) + f_y \frac{1}{2}(1 + \cos 2\theta) + q \sin 2\theta$$

$$f_n = \left(\frac{f_x + f_y}{2}\right) - \left(\frac{f_x - f_y}{2}\right)\cos 2\theta + q \sin 2\theta \qquad \text{-------- (1)}$$

Equation (1) is the desired expression for normal component of stress on a given plane, inclined at an angle ' θ ' measured counter clockwise with respect to the plane of f_Y or X– axis

To derive expression for f_s

Consider the Free Body Diagram of portion FBE shown in figure above. For equilibrium along T direction, we have



$$\Sigma F_T = 0 \; [\swarrow +ve]$$

$$f_s(EF.1) - f_x(BE.1)\cos\theta + q\,(BE.1)\sin\theta + f_y\,(BF.1)\sin\theta - q(BF.1)\cos\theta = 0$$

$$f_s = f_x\frac{BE}{EF}\cos\theta - q\frac{BE}{EF}\sin\theta - f_y\frac{BF}{EF}\sin\theta + q\frac{BF}{EF}\cos\theta$$

Since $\frac{BE}{EF} = \sin\theta \qquad \frac{BF}{EF} = \cos\theta$

$$\therefore f_s = f_x\sin\theta\cos\theta - q\sin^2\theta - f_y\cos\theta\sin\theta + q\cos^2\theta$$

$$\therefore f_s = (f_x - f_y)\sin\theta\cos\theta + q\,(\cos^2\theta - \sin^2\theta)$$

Since $\sin 2\theta = 2\sin\theta\cos\theta$ and $\cos 2\theta = \cos^2\theta - \sin^2\theta$

$$f_s = \left(\frac{fx - fy}{2}\right)\sin 2\theta + q\cos 2\theta \qquad \text{-------- (2)}$$

Equation (2) is the desired expression for tangential component of stress on a given plane, inclined at an angle ' θ ' measured counter clockwise with respect to the plane of fy or X– axis.

Note:

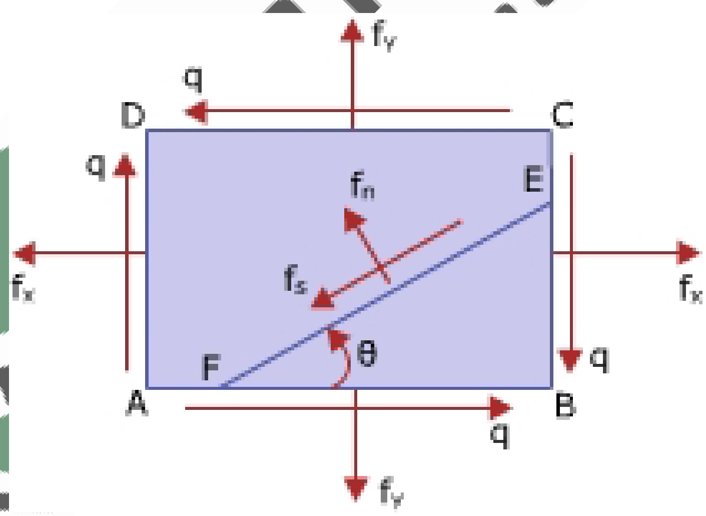
The resultant stress 'R', and its inclination ' α ' on the given plane EF which is inclined at an angle ' θ ' measured counter clockwise with respect to the plane of fy or X–axis, can be determined from the normal (fn) and tangential (fs) components obtained from eqns. (1) and (2).

$$R = \sqrt{f_n^2 + f_s^2}$$

$$\alpha = \tan^{-1}\left(\frac{f_n}{f_s}\right)$$

2.3.1 Expresions for Principal stresses and Principal planes

Consider a rectangular element ABCD of unit thickness subjected to general 2-D stress system as shown in figure. Let f_n and f_s represent the normal and tangential components of stress on any plane EF which is inclined at an angle ' θ ' measured counter clockwise with respect to the plane of f_y or X–axis



The expression for normal component of stress f_n on any given plane EF is given by

$$f_n = \frac{f_x + f_y}{2} - \frac{f_x - f_y}{2}\cos 2\theta + q\sin 2\theta \qquad \text{-------- (1)}$$

To find values of θ at which f n is maximum or minimum, the necessary condition is

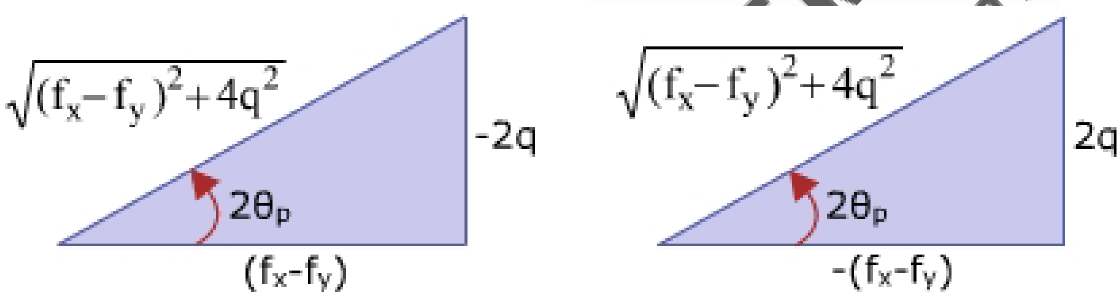
$$\frac{df_n}{d\theta} = 0$$

From eqn. (1) $$-\left(\frac{f_x - f_y}{2}\right)(-2\sin 2\theta) + 2q\cos 2\theta = 0$$

$$\therefore \tan 2\theta_p = -\frac{2q}{f_x - f_y} \qquad \text{------- (2)}$$

Inclination of principal planes can be obtained from eqn. (2). It gives two values of θ differing by 90°. Hence, Principal planes are mutually perpendicular. Here, the two principal planes are designated as θ_{p1} and θ_{p2}.

Graphical representation of eqn. (2) leads to the following



From the above figures,

$$\sin 2\theta_p = \pm\frac{2q}{\sqrt{(f_x - f_y)^2 + 4q^2}} \qquad \cos 2\theta_p = \pm\frac{(f_x - f_y)}{\sqrt{(f_x - f_y)^2 + 4q^2}}$$

Substituting in eqn.(1)

$$f_n = \frac{f_x + f_y}{2} \pm \frac{f_x - f_y}{2}\,\frac{fx - fy}{\sqrt{(fx - fy)^2 + 4q^2}} \pm q\,\frac{2q}{\sqrt{(fx - fy)^2 + 4q^2}}$$

On simplification,

$$f_{n1,2} = \frac{fx + fy}{2} \pm \frac{1}{2}\sqrt{(fx - fy)^2 + 4q^2} \qquad \text{--------- (3)}$$

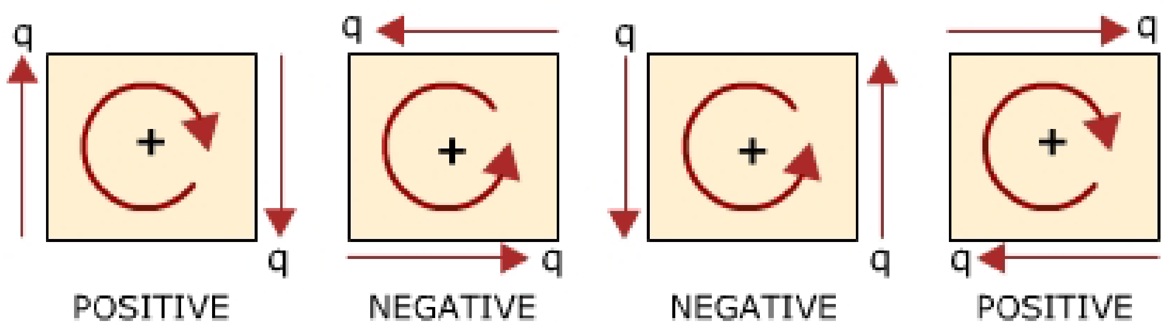
Equation (3) is the desired expression for Principal stresses. Here, the Principal stresses are represented by f_{n1} and f_{n2}.

2.4 Mohr's Circle

The formulae developed so far (to find f_n, f_s, f_{n-max} , f_{n-min} , θ_{p1} , θ_{p2} , $f_{s\ max}$, θ_{s1} , θ_{s2}) may be used for any case of plane stress. A visual interpretation of these relations, devised by the German Engineer Christian Otto Mohr in 1882, eliminates the necessity of remembering them. In this interpretation a circle is used; accordingly, the construction is called **Mohr's Circle.** If this construction is plotted to scale the results can be obtained graphically; usually, however, only a rough sketch is drawn and results are obtained from it analytically.

Rules for applying Mohr's Circle to compound stresses

1. The normal stresses f_x and f_y are plotted along X-axis. Tensile stresses are treated as positive and compressive stresses are treated as negative.
2. The shear stress q is plotted along Y-axis. It is consider positive when its moment about the center of the element is clockwise and negative when its moment about the center of the element is anti-clockwise.

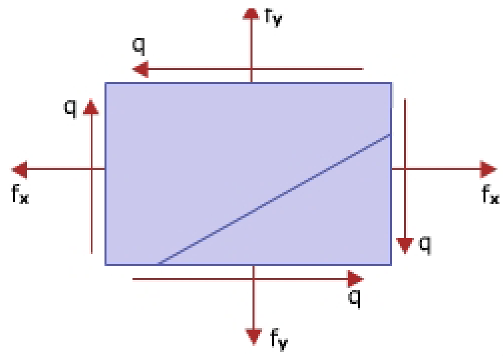


3. Positive angles in the circle are obtained when measured in counter clockwise sense. Further, an angle of '2θ' in the circle corresponds to an angle θ in the element.

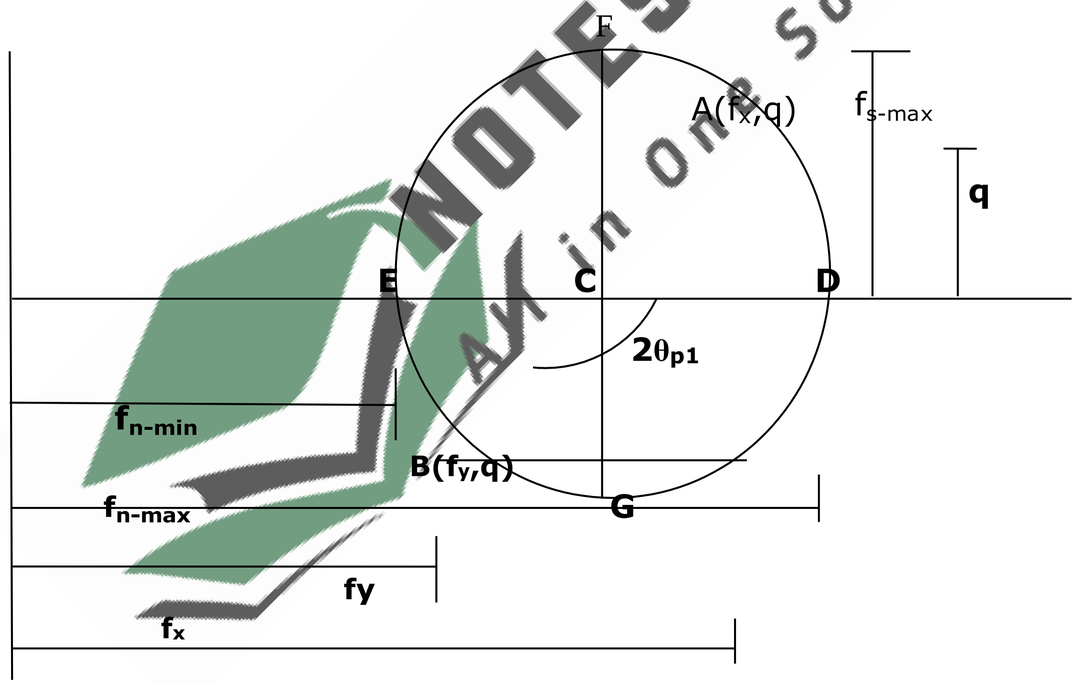
4. A plane in the given element corresponds to a point on the Mohr's circle. Further, the coordinates of the point on the Mohr's circle represent the stresses acting on the plane

Procedure to construct Mohr's circle

Consider an element subjected to normal stresses f_x and f_y accompanied by shear stress q as shown in figure. Let f_x be greater than f_y.



1. In the rectangular coordinate system, locate point A which will be should be a point on the circle representing the stress condition on the plane f_x of the element. The coordinates of point A are (f_x , q).
2. Similarly locate point B, representing stress conditions on plane fy of the element. The coordinates of point B are (f_y – q).
3. Join AB to cut X-axis at point C. Point C corresponds to the center of Mohr's circle.
4. With C as center and CA as radius, draw a circle.



Fig

From figure, it can be seen that OD and OE represent maximum and minimum normal stresses which are nothing but principal stresses. The coordinates of points D and E give the stress condition on principal planes. It can be seen that the value of shear stress is ‘0’ on these two planes. Further, angles BCD = 2 θ_{p1} and BCE =2 θ_{p2} (measured counter clockwise) give

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inclinations of the principal planes with respect to plane of f_y or X-axis. It is seen that $2\theta_{p1} \sim 2\theta_{p2} = 180^0$.

Hence, $\theta_{p1} \sim \theta_{p2} = 90^0$.

It can be observed that shear stress reach maximum values on planes corresponding two points F and G on the Mohr's circle. The coordinates of points F and G represents the stress conditions on the planes carrying maximum shear stress. The ordinate CF and CG represent the maximum shear stresses. The angles BCG = $2\theta_{s1}$ and BCF $=2\theta_{s2}$ (measured counter clockwise) give inclinations of planes carrying maximum shear stress with respect to plane of f_y or X-axis. It is seen that $2\theta_{s1} \sim 2\theta_{s2} = 180^0$.

Hence, $\theta_{s1} \sim \theta_{s2} =90^0$.

Also it is seen that $2\theta_{p1} \sim 2\theta_{s1} \sim 2\theta_{p2} \sim 2\theta_{s2} = 90^0$. Hence, $\theta_{p1} \sim \theta_{s1} \sim \theta_{p2} \sim \theta_{s2} = 45^0$.

