Module-1
Introduction to Linear Programming

Origin of Operation Research:

Operation Research is a scientific way to decision making which seek to determine how best to design and operate a system under scared resource. This subject came into existence into second world war. OR is defined as an experimental science which is devoted to observing understanding and predicting the behavior of purposeful man-machine systems.

Nature and Impact of OR:

OR involves 'research on operations'. Thus operation research is applied to problems that concern how to conduct and co-ordinate the operations within an organizations. The nature of organization is immaterial and in fact OR has been applied extensively in such diverse areas as manufacturing, transportation, construction, tele communication, financial planning and health care. Therefore the breadth of application is usually wide. OR resembles the way research is conducted in established scientific field. It frequently attempts to find the best possible solution to the problem.

Operation Research has had an impressive impact on improving the efficiency of numerous organizations around the world. In the process, OR has made a significant contribution to increasing the productivity of the economics of various countries.

Main Phases of OR:

Phase 1: Formulation

This phase requires the problem to be formulated in the form of an appropriate model. This includes finding objective functions, constraints or restrictions, inter-relationships, possible alternate course of action, time limits for making decisions, ranges of controllable and uncontrollable variables which might affect the possible solutions. Hence one must be very careful while executing this phase.

Phase 2: Construction of a mathematical model

This phase is concerned with reformation of problem in an appropriate form which is useful in analysis. The most suitable model is a mathematical model representing the problem under study. A mathematical model should include decision variables, objective functions and constraints. The advantage of a mathematical model is that it describes the problem more concisely which makes the overall structure of the problem more comprehensible and it also helps to reveal important cause and effect relation.

Phase 3: Derivation of solutions from mathematical model

This phase is devoted to computation of those values of decision variables which maximize or minimize the objective function. It is always important to arrive at the optimal solution of the problem.

Phase 4: Testing the mathematical model and its solution

The completed model is tested for errors if any. The principle of judging the validity of the model is whether or not it predicts the relative effects of the alternative courses of action with sufficient accuracy to permit a sound decision. A good model should be applicable for a longer time and thus updates the model time to time taking into account the past, present and future specifications of the problem.

Phase 5: Establishing control over the solution

After the testing phase the next step is to install a well documented system for applying the model. It includes the solution procedure and operating procedure for implementation. This phase establishes a control over the solution with some degree of satisfaction. This phase also establishes a systematic procedure for detecting changes and controlling the situation.

Phase 6: Implementation

The implementation of controlled solution involves, the translation of models which results into operating instructions. It is important in OR to ensure that the solution is accurately translated into an operating procedure to rectify faults in the solution.

Advantages of OR:

Better Systems: Often, an O.R. approach is initiated to analyze a particular problem of decision making such as best location for factories, whether to open a new warehouse, etc. It also helps in selecting economical means of transportation, jobs sequencing, production scheduling, replacement of old machinery, etc.

Better Control: The management of large organizations recognize that it is a difficult and costly affair to provide continuous executive supervision to every routine work. An O.R. approach may provide the executive with an analytical and quantitative basis to identify the problem area. The most frequently adopted applications in this category deal with production scheduling and inventory replenishment.

Better Decisions: O.R. models help in improved decision making and reduce the risk of making erroneous decisions. O.R. approach gives the executive an improved insight into how he makes his decisions.

Better Co-ordination: An operations-research-oriented planning model helps in co-ordinating different divisions of a company.

Disadvantages of OR:

Dependence on an Electronic Computer: O.R. techniques try to find out an optimal solution taking into account all the factors. In the modern society, these factors are enormous and expressing them in quantity and establishing relationships among these require voluminous calculations that can only be handled by computers.

Non-Quantifiable Factors: OR techniques provide a solution only when all the elements related to a problem can be quantified. All relevant variables do not lend themselves to quantification. Factors that cannot be quantified find no place in O.R. models.

Distance between Manager and Operations Researcher: O.R. being specialist's job requires a mathematician or a statistician, who might not be aware of the business problems. Similarly, a manager fails to understand the complex working of O.R. Thus, there is a gap between the two.

Money and Time Costs: When the basic data are subjected to frequent changes, incorporating them into the O.R. models is a costly affair. Moreover, a fairly good solution at present may be more desirable than a perfect O.R. solution available after sometime.

Implementation: Implementation of decisions is a delicate task. It must take into account the complexities of human relations and behavior.

Linear Programming:

It is a decision making technique under a given constraint that the relationship among the variable involved is linear.

Mathematical formulation of a linear programming:

A mathematical problem is an optimization problem in which the objective and constraints are given as mathematical functions and functional relationships. The procedure for mathematical formulation of a LPP consists of the following steps

Step1: write down the decision variables (Products) of the problem

Step2: formulate the objective function to be optimized (maximized or minimized) as linear function of the decision variables

Step3: formulate the other conditions of the problem such as resource limitation, market, constraints, and interrelations between variables etc., linear in equations or equations in terms of the decision variables.

Step4: add non-negativity constraints

The objective function set of constraint and the non-negative constraint together form a Linear Programming Problem.

Problems:

1. Consider a small manufacturer making two products A & B, two resources R1 and R2 are required to make these products. Each unit of product A requires 1 unit of R1 and 3 units of R2. Each units of B requires 1 unit of R1 and 2 units of R2. The manufacturer has 5 units of R1 and 12 units of R2 available. The manufacturer also makes a profit of Rs 6 per unit of product A sold and Rs 5 per unit of product B sold. Formulate the problem.

Solution:
Step1: Let the total number of units of A produced be 'x'.
Let the total number of units of B produced be 'y'.
Given: profit/one unit of A is Rs.6
Profit/x unit of A is Rs.6x
Profit/one unit of B is Rs.5
Profit/x unit of B is Rs.5x
Step2:Total profit z=6x+5y
Objective function is max z=6x+5y
Step3: Given that the products A and B requires 1 and 1 unit of R1 respectively with total availability of 5 units
i.e x+y<=5
Given that the products A and B requires 3 and 2 units of R2 respectively with total availability of 12 units
i.e 3x+2y<=12
Step4: The non negative conditions are:
x, y>=0

LP model:
Max z=6x+5y
STC x+y<=5
3x+2y<=12
x, y>=0

2. A Manufacture produces two types of models M1 and M2 each model of the type M1 requires 4 hrs of grinding and 2 hours of polishing, where as each model of the type M2 requires 2 hours of grinding and 5 hours of polishing. The manufactures have 2 grinders and 3 polishers. Each grinder works 40 hours a week and each polishers works for 60 hours a week. Profit on M1 model is Rs. 3.00 and on Model M2 is Rs 4.00.Whatever produced in week is sold in the market. How should the manufacturer allocate is production capacity to the two types models, so that he may make max in profit in week?

Solutions:

Step1: Let x1 be the number of units of model M1.
Let x2 be the number of units of model M2.
Step2: Objective function: Since, the profit on M1 and M2 is Rs.3.0 and Rs.4.0 Max Z =3x1 + 4x2

Step3: Constraint: there are two constraints one for grinding and other is polishing. No of grinders are 2 and the hours available in grinding machine is 40 hrs per week, therefore, total no of hours available of grinders is 2 X 40 = 80 hours No of polishers are 3 and the hours available in polishing machine is 60 hrs per week, therefore, total no of hours available of polishers is 3 X 60 = 180 hours

The grinding constraint is given by: $4x1+2x2 \le 80$ The Polishing Constraint is given by: $2x1 + 5x2 \le 180$

Non negativity restrictions are $x1, x2 \ge 0$ if the company is not manufacturing any products

The LPP of the given problem is

Max $Z = 3x1+4x2$

STC $4x1+2x2 \le 80$

$2x1+5x2 \le 180$

$x1, x2 \ge 0$

3. A farmer has 100 acre. He can sell all tomatoes. Lettuce or radishes he raise the price. The price he can obtain is Re 1 per kg of tomatoes, Rs 0.75 a head for lettuce and Rs 2 per kg of radishes. The average yield per acre is 2000kg tomatoes, 3000 heads of lettuce and 1000kgs of radishes. Fertilizer is available at Rs 0.5 per kg and the amount required per acre 100kgs each for tomatoes and lettuce, and 50kgs for radishes. Labor required for sowing, cultivating and harvesting per acre is 5 man-days for tomatoes and radishes, 6 man-days for lettuce. A total of 400 man days of labor available at Rs 20 per man day formulate the problem as linear programming problem model to maximize the farmer's total profit.

Solution:

Farmer's problem is to decide how much area should be allotted to each type of crop. He wants to grow to maximize his total profit. Let the farmer decide to allot X1, X2 and X3 acre of his land to grow tomatoes, lettuce and radishes respectively. So the farmer will produce 2000 X1kgs of tomatoes, 3000 X2head of lettuce and 1000 X3kgs of radishes. Profit=sales–cost=sales–(Labor cost +fertilizer cost) Sales = 1 x 2000 X1 + 0.75 x 3000 X2 + 2 x 1000 X3

Labor cost = 5x 20 X1 + 6 x 20 X2 + 5 x 20 X3

Fertilizer cost = 100x0.5 X1 + 0.5x 100 X2 + 0.5x50 X3

The LPP model is:

Max Z= 1850 X1 + 2080 X2 + 1875 X3 STC X1 + X2 + X3 $\le$ 100 5X1 + 6X2 + 5X3 $\le$ 400 X1 , X2 , X3 $\ge$ 0

4. A TV company has to decide on the minimum of 27 inches and 20 inches TV sets to be produced at one of its factories. The market research indicates that atmost 40, 27 inch TV sets and atmost 10, 20inch TV set can be sold per month. The maximum number of work hours available is 500hrs per month. A 27inch TV requires 20 work hours and a 20inch TV requires 10 work hours. Each 27inch TV is sold at a profit of Rs.120 and 20inch TV sold at a profit of Rs. 80, a wholesaler agreed to purchase all the TV sets produced, if the number do not exceed the max indicated by market research. Formulate the problem as an LP model.

Solution:

Let the total number of 27inches TV be ‘x’
Let the total number of 20inches TV be ‘y’
Given 1unit of 27inch TV produces a profit of Rs.120
‘x’ unit of 27inch TV produces a profit of Rs.120x
Given 1 unit of 20inch TV produces a profit of Rs.80
‘y’ unit of 20inch TV produces a profit of Rs.80y
Total profit= 120x+80y
Objective function z=120x+80y
Given that max sales of 27inch TV is 40 i.e x<=40
Given that max sales of 20inch TV is 10 i.e y<=10
One 27inch TV requires 20 work hours
x 27inch TV requires 20x work hours
One 20inch TV requires 10 work hours
y 20inch TV requires 10y work hours
Total work hour available is 500
i.e 20x+10y<=500
max sales/month 40+10=50
Total number of TV sets=x+y
Given wholesaler will purchase all the TV sets if the total does not exceed the maximum
i.e x+y<=50
LP model

Max z=120x+80y
STC x<=40
y<=10
120x+10y<=500
x+y<=50
where x>=0 y>=0

5. Egg contains 6 units of vitamin A per gram and 7 units of vitamin B per gram and cost 12 paise per gram. Milk contains 8 units of vitamin A per gram and 12 units of vitamin B per gram and costs 20 paise per gram. The daily requirements of vitamin A and vitamin B are 100 units and 120 units respectively. Find the optimal product mix.

	Egg	Milk	Min Requirements
Vitamin A	6	8	100
Vitamin B	7	12	120
Cost	12	20	

Solution:

Let x1 and x2 be the total cost of milk and egg produced respectively
The Objective function z=12x1+20x2
Vitamin A contents in egg and milk is 6 and 8 units respectively and minimum requirements is 100
i.e 6x1+8x2 >=100

Vitamin B contents in egg and milk is 7 and 12 units respectively and minimum requirements is 120

i.e 7x1+12x2 >=120

The non negative constraints are: x1,x2>=0

The LP model is:

Max=z=12x1+20x2

STC 6x1+8x2>=100

7x1+12x2>=120

x1, x2>=0

Graphical Method:

The graphical procedure includes two steps

1. Determination of the solution space that defines all feasible solutions of the model.
2. Determination of the optimum solution from among all the feasible points in the solution space.

There are two methods in the solutions for graphical method

1. Extreme point method
2. Objective function line method

Steps involved in graphical method are as follows:

1. Consider each inequality constraint as equation.

2. Plot each equation on the graph as each will geometrically represent a straight line.

3. Mark the region. If the constraint is <= type then region below line lying in the first quadrant (due to non negativity variables) is shaded. If the constraint is >= type then region above line lying in the first quadrant is shaded.

4. Assign an arbitrary value say zero for the objective function.

5 Draw the straight line to represent the objective function with the arbitrary value.

6. Stretch the objective function line till the extreme points of the feasible region. In the maximization case this line will stop farthest from the origin and passing through at least one corner of the feasible region.

7. In the minimization case, this line will stop nearest to the origin and passing through at least one corner of the feasible region.

8. Find the co-ordination of the extreme points selected in step 6 and find the maximum or minimum value of Z.

Problems:

1. Solve the following LP problem using graphical method

Max: $z=6x+8y$

$5x+10y<=60$

$4x+4y<=40$

$x, y>=0$

Solution:

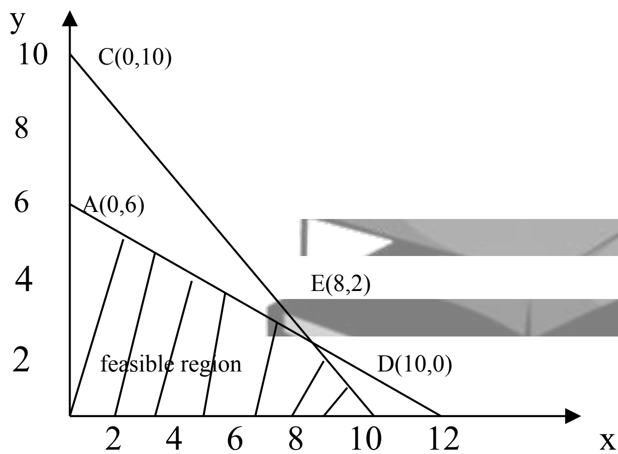
Replace all inequality by equality

$5x+10y=60$ =>when $x=0$ $y=6$

when $y=0$ $x=12$ The points are: A(0,6) and B(12,0)

$4x+4y=40$ => when $x=0$ $y=10$

when $y=0$ $x=10$ The points are C(0,10) and D(10,0)



Corner points	Z=6x+8y
A(0,6)	48
D(10,0)	60
E(8,2)	64

Here the maximum value of z is attained at the corner point E(8,2), which is the point of intersection of lines $5x+10y=60$ and $4x+4y<=40$. Hence the required solution is $x=8,y=2$ and the max value $z=64$

2. Solve the following LPP by graphical method:

Minimize $z=20x+10y$

$x+2y <= 40$

$3x+y >= 30$

4x+3y >=60

Solution:

Replace all inequalities by equality

x+2y = 40 when x=0, y=20

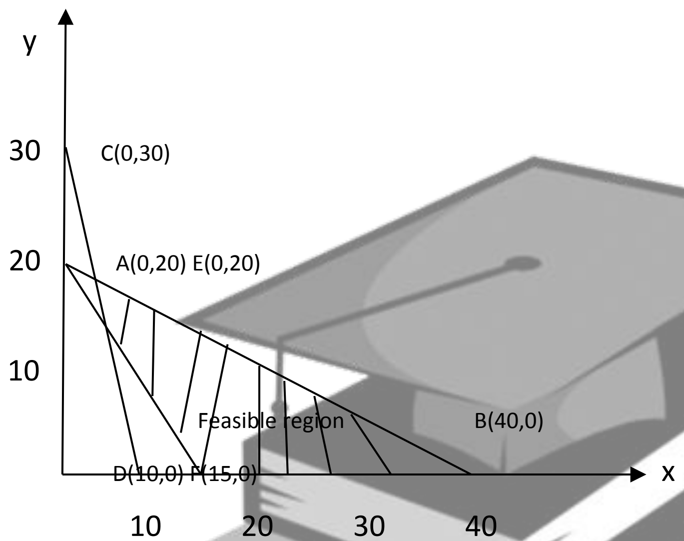
when y=0, x=40 The points are A(0,20) and B(40,0)

3x+y =30 when x=0, y=30

when y=0, x=10 The points are C(0,30) and D(10,0)

4x+3y =60 when x=0, y=20

when y=0, x=15 The points are E(0,20) and F(15,0)



Corner points	Z=20x+10y
G(4,18)	260
F(15,0)	300
B(40,0)	800
H(6,12)	240

Here the minimum value of z is attained at the corner point H(6,12), which is the point of intersection of lines 3x+y=30 and 4x+3y=60. Hence the required solution is x=6, y=12 and the min value z=240

3. Solve the following LPP

Maximize z=3x+2y

x-y >= 1

x-y >= 3

x,y =0

⇨ The solution space is unbounded. In fact the maximum value of Z occurs at infinity. Hence the problem does not have a feasible solution.

