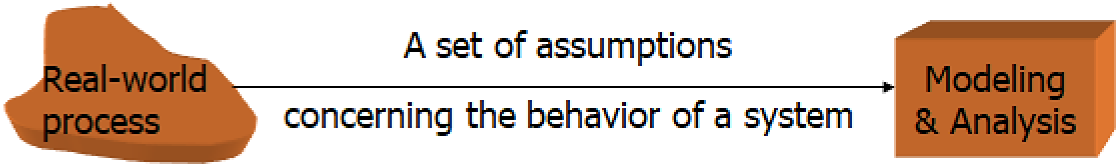
SYSTEM MODELLING AND SIMULATION

Module 1

Introduction: When simulation is the appropriate tool and when it is not appropriate, Advantages and disadvantages of Simulation, Areas of application, Systems and system environment; Components of a system, Discrete and continuous systems, Model of a system, Types of Models, Discrete-Event System Simulation, Simulation examples, Simulation of queuing systems. General Principles, Simulation Software, Concepts in Discrete-Event Simulation, The Event-Scheduling / Time-Advance Algorithm, Manual simulation Using Event Scheduling

Introduction to simulation

- A simulation: imitation of the operation of a real-world process or system over time:
  - Involves generation of an artificial history of a system.
  - Observes that history and draws inferences about system characteristics.
- Can be used as:
  - Analysis tool for predicting the effect of changes to existing systems.
  - Design tool to predict performance of new systems.
- Many real-world systems are very complex that cannot be solved mathematically.
  - Hence, numerical, computer-based simulation can be used to imitate the system behavior.



When to use Simulation?

- Simulation can be used for the purposes of:
  - Study and experiment with internal interactions of a complex system.
  - Observe the effect of system alterations on model behavior.
  - Gain knowledge about the system through design of simulation model.
  - Use as a pedagogical device to reinforce analytic solution methodologies, also to verify analytic solutions.
  - Experiment with new designs or policies before implementation.
  - Determine machine requirements through simulating different capabilities.
  - For training and learning.
  - Show animation.
  - Model complex system.

When Not to Use Simulation?

- Simulation should not be used when:
    - Problem can be solved by common sense.
    - Problem can be solved analytically.
    - If it is easier to perform direct experiments.
    - If the costs exceed the savings.
    - If the resources or time to perform simulation studies are not available.
    - If no data, not even estimates, is available.
    - If there is not enough time or personnel to verify/validate the model.
    - If managers have unreasonable expectations: overestimate the power of simulation.
    - If system behavior is too complex or cannot be defined.

Advantages and Disadvantages of Simulation

- **Advantages**
    - New polices, operating procedures, decision rules, information flows, organizational procedures, and so on can be explored without disrupting ongoing operations of the real system.
    - New hardware designs, physical layouts, transportation systems, and so on, can be tested without committing resources for their acquisition.
    - Hypotheses about how or why certain phenomena occur can be tested for feasibility.
    - Insight can be obtained about the interaction of variables.
    - Insight can be obtained about the importance of variables to the performance of the system.
    - Bottleneck analysis can be performed indicating where work-in-process, information, materials, and so on are being excessively delayed.
    - A simulation study can help in understanding how the system operates rather than how individuals think the system operates.
    - "What-if" questions can be answered. This is particularly useful in the design of new system
- **Disadvantages**
    - Model building requires special training. It is an art that is learned over time and through experience. Furthermore, if two models are constructed by two competent individuals, they may have similarities, but it is highly unlikely that they will be the same.
    - Simulation results may be difficult to interpret. Since most simulation outputs are essentially random variables (they are usually based on random inputs), it may be

hard to determine whether an observation is a result of system interrelationships or randomness.

- Simulation modeling and analysis can be time consuming and expensive. Skimping on resources for modeling and analysis may result in a simulation model or analysis that is not sufficient for the task.
- Simulation is used in some cases when an analytical solution is possible, or even preferable, as discussed in Section 1.2. This might be particularly true in the simulation of some waiting lines where closed-form queuing models are available.