To find the normal and tangential stresses on a plane inclined at θ to the plane of f_y , first locate point M on the circle such that angle BCM = 2θ(measured counter clockwise) as shown in figure. The coordinates of point M represents normal and shear stresses on that plane. From figure, ON is the normal stress and MN is the shear stress.

2.5 Problems:

1. **In a 2-D stress system compressive stresses of magnitudes 100 MPa and 150 MPa act in two perpendicular directions. Shear stresses on these planes have magnitude of 80 MPa. Use Mohr's circle to find,**

(i) Principal stresses and their planes
(ii) Maximum shears stress and their planes and
(iii) Normal and shear stresses on a plane inclined at 45^0 to 150 MPa stress.

Given, $f_x = -150$ MPa
$f_y = -100$ MPa
$q = 80$ MPa

If Mohr's circle is drawn to scale, all the quantities can be obtained graphically. However, the present example has been solved analytically using Mohr's circle.

Construct Mohr's circle with earlier fig

From figure

$$OC = \frac{f_x + f_y}{2} = -125\,MPa$$

To find Radius of Circle

$$CH = \frac{f_x - f_y}{2} = 25\,MPa$$

$$CA = \sqrt{CH^2 + HA^2} = 83.82$$

$$\therefore \text{ Radius} = CD = CE = CF = CG = CA = 83.82 \text{ Units}$$

To find Principal Stress and Principal Planes

f_{n-max} = OC + CD

= – 125 – 83.82

= – 208.82 MPa

$f_{n\ min}$ = OC – CE

= – 125 – (–83.82)

= – 41.18 MPa

$$\alpha = \tan^{-1}\left(\frac{AH}{MC}\right) = 72^0.65$$

$$\text{But } 2\theta_{P1} = \angle ACH = \alpha = 72^0.65$$

$$\text{Hence, } \theta_{P1} = 36^0.32$$

$$\text{Further, } 2\theta_{P2} = \angle ACE = 180 + \alpha = 252^0.65$$

$$\text{Hence, } \theta_{P2} = 126^0.32$$

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2.6 Thick Cylinders

2.6.1 Difference in treatment between thin and thick cylinders - basic assumptions:

The theoretical treatment of thin cylinders assumes that the hoop stress is constant across the thickness of the cylinder wall (Fig. 6.1), and also that there is no pressure gradient across the wall. Neither of these assumptions can be used for thick

cylinders for which the variation of hoop and radial stresses is shown in (Fig. 6.2), their values being given by the Lame equations: -

$$\sigma_H = A + \frac{B}{r^2}$$

$$\sigma_r = A - \frac{B}{r^2}$$

Where: -

σ_H= Hoop stress ($\frac{N}{m^2} = Pa$).

σ_r= Radial stress ($\frac{N}{m^2} = Pa$).

r= Radius (m). A and B are Constants.

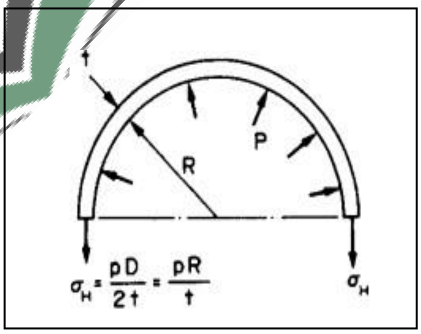

Figure 6.1: - Thin cylinder subjected to internal pressure.

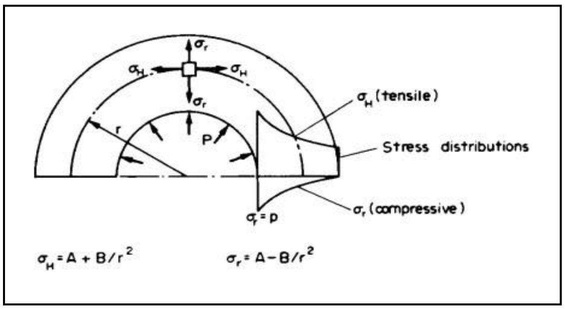

Figure - Thick cylinder subjected to internal pressure.

2.6.2 Thick cylinder- internal pressure only: -

Consider now the thick cylinder shown in (Fig. 6.3) subjected to an internal pressure P, the external pressure being zero.

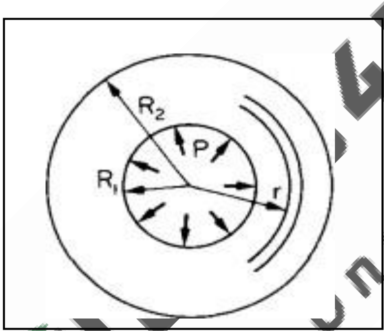

Figure: - Cylinder cross section.

The two known conditions of stress which enable the Lame constants A and B to be determined are:

At $r = R_1$, $\sigma_r = -P$ and at $r = R_2$, $\sigma_r = 0$

Note: -The internal pressure is considered as a negative radial stress since it will produce a radial compression (i.e. thinning) of the cylinder walls and the normal stress convention takes compression as negative.

Substituting the above conditions in eqn. (.2),

$$\sigma_r = A - \frac{B}{r^2}$$

$$-P = A - \frac{B}{R_1^2} \text{ and } 0 = A - \frac{B}{R_2^2}$$

Then $A = \frac{PR_1^2}{(R_2^2 - R_1^2)}$ and $B = \frac{PR_1^2 R_2^2}{(R_2^2 - R_1^2)}$

Substituting A and B in equations 6.1 and 6.2,

$$\sigma_r = \frac{PR_1^2}{(R_2^2 - R_1^2)}\left[1 - \frac{R_2^2}{r^2}\right]$$

$$\sigma_H = \frac{PR_1^2}{(R_2^2 - R_1^2)}\left[1 + \frac{R_2^2}{r^2}\right]$$

2.6.3 Longitudinal stress: -

Consider now the cross-section of a thick cylinder with closed ends subjected to an internal pressure P_1 and an external pressure P_2, (Fig).

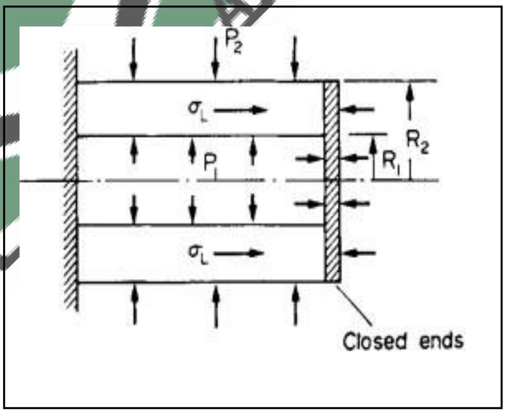

Figure: - Cylinder longitudinal section.

For horizontal equilibrium:

$$P_1{}^2 * \pi R_1{}^2 - P_2{}^2 * \pi R_2{}^2 = \sigma_L * \pi\left[R_2{}^2 - R_1{}^2\right]$$

Where σ_L is the longitudinal stress set up in the cylinder walls,

= A, constant of the Lame equations. ….6.6

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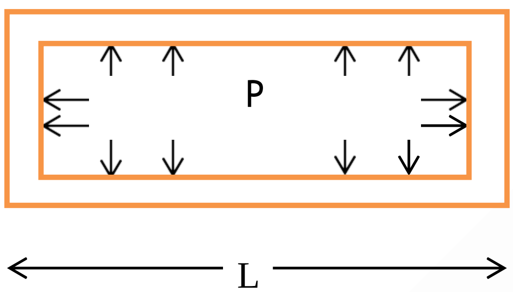
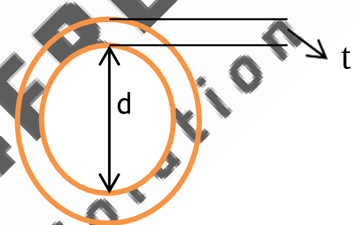
2.7 Thin Cylinders

2.7.1 Introduction

When the thickness of the wall of the cylinder is less than $\frac{1}{1}$ to $\frac{1}{1}$ of the diameter of cylinder then the cylinder is considered as **thin cylinder**.

Otherwise it is termed as thick cylinder.





L=Length of the
cylinder d= Diameter of
cylinder
t = thickness of cylinder
P= Internal Pressure due to fluid

Generally, cylinders are employed for transporting or storing fluids i.e. liquids and gases. Examples-: LPG cylinders, boilers, storage tanks etc.

Due to the fluids inside a cylinder, these are subjected to fluid pressure or internal pressure (Say P). Hence at any point on the wall of the cylinder, three types of stresses are developed in three perpendicular directions. These are:-

1. Circumferential Stress or Hoop Stress (σ_h)
2. Longitudinal Stress (σ_L)
3. Radial Stress (σ_r)

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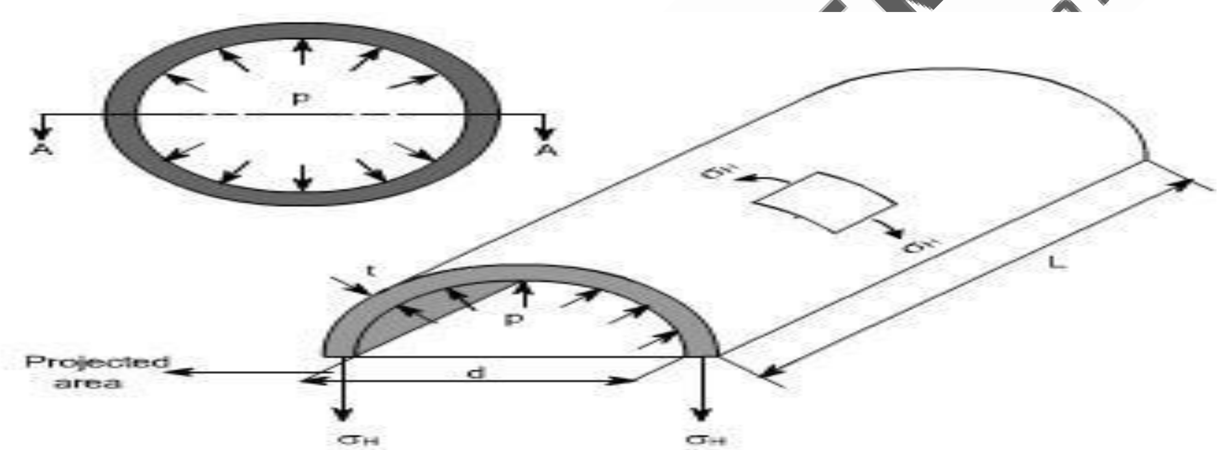
2.7.2 Assumptions in Thin Cylinders

1. It is assumed that the stresses are uniformly distributed throughout the thickness of the wall.

2. As the magnitude of radial stresses is very small in thin cylinders, they are neglected while analyzing thin cylinders i.e. $\sigma_r = 0$

2.7.3 Stresses in Thin Cylinder

1. Circumferential Stress (σ_h):- This stress is directed along the tangent to the circumference of the cylinder. This stress is tensile in nature. This stress tends to increase the diameter.



The bursting in the cylinder will takes place if the force due to internal fluid pressure(P) acting vertically upwards and downwards becomes more than the resisting force due to circumferential stress (σ_h) developed in the cylinder.

Total diametrical Bursting force= P * Projected area of the curved surface
= P * d * L

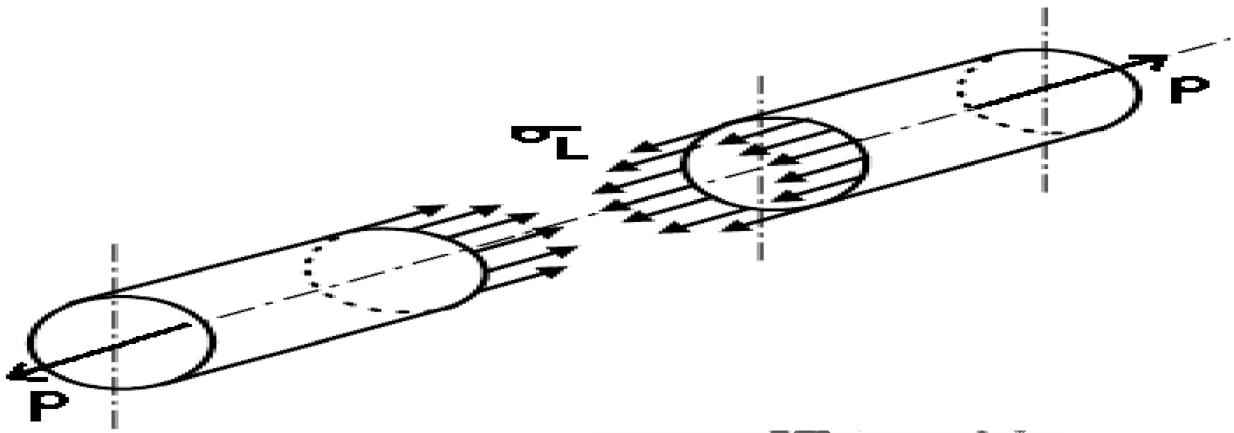
Resisting force due to circumferential stress= $2 * \sigma_h * t * L$

Under equilibrium, Resisting force = Total diametrical Bursting force

$$2 * \sigma_h * t * L = P * d * L$$

Circumferential stress, $\sigma_h = \frac{Pd}{2t}$

2. Longitudinal Stress (σ_L)- This stress is directed along the length of the cylinder. This stress is also tensile in nature. This stress tends to increase the length.