Module - 2

Simplex Methods

Simplex Method is an iterative procedure for solving lpp in a finite number of steps. It provides an algorithm which consists of moving from the region of one vertex of feasible solution to another in such a manner that the value of objective function at the vertex is less or more. This procedure is repeated since the number of vertices is finite.

Problems:

Maximize $z = 10x_1 + 5x_2$

STC $3x_1 + 3x_2 \le 36$

$2x_1 + 6x_2 \le 60$

$5x_1 + 2x_2 \le 50$

where x_1 & $x_2 \ge 0$. Solve the lpp model by applying simplex methods.

$\Rightarrow$ Step 1: The simplex method is applied only for maximization problem. If the objective function is to minimize, convert it to maximization.

Step 2: Convert each inequality to equality and introduce a slak variable.

$$3x_1 + 3x_2 + S_1 = 36$$
$$2x_1 + 6x_2 + S_2 = 60$$
$$5x_1 + 2x_2 + S_3 = 50$$

Step 3: Represent the equation in matrix format the standard matrix format is $A \cdot X = B$

$$\begin{array}{ccccc} x_1 & x_2 & S_1 & S_2 & S_3 \end{array}$$
$$\begin{bmatrix} 3 & 3 & 1 & 0 & 0 \\ 2 & 6 & 0 & 1 & 0 \\ 5 & 2 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ S_1 \\ S_2 \\ S_3 \end{bmatrix} = \begin{bmatrix} 36 \\ 60 \\ 50 \end{bmatrix}.$$

Step 4: Modified objective function is

$$Z = 10x_1 + 5x_2 + 0 \cdot S_1 + 0 \cdot S_2 + 0 \cdot S_3$$

Step 5: C_B → introduced variable cost

Y_B → introduced variables

X_B → RHS value in matrix

Ist Iteration

		C_j	10	5	0	0	0	X_B/x_j
C_B	Y_B	X_B	x_1	x_2	S_1	S_2	S_3	
0	S1	36	3	3	1	0	0	$36/3 = 12$
0	S2	60	2	6	0	1	0	$60/2 = 30$
0	S3	50	$\boxed{5}$	2	0	0	1	$50/5 = 10$ ← (+min)
$Z_j = \Sigma C_B \cdot x_j$			0	0	0	0	0	
$Z_j - C_j \geq 0$			-10	-5	0	0	0	
			↑ -max					

[5 is the key element]

Calculate $Z_j = \Sigma C_B \cdot x_j$

Check if $(Z_j - C_j \geq 0)$ carry out iterations until the condition is satisfied

Select the column which has (-max) and calculate X_B/x_j for the corresponding column. Find out (+min) value in X_B/x_j. Select the key element at the point of intersection.

Convert the key element to unity by performing certain row operations

$R_3 : (50 \quad 5 \quad 2 \quad 0 \quad 0 \quad 1) \times 1/5$

$R_3 : 10 \quad 1 \quad 2/5 \quad 0 \quad 0 \quad 1/5$

Convert the corresponding elements in the column of key elements to zero by applying suitable row operation.

$R3 = R3 \times 2 = (10 \quad 1 \quad 2/5 \quad 0 \quad 0 \quad 1/5) \times 2$

$\quad 20 \quad 2 \quad 4/5 \quad 0 \quad 0 \quad 2/5$

$R2 - R3 = 40 \quad 0 \quad 26/5 \quad 0 \quad 1 \quad -2/5$

$R2 : 40 \quad 0 \quad 26/5 \quad 0 \quad 1 \quad -2/5$

$R3 \times 3 : (10 \quad 1 \quad 2/5 \quad 0 \quad 0 \quad 1/5) \times 3$

$\quad 30 \quad 3 \quad 6/5 \quad 0 \quad 0 \quad 3/5$

$-$

$R1 : 36 \quad 3 \quad 3 \quad 1 \quad 0 \quad 0$

$R1 = R3 - R1$

$R1 : 6 \quad 0 \quad 9/5 \quad 1 \quad 0 \quad -3/5$

IInd Iteration.

		C_j	10	5	0	0	0	x_B/x_j
C_B	Y_B	X_B	x_1	x_2	s_1	s_2	s_3	
0	s_1	6	0	9/5	1	0	-3/5	$6 \times 5/9 = 10/3 \leftarrow$ (min)
0	s_2	40	0	26/5	0	1	-2/5	$40 \times 5/26 = 100/13$
10	x_1	10	1	2/5	0	0	1/5	$10 \times 5/2 = 25$
$Z_j = \sum C_B \cdot x_j$			10	4	0	0	2	
			0	-1	0	0	2	
				(-max)				

Go paperless! Save Earth!

$R_1 : (6 \quad 0 \quad 9/5 \quad 1 \quad 0 \quad -3/5) \times 5/9$

$R_1 : \quad 10/3 \quad 0 \quad 1 \quad 5/9 \quad 0 \quad -1/3$

$R_2 = R_2 - (R_1 \times 26/5)$

$R_1 \times 26/5 : (10/3 \quad 0 \quad 1 \quad 5/9 \quad 0 \quad -1/3) \times 26/5$

$52/3 \quad 0 \quad 26/5 \quad 26/9 \quad 0 \quad -26/27$

$40 \quad 0 \quad 26/5 \quad 0 \quad 1 \quad -2/5$

$R_2 : \quad \frac{68}{3} \quad 0 \quad 0 \quad -\frac{26}{9} \quad 1 \quad \frac{76}{135}$

$R_3 = R_3 - (R_1 \times 2/5)$

$(10/3 \quad 0 \quad 1 \quad 5/9 \quad 0 \quad -1/3) \times 2/5$

$\frac{4}{3} \quad 0 \quad 2/5 \quad 2/9 \quad 0 \quad -2/15$

$- \quad 10 \quad 1 \quad 2/5 \quad 0 \quad 0 \quad 1/5$

$-2/9 \quad 0 \quad 3/5$

$R_3 : \quad 26/3 \quad 1 \quad 0$

III rd Iteration

C_B	Y_B	C_j / X_B	10 / x_1	5 / x_2	0 / s_1	0 / s_2	0 / s_3
5	x_2	10/3	0	1	5/9	0	-1/3
0	s_2	68/3	0	0	-26/9	1	76/135
10	x_1	26/3	1	0	-2/9	0	3/5
$Z_j = \sum C_B \cdot x_j$			10	5	5/9	0	13/3
$Z_j - C_j \geq 0$			0	0	5/9	0	4/3

The condition $z_j - c_j \geq 0$ is satisfied

$\max z = \sum C_B \cdot X_B$

$= (5 \times 10/3) + (10 \times 26/3)$

$= 50/3 + 260/3$

$= 310/3$ //

To verify, substitute the value of x_1, x_2, x_3 in the objective function

$z = 10x_1 + 5x_2$

$= 10\left(\frac{26}{3}\right) + 5\left(10/3\right)$

$= \frac{310}{3}$ //

2) Use simplex method to solve LPP

$\max z = 3x_1 + 2x_2$

STC $\quad x_1 + x_2 \leq 4$

$x_1 - x_2 \leq 2$

where $x_1 \,\&\, x_2 \geq 0$

$\Rightarrow$ Convert inequality to equality

$x_1 + x_2 + S_1 = 4$

$x_1 - x_2 + S_2 = 2$

Represent the equation in matrix format

$A \cdot X = B$

$$\begin{array}{c} \begin{array}{cccc} x_1 & x_2 & s_1 & s_2 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & -1 & 0 & 1 \end{bmatrix} \end{array} \begin{bmatrix} x_1 \\ x_2 \\ s_1 \\ s_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

The modified objective function is

$Z = 3x_1 + 2x_2 + 0 \cdot s_1 + 0 \cdot s_2$

Ist Iteration

		C_j	3	2	0	0	x_B/x_j
C_B	Y_B	X_B	x_1	x_2	s_1	s_2	
0	s_1	4	1	1	1	0	4
0	s_2	2	$\boxed{1}$	-1	0	1	2 ← min
$Z_j = \sum C_B \cdot x_j$			0	0	0	0	
$Z_j - C_j \geq 0$			-3	-2	0	0	
			↑ (−max)				

$R_1 = R_1 - R_2$

4	1	1	1	0
− 2	1	-1	0	1
R_1: 2	0	2	1	-1

IInd Iteration

		C_j	3	2	0	0	x_B/x_i
C_B	Y_B	X_B	x_1	x_2	S_1	S_2	
0	S_1	2	0	2	1	-1	1 ← +min
3	x_1	2	1	-1	0	1	-2
$Z_j = \Sigma C_B \cdot x_j$			3	-3	0	3	
$Z_j - C_j \geq 0$			0	-5	0	3	

↑ (max)

$R_1 : (2 \quad 0 \quad 2 \quad 1 \quad -1) \times 1/2$

$\quad 1 \quad 0 \quad 1 \quad 1/2 \quad -1/2$

$R_2 : \quad 2 \quad 1 \quad -1 \quad 0 \quad 1$

$+ R_1 : \quad 1 \quad 0 \quad 1 \quad 1/2 \quad -1/2$

$R_2 : \quad 3 \quad 1 \quad 0 \quad 1/2 \quad 1/2$

III iteration

		C_i	3	2	0	0
C_B	Y_B	X_B	x_1	x_2	S_1	S_2
2	x_2	1	0	1	1/2	-1/2
3	x_1	3	1	0	1/2	-1/2
$Z_j = \Sigma C_B \cdot x_j$			3	2	5/2	3/4
$Z_j - C_j \geq 0$			0	0	5/2	3/4

the condition $z_j - c_j \geq 0$ is satisfied

max $z = (2\times1) + (3\times3)$

$= 2 + 9$ $\quad x_1 = 3 \quad x_2 = 1$

$= 11$ //

To verify $\quad z = 3x_1 + 2x_2$

$= 3(3) + 2(1)$

$= 9 + 2$

$= 11$ //

Big-M method

1) Use Big-M method to solve the following lpp minimize $z = 5x + 3y$

$$2x + 4y \leq 12$$
$$2x + 2y = 10$$
$$5x + 2y \geq 10$$

⇒ Convert the inequality to equality and introduce slak variable

$$2x + 4y + S_1 = 12$$
$$2x + 2y + A_1 = 10$$
$$5x + 2y - S_2 + A_2 = 10$$

[Since the 3rd constaint does not have a boundry introduce a surplus (-S) variable and an artificial variable (A)]

Convert the given problem to maximization

$$Z = -5x - 3y$$

Write the equation in matrix format

$$\begin{bmatrix} x & y & S_1 & S_2 & A_1 & A_2 \\ 2 & 4 & 1 & 0 & 0 & 0 \\ 2 & 2 & 0 & 0 & 1 & 0 \\ 5 & 2 & 0 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ S_1 \\ S_2 \\ A_1 \\ A_2 \end{bmatrix} \begin{bmatrix} 12 \\ 10 \\ 10 \end{bmatrix}$$

modified objective function:

$$Z = -5x - 3y + 0 \cdot S_1 + 0 \cdot S_2 - MA_1 - MA_2$$

1st Iteration

		C_j	-5	-3	0	0	$-M$	$-M$	x_B/x_j
C_B	Y_B	X_B	x	y	S_1	S_2	A_1	A_2	
0	S_1	12	2	4	1	0	0	0	6
$-M$	A_1	10	2	2	0	0	1	0	5
$-M$	A_2	10	5	2	0	-1	0	1	$2 \leftarrow$ (+min)
$Z_j = \Sigma C_B \cdot x_j$			$-7M$	$-4M$	0	$+M$	$-M$	$-M$	
$Z_j - C_j \geq 0$			$-7M-5$	$-4M-3$	0	M	0	0	
			$\uparrow$ (-max)						

$R3 : (10 \quad 5 \quad 2 \quad 0 \quad -1 \quad 0 \quad 1) \; 1/5$

$\boxed{R3} : 2 \quad 1 \quad 2/5 \quad 0 \quad -1/5 \quad 0 \quad 1/5$

$R3 \times 2 : 4 \quad 2 \quad 4/5 \quad 0 \quad -2/5 \quad 0 \quad 2/5$

$R2 : 10 \quad 2 \quad 2 \quad 0 \quad 0 \quad 1 \quad 0$

$\boxed{R2} : 6 \quad 0 \quad 6/5 \quad 0 \quad 2/5 \quad 1 \quad -2/5$

$R1 : 4 \quad 2 \quad 4/5 \quad 0 \quad -2/5 \quad 0 \quad 2/5$

$12 \quad 2 \quad 4 \quad 1 \quad 0 \quad 0 \quad 0$

$\boxed{R1} : 8 \quad 0 \quad 16/5 \quad 1 \quad 2/5 \quad 0 \quad -2/5$

IInd Iteration

		C_j	-5	-3	0	0	-M	-M	x_B/x_1
C_B	Y_B	X_B	x	y	S1	S2	A1	A2	
0	S1	8	0	$\boxed{16/5}$	1	2/5	0	-2/5	5/2 ← (+min)
-M	A1	6	0	6/5	0	2/5	1	-2/5	5
-5	x	2	1	2/5	0	-1/5	0	1/5	5
$Z_j = \Sigma C_B \cdot X_j$			-5	$\frac{-6M}{5} - 2$	0	$\frac{-2M}{5} + 1$	-M		
$Z_j - C_j \geq 0$			0	$\frac{-6M}{5} + 1$	0	$-\frac{2M}{5} + 1$	0		
				↑ (-max)					

$R1 : (8 \quad 0 \quad 16/5 \quad 1 \quad 2/5 \quad 0) \times 5/16$

$\boxed{R1} : 5/2 \quad 0 \quad 1 \quad 5/16 \quad 1/8 \quad 0$

$R_1 \times 6/5$ $\left(\frac{5}{2} \quad 0 \quad 1 \quad \frac{5}{16} \quad 1/8 \quad 0\right) \times 6/5$

	3	0	6/5	3/8	3/20	0
R_2 −	3	0	6/5	3/8	2/5	1
−	6	0	6/5	0	2/5	1
$\boxed{R_2}$:	3	0	0	−3/8	1/4	1

				1/8	1/20	0
$R_1 \times 2/5$	1	0	2/5	1/8	1/20	0
− R_3	2	1	2/5	0	−1/5	0
R_3 :	1	1	0	−1/8	−1/4	0

		C_j	−5	−3	0	0	−M
C_B	Y_B	X_B	x	y	S_1	S_2	A_1
−3	y	5/2	0	1	5/16	1/8	0
−M	A_1	3	0	0	−3/8	1/4	1
−5	x	1	1	0	−1/8	−1/4	0
$Z_j = \sum C_B \cdot x_j$			−5	−3	$+\frac{3M}{8} - \frac{5}{16}$	$\frac{7}{8} - \frac{M}{4}$	−M
$Z_j - C_j \geq 0$			0	0	1/16	5/8	0

$$Z = (-5 \times 1) + (-3 \times 5/2)$$

$$= -5 - \frac{15}{2}$$

$$= -25/2$$

To verify: $Z = -5x - 3y$

$$= -5(1) - 3(5/2)$$

$$= -5 - 15/2 = -25/2 \;//$$

Two Phase Simplex Method.

The Two Phase Simplex Method is another method to solve the given LPP involving some artificial variables.

Phase 1:- In this phase we construct an auxillary LPP to a final simplex table containing a basic feasible solution to the original problem

Step 1: Assign a cost(-1) to each artificial variable and a cost (0) to all other variables and get a new objective function.

Step 2: Write down the auxillary LPP in which the new objective function is to be maximized subject to the given set of constraints.

Step 3: Solve the auxillary LPP by simplex method either of the following cases arise.

1) max $z < 0$ and atleast 1 artificial variable appears at +ve level

2) max $z = 0$ and atleast 1 artificial variable appears at 0 level

3) max $z = 0$ and no artificial variables appears

Note: In case 1, given LPP does not process any feasible solution where as in case 2 and 3 we go to phase $\neq$ 2.

Phase 2 : Use the optimum basic feasible solution of phase 1 as a starting solution for the original LPP. Assign the actual cost to the variable in the objective function, and a zero cost to every artificial variable at zero level, delete the artificial variable column that is eliminated from the phase 1. Apply simplex method to the modified simplex table obtain at the end of phase 1 till an optimum basic feasible solution is obtained.