Areas of Application

- The applications of simulation are vast.
- The Winter Simulation Conference: an excellent way to learn more about the latest in simulation applications and theory.
- Some areas of applications:
- Manufacturing
  - Construction engineering and project management.
  - Military.
  - Logistics, supply chain, and distribution.
  - Transportation modes and traffic.
  - Business process simulation.
  - Healthcare.
  - Computer and communication systems.
- WSC(Winter Simulation Conference) : http://www.wintersim.org
  - Manufacturing Applications
    - Analysis of electronics assembly operations
    - Design and evaluation of a selective assembly station for high-precision scroll compressor shells
    - Comparison of dispatching rules for semiconductor manufacturing using large-facility models
    - Evaluation of cluster tool throughput for thin-film head production
    - Determining optimal lot size for a semiconductor back-end factory
    - Optimization of cycle time and utilization in semiconductor test manufacturing
    - Analysis of storage and retrieval strategies in a warehouse
    - Investigation of dynamics in a service-oriented supply chain
    - Model for an Army chemical munitions disposal facility
  - Semiconductor Manufacturing
    - Comparison of dispatching rules using large-facility models
    - The corrupting influence of variability

- A new lot-release rule for wafer fabs
- Assessment of potential gains in productivity due to proactive reticle management
- Comparison of a 200-mm and 300-mm X-ray lithography cell
- Capacity planning with time constraints between operations
- 300-mm logistic system risk reduction

- Construction Engineering
  - Construction of a dam embankment
  - Trenchless renewal of underground urban infrastructures
  - Activity scheduling in a dynamic, multiproject setting
  - Investigation of the structural steel erection process
  - Special-purpose template for utility tunnel construction
- Military Application
  - Modeling leadership effects and recruit type in an Army recruiting station
  - Design and test of an intelligent controller for autonomous underwater vehicles
  - Modeling military requirements for nonwarfighting operations
  - Multitrajectory performance for varying scenario sizes
  - Using adaptive agent in U.S Air Force pilot retention
- Logistics, Transportation, and Distribution Applications
  - Evaluating the potential benefits of a rail-traffic planning algorithm
  - Evaluating strategies to improve railroad performance
  - Parametric modeling in rail-capacity planning
  - Analysis of passenger flows in an airport terminal
  - Proactive flight-schedule evaluation
  - Logistics issues in autonomous food production systems for extended-duration space exploration
  - Sizing industrial rail-car fleets
  - Product distribution in the newspaper industry
  - Design of a toll plaza
  - Choosing between rental-car locations
  - Quick-response replenishment
- Business Process Simulation
  - Impact of connection bank redesign on airport gate assignment
  - Product development program planning
  - Reconciliation of business and systems modeling
  - Personnel forecasting and strategic workforce planning
- Human Systems
  - Modeling human performance in complex systems
  - Studying the human element in air traffic control

Systems and System Environment

- A system is a group of objects joined together in some regular interaction or interdependence to accomplish some purpose.
  - e.g., a production system: machines, component parts & workers operate jointly along an assembly line to produce vehicle.
  - Affected by changes occurring outside the system.
- System environment: “outside the system”, defining the boundary between system and it environment is important.

Components of a System

- Entity: an object of interest in the system.
- Attribute: a property of an entity.
- Activity: a time period of specified length.
- State: the collection of variables necessary to describe the system at any time, relative to the objectives of the study.
- Event: an instantaneous occurrence that may change the state of the system.
- Endogenous: to describe activities and events occurring within a system.
- Exogenous: to describe activities and events in an environment that affects the system.

Table 1.1. Examples of Systems and Components

System	*Entities*	*Attributes*	*Activities*	*Events*	*State Variables*
Banking	Customers	Checking account balance	Making deposits	Arrival; departure	Number of busy tellers; number of customers waiting
Rapid rail	Riders	Origination; destination	Traveling	Arrival at station; arrival at destination	Number of riders waiting at each station; number of riders in transit
Production	Machines	Speed; capacity; breakdown rate	Welding; stamping	Breakdown	Status of machines (busy, idle, or down)
Communications	Messages	Length; destination	Transmitting	Arrival at destination	Number waiting to be transmitted
Inventory	Warehouse	Capacity	Withdrawing	Demand	Levels of inventory; backlogged demands

Discrete and Continuous Systems

- Systems can be categorized as discrete or continuous.
  - Bank : a discrete system
  - The head of water behind a dam : a continuous system

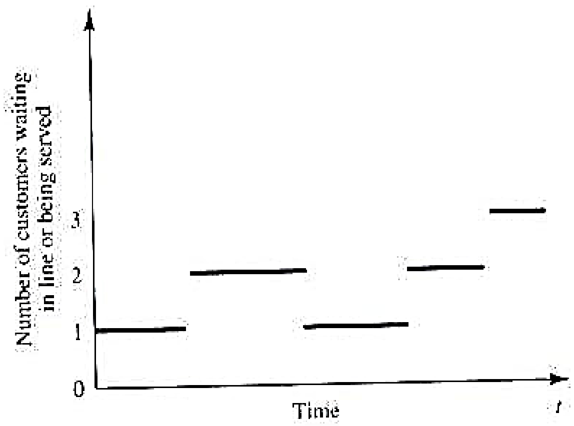


Figure 1.1. Discrete-system state variable.