Total longitudinal bursting force (on the ends of cylinder) $= P * \frac{\Pi}{4} * d^2$

Area of crossection where longitudinal stress is developed= $\Pi * d * t$

Resisting force due to longitudinal stress = $\sigma_L * \Pi * d * t$

Under equilibrium, Resisting force = Total longitudinal Bursting force

$$\sigma_L * \Pi * d * t = P * \frac{\Pi}{4} * d^2$$

$$\text{Longitudinal stress, } \sigma_L = \frac{Pd}{4t}$$

Note:- Due to the presence of longitudinal stress and hoop stress, there is shear stress developed in the cylinder. Maximum in-plane shear stress is given by

$$(\tau_{max})_{inplane} = \frac{\sigma_h - \sigma_L}{2} = \frac{Pd}{8t}$$

2.7.4 Strains in Thin Cylinder

1. Strain in longitudinal direction, $\varepsilon_L = \frac{\sigma_L}{E} - \mu \frac{\sigma_h}{E}$

Longitudinal strain = $\varepsilon_L = \frac{Pd}{4tE}(1-2\mu)$

2. Strain in circumferential direction, $\varepsilon_h = \frac{\sigma_h}{E} - \mu \frac{\sigma_L}{E}$

Circumferential strain = $\varepsilon_h = \frac{Pd}{4TE}(2-\mu)$

3. Volumetric strain = $\varepsilon_v = \frac{Pd}{4tE}(5-4\mu)$

Where μ = Poisson's ratio

E= Modulus of Elasticity

2.7.5 For Objective Questions

1. (a) Major principal stress= Hoop stress or circumferential stress (σ_h)

(b)Minor principal stress= Longitudinal stress (σ_L)

2. If σ_t is the permissible stress for the cylinder material, then major principal stress (σ_h) should be less than or equal to σ_t.

$$\sigma_h \leq \sigma_t$$

$$\frac{Pd}{2t} \leq \sigma_t$$

$$t \geq \frac{Pd}{2\sigma_t}$$

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3. In order to produce pure shear state of stress in thin walled cylinders,

$$\sigma_h = -(\sigma_L)$$

4. Maximum shear stress in the wall of the cylinder (**not in-plane shear stress**) is given by :

$$\tau_{max} = \frac{\sigma_h}{2} = \frac{Pd}{4t}$$

5. In case of thin spherical shell, longitudinal stress and circumferential stress are equal and given by

$$\sigma_L = \sigma_h = \frac{Pd}{4t} \text{ (tensile)}$$

$$(\tau_{max})_{inplane} = \frac{\sigma_h - \sigma_L}{2} = 0$$

Module 3

Bending Moment and Shear Force

Objectives:

Determine the shear force, bending moment and draw shear force and bending moment diagrams, describe behaviour of beams under lateral loads. Stresses induced in beams, bending equation derivation & Deflection behaviour of beams

Learning Structure

- 3.1 Types Of Beams
- 3.2 Shear Force
- 3.3 Bending Moment
- 3.4 Shear Force Diagram And Bending Moment
- 3.5 Relations Between Load, Shear And Moment
- 3.6 Problems
- 3.7 Pure Bending
- 3.8 Effect Of Bending In Beams
- 3.9 Assumptions Made In Simple Bending Theory
- 3.10 Problems
- 3.11 Deflection Of Beams
- Outcomes
- Further Reading

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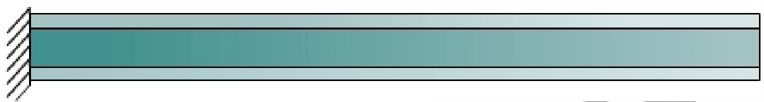
3.1 TYPES OF BEAMS

a) Simple Beam

A simple beam is supported by a hinged support at one end and a roller support at the other end.

b) Cantilever beam



A cantilever beam is supported at one end only by a fixed support.

c) Overhanging beam.

An overhanging beam is supported by a hinge and a roller support with either or both ends extending beyond the supports.

Note: All the beams shown above are the statically determinate beams.

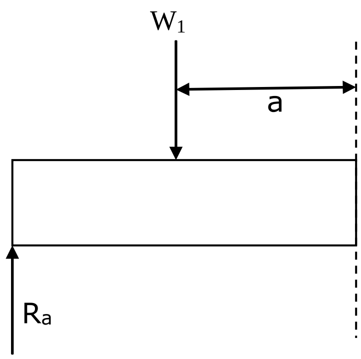


Fig 2 :Shear Force

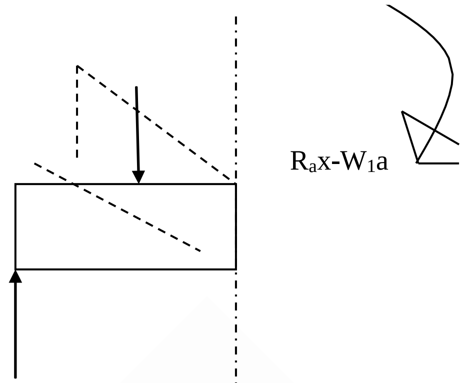


Fig 3 : Bending Moment

Consider a simply supported beam subjected to loads W_1 and W_2. Let R_A and R_B be the reactions at supports. To determine the internal forces at C pass a section at C. The effects of R_A and W_1 to the left of section are shown in Fig (b) and (c). In each case the effect of applied load has been transferred to the section by adding a pair of equal and opposite forces at that section. Thus at the section, moment $M = (W_1a-R_ax)$ and shear force $F = (R_A-W_1)$, exists. The moment M which tend to bends the beam is called bending moment and F which tends to shear the beam is called shear force.

Thus the resultant effect of the forces at one side of the section reduces to a single force and a couple which are respectively the vertical shear and the bending moment at that section. Similarly, if the equilibrium of the right hand side portion is considered, the loading is reduced to a vertical force and a couple acting in the opposite direction. Applying these forces to a free body diagram of a beam segment, the segments to the left and right of section are held in equilibrium by the shear and moment at section.

Thus the shear force at any section can be obtained by considering the algebraic sum of all the vertical forces acting on any one side of the section

Bending moment at any section can be obtained by considering the algebraic sum of all the moments of vertical forces acting on any one side of the section.

3.2 Shear Force

It is a single vertical force developed internally at any point on the beam to balance the external vertical forces and keep the point in equilibrium. It is therefore equal to algebraic sum of all external forces acting to either left or right of the section.

3.3 Bending Moment

It is a moment developed internally at each point in a beam that balances the external moments due to forces and keeps the point in equilibrium. It is the algebraic sum of moments to section of all forces either on left or on right of the section.

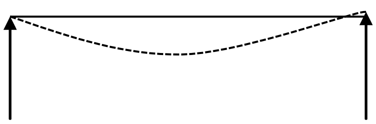
3.3.1 Types of Bending Moment

1) Sagging bending moment

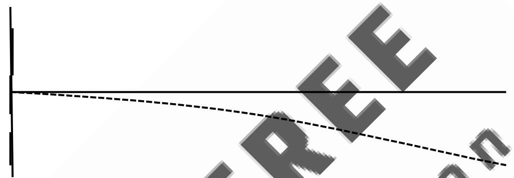
The top fibers are in compression and bottom fibers are in tension.

2) Hogging bending moment

The top fibers are in tension and bottom fibers are in compression.



Sagging Bending Moment



Hogging Bending Moment

3.4 Shear Force Diagram and Bending Moment

3.4.1 Diagram Shear Forces Diagram (SFD)

The SFD is one which shows the variation of shear force from section to section along the length of the beam. Thus the ordinate of the diagram at any section gives the Shear Force at that section.

3.4.2 Bending Moment Diagram (BMD)

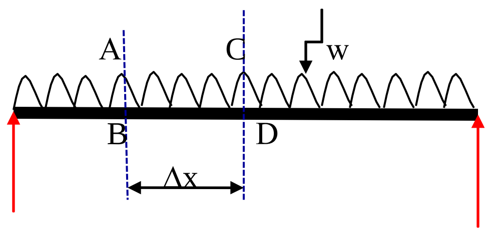
The BMD is one which shows the variation of Bending Moment from section to section along the length of the beam. The ordinate of the diagram at any section gives the Bending Moment at that section.

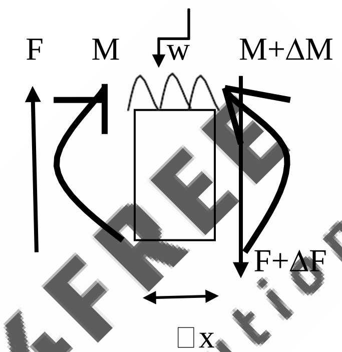
3.4.3 Point of Contraflexure

When there is an overhang portion, the beam is subjected to a combination of Sagging and Hogging moment. The point on the BMD where the nature of bending moment changes from hogging to sagging or sagging to hogging is known as point of contraflexure. Hence, at point of contraflexure BM is zero. The point corresponding to point of contraflexure on the beam is called as point of inflection.

3.5 RELATIONS BETWEEN LOAD, SHEAR AND MOMENT

Consider a simply supported beam subjected to a Uniformly Distributed Load w/m. Let us assume that a portion PQRS of length Δx is cut and taken out. Consider the equilibrium of this portion





$\sum V=0$

F – (F+□F)-w □x =0

Limit □x □ 0, then or F =

Taking moments about section CD for equilibrium

M-(M+□M)+F □x-(w(□x)^2/2) =0

Rate of change of Shear Force or slope of SFD at any point on the beam is equal to the intensity of load at that point.

Properties of BMD and SFD

1) when the load intensity in the region is zero, Shear Force remains constant and Bending Moment varies linearly.

2) When there is Uniformly Distributed Load (UDL), Shear Force varies linearly and BM varies parabolically.

3) When there is Uniformly Varying Load (UVL), Shear Force varies parabolically and Bending Moment varies cubically.

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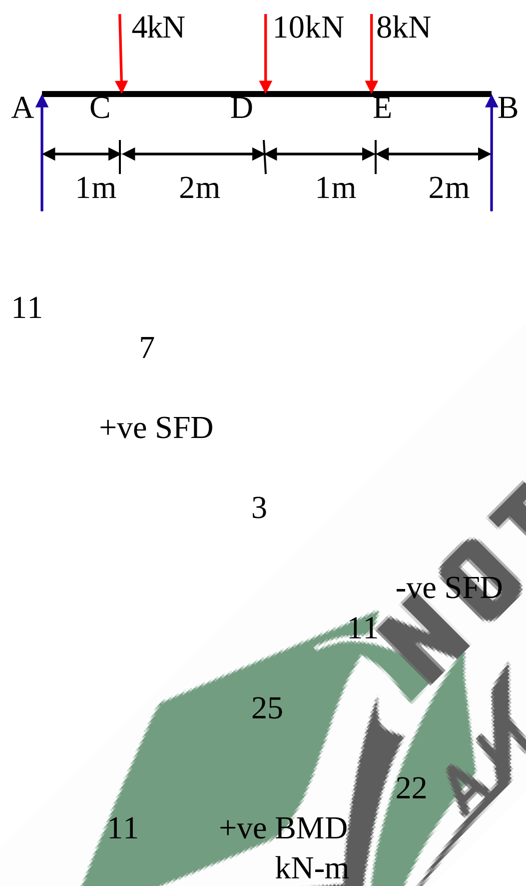
3.6 Problems:

1. **A simply supported beam is carrying point loads, as shown in figure. Draw the SFD and BMD for the beam.**



To draw S.F.D. and B.M.D. we need R_A and R_B.

By taking moment of all the forces about point A, we get

$R_B \times 6 - (8 \times 4) - (10 \times 3) - (4 \times 1) = 0$

$R_B = 11$ kN

From condition of static equilibrium:

$R_A + 11 - 4 - 10 - 8 = 0$

$R_A = 11$ kN

Shear Force Calculations

SF at A $F_A = + R_A = + 11$ kN

SF left of C $F_C = + R_A = +11$ kN

SF right of C $F_C = + 11 - 4 = + 7$ kN

SF left of D $F_D = + 11 - 4 - 10 = 7$ kN

SF right of D $F_D = + 11 - 4 - 10 = - 3$ kN

SF left of E $F_E = + 11 - 4 - 10 = - 3$ kN

SF left of E $F_E = + 11 - 4 - 10 - 8 = - 11$ kN

SF left of B $F_B = + 11 - 4 - 10 - 8 = - 11$ kN

Bending moment Calculations

At x = 0, $M_A = 0$

At x = 1 m; $M_C = + R_A.1 = 11 \times 1 = 11$ kN m

At x = 3 m; $M_D = 11 \times 3 - 4 (3 - 1) = 25$ kN m

At x= 4m

$M_E = 11 \times 4 - 4 (4 - 1) - 10 (4 - 3) = 22$ kN m

At X= 6m $M_B = 0$

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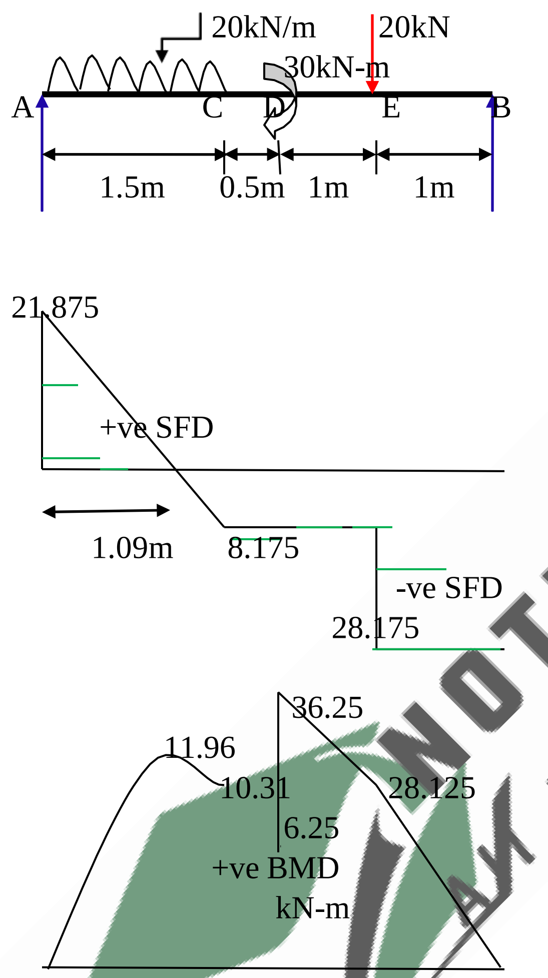
2. Draw the SF and BM diagram for the simply supported beam loaded as shown in fig.



To draw S.F.D. and B.M.D. we need R_A and R_B.