1) Use two phase method to solve

max $z = 3x_1 - x_2$

STC $2x_1 + x_2 \geq 2$

$x_1 + 3x_2 \leq 2$

$x_2 \leq 4$

where $x_1, x_2 \geq 0$

$\Rightarrow$ Step 1: Convert the given inequality to equality

$z = 3x_1 - x_2$

$2x_1 + x_2 - S_1 + A_1 = 2$

$x_1 + 3x_2 + S_2 = 2$

$x_2 + S_3 = 4$

Step 2: Rewrite the equation in matrix format

$$
\begin{array}{cccccc}
x_1 & x_2 & S_1 & S_2 & S_3 & A_1
\end{array}
$$

$$
\begin{bmatrix} 1 & 1 & -1 & 0 & 0 & 1 \\ 1 & 3 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \end{bmatrix}
\begin{bmatrix} x_1 \\ x_2 \\ S_1 \\ S_2 \\ S_3 \\ A_1 \end{bmatrix}
=
\begin{bmatrix} 2 \\ 2 \\ 4 \end{bmatrix}
$$

Phase I

			0	0	0	0	0	-1	x_B/x_1	
C_B	Y_B	X_B	x_1	x_2	S_1	S_2	S_3	A_1		
-1	A_1	2	$\boxed{2}$	1	-1	0	0	1	1	← (+ve min)
0	S_2	2	1	3	0	1	0	0	2	
0	S_3	4	0	1	0	0	1	0	∞	
$Z_j = \Sigma C_B \cdot x_j$			-2	-1	1	0	0	-1		
$Z_j - C_j \geq 0$			-2	-1	1	0	0	0		
			↑ (-max)							

R1: (2 2 1 -1 0 0 1) / 2

1 1 1/2 -1/2 0 0 1/2

R2 2 1 3 0 1 0 0

R2: 1 0 5/2 1/2 1 0 -1/2

R3: 4 0 1 0 0 1 0

			0	0	0	0	0		x_B/x_j
C_B	Y_B	X_B	x_1	x_2	S_1	S_2	S_3	A_1	
0	x_1	1	1	1/2	-1/2	0	0	1/2	
0	S_2	1	0	5/2	1/2	1	0	-1/2	
0	S_3	4	0	1	0	0	1	0	
$Z_j = \Sigma C_B \cdot x_j$			0	0	0	0	0	0	

$z_{max} = 0$ and no artificial variable go to phase II

		C_j	3	-1	0	0	0	X_B/x_j
C_B	Y_B	X_B	x_1	x_2	S_1	S_2	S_3	
3	x_1	1	1	1/2	-1/2	0	0	-2
0	S_2	1	0	5/2	$\boxed{1/2}$	1	0	2 ← +min
0	S_3	4	0	1	0	0	1	∞ ← +min
$Z_j = \sum C_B \cdot X_B$			3	3/2	-3/2	0	0	
$Z_j - C_j \geq 0$			0	5/2	-3/2	0	0	
					↑ (-max)			

R2 : (1 0 5/2 1/2 1 0) × 2

R2 2 0 5 1 2 0

(R2 / 2) 1 0 5/2 1/2 1 0

R1 + ~~1~~ 1 1/2 -1/2 0 0

R1 2 1 3 0 1 0

		C_j	3	-1	0	0	0	X_B/x_j
C_B	Y_B	X_B	x_1	x_2	S_1	S_2	S_3	
3	x_1	2	1	3	0	1	0	
0	S_1	2	0	5	1	2	0	
0	S_3	4	0	1	0	0	1	
$Z_j = \sum C_B \cdot X_j$			3	9	0	3	0	
$Z_j - C_j$			0	10	0	3	0	

$z_{max} = 6$ //

$x_1 = 2 \quad x_2 = 0$

$Z = 3x_1 - x_2$

$= 3(2) - 0$

$= 6$ //

2) min $z = x_1 - 2x_2 - 3x_3$

STC $-2x_1 + x_2 + 3x_3 = 2$

$2x_1 + 3x_2 + 4x_3 = 1$

where $x_1, x_2, x_3 \geq 0$

Step 1: convert min to max problem

$z = -x_1 + 2x_2 + 3x_3$

Step 2: Convert inequality to equality.

$-2x_1 + x_2 + 3x_3 + A_1 = 2$

$+2x_1 + 3x_2 + 4x_3 + A_2 = 1$

Step 3: Write in matrix form.

$$\begin{bmatrix} x_1 & x_2 & x_3 & A_1 & A_2 \\ -2 & 1 & 3 & 1 & 0 \\ 2 & 3 & 4 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ A_1 \\ A_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

		C_j	0	0	0	-1	-1	X_B/x_j
C_B	Y_B	X_B	x_1	x_2	x_3	A_1	A_2	
-1	A_1	2	-2	1	3	1	0	2/3
-1	A_2	1	2	3	$\boxed{4}$	0	1	1/4 ← +min
$z_j = \Sigma C_B \cdot x_j$			-4	-4	-7	-1	-1	
$z_j - C_j$			-4	-4	-7	0	0	

↑ (−max)

R2: (1 2 3 4 0 1) /4

1/4 1/2 3/4 1 0 1/4

(R2 x 3)	3/4	3/2	9/4	3	0	3/4
- R1	2	-2	1	3	1	0
R1 :	5/4	-7/2	-5/4	0	1	-3/4

		C_j	0	0	0	-1	
C_B	Y_B	X_B	x_1	x_2	x_3	A_1	A_2
-1	A_1	5/4	-7/2	-5/4	0	1	-3/4
0	x_3	1/4	1/2	3/4	1	0	1/4
$Z_j = \Sigma C_B \cdot x_j$			7/2	5/4	0	-1	3/4
$Z_j - C_j$			7/2	5/4	0	0	

$\max z = (-1 \times 5/4) + 0$

$= -5/4$

$z_{max} \not< 0$ and one artificial variable appears.

No feasible solution.

Module - 3

Simplex Method - 2 Duality Theory

The essence of duality theory.

Every linear programming problem has been associated with another linear programming problem. The original problem is called "primal" while the other is called its dual. In general either problem can be considered the primal with the remaining one its dual. If the primal is solved it is equivalent to solving its dual.

Definition of the dual problem

Let the primal problem be,

Max $z = c_1x_1 + c_2x_2 + \dots + c_nx_n$

subject to
$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots a_{1n}x_n \le b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \dots a_{2n}x_n \le b_2$$
$$\vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n \le b_m$$
$$x_1, x_2, \dots x_n \ge 0$$

Dual : The dual problem is defined as

Min $z' = b_1w_1 + b_2w_2 + \dots + b_mw_m$

subject to
$$a_{11}w_1 + a_{21}w_2 + \dots + a_{m1}w_m \ge c_1$$
$$a_{12}w_1 + a_{22}w_2 + \dots + a_{m2}w_m \ge c_2$$
$$\vdots$$
$$a_{1n}w_1 + a_{2n}w_2 + \dots + a_{mn}w_m \ge c_n$$
$$w_1, w_2, \dots w_m \ge 0$$

where $w_1, w_2 \dots w_m$ are called dual variables.

Characteristics of the Dual problem.

Duality in linear programming has the following characteristics:

1) Dual of the dual LP is primal.

2) If either the primal or dual of the problem has the optimal solution, then the other one will also have the same.

3) If any of the two problem has an infeasible solution then the value of the objective function on the other is unbounded.

4) The value of the objective function for any feasible solution of the primal is less than the value of the objective function for any feasible solution of the dual.

~~8) If any of the~~

5) If either the primal or the dual has an unbounded objective function value then the solution to the other problem is infeasible.

6) If the primal has a feasible solution but the dual does not have, then the primal will not have finite optimal solution & vice versa.

Formulation of Dual problems

i) Change the objective function of maximization in the primal into minimization in the dual and vice versa.

ii) The number of variables in the primal will be the number of constraint in the dual and vice versa

iii) The cost coefficients $C_1, C_2 \dots C_n$ in the objective function of the primal will be the RHS constant of the constraint in the dual and vice versa.

iv) In forming the constraints for the dual, we consider the transpose of the body matrix of the primal problem.

v) The variables in both problems are non negative

vi) If the variable in the primal is unrestricted in sign, then the corresponding constraints in the dual will be an equation and vice versa.

Problems:

i) Write the dual for the following primal LP problem

Max $\quad z = x_1 + 2x_2 + x_3$

Subject to
$$2x_1 + x_2 + x_3 \le 2$$
$$-2x_1 + x_2 - 5x_3 \ge -6$$
$$4x_1 + x_2 + x_3 \le 6$$
$$x_1, x_2, x_3 \ge 0$$

$\Rightarrow$ Since the problem is not in canonical form we interchange the inequality of the second constraint

Max $\quad z = x_1 + 2x_2 + x_3$

Subject to
$$2x_1 + x_2 - x_3 \le 2$$
$$2x_1 - x_2 + 5x_3 \le 6$$
$$4x_1 + x_2 + x_3 \le 6$$
$$x_1, x_2, x_3 \ge 0$$

Dual: Let w_1, w_2, w_3 be the dual variables

$$z' = 2w_1 + 6w_2 + 6w_3$$

Subject to

$$2w_1 + 2w_2 + 4w_3 \geq 1$$

$$+w_1 - w_2 + w_3 \geq 2$$

$$-w_1 + 5w_2 + w_3 \geq 1$$

$$\text{With } w_1, w_2, w_3 \geq 0$$

2) Find the dual of the following LPP

max $z = 3x_1 - x_2 + x_3$

Subject to

$$4x_1 - x_2 \leq 8$$

$$8x_1 + x_2 + 3x_3 \geq 12$$

$$5x_1 - 6x_3 \leq 13$$

$$x_1, x_2, x_3 \geq 0$$

⇒ Interchange the inequality of the second constraint.

MAX $z = 3x_1 - x_2 + x_3$

$$4x_1 - x_2 + 0x_3 \leq 8$$

$$-8x_1 - x_2 - 3x_3 \leq -12$$

$$5x_1 + 0x_2 - 6x_3 \leq 13$$

$$x_1, x_2, x_3 \geq 0$$

Dual: $z' = 8w_1 - 12w_2 + 13w_3$

$$4w_1 - 8w_2 + 5w_3 \geq 3$$

$$-w_1 - w_2 \geq -1$$

$$-3w_2 + 6w_3 \geq 1$$

$$w_1, w_2, w_3 \geq 0$$

3) Write the dual of the following LPP

$\max z = 40x_1 + 35x_2$

S.T $2x_1 + 3x_2 \le 60$

$4x_1 + 3x_2 \le 96$

$x_1, x_2 \ge 0$

$\Rightarrow$ Dual: $\min z = 60w_1 + 96w_2$

$2w_1 + 4w_2 \ge 40$

$3w_1 + 3w_2 \ge 35$

$w_1, w_2 \ge 0$

Dual of the above dual

$\max z = 40x_1 + 35x_2$

S.T $2x_1 + 3x_2 \le 60$

$4x_1 + 3x_2 \le 96$

$x_1, x_2 \ge 0$

4) Write the dual of the following LPP

$\max z = 3x_1 + 4x_2 + 7x_3$

S.T $x_1 + x_2 + x_3 \le 10$

$4x_1 - x_2 - x_3 \ge 15$

$x_1 + x_2 + x_3 = 7$

$x_1, x_2 \ge 0$

$\Rightarrow$ Convert the second constraint to standard form

$-4x_1 + x_2 + x_3 \le -15$

The third constraint can be expressed as a pair of inequalities.

$\cancel{x_1 + x_2 + y_3}$

$x_1 + x_2 + x_3 \le 7$

$-x_1 - x_2 - x_3 \le -7$

Let $y_3 = y_3' - y_3''$

Dual : $z' = 10y_1 + 15y_2 + 7(y_3' - y_3'')$

$y_1 - 4y_2 + (y_3' - y_3'') \geq 3$

$y_1 - y_2 + (y_3' - y_3'') \geq 4$

$y_1 + y_2 + (y_3' - y_3'') \geq 7$

$\therefore$ $z' = 10y_1 + 15y_2 + 7y_3$

$y_1 - 4y_2 + y_3 \geq 3$

$y_1 - y_2 + y_3 \geq 4$

$y_1 + y_2 + y_3 \geq 7.$

The Dual Simplex Method.

The algorithm is designed to solve a class of LP models efficiently. It is used to solve problems which start dual feasible. i.e., whose primal is optimal but infeasible. In this method the solution starts better than optimum but infeasible and remains infeasible until the true optimum is reached at which the solution becomes feasible.

Application of Dual simplex method.

1. Parametric programming.
2. Integer programming algorithms
3. Some non linear programing algorithms
4. It eliminates the introduction of artificial variables in the LP problems.

Dual Simplex Algorithm.

Step 1: Convert the problem into maximization problem if it is initially in the minimization form.

Step 2: Convert $\geq$ type constraints, if any, into $\leq$ type by multiplying both sides of such constraints by -1.

Step 3: Convert the inequality constraints into equalities by addition of slack variables and obtain the initial solution. Express this in the form of a table.

Step 4: Compute $c_j - z_j$ for every coloumn. Three cases arise:

a) If all $c_j - z_j$ are either negative or zero and all b_i are non negative, the solution obtained above is the optimal basic feasible solution.

b) If all $c_j - z_j$ are either negative or zero and at least one b_i is negative, then proceed to step 5

c) If any $c_j - z_j$ is positive, the method fails.

Step 5: Select the row that contains the most negative b_i. This row is called the key row or the pivot row. The corresponding basic variable leaves basis. This is called dual feasibility condition.

Step 6: Look at the elements of the key row.

a) If all the elements are non negative, the problem does not have a feasible solution.

b) If atleast one element is negative, find the ratio of the corresponding elements of $c_j - z_j$ row to these elements. Ignore the ratio's associated with positive or zero elements of the key row. Choose the smallest of these ratio's. The corresponding coloumn is the key column and the associated variable is the entering variable. This is called Dual optimality condition. Mark the key element or the pivot element.

Step 7: Make the key element unity. Perform the row operation as in the regular simplex method and repeat iterations until either an optimal feasible solution is obtained in a finite number of steps or there is an indication of the non existence of the feasible solution.

Problems:

1) Solve the dual simplex method for the following LPP

min $\quad z = 2x_1 + 2x_2 + 4x_3$

S.T $\quad 2x_1 + 3x_2 + 5x_3 \geq 2$

$3x_1 + x_2 + 7x_3 \leq 3$

$x_1 + 4x_2 + 6x_3 \leq 5$

$x_1, x_2, x_3 \geq 0$

⇒ Step 1: The given problem is converted to minimization

$z = -2x_1 - 2x_2 - 4x_3$

Step 2: The constraint of type $\geq$ is converted to $\leq$ type

$-2x_1 - 3x_2 - 5x_3 \leq -2$

Step 3: Add slack variable to convert the given problem to standard form.

$$z = -2x_1 - 2x_2 - 4x_3 + 0s_1 + 0s_2 + 0s_3$$

$$-2x_1 - 3x_2 - 5x_3 + s_1 = -2$$

$$3x_1 + x_2 + 7x_3 + s_2 = 3$$

$$x_1 + 4x_2 + 6x_3 + s_3 = 5$$

$$x_1, x_2, x_3, s_1, s_2, s_3 \geq 0$$

	c_j	-2	-2	-4	0	0	0	
c_B	Basis	x_1	x_2	x_3	s_1	s_2	s_3	b
0	s_1	-2	$\boxed{-3}$	-5	1	0	0	-2 ←
0	s_2	3	1	7	0	1	0	3
0	s_3	1	4	6	0	0	1	5
$z_j = \sum c_B \cdot x_{ij}$		0	0	0	0	0	0	0
$c_j - z_j$		-2	-2 ↑	-4	0	0	0	

Step 4: Compute $c_j - z_j$ where $z_j = \sum c_B a_{ij}$. As all $c_j - z_j$ are either negative or zero and b_1 is negative the solution is optimal but infeasible.

Step 5: As $b_1 = -2$, the first row is the key row and s_1 is the outgoing variable.

Step 6: Find the ratio of elements of $c_j - z_j$ row to the elements of the key row. Neglect the ratio corresponding to positive or zero elements of key row.

$$\frac{-2}{-2} = 1, \quad \frac{-2}{-3} = \frac{2}{3}, \quad \frac{-4}{-5} = \frac{4}{5}$$

Since $\frac{2}{3}$ is the smallest ratio, 'x_2' column is the key column, x_2 is the incoming variable and -3 is the key element.