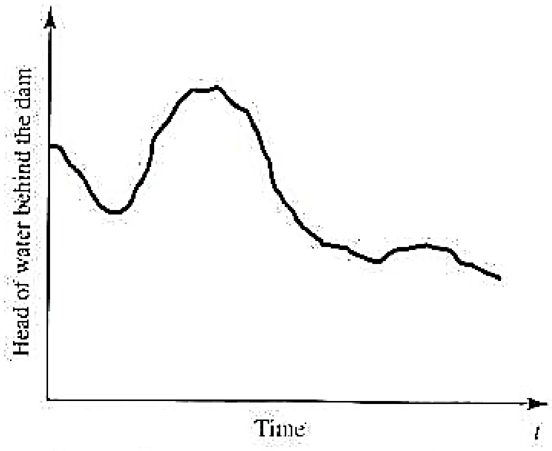


Figure 1.2. Continuous-system state variable.

Discrete System:

- Is one in which the state variable change only at a discrete set of points in time.
- The bank is an example, since the state variable the number of customer in the bank changes only when a customer arrives or when the service provided a customer is completed.

Continuous system:

- Is one in which the state variable change continuous over time.
- Head of water behind a dam, during and for some time after a rain storm water flow into the lake behind the dam.

Model of a System

- Studies of systems are often accomplished with a model of a system.
- A model: a representation of a system for the purpose of studying the system.
  - A simplification of the system.
  - Should be sufficiently detailed to permit valid conclusions to be drawn about the real system.
  - Should contain only the components that are relevant to the study.

Types of Models

- Two types of models: mathematical or physical.
- Mathematical model: uses symbolic notation and mathematical equations to represent a system.
  - Simulation is a type of mathematical model.
- Simulation models:
  - Static or dynamic.
  - Deterministic or stochastic.
  - Discrete or continuous.
- Our focus: discrete, dynamic, and stochastic models.

- Static or Dynamic Simulation Models
  - Static simulation model (called Monte Carlo simulation) represents a system at a particular point in time.
  - Dynamic simulation model represents systems as they change over time
- Deterministic or Stochastic Simulation Models
  - Deterministic simulation models contain no random variables and have a known set of inputs which will result in a unique set of outputs
  - Stochastic simulation model has one or more random variables as inputs. Random inputs lead to random outputs.
- The model of interest in this class is discrete, dynamic, and stochastic.

Discrete-Event System Simulation

- The simulation models are analyzed by numerical rather than by analytical methods
  - Analytical methods employ the deductive reasoning of mathematics to solve the model.
  - Numerical methods employ computational procedures to solve mathematical models.

Steps in a Simulation Study

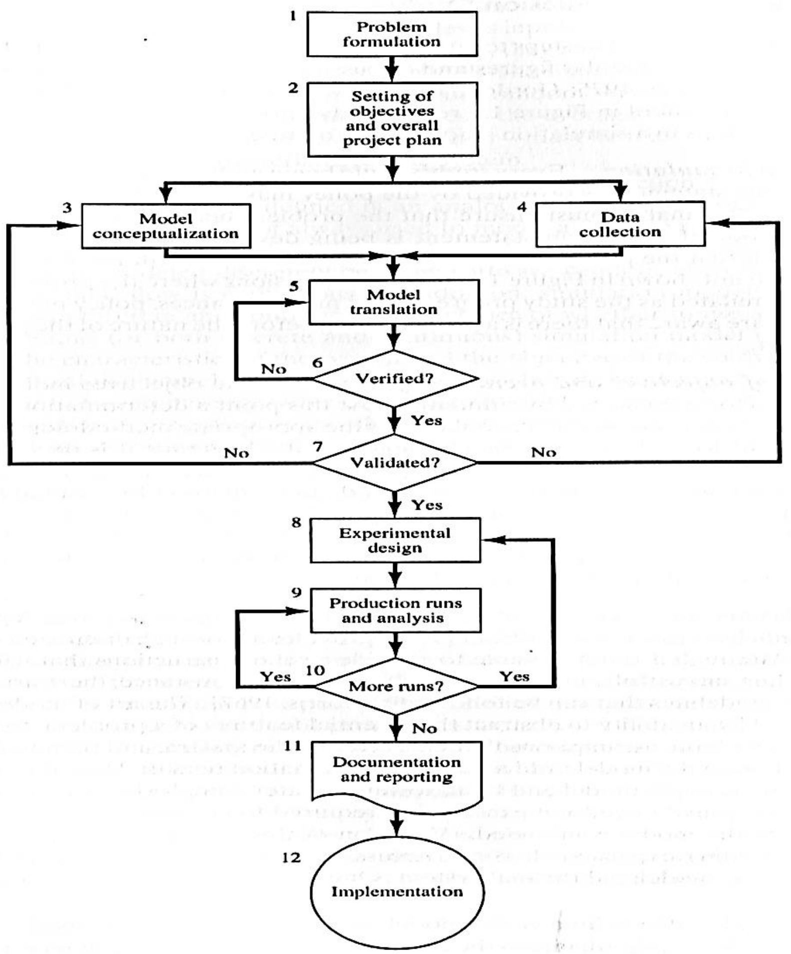


Figure 1.3. Steps in a simulation study.