By taking moment of all the forces about point A, we get

$R_B \times 4 - (20 \times 3) - (20 \times 1.5^2/2) - 30 = 0$

$R_B = 28.125$ kN

From condition of static equilibrium:

$R_A + 28.125 - 30 - 20 = 0$

$R_A = 21.875$ kN

Shear Force Calculations

SF at A $F_A = + R_A = + 21.875$ kN

SF left of C $F_C = +21.875 - 30 = -8.175$ kN SF

right of C $F_C = +21.875 - 30 = -8.175$ kN SF

left of D $F_D = +21.875 - 30 = -8.175$ kN SF

right of D $F_D = +21.875 - 30 = -8.175$ kN SF

left of E $F_E = +21.875 - 30 = -8.175$ kN SF

left of E $F_E = +21.875-30-20 = -28.125$ kN SF

left of B $F_B = +21.875-30-20 = -28.125$ kN

SF is zero, between A & C at x = 1.09m

Bending moment Calculations

At x = 0, $M_A = 0$

At x = 1.5 m;

$M_C = 21.875 \times 1.5 - 20 \times 1.5^2/2 = 10.31$ kN m At x = 1.09 m;

$M_{max} = 21.875 \times 1.09 - 20 \times 1.09^2/2 = 11.96$ kN m

At x = 2 m;

$M_D = 21.875 \times 2 - 20 \times 1.5(0.75+0.5) = 6.25$ kN m

At x = 2 m;

$M_D = 21.875 \times 2 - 20 \times 1.5(0.75+0.5)+30$

$= 36.25$ kN m

At x= 3m

$M_E = 21.875 \times 3 - 20 \times 1.5(0.75+1.5)+30$

$= 28.125$ kN m

At X= 4m $M_B = 0$

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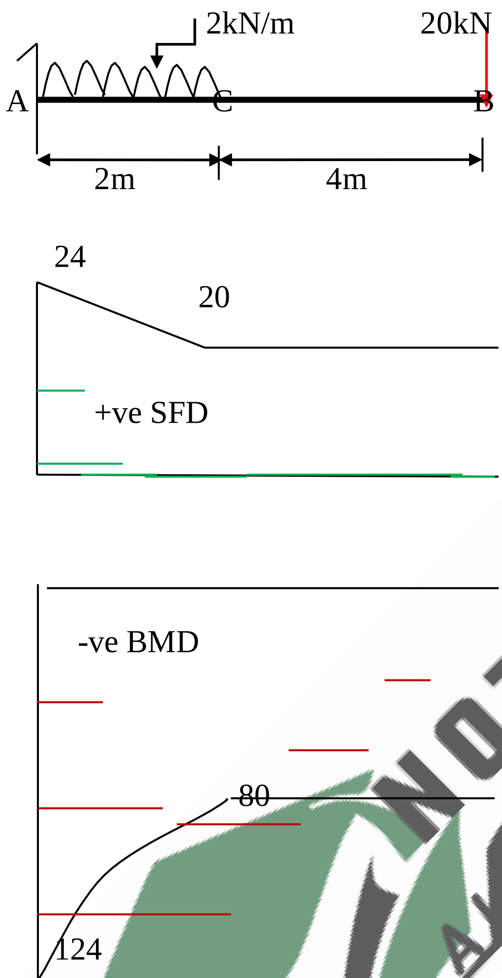
3. A cantilever is shown in fig. Draw the BMD and SFD. What is the reaction at supports?



To draw S.F.D. and B.M.D. we need R_A and R_B.

By taking moment of all the forces about point A, we get

$R_B \times 4 - (20 \times 3) - (20 \times 1.5^2/2) - 30 = 0$

$R_B = 28.125$ kN

From condition of static equilibrium:

$R_A + 28.125 - 30 - 20 = 0$

$R_A = 21.875$ kN

Shear Force Calculations

SF at B $\quad F_B = + 20$ kN

SF right of C $\quad F_C = + 20$ kN

SF left of C $\quad F_C = + 20$ kN

SF right of A $\quad F_A = +20+ 2\text{x}\ 2 = 24$ kN

Since the moment is considered from right to left the anticlockwise moments are considered to be +ve and clockwise moments are considered to be -ve

Bending moment Calculations

At x = 0, $\quad M_B = 0$

At x = 4 m; $M_c = -20\text{x}\ 4 = - 80$ kN-m At x = 6 m;

$M_A = -(20 \times 6 + 2 \times 2^2/2) = 124$ kN m

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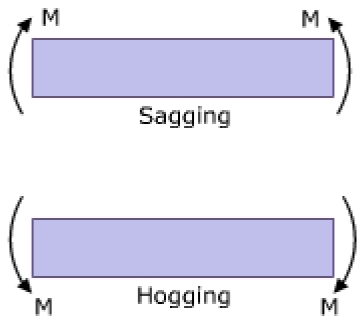
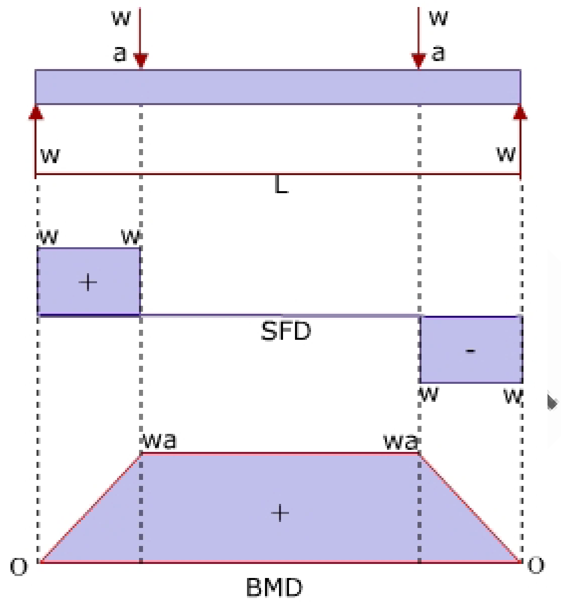
Stresses in Beams

3.7 Pure Bending





A beam or a part of a beam is said to be under pure bending if it is subjected to only Bending Moment and no Shear Force.

3.8 Effect of Bending in Beams

The figure shows a beam subjected to sagging Bending Movement. The topmost layer is under maximum compressive stress and bottom most layer is under maximum tensile stress. In between there should be a layer, which is neither subjected to tension nor to compression. Such a layer is called "Neutral Layer". The projection of Neutral Layer over the cross section of the beam is called "Neutral Axis".

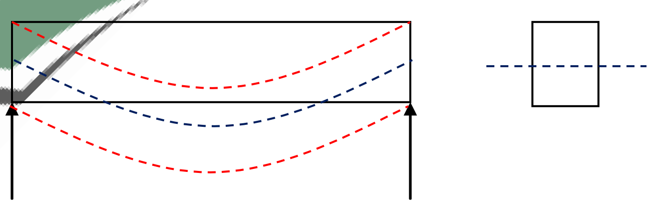


Fig-1

When the beam is subjected to sagging, all layers below the neutral layer will be under tension and all layers above neutral layer will be under compression. When the beam is subjected to hogging, all layers above the neutral layer will be under tension and all the layers below neutral layer will be under compression and vice versa if it is hogging bending moment

3.9 Assumptions made in simple bending theory

- The material is isotropic and homogenous.
- The material is perfectly elastic and obeys Hooke's Law i.e., the stresses are within the limit of proportionality.
- Initially the beam is straight and stress free.
- Beam is made up of number of layers and they undergo bending independently.
- Bending takes place over an arc of a circle and the radius of curvature is very large when compared to the dimensions of the beam.
- Normal plane sections before bending remain normal and plane even after bending.
- Young's Modulus of Elasticity is same under tension a ndcompression.

3.9.1 Euler- Bernoulli bending Equation (Flexure Formula)

$$\frac{M}{I}=\frac{f}{y}=\frac{E}{R}$$

where,

M = Resisting moment developed inside the material against applied bending movement and is numerically equal to bending moment applied (Nmm)

I = Moment of Inertia of cross section of beam about the Neutral Angle. (mm^4)

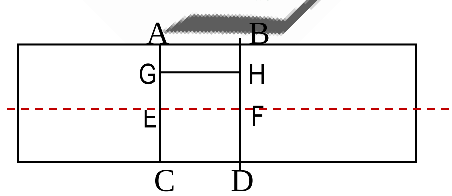
F = Direct Stress (Tensile or Compression) developed in any layer of the beam (N/mm^2)

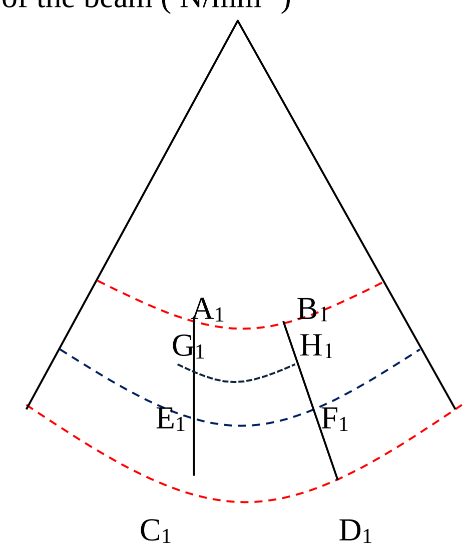
Y = Distance of the layer from the neutral axis (mm)

E = Young's Modulus of Elasticity of the material of the beam (N/mm^2)

R = Radius of curvature of neutral layer (mm)

Euler- Bernoulli's Equation





Consider two section very close together (AB and CD). After bending the sections will be at $A_1 B_1$ and $C_1 D_1$ and are no longer parallel. AC will have extended to A_1 C1 and $B_1 D_1$ will have compressed to B_1D_1. The line EF will be located such that it will not change in length. This surface is called neutral surface and its intersection with Z-Z is called the neutral axis.

The development lines of A'B' and C'D' intersect at a point 0 at an angle of θ radians and the radius of $E_1F_1 = R$.

Let y be the distance(E'G') of any layer H_1G_1 originally parallel to EF.

Then $H_1G_1/ E_1F_1 =(R+y)\theta /R \theta = (R+y)/R$

and the strain at layer H_1G_1= = $(H_1G_1'- HG) / HG = (H_1G_1- HG) / EF$

$$= [(R+y)\theta - R \theta] /R \theta$$

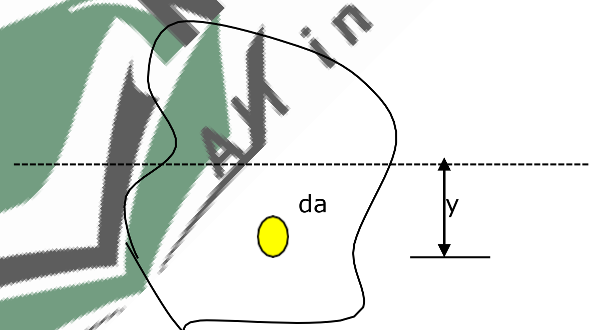
$$= y /R.$$

The relation between stress and strain is σ= E. Therefore

$$\sigma = E. = E. y /R$$

$$\sigma / E = y / R$$

Let us consider an elemental area ‘da ‘at a distance y, from the Neutral Axis.



Section Modulus(Z)

$$F = \frac{M}{I} . y$$

$$\Rightarrow f_{max} = \frac{M_{max}}{I} . y_{max}$$

$$i,e., M_{max} = f_{max} . \frac{I}{y_{max}}$$

$$\text{Therefore,} \quad M_{max} = f_{max} . Z$$

Section modulus of a beam is the ratio of moment of inertia of the cross section of the bea m about the neutral axis to the distance of the farthest fiber from neutral axis.

$$\text{Therefore,} \quad Z = \frac{I}{y_{max}} \quad \text{unit} = mm^3$$

More the section modulus more will be the moment of resistive (or) moment carrying capacity of the beam. For the strongest beam, the section modulus must be maximum.

3.10 Problems

1. **A steel bar 10 cm wide and 8 mm thick is subjected to bending mome nt. The radius of neutral surface is 100 cm. Determine maximum and minimum bending stress in the beam.**

Solution : Assume for steel bar $E = 2 \times 10^5$ N/mm^2

y_{max} = 4mm

R = 1000mm

$f_{max} = E.y_{max}/R = (2 \times 105 \times 4)/1000$

We get maximum bending moment at lower most fiber, Because for a simply supported beam tensile stress (+ve value) is at lower most fiber, while compressive stress is at top most fiber (–ve value).

F_{max} = 800 N/mm^2

f_{min} occurs at a distance of – 4mm

R = 1000mm

$f_{min} = E.y_{min}/R = (2 \times 105\ x - 4$

)/1000 f_{min}**= –800 N/mm^2**

2. **A simply supported rectangular beam with symmetrical section 200mm in dept h has mome nt of inertia of 2.26 x 10^{-5} m^4 about its neutral axis. Determine the longest span over which the beam would carry a uniformly distributed load of 4kN/m run such that the stress due to bending does not exceed 125 MN/m^2.**

Solution: Given data:

Depth d = 200mm = 0.2m
I = Moment of inertia = 2.26 × 10-5 m4
UDL = 4kN/m
Bending stress s = 125 MN/m^2 = 125 × 10^6 N/m^2
Span = ?

Since we know that Maximum bending moment for a simply supported beam with UDL on its entire span is given by = $WL^2/8$

i.e; $M = WL^2/8$ -------------(A)

From bending equation $M/I = f/y_{max}$

$y_{max} = d/2 = 0.2/2 = 0.1$m

M = f.I/$y$max = [(125 × 106) × (2.26 × 10^{-5})]/ 0.1 = 28250 Nm

Substituting this value in equation (A); we get

28250 = (4 × 103)L^2/8

***L* = 7.52m**

3. **Find the dime nsion of the strongest rectangular beam that can be cut out of a log of 25 mm diameter.**

Solution: $b^2 + d^2 = 25^2$

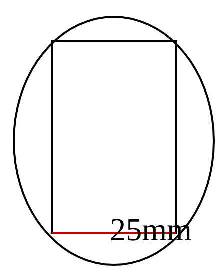
$d^2 = 25^2 - b^2$

we Know — ; —

M = f (I/y) = f.Z

M will be maximum when Z will be maximum

Z = I/y = $(bd^3/12)/(d/2) = bd^2/6 = b.(25^2 - b^2)/6$



The value of Z maximum at dZ/db = 0;

i.e.; $d/db[25^2b/6 - b^3/6] = 0$

$25^2/6 - 3b^2/6 = 0$

$b^2 = 25^2/3$

b = 14.43 mm

d = 20.41 mm

NOTES4FREE All in One Solution

3.11 Deflection of Beams

3.11.1 INTRODUCTION

Under the action of external loads, the beam is subjected to stresses and deformation at various points along the length. The deformation is caused due to bending moment and shear force. Since the deformation caused due to shear force in shallow beams is very small, it is generally neglected.