Step 7 : Replace s_1 by x_2. Apply corresponding row operation

$R_1 = R_1/-3$

R_1: $\frac{2}{3}$ 1 $5/3$ $-1/3$ 0   0 $2/3$

$R_2 = R_2 - R_1$: 3   1   7   0   1   0   3

$-$ $2/3$ 1 $5/3$ $-1/3$ 0   0 $2/3$

R_2: $7/3$ 0 $16/3$ $1/3$ 1   0 $7/3$

$R_3 = (R_1 \times 4) - R_3$

$\frac{8}{3}$ 4 $\frac{20}{3}$ $\frac{-4}{3}$ 0   0 $\frac{8}{3}$

$-$ 1   4   6   0   0   1   5

R_3: $-5/3$ 0 $-2/3$ $4/3$ 0 -1 $7/3$

	C_j	-2	-2	-4	0	0	0	
C_B	Basis	x_1	x_2	x_3	s_1	s_2	s_3	b
-2	x_2	$2/3$	1	$5/3$	$-1/3$	0	0	$2/3$
0	s_2	$7/3$	0	$16/3$	$1/3$	1	0	$7/3$
0	s_3	$-5/3$	0	$-2/3$	$4/3$	0	1	$7/3$
$Z_j = \sum C_B \cdot a_{ij}$		$-4/3$	-2	$-10/3$	$2/3$	0	0	$-4/3$
$C_j - Z_j$		$-2/3$	0	$-2/3$	$-2/3$	0	0	

optimal basic feasible solution.

As all $c_j - z_j$ are negative or zero and all b_i are positive, the given solution is optimal

$x_1 = 0$, $x_2 = 2/3$ $x_3 = 0$

~~max Z = -2 × (0 × -2) × 2/3~~

max : $(-2 \times 0) - (2 \times \frac{2}{3}) - (4 \times 0) = -4/3$

or min $z = 4/3$.

---

2) Use dual simplex method to

maximize $z = -3x_1 - 2x_2$

ST $x_1 + x_2 \geq 1$

$x_1 + x_2 \leq 7$

$x_1 + 2x_2 \geq 10$

$x_2 \leq 3$

$x_1, x_2 \geq 0$

⇒ Step 1: The given problem is maximization

Step 2: Convert the constraint of ≥ type to ≤ type.

$-x_1 - x_2 \leq -1$

$x_1 + x_2 \leq 7$

$-x_1 - 2x_2 \leq -10$

$x_2 \leq 3$

Step 3: Add slack variable to express the given problem in standard form.

$z = -3x_1 - 2x_2 + 0s_1 + 0s_2 + 0s_3 + 0s_4$

$-x_1 - x_2 + s_1 = -1$

$x_1 + x_2 + s_2 = 7$

$-x_1 - 2x_2 + s_3 = -10$

$x_2 + s_4 = 3$

	C_j	-3	-2	0	0	0	0	
C_B	Basis	x_1	x_2	s_1	s_2	s_3	s_4	b
0	s_1	-1	-1	1	0	0	0	-1
0	s_2	1	1	0	1	0	0	7
0	s_3	-1	$\boxed{-2}$	0	0	1	0	-10 ←
0	s_4	0	1	0	0	0	1	3
$z_j = \Sigma c_B \cdot x_{ij}$		0	0	0	0	0	0	0
$C_j - z_j$		-3	-2 ↑	0	0	0	0	

Step 4: Compute $C_j - z_j$ where $z_j = \Sigma c_B \cdot x_{ij}$. As all $C_j - z_j$ are either negative or zero and b_1 and b_3 are negative the solution is optimal but infeasible. We proceed to step 5.

Step 5: $b_3 = -10$ is the key row and s_3 is the outgoing variable

Step 6: Find the ratio of elements of $C_j - z_j$ row to the elements of key row.

$\frac{-3}{-1} = 3$ $\quad \frac{-2}{-2} = 1$

x_2 column is the key column and (-2) is the key element. s_3 is replaced by x_2.

R3 = B R3 | -2

R3 :	$\frac{1}{2}$	1	0	0	-1/2	0	5

R1 = R1 + R3

	-1	-1	1	0	0	0	-1
	1/2	1	0	0	-1/2	0	5
R1 :	-1/2	0	1	0	-1/2	0	4

R2 = R2 - R3

	1	1	0	1	0	0	7
	1/2	1	0	0	-1/2	0	5
R2 :	1/2	0	0	1	1/2	0	2

R4 = ~~R3 + R4~~ R4 - R3

$\frac{1}{2}$	1	0	0	-1/2	0	5
0	1	0	0	0	1	3
-1/2	0	0	0	1/2	1	-2

	C_j	-3	-2	0	0	0	0	
C_B	Basis	x_1	x_2	S_1	S_2	S_3	S_4	b
0	S_1	-1/2	0	1	0	-1/2	0	4
0	S_2	1/2	0	0	1	1/2	0	2
-2	x_2	1/2	1	0	0	-1/2	0	5
0	S_4	$\boxed{-1/2}$	0	0	0	1/2	1	-2 ←
$Z_j = \Sigma C_B \cdot x_{ij}$		-1	-2	0	0	1	0	-10
$C_j - Z_j$		-2	0	0	0	-1	0	
$\frac{C_j - Z_j}{a_{ij}}$		4 ↑	~~0~~ -	~~0~~ -	~~0~~ -	-	-	

Replace S4 and x1

$R4 = (R4)/(-1/2)$

$= (-1/2 \quad 0 \quad 0 \quad 0 \quad 1/2 \quad 1 \quad -2)/(-1/2)$

$= 1 \quad 0 \quad 0 \quad 0 \quad -1 \quad -2 \quad 4$

$R3 = (R4 \times (-1/2)) + R3$

	$\frac{-1}{2}$	0	0	0	1/2	1	-2
+	$\frac{1}{2}$	1	0	0	-1/2	0	5
R3 :	0	1	0	0	0	1	3

$R2 = (R4 \times (-1/2)) + R2$

	$\frac{-1}{2}$	0	0	0	1/2	1	-2
+	1/2	0	0	1	1/2	0	2
R2	0	0	0	1	1	1	0

$R1 = (R4 \times (-1/2)) - R1$

	$\frac{-1}{2}$	0	0	0	1/2	1	-2
-	-1/2	0	1	0	-1/2	0	4
R1 =	0	0	1	0	-1	-1	6

	C_j	-3	-2	0	0	0	0	
C_B	Basis	x_1	x_2	s_1	s_2	s_3	s_4	b
0	s_1	0	0	1	0	-1	-1	6
0	s_2	0	0	0	1	1	1	0
-2	x_2	0	1	0	0	0	1	3
-3	x_1	1	0	0	0	-1	-2	4
$Z_j = \sum C_B a_{ij}$		-3	-2	0	0	3	4	-18
$C_j - Z_j$		0	0	0	0	-3	-4	

The table gives optimal feasible solution

$x_1 = 4$, $x_2 = 3$

$$Z_{max} = -(3 \times 4) - (2 \times 3)$$

$$= \underline{\underline{-18}}$$

Module - 4

Transportation and Assignment Problems

The transportation problem:

The transportation problem is to transport various amounts of a single homogeneous comodity, that are initially stored at various origins, to different destinations in such a way that the total transportation cost is minimum.

Methods to find initial feasible solution:

1. Northwest corner method (NWC)
2. Matrix Minima method
3. Vogels Approximation method. (VAM)

North West Corner Method (NWC)

Step 1: Identify the northwest corner of the table

Allocate $x_{11} = \min(a_1, b_1)$

case 1: If $a_1 < b_1$, then first row gets completed.

case 2: If $b_1 < a_1$, then first column gets completed

case 3: If $a_1 = b_1$, then there is a tie and allocation can be made arbitrarily.

Step 2: Start from the northwest corner and repeat step 1 until all the requirements are satisfied.

Q1: Find the initial feasible solution for the following transportation problem by ~~us~~ using north west corner method.

	C1	C2	C3	Supply
B1	3	2	1	20
B2	2	4	1	50
B3	3	5	2	30
B4	4	6	7	25
Demand	40	30	55	

Step 1: Supply = 20 + 50 + 30 + 25 = 125

Demand = 40 + 30 + 55 = 125

Supply = Demand, Hence the given transportation problem is balanced.

	C_1	C2	C3	
B1	$\boxed{20}$ 3	2	1	~~20~~ 0
B2	$\boxed{20}$ 2	$\boxed{30}$ 4	1	~~50~~ ~~30~~ 0
B3	3	5	$\boxed{30}$ 2	~~30~~ 0
B4	4	6	$\boxed{25}$ 7	~~25~~ 0
	~~40~~ ~~20~~ 0	~~30~~ 0	~~55~~ ~~25~~ 0	

Step 2: The NWC is (1, 1), x_{11} = min(20, 40) = 20

B1 completes

20 is allocated to x_{11} (1, 1)

Step 3: The NWC = (2, 1) x_{21} = min (20, 50) = 20

20 is allocated to (2, 1), C1 completes.

Step 4: The NWC is (2, 2) x_{22} = min (30, 30) = 30

30 is allocated to (2, 2) C2 and B2 are complete.

Step 5: The NWC is (3, 3) x_{33} = min (30, 55) = 30

30 is allocated to (3, 3) B3 is complete.

Step 6: The NWC is (4, 3) x_{43} = min (25, 25) = 25

25 is allocated to (4, 3) C3 and B4 are complete.

∴ The total cost

TC = (20 × 3) + (20 × 2) + (30 × 4) + (30 × 2) + (25 × 7)

= 455

Q2: Find initial feasible solution by applying northwest corner method.

	D_1	D_2	D_3	Supply
O1	5	7	8	70
O2	4	4	6	30
O3	6	7	7	50
Demand	65	42	43	

Step 1: Supply = 70 + 30 + 50 = 150

Demand = 65 + 42 + 43 = 150

Supply = Demand, Hence given transportation problem is balanced.

	D_1	D_2	D_3	Supply
O_1	$\boxed{65}$ 5	$\boxed{5}$ 7	8	~~70~~ ~~5~~ 0
O_2	4	$\boxed{30}$ 4	6	~~30~~ 0
O_3	6	$\boxed{7}$ 7	$\boxed{43}$ 7	~~50~~ ~~43~~ 0
Demand	~~65~~ 0	~~42~~ ~~37~~ ~~7~~ 0	~~43~~ 0	

Step 2: The NWC is (1, 1), $x_{11} = \min(70, 65) = 65$

65 is allocated to (1, 1), D1 is complete

Step 3: The NWC is (1, 2) $x_{12} = \min(5, 42) = 5$

5 is allocated to (1, 2), O_1 is complete

Step 4: The NWC is (2, 2) $x_{22} = \min(30, 37) = 30$

30 is allocated to (2, 2) O_2 is complete

Step 5: The NWC is (3, 2) $x_{32} = \min(50, 7) = 7$

7 is allocated to (3, 2), D_2 is complete

Step 6: The NWC is (3, 3) $x_{33} = \min(43, 43) = 43$

43 is allocated to (3, 3) O_3 and D_3 are complete

∴ The total cost is

$TC = (65 \times 5) + (5 \times 7) + (30 \times 4) + (7 \times 7) + (43 \times 7)$

$= \underline{\underline{830}}$

Q 3). Find the feasible solution by applying northwest corner method.

	D_1	D_2	D_3	D_4	Supply
O_1	6	4	1	5	14
O_2	8	9	2	7	6
O_3	4	3	6	2	3
Demand	6	10	15	4	

Step 1: Supply $= 14 + 6 + 3 = 23$

Demand $= 6 + 10 + 15 + 4 = 35$

Supply ≠ Demand. The problem is unbalanced.

Add a dummy row O_4 with cost 12 to (35 - 23) to balance supply and demand.

	D_1	D_2	D_3	D_4	
O_1	$\boxed{6}$ 6	$\boxed{8}$ 4	1	5	~~14~~ ~~8~~ 0
O_2	8	$\boxed{2}$ 9	$\boxed{4}$ 2	7	~~6~~ ~~4~~ 0
O_3	4	3	$\boxed{3}$ 6	2	~~3~~ 0
O_4	0	0	$\boxed{8}$ 0	$\boxed{4}$ 0	~~12~~ ~~4~~ 0
	~~6~~ 0	~~10~~ ~~2~~ 0	~~15~~ ~~11~~ ~~8~~ 0	~~4~~ 0	

Step 2: The NWC is (1,1), $x_{11} = \min(14, 6) = 6$

6 is allocated to (1,1), D_1 is complete

Step 3: The NWC is (1,2) $x_{12} = \min(8, 10) = 8$

8 is allocated to (1,2) O_1 is complete

Step 4: The NWC is (2,2) $x_{22} = \min(2, 6) = 2$

2 is allocated to (2,2) D_2 is complete

Step 5: The NWC is (2,3) $x_{23} = \min(15, 4) = 4$

4 is allocated to (2,3) O_2 is complete

Step 6: The NWC is (3,3) $x_{33} = \min(3, 11) = 3$

3 is allocated to (3,3) O_3 is complete

Step 7: The NWC is (4,3) $x_{43} = \min(8, 12) = 8$

8 is allocated to (4,3) D_3 is complete

Step 8: The NWC is (4,4) $x_{44} = \min(4, 4) = 4$

4 is allocated to (4,4) O_4 and D_4 are complete.

$\therefore$ The total cost is

$$TC = (6 \times 6) + (8 \times 4) + (2 \times 9) + (4 \times 2) + (6 \times 3) + (8 \times 0) + (4 \times 0)$$

$$= \underline{\underline{112}}$$

Least cost method (Matrix Minima Method)

Step 1: Determine the smallest cost in the transportation table. Let it be c_{ij}. Allocate $= \min(a_i, b_j)$

Step 2: i) If $x_{ij} = a_i$, then cross out ith row. Goto step 3.

ii) If $x_{ij} = b_j$, then cross out jth coloumn. Goto step 3

iii) If $x_{ij} = a_i = b_j$, then cross out ith row or jth coloumn, but not both

Step 3: Repeat steps 1 and 2 for resulting transportation table until all requirements are satisfied.

Step 4: Whenever minimum cost is not unique, make an arbitary choice among the minima.

Q1)

	D_1	D_2	D_3	Supply
S_1	3	2	1	20
S_2	2	4	1	50
S_3	3	5	2	30
S_4	4	6	7	25
Demand	40	30	55	

Step 1: Supply = 20 + 50 + 30 + 25 = 125

Demand = 40 + 30 + 55 = 125

Demand = Supply. Hence the given transportation problem is balanced.

	D_1	D_2	D_3	Supply
S_1	3	$\boxed{15}$ 2	$\boxed{5}$ 1	~~20~~ ~~15~~ 0
S_2	2	4	$\boxed{50}$ 1	~~50~~ 0
S_3	$\boxed{30}$ 3	5	2	~~30~~ 0
S_4	$\boxed{10}$ 4	$\boxed{15}$ 6	7	~~25~~ ~~15~~ 0
Demand	~~40~~ ~~10~~ 0	~~30~~ ~~15~~ 0	~~55~~ ~~50~~ 0	

Step 2: The least cost is 1, there is a tie between (1, 3) and (2, 3). Find out the cell to which maximum cost can be allocated i.e (2, 3) = 50, S2 is completed.

Step 3: The least cost is 1. Allocate min $\min_{13}$ = (5, 20) = 5 Allocate 5 to (1, 3)
D3 is completed.

Step 4: The least cost is 2. x_{ij} $x_{12} = \min(30, 15) = 15$
Allocate 15 to (1, 2)
S1 is completed.

Step 5: The least cost is 3. $x_{31} = \min(30, 40) = 30$
Allocate 30 to (3, 1), S3 is complete

Step 6: The least cost is 4 $x_{41} = \min(10, 25) = 10$
Allocate 10 to (4, 1) D_1 is complete.

Step 7: The least cost is 6 $x_{42} = \min(15, 15) = 15$
Allocate 15 to (4, 2) D_2 is complete.

$\therefore$ Total cost

$TC = (15 \times 2) + (5 \times 1) + (50 \times 1) + (30 \times 3) + (10 \times 4)$
$+ (15 \times 6)$
$= \underline{\underline{305}}$

Q2)

	D1	D2	D3	Supply
O1	5	7	8	70
O2	4	4	6	30
O3	6	7	7	50
Demand	65	42	43	

Step 1: Supply = 70 + 30 + 50 = 150
Demand = 65 + 42 + 43 = 150

Demand = Supply. Hence the given transportation problem is balanced.