3.11.1.1 Elastic Line:

It is a line which represents the deformed shape of the beam. Hence, it is the line along which the longitudinal axis of the beam bends.

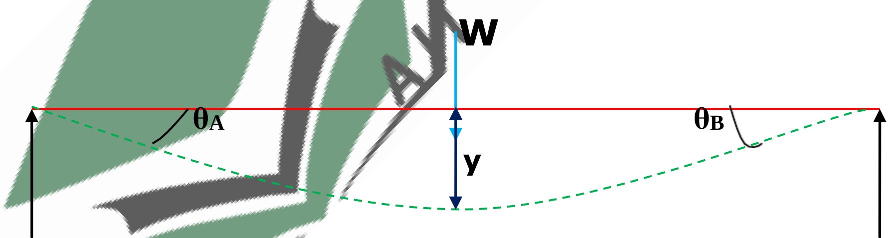
3.11.1.2 Deflection:

Vertical displacement measured from original neutral surface (refer to earlier chapter) to the neutral surface of the deformed beam.

3.11.1.3 Slope:

Angle made by the tangent to the elastic curve with respect to horizontal

The designers have to decide the dimensions of beam not only based on strength requirement but also based on considering deflection. In mechanical components excessive deflection causes mis-alignment and non performance of machine. In building it give rise to psychological unrest and sometimes cracks in roofing materials. Deflection calculations are required to impose consistency conditions in the analysis of indeterminate structures.



3.11.1.4 Strength:

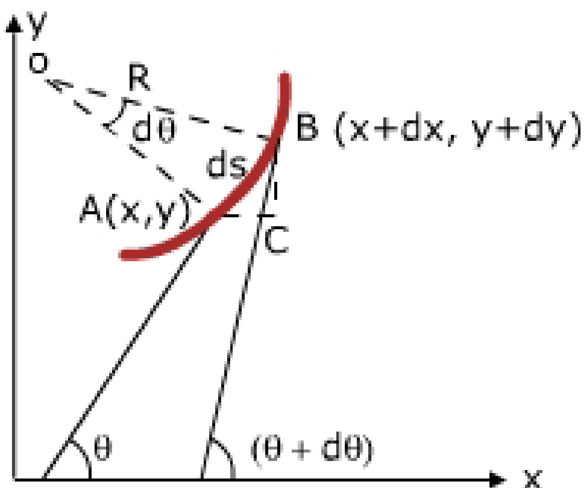
It is a measure of the resistance offered by the beam to load

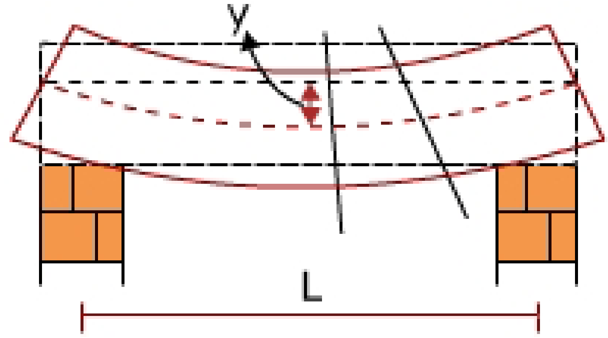
3.11.1.5 Stiffness:

It is a measure at the resistance offered by the beam to deformation. Usually span / deflection is used to denote the stiffness. Greater the stiffness, smaller will be the deflection. The term (EI) called "flexural rigidity" and is used to denote the stiffness.

3.11.2 Flexural Rigidity

The product of Young's modulus and moment of inertia (EI) is used to denote the flexural rigidity.





Let AB be the part of the beam which is bent into an arc of the circle. Let (x,y) be co- ordinates of A and (x + dx, y + dy) be the co-ordinates of B. Let the length of arc AB = ds. Let the tangents at A and B make angles q and (q + d q) with respect to x-axis.

We have __

Differentiating both sides with respect of x;

$$\sec^2\theta \cdot \frac{d\theta}{dx} = \frac{d^2y}{dx^2}$$

$$\sec^2\theta \quad \frac{d\theta}{ds} \quad \frac{ds}{dx} = \frac{d^2y}{dx^2} \qquad \text{----- (1)}$$

we have from figure $ds = R\,d\theta$; $\frac{d\theta}{ds} = \frac{1}{R}$

again in Δ^{le} ABC, $\frac{ds}{dx} = \sec\theta$

From eq. 1; $\frac{d^2y}{dx^2} = \sec^2\theta \, \frac{1}{R} \, \sec\theta$

$$\frac{1}{R} = \frac{\frac{d^2y}{dx^2}}{\sec^2\theta \cdot \sec\theta} = \frac{\frac{d^2y}{dx^2}}{(1+\tan^2\theta)^{\frac{3}{2}}}$$

$$\frac{1}{R} = \frac{\frac{d^2y}{dx^2}}{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}}$$

Since dy/dx is small, its square is still small, neglecting $(dy/dx)^2$; we have

$$\frac{1}{R} = \frac{d^2y}{dx^2}$$

From bending theory $\frac{M}{I} = \frac{E}{R}$

$$\frac{M}{EI} = \frac{1}{R} \qquad \text{or}$$

$$\frac{M}{EI} = \frac{d^2y}{dx^2}$$

$$M = EI\,\frac{d^2y}{dx^2}$$

This is also known as Euler - Bernoulli's equation.

NOTE:

- While deriving Y-axis is taken upwards
- Curvature is concave towards the positive y axis.
- This occurs for sagging BM, which is positive.

Sign Convention

Bending moment  Sagging +vc

If Y is $+^{ve}$ - Deflection is upwards

Y is $-^{ve}$ - Deflection is downwards

If θ is $+^{ve}$ – Slope is Anticlockwise

θ is $-^{ve}$ – Slope is clockwise

Methods of Calculating Deflection and Slope

- Double Integration method
- Macaulay's method
- Strain energy method
- Moment area method
- Conjugate Beam method

Each method has certain advantages and disadvantages.

Relationship between Loading, S.F, BM, Slope and Deflection

If	Y	-	deflection
Differentiating	$\frac{dy}{dx}$	-	Slope (θ)
Differentiating	$\frac{d^2y}{dx^2}$	-	M. Bending moment
Differentiating	$\frac{dM}{dx}$	=	$\frac{d^3y}{dx^3}$ = Shear force (F)
Differentiating	$\frac{dF}{dx}$	=	$\frac{d^4y}{dx^4}$ = Loading (W)

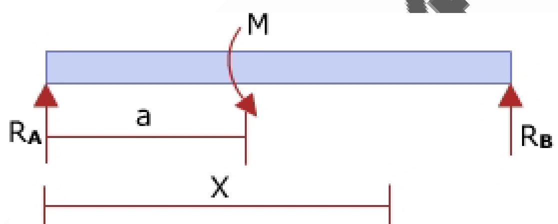
NOTES4FREE
All in One Solution

3.11.3 Macaulay's Method

1. Take the origin on the extreme left.
2. Take a section in the last segment of the beam and calculate BM by considering left portion.
3. Integrate (x-a) using the formula

$$\int (x\text{-}a)\,dx = \frac{(x-a)^2}{2}$$

4. If the expression $(x\text{-}a)^n$ becomes negative on substituting the value of x, neglect the terms containing the factor $(x\text{-}a)^n$
5. If the beam carries UDL and if the section doesn't cuts the UDL, extend the UDL upto the section and impose a UDL in the opposite direction to counteract it.
6. If a couple is acting, the BM equation is modified as; $M = R_A x + M (x\text{-}a)^0$.



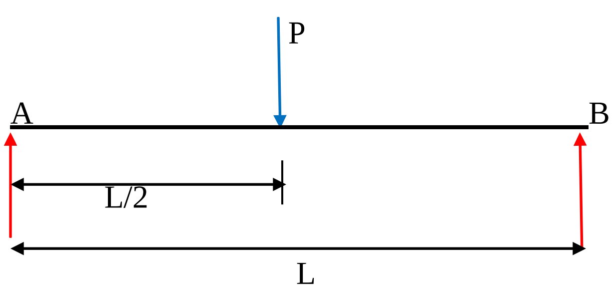
7. The constant C_1 and C_2 all determined using boundary conditions.

a) S.S. Beam – Deflection is zero at supports

b) Cantilever – Deflection and slope are zero at support.

3.11.4 Problems:

1. **Determine the maximum deflection in a simply supported beam of length L carrying a concentrated load P at its midspan**.



$$EI\,y'' = \tfrac{1}{2}Px - P\langle x - \tfrac{1}{2}L\rangle$$

$$EI\,y' = \tfrac{1}{4}Px^2 - \tfrac{1}{2}P\langle x - \tfrac{1}{2}L\rangle^2 + C_1 \text{.....................(1)}$$

$$EI\,y = \tfrac{1}{12}Px^3 - \tfrac{1}{6}P\langle x - \tfrac{1}{2}L\rangle^3 + C_1x + C_2 \text{.....................(2)}$$

At x =0; y =0 □ C_2 =0
At x = L y =0

$$0 = \tfrac{1}{12}PL^3 - \tfrac{1}{48}PL^3 + C_1L$$
$$C_1 = -\tfrac{1}{16}PL^2$$

Maximum deflection occurs at x = L/2
Substituting the values of x and C_1 in equation…. (2)

$$EI\,y_{max} = \tfrac{1}{12}P(\tfrac{1}{2}L)^3 - \tfrac{1}{6}P(\tfrac{1}{2}L - \tfrac{1}{2}L)^3 - \tfrac{1}{16}PL^2(\tfrac{1}{2}L)$$

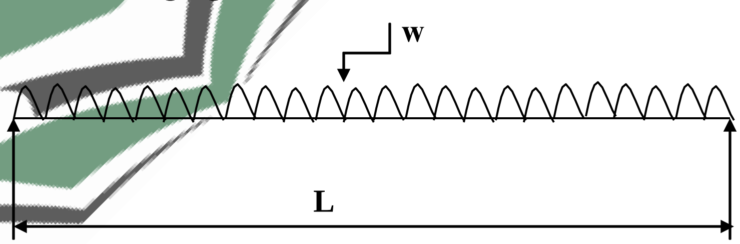
$$y_{max} = -\frac{PL^3}{48EI}$$

The negative sign indicates that the deflection is below the undeformed neural axis

$$\delta_{max} = \frac{PL^3}{48EI}$$

3. Determine the maximum deflection in a simply supported beam of length L carrying a uniformly distributed load 'w' for the entire length of the beam.

Solution : From the following fig



$$EI\,y'' = \tfrac{1}{2}w_oLx - \tfrac{1}{2}w_ox^2$$
$$EI\,y' = \tfrac{1}{4}w_oLx^2 - \tfrac{1}{6}w_ox^3 + C_1 \text{ ....(1)}$$
$$EI\,y = \tfrac{1}{12}w_oLx^3 - \tfrac{1}{24}w_ox^4 + C_1x + C_2 \text{............(2)}$$

At x =0 y=0 and C_2 =0

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At x =L y =0

$0 = \frac{1}{12}w_oL^4 - \frac{1}{24}w_oL^4 + C_1L$

$C_1 = -\frac{1}{24}w_oL^3$

Substituting the C_1 values in equation 2 we get

$$EI\,y = \frac{1}{12}w_oLx^3 - \frac{1}{24}w_ox^4 - \frac{1}{24}w_oL^3x$$

x = L/2, y is maximum due to symmetric loading

$$EI\,y_{max} = \frac{1}{12}w_oL(\frac{1}{2}L)^3 - \frac{1}{24}w_o(\frac{1}{2}L)^4 - \frac{1}{24}w_oL^3(\frac{1}{2}L)$$

$$EI\,y_{max} = -\frac{5}{384}w_oL^4$$

$$\delta_{max} = \frac{5w_oL^4}{384EI}$$

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Module 4

TORSION OF SHAFTS

Objectives:

Explain the structural behavior of members subjected to torque, Calculate twist and stress induced in shafts subjected to bending and torsion. & Understand the concept of stability and derive crippling loads for columns

Learning Structure

- 4.1 Bending Moment
- 4.2 ASSUMPTIONS IN TORSION THEORY
- 4.3 Problems
- 4.4 Columns and Struts:
- 4.5 SLENDERNESS RATIO
- 4.6 EFFECTIVE LENGTH OF COLUMN
- .7 Euler's Theorem
- Outcomes
- Further Reading

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4.1 Bending Moment

The moment applied in a vertical plane containing the longitudinal axis is resisted by longitudinal tensile and compressive stresses of varying intensities across the depth of bea m and are called as bending stresses. The moment applied is called Bending Moment.

4.1.1 Torsional Moment

The moment applied in a vertical plane perpendicular to the longitudinal axis i.e., in the plane of the cross section of the member, it causes twisting of layers which will be resisted by the shear stresses. The moment applied is called Torsion Moment or Torsional Moment. Torsion is useful form of transmitting power and its application is seen in screws and shafts.

4.2 ASSUMPTIONS IN TORSION THEORY

1. Material is homogenous and isotropic
2. Plane section remain plane before and after twisting i.e., no warpage of planes.
3. Twist along the shaft is uniform.
4. Radii which are straight before twisting remain straight after twisting.
5. Stresses are within the proportional limit.

4.2.1 DERIVATION OF TORSIONAL EQUATION:

Torsional Rigidity

We have $$\theta = \frac{TL}{CI_p}$$

As product (CI_P) is increased deformation q reduces. This product gives the strength of the section to resist torque and is called Torsional rigidity.

NOTES4FREE in One Solution

Polar Modulus : (Z_P)

$$\text{We have} \qquad \frac{T}{I_p} = \frac{f}{r}$$

Maximum shear stress occurs at surface

$$T = f_s \cdot \frac{I_p}{R}$$

$$T = f_s \cdot Z_p$$

$$\text{Where } Z_P \text{ is called polar modulus} \qquad Z_p = \frac{I_p}{R}$$

POWER TRANSMITTED BY SHAFT

Power transmitted = Torsional moment x Angle through which the torsional moment rotates / unit tank

If the shaft rotates with 'N' rpm

$$= T\left(\frac{N \,.\, 2\pi}{60}\right)$$

$$\text{Power transmitted} = \frac{2\pi NT}{60} \ \text{N.m / sec}$$

$$\text{Power transmitted in kw} = \frac{2\pi NT}{60 \times 1000} = \frac{\pi NT}{30{,}000}$$

Note:

N is in rpm and T is in N-m

4.3 Problems:

1. **Find the maximum shear stress induced in a solid circular shaft of diameter 200 mm when the shaft transmits 190 kW power at 200 rpm**

Given data: Power transmitted, P = 190 kW, $I_p = 1.57 \times 10^8$ mm^4

speed N = 200 rpm and diameter of shaft = 200 mm.

Substituting all the values $f_s = 5.78 N/mm^2$.

2. **A solid shaft of mild steel 200 mm in diameter is to be replaced by hollow shaft of allowable shear stress is 22% greater. If the power to be transmitted is to be increased by 20% and the speed of rotation increased by 6%, determine the maximum internal diameter of the hollow shaft. The external diameter of the hollow shaft is to be 200 mm.**

Solution: Given that:

Diameter of solid shaft $d = 200$ mm

For hollow shaft diameter, $d_0 = 200$ mm

Shear stress; $t_H = 1.22\, t_s$

Power transmitted; $P_H = 1.20\, P_s$

Speed $N_H = 1.06\, N_s$

As the power transmitted by hollow shaft

$P_H = 1.20\, P_s$

$(2\pi.N_H.T_H)/60 = (2\pi.N_s.T_s)/60 \times 1.20$

$N_H.T_H = 1.20\, N_s.T_s$

$1.06\, N_s.T_H = 1.20\, N_sT_s$

$1.06/1.20\, T_H = T_s$

$1.06/1.20 \times \pi/16\, t_H\, [(d_0)^4 - (d_i)^4/d_0] = \pi/16\, t_s.[d]^3$

$1.06/1.20 \times 1.22\, t_s\, [(200)^4 - (d_i)^4/200] = t_s \times [200]^3$

$d_i = 104$ mm

3. **A solid shaft is subjected to a maximum torque of 1.5 MN.cm Estimate the diameter for the shaft, if the allowable shearing stress and the twist are limited to 1 kN/cm² and 1o respectively for 200 cm length of shaft. Take G = 80 × 10⁵ N/cm²**

Solution: Since we have

$T/I_p = f_s/r = C.\theta/L$

$f_s = T.I_p\, r = 1.5 \times 10^6 / \theta/32.d^4\, .\, d/2$

$1 \times 10^3 * 2\pi /1.5 \times 10^6 * 32 = 1/d^3$

$d = 19.69$ cm

$\theta = T.L / C.I_p$

$1.5 \times 10^6 * 2\pi / 1.5 \times 10^6 * 32 = 1 / d^3$

$d = 19.69$ cm

$\theta = T.L / C.Ip$

$1.5 \times 10^6 * 200\, d/80 * 10^5 * \pi/32\, d^4 = \pi/180$

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All in One Solution

$d^3 = 1.5 \times 10^6 * 180 * 200 * 32 / (80 * 10^5 * \pi * \pi)$

$d = 27.97$ cm

4. **A hollow circular shaft of 20 mm thickness transmits 300 kW power at 200 r.p.m. Determine the external diameter of the shaft if the shear strain due to torsion is not to exceed 0.00086. Take modulus of rigidity = 0.8×10^5 N/mm^2.**

Solution: Let d_i = inner diameter of circular shaft

d_0 = outer diameter of circular shaft

Then $d_0 = d_i + 2t$ where t = thickness

$d_0 = d_i + 2 * 20$

$d_0 = d_i + 40$

$d_i = d_0 - 40$

Since we have

Power transmitted = 2π NT/60

$300{,}000 = 2\pi * 200 * T / 60$

→ $T = 14323900$ N mm

Also, we have $C = f_s/y$

→ $0.8 * 10^5 = f_s / 0.00086$

→ $f_s = 68.8$ N/mm^2

Now $T = \pi/16.\ f_s.(d_0^4 - d_i^4 / d_0)$

$14323900 = f_s /16 * 68.8 (d_0^4 - (d_0 - 40)^4 / d_0)$

$1060334.6\ d_0 = d_0^4 - (d_0 - 40)^4$

$= (d_0^2 - d_0^2 + 80d_0 - 1600)*(d_0^2 + d_0^2 - 80d_0 + 1600)$

$= (80d_0 - 1600)(2d_0^2 - 80d_0 + 1600)$

$= 80 (d_0 - 2_0) * 2 * (d_0^2 - 40 d_0 + 800)$

$= 160 (d_0^3 - 40d_0^2 + 800 d_0 - 20 d_0^2 + 800 d_0 - 16000)$

→ $1060334.6\ d_0 / 160 = d_0^3 - 60d_0^2 + 1600d_0 - 16000$

→ $6627\ d_0 = d_0^3 - 60d_0^2 + 1600\ d_0 - 1600$

→ $d_0^3 - 60d_0^2 + 1600d_0 - 6627\ d_0 - 16000 = 0$

→ $d_0^3 - 60d_0^2 - 5027\ d_0 - 16000 = 0$

Using trial and error method to solve the above equation for d_0, we get $d_0 = 107.5$ mm.

NOTES4FREE
All in One Solution

Elastic Stability of Columns

4.4 Columns and Struts:

Columns and struts are structural members subjected to compressive forces. Theses members are often subjected to axial forces, although they may be loaded eccentrically. The lengths of these members are large compared to their lateral dimensions. In general vertical compressive members called columns and inclined compressive members are called struts.

4.4.1 CLASSIFICATION OF COLUMNS:

Columns are generally classified in to three general types. The distinction between types of columns is not well, but a generally accepted measure is based on the slenderness ratio (l_e/r_{min}).

4.4.1 .1 Short Column :

A short column essentially fails by crushing and not by buckling. A column is said to be short, if $l_e/b \leq 15$ or $l_e/r_{min} \leq 50$, where l_e = effective length, b = least lateral dimension and r_{min}= minimum radius of gyration.

4.4.1 .2 Long Column :

A long column essentially fails by buckling and not by crushing. In long columns, the stress at failure is less than the yield stress. A column is said to be long $l_e/b > 15$ or $l_e/r_{min} > 50$.

4.4.1 .3 Intermediate Column :

An intermediate column is one which fails by a combination of crushing and buckling.

4.4.1.4 Elastic Stability of Column

Consider a long column subjected to an axial load P as shown in figure. The column deflects laterally when a small test load F is applied in lateral direction. If the axial load is small, the column regains its stable position when the test load is removed. At a certain value of the axial load, the column fails to regain its stable position even after the removal of the test load. The column is then said to have failed by buckling and the corresponding axial load is called Critical Load or failure Load or Crippling Load

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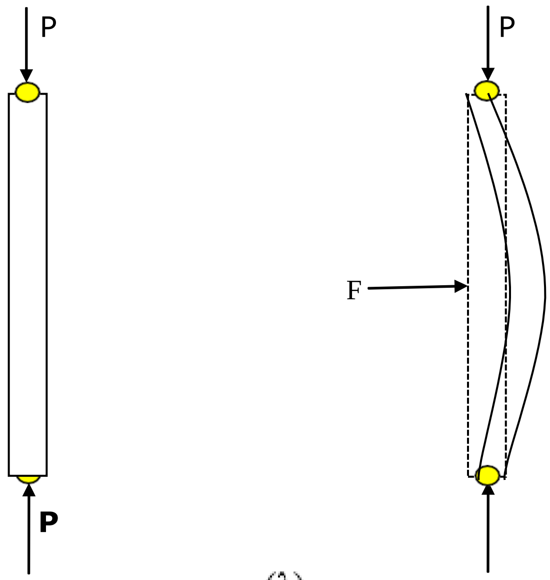


4.5 SLENDERNESS RATIO (λ)

Slenderness ratio is defined as the ratio of effective length (l_e) of the column to the minimum radius of gyration (r_{min}) of the cross section.

$$\lambda = \frac{l_e}{r_{min}}$$

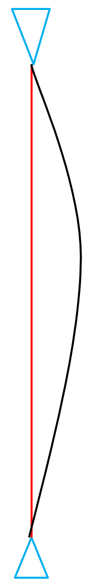
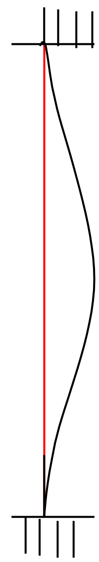
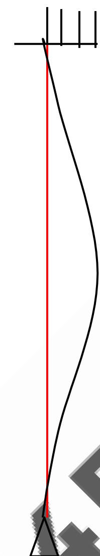
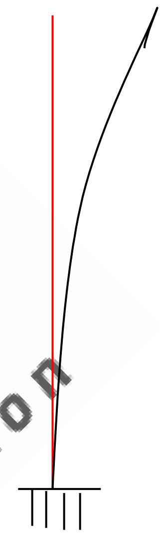
Since an axially loaded column tends to buckle about the axis of minimum moment of inertia (I_{min}), the minimum radius of gyration is used to calculate slenderness ratio.

Further, —, where A is the cross sectional area of column.

4.6 EFFECTIVE LENGTH OF COLUMN (*le*)

Effective length is the length of an imaginary column with both ends hinged and whose critical load is the same as the column with given end conditions. It should be noted that the material and geometric properties should be the same in the above columns. The effective length of a column depends on its end condition. Following are the effective lengths for some standard cases.

NOTES4FREE
All in One Solution

Both endsare hinged	Both ends arefixed	One end fixed and other end hinged	One end fixed and other end is free
			
Effective Length L_e = L	Effective Length L_e = _	Effective Length L_e = _	Effective Length L_e = 2L

4.7 Euler's Theorem

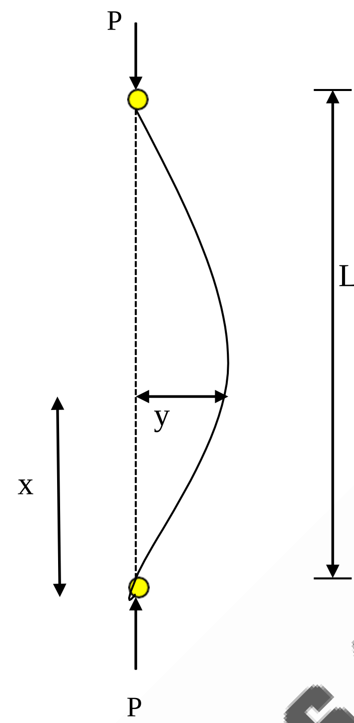
Theoretical analysis of the critical load for long columns was made by the great Swiss mathematician Leonard Euler (pronounced as Oiler). The assumptions made in the analysis are as follows:

- The column is long and fails by buckling.
- The column is axially loaded.
- The column is perfectly straight and the cross sections are uniform (prismatic).
- The column is initially free from stress.
- The column is perfectly elastic, homogeneous and isotropic.

4.7.1 Eulers Critical Load for Long Columns

Case (1) Both ends hinged

Consider a long column with both ends hinged subjected to critical load P as shown.



Consider a section at a distance x from the origin. Let y be the deflection of the column at this section. Bending moment in terms of load P and deflection y is given by

$$M = -P\,y \qquad \text{------------ (1)}$$

We can also write that for beams/columns the bending moment is proportional to the curvature of the beam, which, for small deflection can be expressed as

or ……………(2)

where E is the Young's modulus and I is the moment of Inertia.

Substituting eq.(1) in eq.(2)

$$-P\,y = EI\,\frac{d^2y}{dx^2}$$

or

$$\frac{d^2y}{dx^2} + \left(\frac{P}{EI}\right) y = 0$$

NOTES4FREE

This is a second order differential equation, which has a general solution form of

$$y = C_1 \sin\left(x\sqrt{\frac{P}{EI}}\right) + C_2 \cos\left(x\sqrt{\frac{P}{EI}}\right) \qquad \text{------(3)}$$

where C_1 and C_2 are constants. The values of constants can be obtained by applying the boundary conditions:

(i) y = 0 at x = 0. That is, the deflection of the column must be zero at each end since it is pinned at each end. Applying these conditions (putting these values into the eq. (3)) gives us the following results: For y to be zero at x =0, the value of C_2 must be zero (since cos (0) = 1).

(i) Substituting y = 0 at x = L in eq. (3) lead to the following.

$$0 = C_1 \sin\left(L\sqrt{\frac{P}{EI}}\right)$$

While for y to be zero at x = L, then either C_1 must be zero (which leaves us with no equation at all, if C_1 and C_2 are both zero), or

$$\sin\left(L\sqrt{\frac{P}{EI}}\right) = 0$$

which results in the fact that

$$\left(L\sqrt{\frac{P}{EI}}\right) = n\pi$$

$$\text{or} \qquad L\sqrt{\frac{P}{EI}} = n\pi \qquad \text{where } n = 0,1,2,2\ldots$$

$$\text{or} \qquad P = \frac{n^2\pi^2 EI}{L^2}$$

Taking least significant value of n, i.e. n= 1

NOTES4FREE
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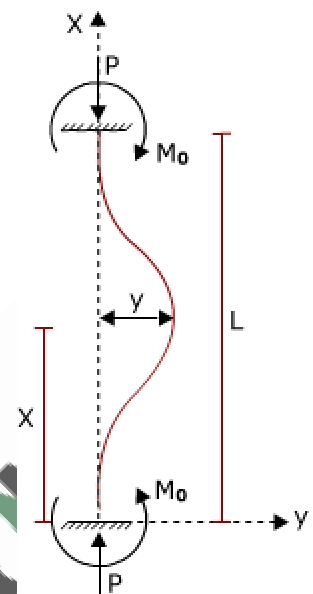
We have $$P = \frac{\pi^2 EI}{L^2}$$

or $$P_E = \frac{\pi^2 EI}{l_e^2}$$

where $l_e = L$.

Case (2) Both ends fixed

Consider a long column with both ends fixed subjected to critical load P as shown.



Consider a section at a distance x from the origin. Let y be the deflection of the column at this section. Bending moment in terms of load P, fixed end moment M 0 and deflection y is given by

$$M = -P\,y + M_0 \qquad \text{-------------(1)}$$

We can also write that for beams/columns the bending moment is proportional to the curvature of the beam, which, for small deflection can be expressed as

$$\frac{M}{EI} = \frac{d^2y}{dx^2}$$

or $$M = EI\frac{d^2y}{dx^2} \qquad \text{-------------(2)}$$

where E is the Young's modulus and I is the moment of Inertia.