	D_1	D_2	D_3	Supply
O_1	$\boxed{35}$ 5	$\boxed{35}$ 7	8	~~70~~ ~~35~~ 0
O_2	$\boxed{30}$ 4	4	6	~~30~~ 0
O_3	6	$\boxed{7}$ 7	$\boxed{43}$ 7	~~50~~ ~~7~~ 0
Demand	~~65~~ ~~35~~ 0	~~42~~ ~~7~~ 0	~~43~~ 0	

Step 2: The least cost is 4. maximum cost can be allocated to (2, 1) $x_{21} = \min(30, 65) = 30$
30 is allocated to (2, 1) O_2 is completed.

Step 3: The least cost is 5. $x_{11} = \min(70, 35) = 35$
35 is allocated to (1, 1) D_1 is completed.

Step 4: The least cost is 7, maximum cost can be allocated to (3, 3) $x_{33} = \min(50, 43) = 43$
43 is allocated to (3, 3) D_3 is completed.

Step 5: The least cost is 7. maximum cost can be allocated to (1, 2) $x_{12} = \min(35, 42) = 35$
O_1 is completed

Step 6: The least cost is 7 ~~$x_{32} = (7, 7)$~~
$x_{32} = \min(7, 7) = 7$ 7 is allocated to (3, 2)
D_2 and O_3 are completed.

∴ The total cost is :

$TC = (35 \times 5) + (35 \times 7) + (30 \times 4) + (7 \times 7) +$
(43×7)
$= 890$

Vogel's Approximation Method (VAM)

1. Find initial feasible solution by applying VAM method.

	C_1	C_2	C_3	Supply
B_1	3	2	1	20
B_2	2	4	1	50
B_3	3	5	2	30
B_4	4	6	7	25
Demand	40	30	55	

⇒ Supply = 20 + 50 + 30 + 25 = 125
Demand = 40 + 30 + 55 = 125

Supply = Demand. Hence the given transportation problem is balanced.

Step 2: Add a penalty column. Find 2 least cell in the row and find the difference. The result is added to the penalty coloumn.

	C1	C2	C3	Supply	Penalty			
B1	3	$\boxed{20}$ 2	1	~~20~~ 0	1			
B2	2	4	$\boxed{50}$ 1	~~50~~ 0	1	1	1	
B3	$\boxed{15}$ 3	$\boxed{10}$ 5	$\boxed{5}$ 2	~~30~~ ~~20~~ ~~5~~ 0	1	1	1	1
B4	$\boxed{25}$ 4	6	7	~~25~~ 0	2	2		
Demand	~~40~~ ~~15~~ 0	~~30~~ ~~10~~ 0	~~55~~ ~~5~~ 0					
Penalty	1	2	0					
	1	1	1					
	1	1	1					
	3	5	2					

Step 2

Step 3: Find the maximum penalty in both row and coloumn. Here the maximum penalty is 2 for both B4 and C2. Find the least cell in B4 and C2 and assign the cost. Here 20 is assigned to (1, 2), B1 is completed.

Step 4: Calculate new penalty for the remaining rows and column. Repeat step 2 to

Repeat steps 2 to 4 until all the rows and column are completed.

Step 5: Calculate the total cost for all the allocated steps

Total cost:

$TC = (20\times2) + (50\times1) + (15\times3) + (10\times5) + (5\times2)$
$+ (25\times4)$
$= \underline{\underline{295}}$

Q 2)

	C1	C2	C3	Supply
B1	5	7	8	70
B2	4	4	6	30
B3	6	7	7	50
Demand	65	42	43	

$\Rightarrow$ Step 1: Supply = 70 + 30 + 50 = 150

Demand = 65 + 42 + 43 = 150

Supply = Demand. Hence the given transportation problem is balanced.

Step 2: Add a penalty column. Find 2 least cell in the row and find the difference. The result is added to the penalty column.

	C1	C2	C3	Supply	Penalty			
B1	3	$\boxed{20}$ 2	1	~~20~~ 0	1			
B2	2	4	$\boxed{50}$ 1	~~50~~ 0	1	1	1	
B3	$\boxed{15}$ 3	$\boxed{10}$ 5	$\boxed{5}$ 2	~~30~~ ~~20~~ ~~5~~ 0	1	1	1	1
B4	$\boxed{25}$ 4	6	7	~~25~~ 0	2	2		
Demand	~~40~~ ~~15~~ 0	~~30~~ ~~10~~ 0	~~55~~ ~~5~~ 0					
Penalty	1	2	0					
Step 2	1	1	1					
	1	1	1					
	3	5	2					

Step 3: Find the maximum penalty in both row and coloumn. Here the maximum penalty is 2 for both B4 and C2. Find the least cell in B4 and C2 and assign the cost. Here 20 is assigned to (1, 2), B1 is completed.

Step 4: Calculate new penalty for the remaining rows and column. Repeat ~~step 2 to~~

Repeat steps 2 to 4 until all the rows and column are completed.

Modified Distribution Method.

1) Solve the following transportation problem by applying Vogus method and also check optimality test

	S1	S2	S3	S4	Supply
O1	6	1	9	3	70
O2	11	5	2	8	55
O3	10	12	4	7	90
Demand	85	35	50	45	

Step 1:: Apply Vogus approximation method and find the total cost.

Supply = 70 + 55 + 90 = 215

Demand = 85 + 35 + 50 + 45 = 215

Supply = Demand. Hence the given transportation problem is balanced.

	S1	S2	S3	S4	Supply	Penalty			
O1	6	$\boxed{35}$ 1	9	$\boxed{35}$ 3	~~70~~ ~~35~~ 0	2	3	3	
O2	$\boxed{5}$ 11	5	$\boxed{50}$ 2	8	~~55~~ ~~50~~	3	6	3	3
O3	$\boxed{80}$ 10	12	4	$\boxed{10}$ 7	~~90~~ ~~80~~ 0	3	3	3	3
Demand	~~85~~ 5	~~35~~ 0	~~50~~ 0	~~45~~ 0					
Penalty	4	4	2	4					
	4		2	4					
	4			4					
	1			1					

∴ The total cost is

$TC = 35 + (35 \times 3) + (5 \times 11) + (50 \times 2) + (10 \times 8) + (10 \times 7)$

$= 1165$

Phase - II : MODI / UV , LOOP METHOD

Check if the total numbers of allocations is equal to $m+n-1$ m = no of rows , n = no of column

$m+n-1$ = total no of allocation

$3+4-1 = 6$

$7-1 = 6$

$6 = 6$

Consider the occupied cell

	S1	S2	S3	S4	u_i
O1		1		3	0
O2	11			2	5
O3	10			7	4
v_j	6	1	-3	3	

Calculate the values of u_i and v_j such that $u_i + v_j = C_{ij}$. Start by initializing any one of the row or column value as 0

Consider the unoccupied cells

				u_i
$\boxed{6}$ 6		$\boxed{-3}$ 9		0
	$\boxed{6}$ 5		$\boxed{8}$ 8	5
	$\boxed{5}$ 12	$\boxed{1}$ 4		4
v_j: 6	1	-3	3	

Calculate z_j for each unoccupied cell such that $z_j = v_j + u_i$

Calculate $(C_{ij} - z_j)$ for each ~~ea~~ cell and check if the condition $C_{ij} - z_j \geq 0$. If the condition is not statisfied then $TC = 1165$ is not optimum solution.

0		12	
	-1		0
	7	3	

Here the cell $(2,2)$ has a negative value. Hence the condition $C_{ij} - z_j \geq 0$ is not satisfied.

Now consider the cell with the negative value i.e $(2,2)$ and form a closed loop to the of occupied cells and assign $+/- \theta$ to the alternate cells.

6	$\boxed{35}$ $-\theta$ 1	9	$\boxed{35}$ $+\theta$ 3
$\boxed{5}$ $-\theta$ 11	$+\theta$ 5	$\boxed{50}$ 2	8
$\boxed{80}$ $+\theta$ 10	12	4	$\boxed{10}$ $-\theta$ 7

To calculate the value of θ. consider the cell with negative theta values and find the minimum among them

$$\theta = \min(35-\theta, 5-\theta, 10-\theta) = 0$$

$$5-\theta = 0$$

$$\theta = 5$$

Substitute the theta values to the corresponding cells occupied cells and and calculate the total cost.

6	$\boxed{30}$ 1	9	$\boxed{40}$ 3
$\boxed{0}$ 11	$\boxed{5}$ 5	$\boxed{50}$ 2	8
$\boxed{85}$ 10	12	4	$\boxed{5}$ 7

$$TC = (30\times1) + (40\times3) + (0\times11) + (5\times5) + (50\times2)$$
$$+ (85\times10) + (5\times7)$$
$$= 1160$$

Apply MODI/UV or LOOP method again to check if the solution is optimum or not.

$m+n-1 =$ total no of allocations

$3+4-1 = 6$

$6 = 6$

Consider the occupied cells

Calculate the values of u_i and v_j such that $u_i + v_j = c_{ij}$. Start by initializing any one of the row or column value as 0

				u_i
	1		3	-4
	5	2		0
10			7	0
v_j 10	5	2	7	

Consider the unoccupied cells.

Calculate z_{ij} for each unoccupied cells such that $z_{ij} = v_j + u_i$ and calculate $(c_{ij} - z_{ij})$ for each cell and check if the condition $(c_{ij} - z_{ij}) \geq 0$ is satisfied.

				u_i
$\boxed{6}$ 6		$\boxed{-2}$ 9		-4
$\boxed{10}$ 11			$\boxed{7}$ 8	0
	$\boxed{5}$ 12	$\boxed{2}$ 4		0
v_j 10	5	2	7	

0		11	
1			1
	7	2	

The condition is satisfied $(c_{ij} - z_j \geq 0)$

Hence $TC = 1160$ is the optimum solution.

2)

	D1	D2	D3	D4	Supply
S1	21	16	25	13	11
S2	17	18	14	23	13
S3	32	17	18	41	19
Demand	6	10	12	15	

⟹ Apply Vogus Approximation method and find the total cost

Supply = 11 + 13 + 19 = 43

Demand = 6 + 10 + 12 + 15 = 43

Supply = Demand. Hence the given problem is balanced.

	D1	D2	D3	D4	Supply	Penalty				
S1	21	16	25	$\boxed{11}$ 13	~~11~~ 0	3				
S2	$\boxed{6}$ 17	18	$\boxed{3}$ 14	$\boxed{4}$ 23	~~13~~ ~~9~~ ~~8~~ 0	3	3	3	4	
S3	32	$\boxed{10}$ 17	$\boxed{9}$ 18	41	~~19~~ 10	1	1	1	1	1
Demand	~~6~~ 0	~~10~~ 0	~~12~~ ~~9~~ 0	~~15~~ ~~4~~ 0						
Penalty	4	1	4	10						
	15	1	4	18						
	15	1	4							
		1	4							
		17	18							

$TC = (11 \times 13) + (6 \times 17) + (3 \times 14) + (4 \times 23) + (10 \times 17) + (9 \times 18)$

$= 711$

Phase II : MODI / UV, LOOP Method.

Check if the total number of allocations is equal to m+n-1 m = no of rows, n = no of columns

m+n-1 = total no of allocations

3+4-1 = 6

6 = 6

Consider the occupied cells

				U_i
			13	0
17		14	23	10
	17	18		14
V_j 7	3	4	13	

Calculate the values of U_i and V_j such that $U_i + V_j = C_{ij}$. Start by initializing any one of the row or column value by 0

Consider the unoccupied cells

				U_i
$\boxed{7}$ 21	$\boxed{3}$ 16	$\boxed{4}$ 25		0
	$\boxed{13}$ 18			10
$\boxed{21}$ 32			$\boxed{27}$ 41	14
V_j 7	3	4	13	

Calculate Z_j for each unoccupied cell such that $Z_j = V_j + U_i$

Calculate $(C_{ij} - Z_j)$ for each cell and check if the condition $C_{ij} - Z_j \geq 0$. Here the condition is satisfied and hence TC = 711 is the optimal solution.

14	13	21	
	5		
11			14

$$C_{ij} - z_j \geq 0$$

Hence TC = 711 is the optimal solution.

Assignment Problems :-

1)

$$\begin{array}{c} \\ A \\ B \\ C \\ D \\ E \end{array} \begin{array}{c} \begin{array}{ccccc} S1 & S2 & S3 & S4 & S5 \end{array} \\ \begin{bmatrix} 10 & 3 & 3 & 2 & 8 \\ 9 & 7 & 8 & 2 & 7 \\ 7 & 5 & 6 & 2 & 4 \\ 3 & 5 & 2 & 2 & 4 \\ 9 & 10 & 9 & 6 & 10 \end{bmatrix} \end{array}$$

⇒ Step 1 :

Row operation : Find the minimum element in each row and subtract it with other element of the row.

$$\begin{array}{c} \\ A \\ B \\ C \\ D \\ E \end{array} \begin{array}{c} \begin{array}{ccccc} S1 & S2 & S3 & S4 & S5 \end{array} \\ \begin{bmatrix} 8 & 1 & 1 & 0 & 6 \\ 7 & 5 & 6 & 0 & 5 \\ 5 & 3 & 4 & 0 & 2 \\ 1 & 3 & 6 & 0 & 2 \\ 3 & 4 & 3 & 0 & 4 \end{bmatrix} \end{array}$$

Step 2 :

Column operation : find min in each column & subtract it with other element of the row.

$$\begin{bmatrix} 7 & 0 & 0 & 0 & 4 \\ 6 & 4 & 5 & 0 & 3 \\ 4 & 2 & 3 & 0 & 0 \\ 0 & 2 & 5 & 0 & 0 \\ 2 & 3 & 2 & 0 & 2 \end{bmatrix}$$

Step 3:
Draw minimum horizontal and vertical lines such that it should cover all the zero's.

$$\begin{bmatrix} 7 & 0 & 0 & 0 & 4 \\ 6 & 4 & 5 & 0 & 3 \\ 4 & 2 & 3 & 0 & 0 \\ 0 & 2 & 5 & 0 & 0 \\ 2 & 3 & 2 & 0 & 2 \end{bmatrix}$$

check if no of lines = cost matrix. If equal jump to step 5. If not equal jump to step 4

no of lines $\neq$ cost matrix

$4 \neq 5$

step 4: Consider unallocated elements and find the smallest cost. Substract remaining elements with the cost and add the cost to intersection points

$$\begin{bmatrix} 7 & 0 & 0 & 2 & 4 \\ 4 & 2 & 3 & 0 & 1 \\ 4 & 2 & 3 & 2 & 0 \\ 0 & 2 & 5 & 2 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

Repeat step 3

$\therefore$ no of lines = cost matrix

$5 = 5$

Now R1

step 5:

$$\begin{bmatrix} 7 & \boxed{0} & \cancel{0} & 2 & 4 \\ 4 & 2 & 3 & \boxed{0} & 1 \\ 4 & 2 & 3 & 2 & \boxed{0} \\ \boxed{0} & 2 & 5 & 2 & \cancel{0} \\ \cancel{0} & 1 & \boxed{0} & \cancel{0} & \cancel{0} \end{bmatrix}$$

Consider the row or column with 1 zero and strike out the other zero's of that the allocated row or column.

Job		M/c		
A	→	S2	=	3
B	→	S4	=	+ 2
C	→	S5	=	+ 4
D	→	S1	=	+ 3
E	→	S3	=	+ 9
				21

Maximization in assignment problem:

The objective is to maximize the profit to solve this we first convert the given profit matrix into the loss matrix by substracting all the elements from the highest element. For this converted loss matrix we apply the steps in hungarian method to get optimum assignment.