Substituting eq.(1) in eq.(2)

$$-P\,y + M_0 = E\,I\,\frac{d^2y}{dx^2}$$

or
$$\frac{d^2y}{dx^2} + \left(\frac{P}{EI}\right)y = \frac{M_0}{EI}$$

This is a second order differential equation, which has a general solution form of

$$y = C_1 \sin\left(x\sqrt{\frac{P}{EI}}\right) + C_2 \cos\left(x\sqrt{\frac{P}{EI}}\right) + \frac{M_0}{P} \quad \text{---------} \;(3)$$

where C_1 and C_2 are constants. The values of constants can be obtained by applying the boundary conditions:

(i) y = 0 at x = 0. That is, the deflection of the column must be zero at near end since it is fixed. Applying this condition (putting these values into the eq. (3)) gives us the following result:

$$C_2 = -\frac{M_0}{P}$$

ii) At X = 0 =0, that is, the slope of the column must be zero, since it is fixed.

$$\frac{dy}{dx} = C_1\sqrt{\frac{P}{EI}}\cos\left(x\sqrt{\frac{P}{EI}}\right) - C_2\sqrt{\frac{P}{EI}}\sin\left(x\sqrt{\frac{P}{EI}}\right) \quad \text{-------}(4)$$

Substituting the boundary condition in eq. (4)

$$0 = C_1\sqrt{\frac{P}{EI}}$$

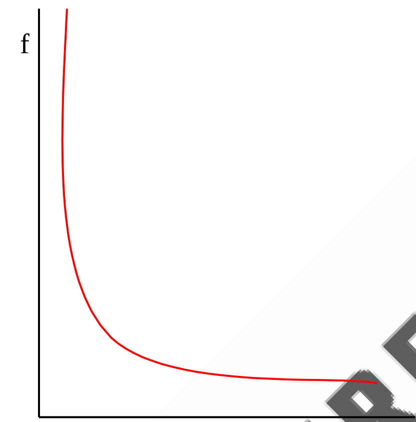
Hence, $C_1 = 0$

Substituting the constants C_1 and C_2 in eq. (3) leads to the following

$$y = -\frac{M_0}{P}\cos\left(x\sqrt{\frac{P}{EI}}\right) + \frac{M_0}{P} \qquad \text{---------(5)}$$

NOTES4FREE
All in One Solution

The variation of limiting stress ‘f' versus slenderness ratio in the above equation is shown below.



The above plot shows that the limiting stress ‘f' decreases as increases. In fact, when very small, limiting stress is is close to infinity, which is not rational. Limiting stress cannot be greater than the yield stress of the material.

1. Eulers formula determines the critical load, not the working load. Suitable factor of safety (which is about 1.7 to 2.5) should be considered to obtain the allowable load.

4.7.2 Rankine's critical Load

$$\frac{1}{P_R} = \frac{1}{P_C} + \frac{1}{P_E} \qquad \ldots\ldots\ldots\ldots \quad (1)$$

Where, P_R = Rankine's critical load

$P_C = f_C\, A$ = Crushing load for short columns

$P_E = \dfrac{\pi^2 E I}{l_e^2}$ = Euler's critical load for long columns

Rankine Gordon Load is given by the following empirical formula,

This relationship is assumed to be valid for short, medium and long columns. This relation can be used to find the load carrying capacity of a column subjected to crushing and/or buckling.

From eq. (1)

NOTES4FREE in one Solution

Substituting P_C and P_E in the above relation

$$P_R = \frac{f_c\ A}{1 + \left[\dfrac{f_c\ A}{\dfrac{\pi^2 E I}{l_e^2}}\right]} = \frac{f_c\ A}{1 + \left(\dfrac{f_c}{\pi^2 E}\right)\left[\dfrac{l_e^2\ A}{I}\right]}$$

Since $\dfrac{I_{min}}{A} = (r_{min})^2$

$$P_R = \frac{f_c\ A}{1 + a\left[\dfrac{l_e}{r_{min}}\right]^2}$$

FS4FREE
Solution

Module 5: Theories of Failure

Objectives:

Various types of theories of failure and its importance

Learning Structure

- 5.0 Introduction
- 5.1 Stress-Strain relationships
- 5.2 Types of Failure
- 5.3 Use of factor of safety in design
- 5.4 Theories of Failure
- 5.5 Problems
- Outcomes
- Further reading

NOTES4FREE
All in One Solution

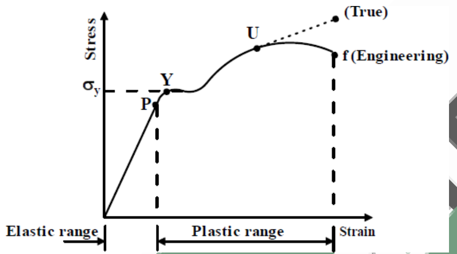
5.0 Introduction:

Failure indicate either fracture or permanent deformation beyond the operational range due to yielding of a member. In the process of designing a machine element or a structural member, precautions has to be taken to avoid failure under service conditions.

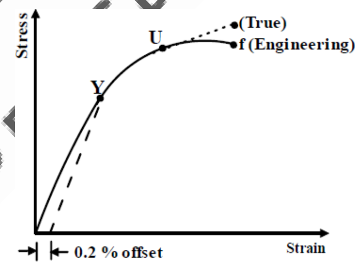
When a member of a structure or a machine element is subjected to a system of complex stress system, prediction of mode of failure is necessary to involve in appropriate design methodology. Theories of failure or also known as failure criteria are developed to aid design.

5.1 Stress-Strain relationships:

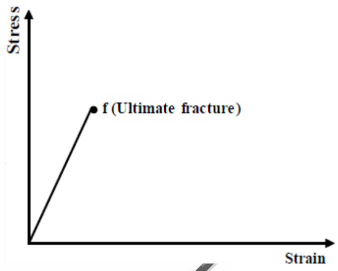
Following Figure-1 represents stress-strain relationship for different type of materials.



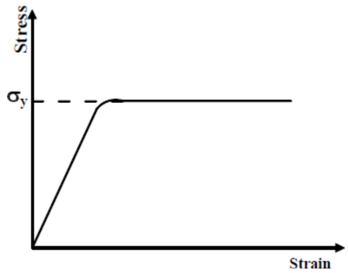
Ductile material e.g. low carbon steel



Low ductility



Brittle material



Elastic – perfectly plastic material

Figure-: Stress-Strain Relationship

Bars of ductile materials subjected to tension show a linear range within which the materials exhibit elastic behaviour whereas for brittle materials yield zone cannot be identified. In general, various materials under similar test conditions reveal different behaviour. The cause of failure of a ductile material need not be same as that of the brittle material.

5.2 Types of Failure:

The two types of failure are,

Yielding - This is due to excessive inelastic deformation rendering the structural member or machine part unsuitable to perform its function. This mostly occurs in ductile materials.

Fracture - In this case, the member or component tears apart in two or more parts. This mostly occurs in brittle materials.

5.3 Use of factor of safety in design:

In designing a member to carry a given load without failure, usually a factor of safety (FS or N) is used. The purpose is to design the member in such a way that it can carry N times the actual working load without failure. Factor of safety is defined as Factor of Safety (FS) = Ultimate Stress/Allowable Stress.

5.4 Theories of Failure:

a) Maximum Principal Stress Theory (Rankine Theory)
b) Maximum Principal Strain Theory (St. Venant's theory)
c) Maximum Shear Stress Theory (Tresca theory)
d) Maximum Strain Energy Theory (Beltrami's theory)

5.4.1 Maximum Principal Stress Theory (Rankine theory)

According to this, if one of the principal stresses σ_1 (maximum principal stress), σ_2 (minimum principal stress) or σ_3 exceeds the yield stress (σ_y), yielding would occur. In a two dimensional loading situation for a ductile material where tensile and compressive yield stress are nearly of same magnitude

$\sigma_1 = \pm \sigma_y \qquad \sigma_2 = \pm\sigma_y$

Yield surface for the situation is, as shown in Figure-2

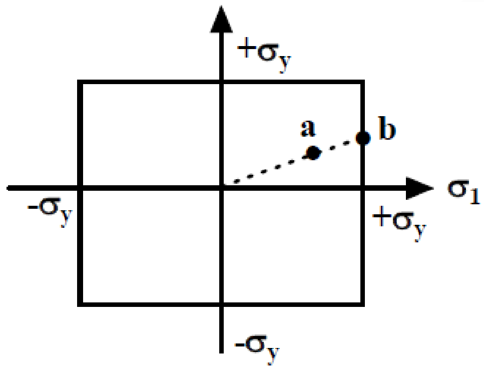


Figure- 2: Yield surface corresponding to maximum principal stress theory

Yielding occurs when the state of stress is at the boundary of the rectangle. Consider, for example, the state of stress of a thin walled pressure vessel. Here $\sigma_1 = 2\sigma_2$, σ_1 being the circumferential or hoop stress and σ_2 the axial stress. As the pressure in the vessel increases, the stress follows the dotted line. At a point (say) a, the stresses are still within the elastic limit but at b, σ_1 reaches σ_y although σ_2 is still less than σ_y. Yielding will then begin at point b. This theory of yielding has very poor agreement with experiment. However, this theory is being used successfully for brittle materials.

5.4.2 Maximum Principal Strain Theory (St. Venant's Theory)

According to this theory, yielding will occur when the maximum principal strain just exceeds the strain at the tensile yield point in either simple tension or compression. If ε_1 and ε_2 are maximum and minimum principal strains corresponding to σ_1 and σ_2, in the limiting case

$\varepsilon_1 = (1/E)(\sigma_1 - \nu\sigma_2) \qquad |\sigma_1| \geq |\sigma_2|$

$\varepsilon_2 = (1/E)(\sigma_2 - \nu\sigma_1) \qquad |\sigma_2| \geq |\sigma_1|$

This results in,

$E\,\varepsilon_1 = \sigma_1 - \nu\sigma_2 = \pm\,\sigma_0$
$E\,\varepsilon_2 = \sigma_2 - \nu\sigma_1 = \pm\,\sigma_0$

The boundary of a yield surface in this case is shown in Figure – 3.

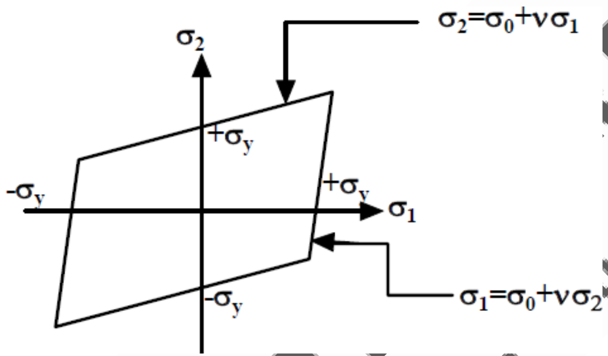


Figure-3: Yield surface corresponding to maximum principal strain theory

5.4.3 Maximum Shear Stress Theory (Tresca theory)

According to this theory, yielding would occur when the maximum shear stress just exceeds the shear stress at the tensile yield point. At the tensile yield point $\sigma_2 = \sigma_3 = 0$ and thus maximum shear stress is $\sigma_y/2$. This gives us six conditions for a three-dimensional stress situation:

$\sigma_1 - \sigma_2 = \pm\,\sigma_y$
$\sigma_2 - \sigma_3 = \pm\,\sigma_y$
$\sigma_3 - \sigma_1 = \pm\,\sigma_y$

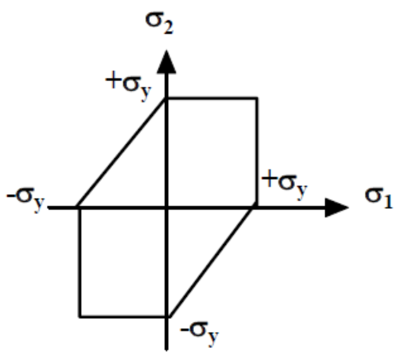


Figure – 4: Yield surface corresponding to maximum shear stress theory

In a biaxial stress situation (Figure - 4) case, $\sigma_3 = 0$ and this gives

$\sigma_1 - \sigma_2 = \sigma_y$	if $\sigma_1 > 0, \sigma_2 < 0$
$\sigma_1 - \sigma_2 = -\sigma_y$	if $\sigma_1 < 0, \sigma_2 > 0$
$\sigma_2 = \sigma_y$	if $\sigma_2 > \sigma_1 > 0$
$\sigma_1 = -\sigma_y$	if $\sigma_1 < \sigma_2 < 0$
$\sigma_1 = -\sigma_y$	if $\sigma_1 > \sigma_2 > 0$
$\sigma_2 = -\sigma_y$	if $\sigma_2 < \sigma_1 < 0$

This criterion agrees well with experiment.

In the case of pure shear, $\sigma_1 = - \sigma_2 = k$ (say), $\sigma_3 = 0$
and this gives $\sigma_1 - \sigma_2 = 2k = \sigma_y$
This indicates that yield stress in pure shear is half the tensile yield stress and this is also seen in the Mohr's circle (Figure - 5) for pure shear.