Q: A marketing manager has 5 salesman and there are 5 districts considering the capability of salesman and nature of districts. The estimates made by the marketing managers for the sales per month for each salesman in each district could be as follows find the assignment of salesman to the districts that will result in the maximum sales

$$\begin{bmatrix} 32 & 38 & 40 & 28 & 40 \\ 40 & 24 & 28 & 21 & 36 \\ 41 & 27 & 33 & 30 & 37 \\ 22 & 38 & 41 & 36 & 36 \\ 29 & 33 & 40 & 35 & 39. \end{bmatrix}$$

Step 1: Find the maximum element. Subtract all the elements of the matrix with the maximum.

max = 40

$$\begin{bmatrix} 9 & 3 & 1 & 13 & 1 \\ 1 & 17 & 13 & 20 & 5 \\ 0 & 14 & 8 & 11 & 4 \\ 19 & 3 & 0 & 5 & 5 \\ 12 & 8 & 1 & 6 & 2 \end{bmatrix}$$

Step 2: Row operation : Find the minimum element in each row and subtract it with other element of the row

min = 1

$$\begin{bmatrix} 8 & 2 & 0 & 12 & 0 \\ 0 & 16 & 12 & 19 & 4 \\ 0 & 14 & 8 & 11 & 4 \\ 19 & 3 & 0 & 5 & 5 \\ 11 & 7 & 0 & 5 & 1 \end{bmatrix}$$

Step 3: Column operation : Find the minimum element in each column and subtract it with other element of the column

$$\begin{bmatrix} 8 & 0 & 0 & 7 & 0 \\ 0 & 14 & 12 & 14 & 4 \\ 0 & 12 & 8 & 6 & 4 \\ 19 & 1 & 0 & 0 & 5 \\ 11 & 5 & 0 & 0 & 1 \end{bmatrix}$$

Step 3: Draw minimum horizontal and vertical lines such that it should cover all the zero's

$$\begin{bmatrix} 8 & 0 & 0 & 7 & 0 \\ 0 & 14 & 12 & 14 & 4 \\ 0 & 12 & 8 & 6 & 4 \\ 19 & 1 & 0 & 0 & 5 \\ 11 & 5 & 0 & 0 & 1 \end{bmatrix}$$

Check if no of lines = cost matrix. If equal jum to step 5. If not equal repeat step 4

Step 4: Consider un allocated elements and find the smallest cost. Subtract minimum element with the cost and add the cost to intersection point

$$\begin{bmatrix} 9 & 0 & 1 & 8 & 0 \\ 0 & 13 & 12 & 14 & 3 \\ 0 & 11 & 8 & 6 & 3 \\ 19 & 0 & 0 & 0 & 4 \\ 11 & 4 & 0 & 0 & 0 \end{bmatrix}$$

no of lines ≠ cost matrix

4 ≠ 5

Repeat step 4

$$\begin{bmatrix} 12 & 0 & 1 & 8 & 0 \\ 0 & 10 & 9 & 11 & 0 \\ 0 & 8 & 5 & 3 & 0 \\ 0 & 0 & 0 & 0 & 4 \\ 14 & 4 & 0 & 0 & 0 \end{bmatrix}$$

no of lines ≠ 0

no of lines = cost matrix

5 = 5

Step 5: Consider row or column with one zero and strike out the other zero's of the allocated row or column.

$$\begin{bmatrix} 12 & \boxed{0} & 1 & 8 & \cancel{0} \\ \boxed{0} & 10 & 9 & 11 & \cancel{0} \\ \cancel{0} & 8 & 5 & 3 & \boxed{0} \\ 3 & \cancel{0} & \boxed{0} & \cancel{0} & 4 \\ 14 & 4 & \cancel{0} & \boxed{0} & \cancel{0} \end{bmatrix}$$

$TC = 38 + 40 + 37 + 41 + 35 = \underline{\underline{191}}$

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Module - 5
Game Theory.

Game Theory:
The term game represents a competition between two or more parties. A situation is termed as game when it posses the following properties: 1) The no of competitors is finite.
2) There is a competition between the participants
3) The rules must known to all players.
4) The outcome of the game is affected by the choices made by all the players.

Strategy: The term strategy is defined as a complete set of plans of action. The players we consider during the play of the game i.e strategy of a player is the decision rule.
Strategy can be classified as 1) pure strategy 2) mixed strategy.

Pure strategy: A strategy is called pure if all the players know the rules

Mixed strategy: The strategy is mixed strategy if the probability of combination of available choices of strategy

Types of Games:

1) ~~2 person games~~
b) n person games

1) 2 person game & n person game:

In two person games the players may have many possible choices to them for each play of the game, but the number of players remain only two. Hence it is called two person game. In case of more than two persons, the game is generally called n person game.

2) Zero sum game:

Zero sum game is one in which the sum of the payments to all the competitors is zero, for every possible outcome of the game if sum of the points scored is equal to sum of the points lost.

3) Two person zero sum game:

The game with two players where the game of one player is equal to loss of other is known as two person zero sum game. It is also called as rectangular game.

Characteristics of 2 player zero sum game.

* Only 2 players participate in the game.
* Each player has a finite number of strategies to use.
* Total pay off to the two players at the

end of each play is zero.

pay-off matrix

$$
\begin{array}{c|c}
 & \text{player B} \\
 & \begin{array}{cccccc} 1 & 2 & 3 & 4 & \cdots & m \end{array} \\
\text{player A} \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ \vdots \\ n \end{array} & \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & \cdots & a_{1m} \\ a_{21} & a_{22} & a_{23} & a_{24} & \cdots & a_{2m} \\ \vdots & & & & & \vdots \\ \vdots & & & & & \vdots \\ \vdots & & & & & \vdots \\ a_{n1} & \cdots & & & & a_{nm} \end{bmatrix}
\end{array}
$$

Ex:

$$
\begin{array}{cc}
 & \text{player A} \\
 & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\
\text{player A} \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 4 & 5 & 6 \\ -7 & -8 & 9 \\ 1 & 2 & -3 \end{bmatrix}
\end{array}
$$

Maximin - Minmax principle:

Defination:
Maximin - Minmax:
This principle is used for the selection of optimal strategies by two players. Consider two players A & B. A is a player who wh wishes to maximize his game while player B wishes to minimize his loss. Since A player would try to maximize his minimum game we obtain for player A a value called maximin value and the corresponding strategy is called maximin strategy.
Since the player B wishes to minimize his

loss, the value is called minimax value which is the minimum of maximum loss. The corresponding strategy is called minmax strategy.

Note: When maximin value is equal to minmax value the corresponding strategy is called optimal strategy, And game and game have "saddle point". The value of the game is given by "saddle point"

Saddle point: A saddle point is a position in the pay off matrix where maximum of row minima considered with minimum of column maximum. The pay off at the saddle point is called the value of the game.

1) Solve the game who's pay off matrix is given below

Player B

$$\text{Player } A \quad \begin{array}{c|cccc|} & B_1 & B_2 & B_3 & B_4 \\ \hline A_1 & 1 & 3 & 1 & \\ A_2 & 0 & -4 & -3 & \\ A_3 & 1 & 5 & -1 & \\ A_4 & & & & \\ \hline \end{array}$$

Gain for player A is loss for player B

Step 1: Find out the row minimum & column maximum

$$\begin{array}{c|ccc|c} A_1 & \boxed{1} & 3 & \boxed{1} & 1 \\ A_2 & 0 & -4 & -3 & -4 \\ A_3 & 1 & 5 & -1 & -1 \\ \hline & 1 & 5 & 1 & \end{array}$$

Step 2: Find out min max

Min = {max} → minimum of maximum
= 1 among (1, 5, 1)

max = {min} → maximum of minimum
= 1 among (1, -4, -1)

∴ maxmin = minmax

1 = 1

The game has optimal strategy.
saddle point is 1. Strategy for A = A_1 & ~~A_3~~
Strategy for B = B_1 & B_3

2) Determine which of the following 2 person zero sum games are optimal strategies

a)
$$\begin{array}{c|cc} & B_1 & B_2 \\ \hline A_1 & -5 & 2 \\ A_2 & -7 & -4 \end{array}$$

b)
$$\begin{array}{c|cc} & B_1 & B_2 \\ \hline A_1 & 1 & 4 \\ A_2 & 4 & -3 \end{array}$$

a) Step 1: Find out row minima & column maxima.

	B_1	B_2	row min
A_1	$\boxed{-5}$	2	-5
A_2	-7	-4	-7
column max	-5	2	

Step 2: Find out min max

Min = {max} = -5

max = {min} = -5

Saddle point = -5

optimal strategies O.S = [A_1 & B_1]

b) Step 1: Find row min and column max

$$\begin{array}{c|cc|c} & B_1 & B_2 & \text{row min} \\ \hline A_1 & 1 & \boxed{1} & 1 \\ A_2 & 4 & -3 & -3 \\ \hline \text{col max} & 4 & 1 & \end{array}$$

Step 2: Min $\{max\}$ = 1

max $\{min\}$ = 1

saddle point = 1

optimal strategies A_1, ~~B_2~~ and B_2

3) Find saddle pt and value of the game

$$\begin{array}{c|ccc|} & B_1 & B_2 & B_3 \\ \hline A_1 & 15 & 2 & 3 \\ A_2 & 6 & 5 & 7 \\ A_3 & -7 & 4 & 0 \end{array}$$

Step 1: Find row min and column max

$$\begin{array}{c|ccc|c} & B_1 & B_2 & B_3 & \text{row min} \\ \hline A_1 & 15 & 2 & 3 & 2 \\ A_2 & 6 & \boxed{5} & 7 & 5 \\ A_3 & -7 & 4 & 0 & 0 \\ \hline \text{col max} & 15 & 5 & 7 & \end{array}$$

Step 2: Min $\{max\}$ = 5

max $\{min\}$ = 5

saddle point = 5

Optimal strategies for A : A_2

for B : B_2

4

$$\begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \end{array}\begin{array}{c} \begin{array}{cccc} B_1 & B_2 & B_3 & B_4 \end{array} \\ \left[\begin{array}{cccc} 1 & 2 & 1 & 20 \\ 5 & 5 & 4 & 6 \\ 4 & -2 & 0 & -5 \end{array}\right] \end{array}$$

Step 1:

$$\begin{array}{c|cccc|c} & B_1 & B_2 & B_3 & B_4 & \text{row min} \\ \hline A_1 & 1 & 2 & 1 & 20 & 1 \\ A_2 & 5 & 5 & \boxed{4} & 6 & 4 \\ A_3 & 4 & -2 & 0 & -5 & -5 \\ \hline \text{col max} & 5 & 5 & 4 & 20 & \end{array}$$

Step 2: min = {max} = 4
max = {min} = 4
saddle point = 4
Optimal strategies for player A : A_2
player B : B_3

5.

$$\begin{array}{c|cccc|c} & B_1 & B_2 & B_3 & B_4 & \text{row min} \\ \hline A_1 & 1 & 7 & 3 & 4 & 1 \\ A_2 & 5 & 6 & \boxed{4} & 5 & 4 \\ A_3 & 7 & 2 & 0 & 3 & 0 \\ \hline \text{col max} & 7 & 7 & 4 & 5 & \end{array}$$

Step 2: Min = {max} = 4
max = {min} = 4
saddle point = 4
for player A = A_2
for player B = B_3

Games without saddle points means mixed Strategies

2x2 games without saddle points:

	b_1	b_2
a_1	a	b
a_2	c	d

$$p_1 = \frac{d-c}{(a+d)-(b+c)} \qquad p_2 = 1 - p_1$$

$$q_1 = \frac{d-b}{(a+d)-(b+c)} \qquad q_2 = 1 - q_1$$

$$v = \frac{ad-bc}{(a+d)-(b+c)}$$

1)

$$\begin{array}{c} \\ A_1 \\ A_2 \end{array} \begin{array}{c} \begin{array}{cc} B_1 & B_2 \end{array} \\ \begin{bmatrix} 8 & -3 \\ -3 & 1 \end{bmatrix} \end{array}$$

Step 1: Check for saddle point

$$\begin{array}{c} \\ A_1 \\ A_2 \\ \text{col max} \end{array} \begin{array}{c} \begin{array}{cc} B_1 & B_2 \end{array} \\ \begin{bmatrix} 8 & -3 \\ -3 & 1 \end{bmatrix} \\ \begin{array}{cc} 8 & 1 \end{array} \end{array} \begin{array}{c} \text{row min} \\ -3 \\ -3 \\ \\ \end{array}$$

min {max} = 1

max {min} = -3

minmax ≠ maxmin No saddle point

Step 2: $p_1 = \frac{d-c}{(a+d)-(b+c)}$

$$= \frac{1-(-3)}{(8+1)-(-3-3)} = \frac{4}{9+6} = \frac{4}{15}$$

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$$p_2 = 1 - p_1$$
$$= 1 - \frac{4}{15}$$
$$= \frac{15-4}{15} = \frac{11}{15}$$

$$q_1 = \frac{d-b}{(a+d)-(b+c)} = \frac{4}{15}$$

$$q_2 = 1 - q_1 = \frac{11}{15}$$

$$A = \left(\frac{4}{15}, \frac{11}{15}\right) \quad B = \left(\frac{4}{15}, \frac{11}{15}\right)$$

$$V = \frac{ad-bc}{(a+d)-(b+c)} = \frac{(8\times1)-(-3\times-3)}{15}$$
$$= \frac{8-9}{15} = \frac{-1}{15}$$

Note: If the value is positive. It is advantage to player A.
If the value is negative It is advantage to player B.

2) Determine optimal strategies and value of the game

$$\begin{array}{cc} & B \\ A & \begin{bmatrix} 5 & 1 \\ 3 & 4 \end{bmatrix} \end{array}$$

⇒ Step 1: Check for saddle point

$$\begin{array}{ccc} & & \text{row min} \\ & \begin{bmatrix} 5 & 1 \\ 3 & 4 \end{bmatrix} & \begin{array}{c} 1 \\ 3 \end{array} \\ \text{col max} & \begin{array}{cc} 5 & 4 \end{array} & \end{array}$$

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min{max} = 4
max{min} = 3
minmax ≠ maxmin   No saddle point

Step 2: $p_1 = \frac{d-c}{(a+d)-(b+c)} = \frac{4-3}{(9)-(4)} = \frac{1}{5}$

$p_2 = 1 - p_1 = 1 - \frac{1}{5} = \frac{4}{5}$

$q_1 = \frac{d-b}{(a+d)-(b+c)} = \frac{3}{5}$

$q_2 = 1 - q_1 = 1 - \frac{3}{5} = \frac{2}{5}$

$A = \left(\frac{1}{5}, \frac{4}{5}\right)$ $B = \left(\frac{3}{5}, \frac{2}{5}\right)$

$v = \frac{ad-bc}{(a+d)-(b+c)} = \frac{20-3}{5} = \frac{17}{5}$ //

∴ Strategy advantage is for A.

3) Determine mixed strategies and value of the game

$$\begin{array}{cc} & B \\ A & \begin{bmatrix} 4 & -4 \\ -4 & 4 \end{bmatrix} \end{array}$$

⇒ Step 1: Check for saddle point

$$\begin{array}{cc} \begin{bmatrix} 4 & -4 \\ -4 & 4 \end{bmatrix} & \begin{array}{c} \text{row min} \\ -4 \\ -4 \end{array} \\ \text{col max} \quad 4 \quad\quad 4 & \end{array}$$

min{max} = 4
max{min} = -4
minmax ≠ maxmin   No saddle point.

Step 2:

$$p_1 = \frac{d-c}{(a+d)-(b+c)} = \frac{4-(-4)}{(8)-(-8)} = \frac{8}{16} = \frac{1}{2}$$

$$p_2 = 1 - p_1 = 1 - \frac{1}{2} = \frac{1}{2}$$

$$q_1 = \frac{d-b}{(a+d)-(b+c)} = \frac{8}{16} = \frac{1}{2}$$

$$q_2 = 1 - q_1 = 1 - \frac{1}{2} = \frac{1}{2}$$

$$A = \left(\frac{1}{2}, \frac{1}{2}\right) \qquad B = \left(\frac{1}{2}, \frac{1}{2}\right)$$

$$v = \frac{ad - bc}{(a+d)-(b+c)} = \frac{16-16}{16} = \frac{0}{16} = 0$$

34) In a game of matching coins with 2 players suppose player A wins 1 unit of value when there are two heads, win nothing when there are 2 tail tossing coin and lose of $\frac{1}{2}$ unit of value when there are 1 head and 1 tail. determine the pay off matrix, the best strategies for each player and value of the game.

$\Rightarrow$

$$\begin{array}{c|cc} & H & T \\ \hline H & 1 & -1/2 \\ T & -1/2 & 0 \end{array}$$

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Step 1: Check for saddle point

$$\begin{array}{c|c} \begin{bmatrix} 1 & -1/2 \\ -1/2 & 0 \end{bmatrix} & \begin{array}{c} \text{row min} \\ -1/2 \\ -1/2 \end{array} \end{array}$$

col max 1 0

min{max} = 0

max{min} = -1/2 No saddle point

Step 2: $p_1 = \frac{d-c}{(a+d)-(b+c)} = \frac{0-(-1/2)}{(1+0)-(-\frac{1}{2}-\frac{1}{2})} = \frac{1/2}{1+1}$

$= \frac{1}{2 \times 2} = \frac{1}{4}$

$p_2 = 1 - p_1 = 1 - \frac{1}{4} = \frac{3}{4}$

$q_1 = \frac{d-b}{(a+d)-(b+c)} = \frac{1/2}{2} = \frac{1}{4}$

$q_2 = 1 - q_1 = 1 - \frac{1}{4} = \frac{3}{4}$

$A = \left(\frac{1}{4}, \frac{3}{4}\right)$ $B = \left(\frac{1}{4}, \frac{3}{4}\right)$

$V = \frac{ad-bc}{(a+d)-(b+c)} = \frac{0-(-1/2 \times -1/2)}{2}$

$= \frac{-\frac{1}{4}}{2} = -\frac{1}{8}$

Strategy advantage for B.

5) Find value of the game

$$\begin{array}{cc} & B \\ A & \begin{bmatrix} 6 & -3 \\ -3 & 3 \end{bmatrix} \end{array}$$

Step 1: Check for saddle point

$$\begin{array}{cc} \begin{bmatrix} 6 & -3 \\ -3 & 3 \end{bmatrix} & \begin{matrix} \text{row min} \\ -3 \\ -3 \end{matrix} \end{array}$$

col max   6   3

min{max} = 3

max{min} = -3   minmax ≠ maxmin

∴ No saddle point

Step 2:

$$p_1 = \frac{d-c}{(a+d)-(b+c)} = \frac{3-(-3)}{(9)-(-6)} = \frac{6}{15}$$

$$p_2 = 1 - p_1 = 1 - \frac{6}{15} = \frac{9}{15}$$

$$q_1 = \frac{d-b}{(a+d)-(b+c)} = \frac{3-(-3)}{15} = \frac{6}{15}$$

$$q_2 = 1 - q_1 = \frac{9}{15}$$

$$A = \left(\frac{\cancel{6}^{2}}{\cancel{15}_{5}}, \frac{\cancel{9}^{3}}{\cancel{15}_{5}}\right) \qquad B = \left(\frac{\cancel{6}^{2}}{\cancel{15}_{5}}, \frac{\cancel{9}^{3}}{\cancel{15}_{5}}\right)$$

$$v = \frac{ad-bc}{(a+d)-(b+c)} = \frac{18-9}{15} = \frac{\cancel{9}^{3}}{\cancel{15}_{5}} = \frac{3}{5}$$

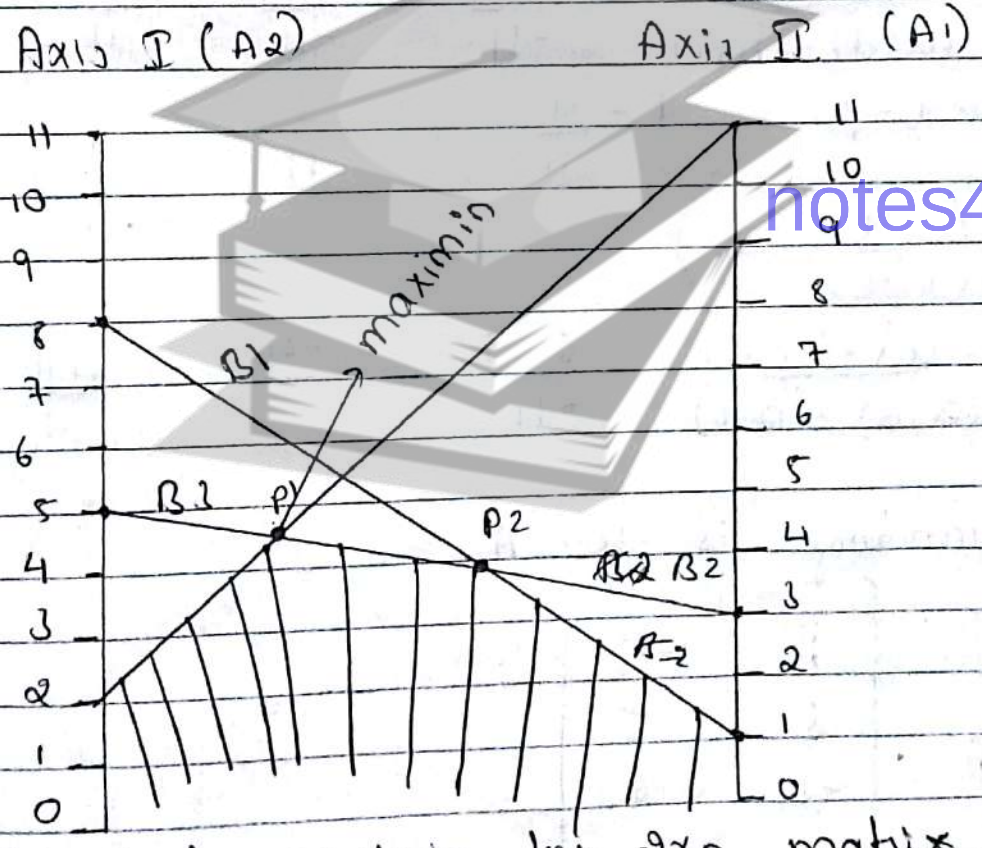
Strategy advantage is for A

Graphical method for 2×n & m×2 matrix

1)
$$A\;\overset{B}{\begin{bmatrix} 1 & 3 & 11 \\ 8 & 5 & 2 \end{bmatrix}}$$

$$\begin{array}{c|ccc|c} & B_1 & B_2 & B_3 & \text{row min} \\ \hline A_1 & 1 & 3 & 11 & 1 \\ A_2 & 8 & 5 & 2 & 2 \\ \hline \text{col max} & 8 & 5 & 11 & \end{array}$$

min{max} = 5
max{min} = 2
minmax ≠ maxmin. No saddle point.



To find maximin for 2×n matrix. Mark the region below the intersection points and find the maximum point.
The 2 intersection points are P1 and P2 and P1 is the maximum, which corresponds to the column B2 and B3

Consider B3 and B2.

$$\begin{bmatrix} 3 & 11 \\ 5 & 2 \end{bmatrix}$$

$p_1 = \frac{d-c}{(a+d)-(b+c)} \quad \frac{2-5}{5-16} = \frac{-3}{-11} = \frac{3}{11}$

$p_2 = 1 - p_1 = 1 - \frac{3}{11} \quad \frac{8}{11}$

$A = \left(\frac{3}{11}, \frac{8}{11}\right)$

$q_1 = \frac{d-b}{(a+d)-(b+c)} = \frac{2-11}{-11} = \frac{-9}{-11} = \frac{9}{11}$

$q_2 = 1 - p_1 = 1 - \frac{9}{11} = \frac{2}{11}$

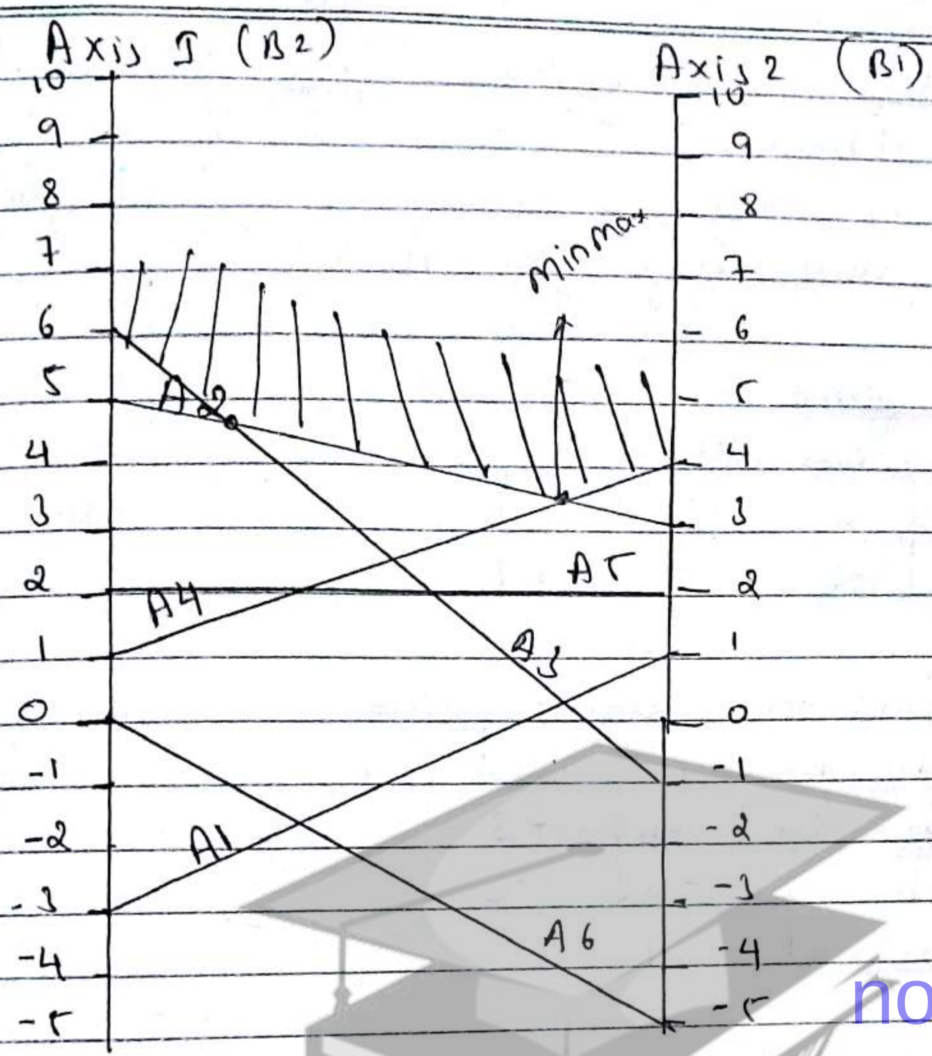
$B = \left(\frac{9}{11}, \frac{2}{11}\right)$

$v = \frac{ad-bc}{(a+d)-(b+c)} = \frac{6-55}{-11} = \frac{-49}{-11} = \frac{49}{11}$

Advantage is for A.

2)

$$\begin{array}{cc} & B \\ A & \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 6 \\ 4 & 1 \\ 2 & 2 \\ -5 & 0 \end{bmatrix} \end{array}$$



The minmax point is A 4 and A 2

$$\begin{array}{cc} & B \\ A & \begin{bmatrix} 3 & 5 \\ 4 & 1 \end{bmatrix} \end{array}$$

$$p_1 = \frac{d-c}{(a+d)-(b+c)} = \frac{1-4}{4-9} = \frac{-3}{-5} = \frac{3}{5}$$

$$p_2 = 1 - p_1 = 1 - \frac{3}{5} = \frac{5-3}{5} = \frac{2}{5}$$

$$q_1 = \frac{d-b}{(a+d)-(b+c)} = \frac{-4}{-5} = \frac{4}{5}$$

$$q_2 = 1 - q_1 = 1 - \frac{4}{5} = \frac{5-4}{5} = \frac{1}{5}$$

$$A = \left(\frac{3}{5}, \frac{2}{5}\right) \qquad B = \left(\frac{4}{5}, \frac{1}{5}\right)$$

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$$v = \frac{ad - bc}{(a+d) - (b+c)} = \frac{3 - 20}{-5} = \frac{17}{5}$$ //

Strategy advantage for A

3) For the game

$$\begin{array}{cc} & B \\ A & \begin{bmatrix} 3 & -3 & 4 \\ -1 & 1 & -3 \end{bmatrix} \end{array}$$

Step 1 : Find the saddle point.

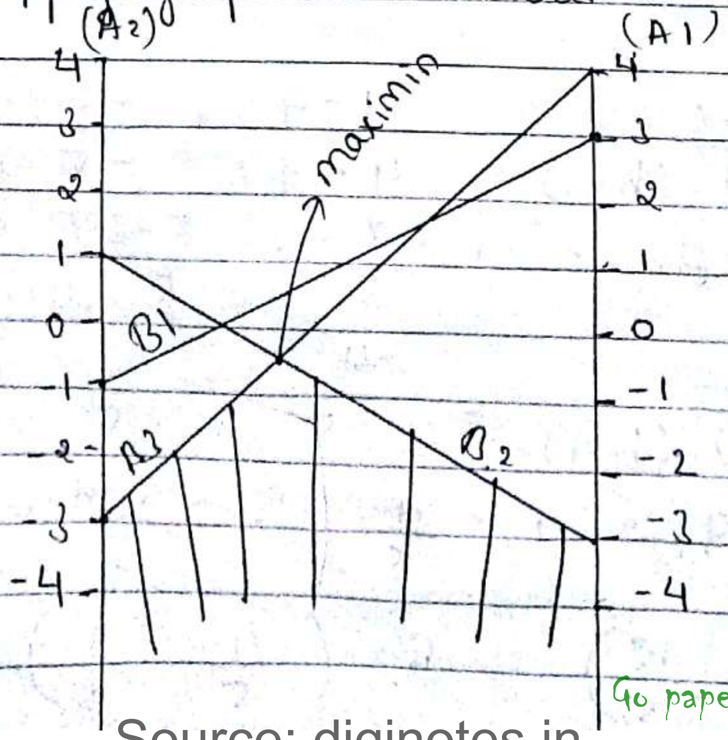
$$\begin{array}{ccccc} & B1 & B2 & B3 & \text{row min} \\ \begin{array}{c} A1 \\ A2 \end{array} & \multicolumn{3}{c}{\begin{bmatrix} 3 & -3 & 4 \\ -1 & 1 & -3 \end{bmatrix}} & \begin{array}{c} -3 \\ -3 \end{array} \\ \text{colmax} & 3 & 1 & 4 & \end{array}$$

min {max} = -3

max {min} = 1

minmax ≠ maxmin No saddle point

Step 2 : Apply graphical method.



The intersecting lines are B_2 and B_3

$$\begin{array}{cc} a & b \\ \end{array}$$
$$\begin{bmatrix} -3 & 4 \\ 1 & -3 \end{bmatrix}$$

$$p_1 = \frac{d-c}{(a+d)-(b+c)} = \frac{-3-1}{(-6)-(5)} = \frac{-4}{-11} = \frac{4}{11}$$

$$p_2 = 1 - p_1 = 1 - \frac{4}{11} = \frac{7}{11}$$

$$q_1 = \frac{d-b}{(a+d)-(b+c)} = \frac{-3-4}{-11} = \frac{-7}{-11} = \frac{7}{11}$$

$$q_2 = 1 - \frac{7}{11} = \frac{4}{11}$$

$$A\left(\frac{4}{11}, \frac{7}{11}\right) \quad B\left(\frac{7}{11}, \frac{4}{11}\right)$$

$$v = \frac{ad-bc}{(a+d)-(b+c)} = \frac{9-4}{-11} = -\frac{5}{11} \;//$$

4)

$$\begin{array}{cc} & B \\ A & \begin{bmatrix} -6 & 7 \\ 4 & -5 \\ -1 & -2 \\ -2 & 5 \\ 7 & 6 \end{bmatrix} \end{array}$$

Step 1: Find out saddle point

$$\begin{array}{lc} & \text{row min} \\ \begin{bmatrix} -6 & 7 \\ 4 & -5 \\ -1 & -2 \\ -2 & 5 \\ 7 & 6 \end{bmatrix} & \begin{array}{c} -6 \\ -5 \\ -2 \\ -2 \\ 6 \end{array} \\ \text{col max} \quad 7 \quad\quad 7 & \end{array}$$

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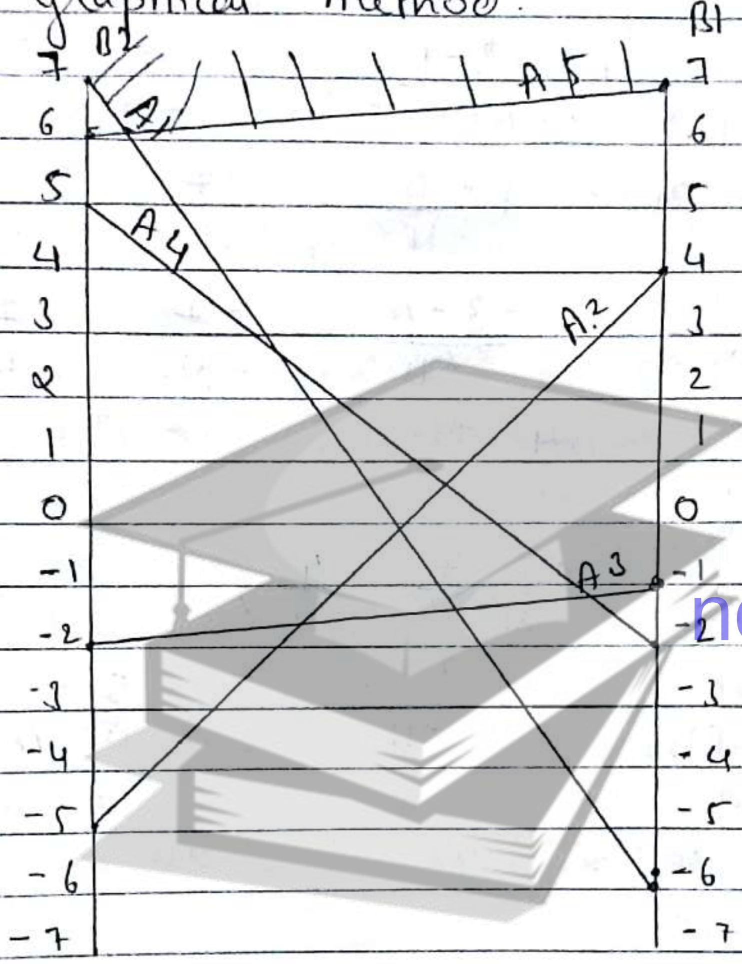
max{min} = 7

min{max} = 6

max{min} ≠ ~~maxmin~~ minmax No saddle point.