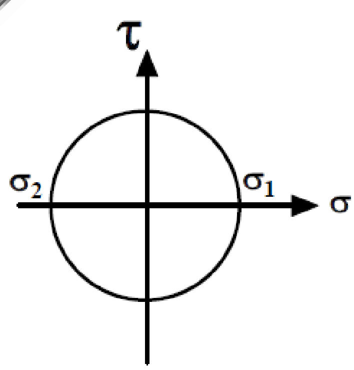


Figure – 5: Mohr's circle for

pure shear

5.4.4 Maximum strain energy theory (Beltrami's theory)

According to this theory failure would occur when the total strain energy absorbed at a point per unit volume exceeds the strain energy absorbed per unit volume at the tensile yield point. This may be expressed as,

$(1/2)(\sigma_1 \varepsilon_1 + \sigma_2 \varepsilon_2 + \sigma_3 \varepsilon_3) = (1/2)\, \sigma_y \varepsilon_y$

Substituting ε_1, ε_2, ε_3 and ε_y in terms of the stresses we have

$\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2\upsilon\,(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1) = \sigma_y^2$

$(\sigma_1/\sigma_y)^2 + (\sigma_2/\sigma_y)^2 - 2\nu(\sigma_1\sigma_2/\sigma_y^2) = 1$

The above equation represents an ellipse and the yield surface is shown in F igure - 6

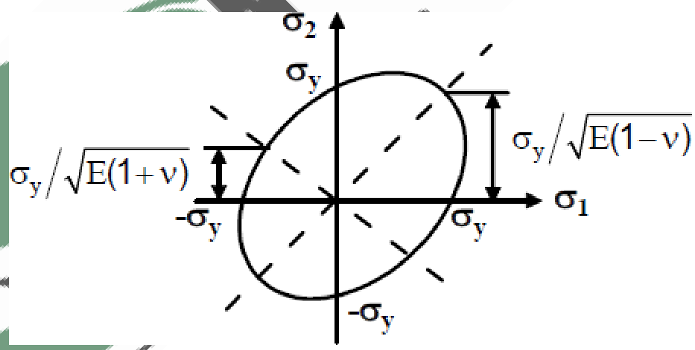


Figure – 6: Yield surface corresponding to Maximum strain energy theory.

It has been shown earlier that only distortion energy can cause yielding but in the above expression at sufficiently high hydrostatic pressure $\sigma_1 = \sigma_2 = \sigma_3 = \sigma$ (say), yielding may also occur. From the above we may write $\sigma^2(3 - 2\nu) = \sigma_y^2$ and if $\nu \sim 0.3$, at stress level lower than yield stress, yielding would occur. This is in contrast to the experimental as well as analytical conclusion and the theory is not appropriate.

NOTES4FREE One Solution

5.4.5 Superposition of yield surfaces of different failure theories:

A comparison among the different failure theories can be made by superposing the yield surfaces as shown in figure – 7. It is clear that an immediate assessment of failure probability can be made just by plotting any experimental in the combined yield surface. Failure of ductile materials is most accurately governed by the distortion energy theory where as the maximum principal strain theory is used for brittle materials.

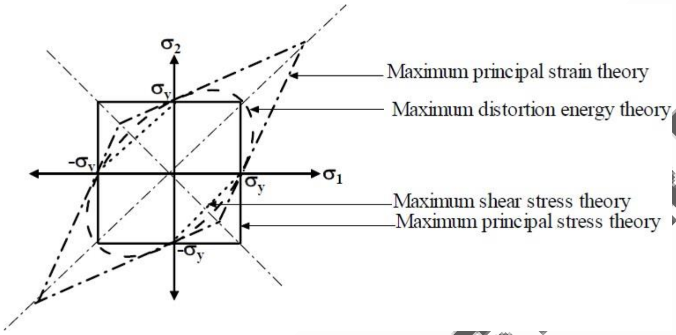


Figure – 7: Comparison of different failure theories

5.5 Problems:

Numerical-1: A shaft is loaded by a torque of 5 KN-m. The material has a yield point of 350 MPa. Find the required diameter using Maximum shear stress theory. Take a factor of safety of 2.5.

Torsional Shear Stress, $\tau = 16T/\pi d^3$, where d represents diameter of the shaft

Maximum Shear Stress theory, √

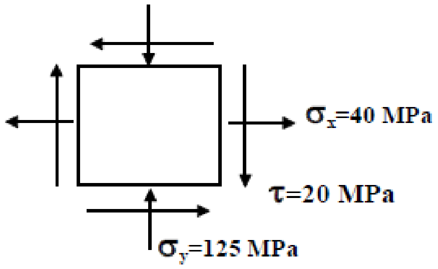
Factor of Safety (FS) = Ultimate Stress/Allowable Stress

Since $\sigma_x = \sigma_y = 0$, $\tau_{max} = 25.46 \times 10^3/d^3$

Therefore $25.46 \times 10^3/d^3 = \sigma_y/(2*FS) = 350*10^6/(2*2.5)$

Hence, d = 71.3 mm

Numerical-2: The state of stress at a point for a material is shown in the following figure Find the factor of safety using (a) Maximum shear stress theory Take the tensile yield strength of the material as 400 MPa.



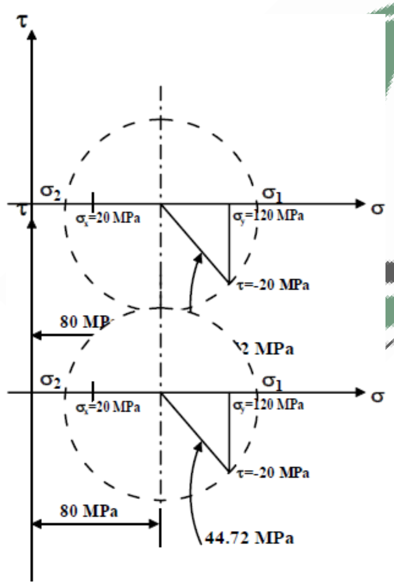
From the Mohr's circle shown below we determine,

σ_1 = 42.38MPa and
σ_2 = -127.38MPa

from Maximum Shear Stress theory

$(\sigma_1 - \sigma_2)/2 = \sigma_y/(2*FS)$

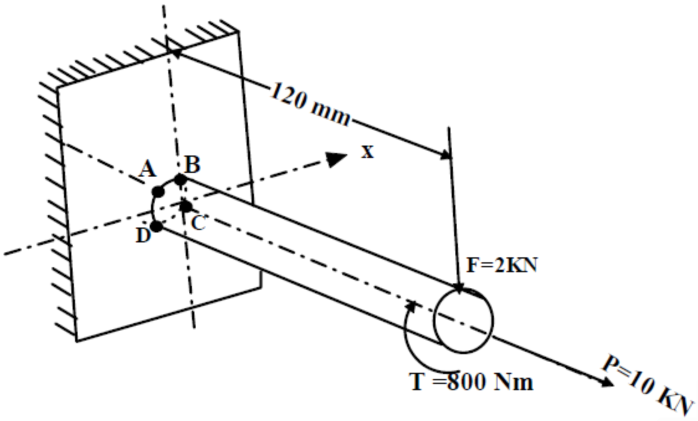
By substitution and calculation factor of safety FS = 2.356



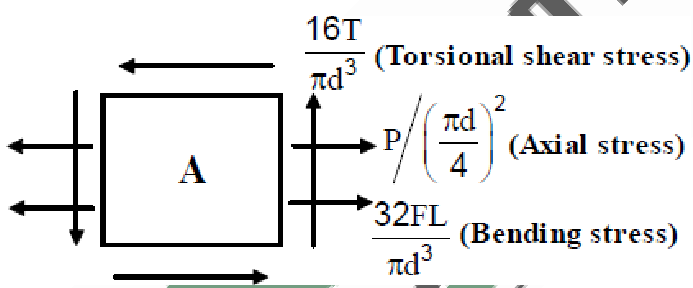
NOTES4FREE
All in One Solution

Numerical-3: A cantilever rod is loaded as shown in the following figure. If the tensile yield strength of the material is 300 MPa determine the

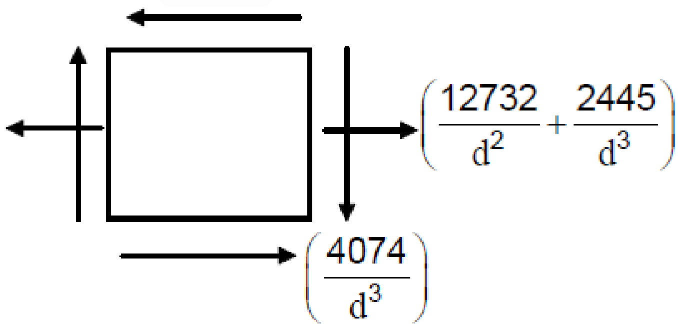
rod diameter using (a) Maximum principal stress theory (b) Maximum shear stress theory



At the outset it is necessary to identify the mostly stressed element. Torsional shear stress as well as axial normal stress is the same throughout the length of the rod but the bearing stress is largest at the welded end. Now among the four corner elements on the rod, the element A is mostly loaded as shown in following **figure**



Shear stress due to bending VQ/It is also developed but this is neglected due to its small value compared to the other stresses. Substituting values of T, P, F and L, the elemental stresses may be shown as in following **figure.**



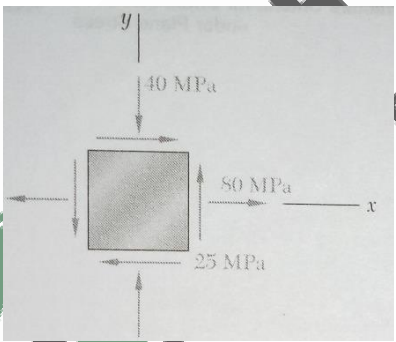
The principal stress for the case is determined by the following equation,

$$\sigma_{1,2} = \frac{1}{2}\left(\frac{12732}{d^2} + \frac{2445}{d^3}\right) \pm \sqrt{\frac{1}{4}\left(\frac{12732}{d^2} + \frac{2445}{d^3}\right)^2 + \left(\frac{4074}{d^3}\right)^2}$$

By Maximum Principal Stress Theory, Setting, $\sigma_1 = \sigma_y$ we get d = 26.67mm

By maximum shear stress theory by setting $(\sigma_1 - \sigma_2)/2 = \sigma_y/2$, we get, d = 30.63mm

Numerical-4: The state of plane stress shown occurs at a critical point of a steel machine component. As a result of several tensile tests it has been found that the tensile yield strength is σ_y=250MPa for the grade of steel used. Determine the factor of safety with respect to yield using maximum shearing stress criterion.

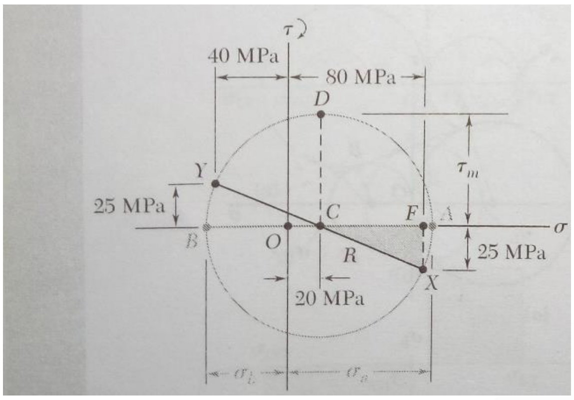


Construction of the Mohr's circle determines

σ_{avg} = ½ (80-40) = 20MPa and $\tau_m = (60^2+25^2)^{1/2}$ = 65MPa
σ_a= 20+65 = 85 MPa and σ_b= 20-65 = -45 MPa

The corresponding shearing stress at yield is τ_y= ½ σ_y = ½ (250) = 125MPa

Factor of safety, FS = τ_m / τ_y = 125/65 = 1.92

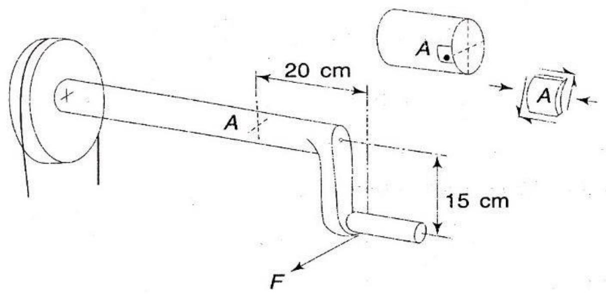


Summary:

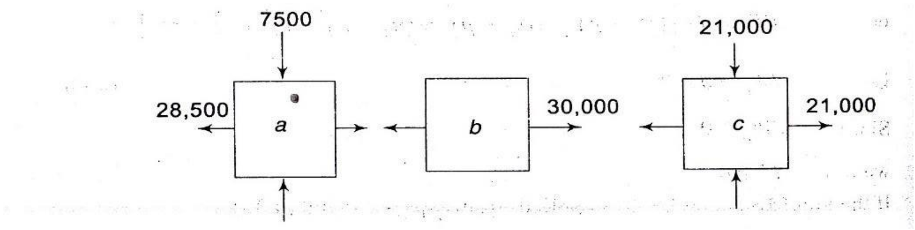
Different types of loading and criterion for design of structural members/machine parts subjected to static loading based on different failure theories have been discussed. Development of yield surface and optimization of design criterion for ductile and brittle materials were illustrated.

Assignments:

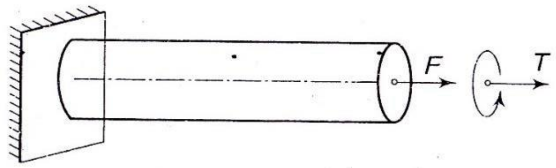
Assignment-1: A Force F = 45,000N is necessary to rotate the shaft shown in the following figure at uniform speed. The crank shaft is made of ductile steel whose elastic limit is 207,000 kPa, both in tension and compression. With E = 207 X 10^6 kPa and ν = 0.25, determine the diameter of the shaft using maximum shear stress theory, using factor of safety = 2. Consider a point on the periphery at section A for analysis **(Answer, d = 10.4 cm)**



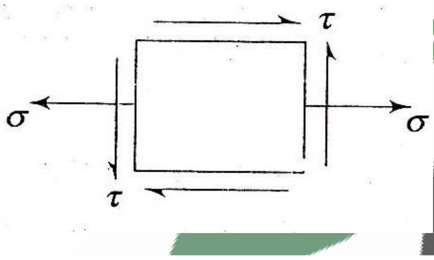
Assignment-2: Following figure shows three elements a, b and c subjected to different states of stress. Which one of these three, do you think will yield first according to i) maximum stress theory, ii) maximum strain theory, and iii) maximum shear stress theory? Assume Poisson's ratio ν = 0.25 **[Answer: i) b, ii) a, and iii) c]**



Assignment-3: Determine the diameter of a ductile steel bar if the tensile load F is 35,000N and the torsional moment T is 1800N.m. Use factor of safety = 1.5. E = $207*10^6$kPa and σ_{yp} = 207,000kPa. Use the maximum shear stress theory. **(Answer: d = 4.1cm)**



Assignment-4: At a pint in a steel member, the state of stress shown in Figure. The tensile elastic limit is 413.7kPa. If the shearing stress at a point is 206.85kPa, when yielding starts, what is the tensile stress σ at the point according to maximum shearing stress theory? **(Answer: Zero)**



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All in One Solution