Step 2: Graphical method.



Insertion lines are A1 and A5

$$\begin{bmatrix} -6 & 7 \\ 7 & 6 \end{bmatrix}$$

$$p_1 = \frac{d-c}{(a+d)-(b+c)} = \frac{6-7}{(0)-(14)} = \frac{-1}{-14} = \frac{1}{14}$$

$$p_2 = 1 - p_1 = 1 - \frac{1}{14} = \frac{13}{14}$$

$$q_1 = \frac{d-b}{(a+d)-(b+c)} = \frac{6-7}{-14} = \frac{1}{14}$$

$$q_2 = \frac{13}{14}$$

$$A\left(\frac{1}{14}, \frac{13}{14}\right) \qquad B\left(\frac{1}{14}, \frac{13}{14}\right)$$

$$V = \frac{ad-bc}{(a+d)-(b+c)} = \frac{(-36)-49}{-14} = \frac{85}{14}$$

5. $\begin{bmatrix} 1 & 3 & -3 & 7 \\ 2 & 5 & 4 & -6 \end{bmatrix}$

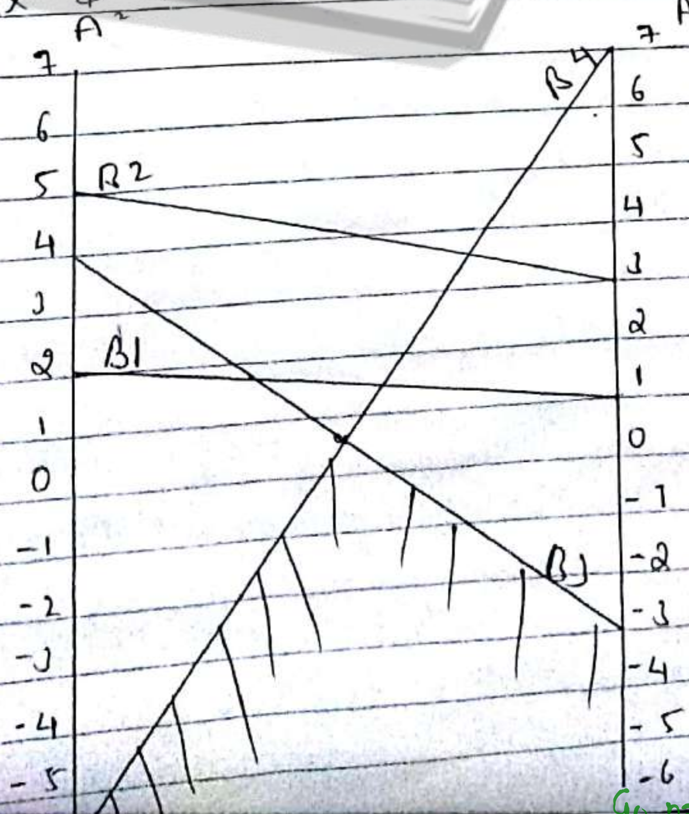
Step 1: Find saddle point

$$\begin{array}{cc} & \text{row min} \\ \begin{bmatrix} 1 & 3 & -3 & 7 \\ 2 & 5 & 4 & -6 \end{bmatrix} & \begin{array}{c} -3 \\ -6 \end{array} \\ \text{col max } \; 2 \quad 5 \quad 4 \quad 7 & \end{array}$$

min{max} = -3

max{min} = 2

minmax ≠ maxmin No saddle point



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Intersecting lines are R1 and R4

$$\begin{bmatrix} -3 & 7 \\ 4 & -6 \end{bmatrix}$$

$p_1 = \frac{d-c}{(a+d)-(b+c)} = \frac{-6-4}{(-3-6)-(\cancel{4}7+4)} = \frac{-10}{-9-11} = \frac{10}{20}$

$= \frac{1}{2}$

$p_2 = 1 - p_1 = 1 - \frac{1}{2} = \frac{1}{2}$

$q_1 = \frac{d-b}{(a+d)-(b+c)} = \frac{-13}{-20} = \frac{13}{20}$

$q_2 = 1 - q_1 = 1 - \frac{13}{20} = \frac{7}{20}$

$v = \frac{ad-bc}{(a+d)-(b+c)} = \frac{+18-28}{-20} = \frac{-\cancel{46}}{-20} = \frac{10}{20} = \frac{1}{2}$

Dominance Property:

We use the following rules to reduce a given matrix to a 2x2 matrix or 1x1 matrix

Rule 1: If all the elements in ith row are less than or equal to the corresponding elements ~~are~~ of the jth row, we say that jth strategy dominates ith strategy and hence we delete ith row

$R_i \leq R_j$ delete R_i

Rule 2: If all the elements of the nth column are greater than or equals corresponding elements of the mth column than we say that mth strategy dominates nth strategy. Hence we ~~go~~ delete nth strategy.

Rule 3: If row dominance and column dominance cannot reduce a matrix then we take average

1. If all the elements of the ith row less than or equals the average of two or more rows than we say that the group of rows dominates ith row. Hence we delete ith row.

2. If all the elements of the nth column are greater than or equals the average of two or more columns than we say that group of columns dominant nth column. Hence we delete nth column.

1) Solve the game by applying dominance

	b_1	b_2	b_3
a_1	5	20	-10
a_2	10	6	2
a_3	20	15	18

	b_1	b_2	b_3
a_1	5	20	-10
~~a_2~~	~~10~~	~~6~~	~~2~~
a_3	20	15	18

Compare all possible combination of rows.

Step 1: $a_2 \leq a_3$ delete a_2

Step 2: Compare all possible combinations of column

$b_3 \leq b_1$ b_3 dominates b_1

delete b_1

$$\begin{array}{c} \\ \\ \text{col max} \end{array}\begin{bmatrix} 20 & -10 \\ 15 & 18 \end{bmatrix}\begin{array}{c} \text{row min} \\ -10 \\ 15 \end{array}$$

col max 20 18

min{max} = 15

max{min} = 18

minmax ≠ maxmin No saddle point

$$\begin{bmatrix} 20 & -10 \\ 15 & 18 \end{bmatrix}$$

$$p_1 = \frac{d-c}{(a+d)-(b+c)} = \frac{18-15}{(38)-(5)} = \frac{3}{33} = \frac{1}{11}$$

$$p_2 = 1 - p_1 = 1 - \frac{1}{11} = \frac{10}{11}$$

$$q_1 = \frac{d-b}{(a+d)-(b+c)} = \frac{18+10}{33} = \frac{28}{33}$$

$$q_2 = 1 - q_1 = 1 - \frac{28}{33} = \frac{\cancel{33}}{\cancel{33}} \quad \frac{5}{33}$$

$$v = \frac{ad-bc}{(a+d)-(b+c)} = \frac{360+150}{33} = \frac{510}{33}$$

Stratgy advantage for game A

2)

	b1	b2	b3	b4	b5
a1	2	4	3	8	4
a2	5	6	3	7	8
a3	6	7	9	8	7
a4	4	2	8	4	3

$\Rightarrow$

	b_1	b_2	b_3	b_4	b_5
a_1	2	4	3	8	4
a_2	5	6	3	7	8
a_3	6	7	9	8	7
a_4	4	2	8	4	8

$a_1 \le a_3$ a_1 is deleted. , a_1 is dominating

$a_4 \le a_3$ a_4 is deleted. , a_4 is dominating

$b_1 \le b_2$ b_2 is deleted b_2 is dominating

$b_1 \le b_5$ b_5 is deleted b_5 is dominating

~~b_4~~ $b_1 \le b_4$ b_4 is deleted b_4 is dominating

$a_2 \le a_3$, a_2 is deleted a_3 is dominating
$b_1 \le b_3$ b_3 is deleted b_1 is dominating.

$\boxed{v = 6}$

3)

	b_1	b_2	b_3
a_1	1	2	0
a_2	2	-3	-2
a_3	0	3	-1

⇒

	b_1	b_2	b_3
a_1	1	2	0
a_2	~~2~~	~~-3~~	~~-2~~
a_3	~~0~~	~~3~~	~~-1~~

$b_1 > b_3$ b_1 is deleted
$a_2 < a_1$ a_2 is deleted
$b_2 > b_3$ b_2 is deleted
$a_3 \le a_1$ a_3 is deleted.

Value of the game is 0.

4) Solve the game using dominance property

$$\begin{array}{c|cccc} & b_1 & b_2 & b_3 & b_4 \\ \hline a_1 & 2 & -2 & 4 & 1 \\ a_2 & 6 & 1 & 12 & 3 \\ a_3 & -3 & 2 & 0 & 6 \\ a_4 & 2 & -3 & 7 & 7 \end{array}$$

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$$\begin{array}{c|cccc} & b_1 & b_2 & b_3 & b_4 \\ \hline a_1 & 2 & -2 & 1 & \\ a_2 & 6 & 1 & 12 & 3 \\ a_3 & -3 & 2 & 0 & 6 \\ a_4 & 2 & -3 & 7 & 7 \end{array}$$

$a_1 < a_2$ delete a_1

$b_4 > b_2$ delete b_4

$b_3 > b_1$ delete b_3

$a_2 \geq a_4$ delete a_4

$$\begin{bmatrix} 6 & 1 \\ -3 & 2 \end{bmatrix} \quad \begin{array}{c} \text{row min} \\ 1 \\ -3 \end{array}$$

col max $\quad 6 \quad 2$

$\min\{\max\} = 2$

$\max\{\min\} = 1$

maxmin $\neq$ min max No saddle point.

$$p_1 = \frac{d-c}{(a+d)-(b+c)} = \frac{2-(-3)}{(6+2)-(1-3)} = \frac{5}{8+2} = \frac{5}{10} = \frac{1}{2}$$

$$p_2 = 1 - p_1 = 1 - \frac{1}{2} = \frac{1}{2} \qquad A\left(\frac{1}{2}, \frac{1}{2}\right)$$

$$q_1 = \frac{d-b}{(a+d)-(b+c)} = \frac{(2-1)}{10} = \frac{1}{10} \qquad B\left(\frac{1}{10}, \frac{9}{10}\right)$$

$$q_2 = 1 - q_1 = 1 - \frac{1}{10} = \frac{9}{10}$$

$$v = \frac{ad-bc}{(a+d)-(b+c)} = \frac{12+3}{10} = \frac{15}{10} = \frac{3}{2} //$$

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5) The following matrix represents the pay off to P_1 in a rectangular game between two persons P1 and P2 by using dominance property reduce the game to 2x4 and solve it graphically.

$$\begin{array}{cc} & P_2 \\ P_1 & \begin{bmatrix} 8 & 15 & -4 & -2 \\ 19 & 15 & 17 & 16 \\ 0 & 20 & 15 & 5 \end{bmatrix} \end{array}$$

R_1 $a_1 < a_2$ delete a_1

$$\begin{bmatrix} \cancel{8} & \cancel{15} & \cancel{-4} & \cancel{-2} \\ 19 & 15 & 17 & 16 \\ 0 & 20 & 5 & 5 \end{bmatrix}$$

$$\begin{array}{cccccc} & b_1 & b_2 & b_3 & b_4 & \text{row min} \\ a_1 & 19 & 15 & 17 & 16 & 15 \\ a_2 & 0 & 20 & 5 & 5 & 0 \\ \text{col max} & 0 & 15 & 5 & 5 & \end{array}$$

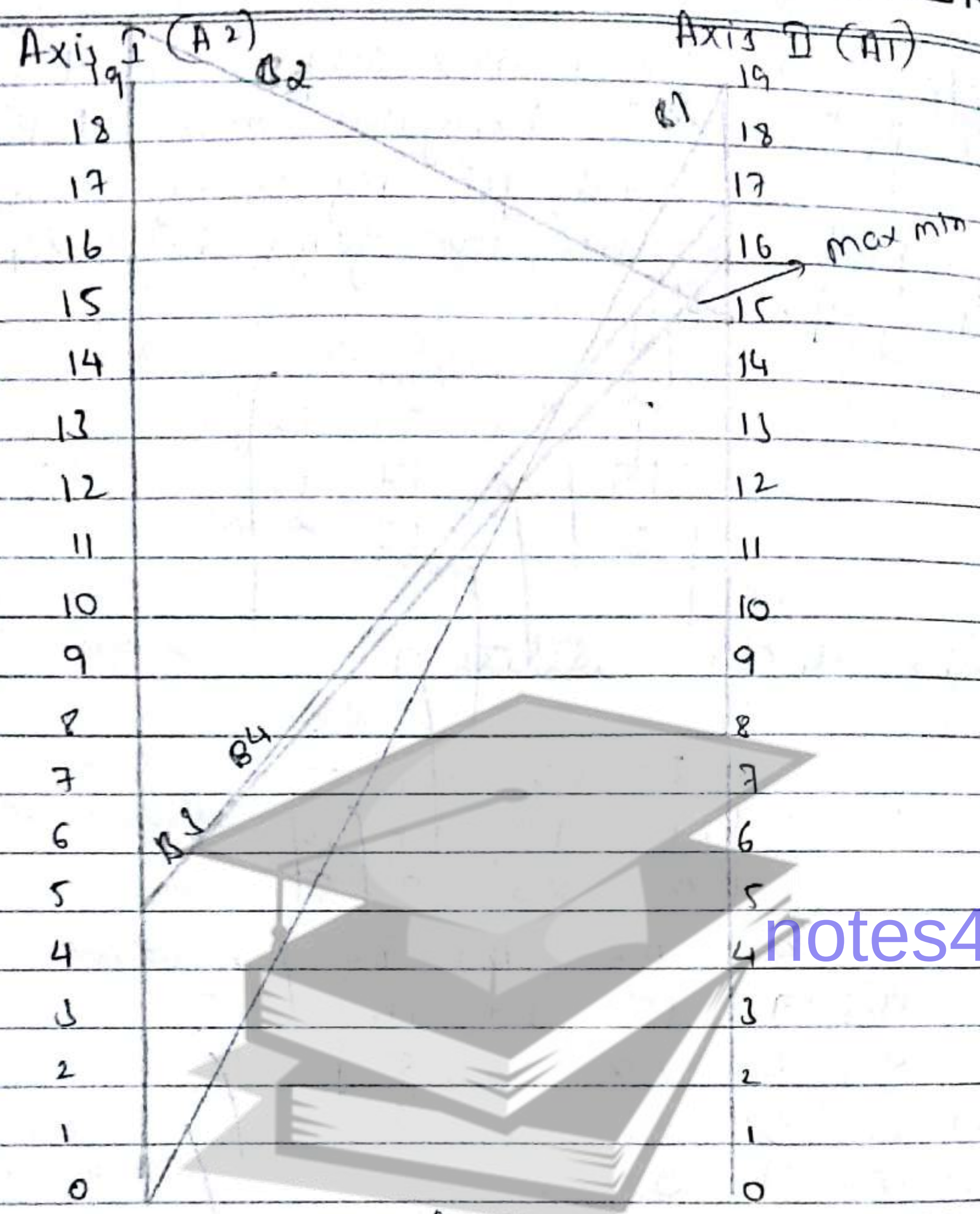
min{max} = 0

max{min} = 15

minmax $\neq$ maxmin No saddle point

Apply graphical method.

6)



The lines are B2, B4

$$\begin{bmatrix} 15 & 16 \\ 20 & 5 \end{bmatrix}$$

$$p_1 = \frac{d-c}{(a+d)-(b+c)} = \frac{5-20}{(15+5)-(16+20)} = \frac{-15}{20-36} = \frac{15}{16}$$

$$p_2 = 1 - p_1 = 1 - \frac{15}{16} = \frac{1}{16}$$

$$q_1 = \frac{d-b}{(a+d)-(b+c)} = \frac{5-16}{-16} = \frac{-11}{-16} = \frac{11}{16}$$

$$q_2 = 1 - q_1 = 1 - \frac{11}{-16} \quad \frac{5}{16}$$

$$V = \frac{ad-bc}{(a+d)-(b+c)} = (15\times5)-(16\times20) \,\Big|\, -16 = \frac{245}{16}$$