- Problem formulation
    - Policy maker/Analyst understand and agree with the formulation.
- Setting of objectives and overall project plan
- Model conceptualization
    - The art of modeling is enhanced by an ability to abstract the essential features of a problem, to select and modify basic assumptions that characterize the system, and then to enrich and elaborate the model until a useful approximation results.
- Data collection
    - As the complexity of the model changes, the required data elements may also change.
- Model translation
    - GPSS/H™ or special-purpose simulation software
- Verified?
    - Is the computer program performing properly?
    - Debugging for correct input parameters and logical structure
- Validated?
    - The determination that a model is an accurate representation of the real system.
    - Validation is achieved through the calibration of the model
- Experimental design
    - The decision on the length of the initialization period, the length of simulation runs, and the number of replications to be made of each run.
- Production runs and analysis
    - To estimate measures of performances
- More runs?
- Documentation and reporting
    - Program documentation : for the relationships between input parameters and output measures of performance, and for a modification
    - Progress documentation: the history of a simulation, a chronology of work done and decision made.
- Implementation
- Four phases according to Figure 1.3
    - First phase : a period of discovery or orientation
      (Step 1, step2)
    - Second phase : a model building and data collection
      (Step 3, step 4, step 5, step 6, step 7)
    - Third phase : running the model
      (Step 8, step 9, step 10)
    - Fourth phase : an implementation
      (Step 11, step 12)

Simulation Examples

- The simulations are carried out by following steps:
    - Determine the characteristics of each of the inputs to the simulation. Quite often, these may be modeled as probability distributions, either continuous or discrete.
    - Construct a simulation table. Each simulation table is different, for each is developed for the problem at hand.
    - For each repetition *i*, generate a value for each of the p inputs, and evaluate the function, calculating a value of the response y_i. The input values may be computed by sampling values from the distributions determined in step 1. A response typically depends on the inputs and one or more previous responses.
- Simulation examples are in queuing, inventory, reliability and network analysis.
- The simulation table provides a systematic method for tracking system state over time.

	Inputs						Response
Repetitions	X_{i1}	X_{i2}	...	X_{ij}	...	X_{ip}	y_i
1							
2							
.							
.							
.							
n							

Simulation of Queuing Systems

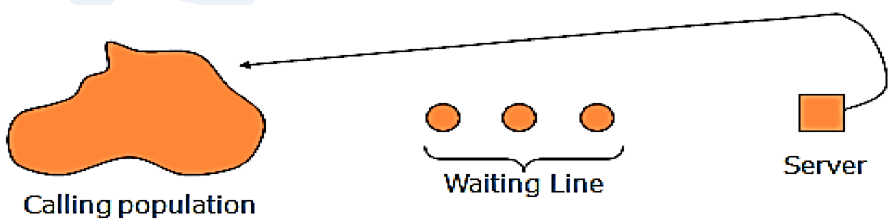


Fig. 2.1 Queuing System

- A queuing system is described by its calling population, the nature of the arrivals, the service mechanism, the system capacity, and the queuing discipline.
- In the single-channel queue, the calling population is infinite.
    - If a unit leaves the calling population and joins the waiting line or enters service, there is no change in the arrival rate of other units that may need service.

- Arrivals for service occur one at a time in a random fashion.
    - Once they join the waiting line, they are eventually served.
- Service times are of some random length according to a probability distribution which does not change over time.
- The system capacity has no limit, meaning that any number of units can wait in line.
    - Finally, units are served in the order of their arrival (often called FIFO: First In, First out) by a single server or channel
- Arrivals and services are defined by the distribution of the time between arrivals and the distribution of service times, respectively.
- For any simple single- or multi-channel queue, the overall effective arrival rate must be less than the total service rate, or the waiting line will grow without bound.
    - In some systems, the condition about arrival rate being less than service rate may not guarantee stability
- System state: the number of units in the system and the status of the server(busy or idle).
- Event: a set of circumstances that cause an instantaneous change in the state of the system.
    - In a single-channel queuing system there are only two possible events that can affect the state of the system.
        - the arrival event: the entry of a unit into the system
        - The departure event: the completion of service on a unit.
- Simulation clock: used to track simulated time.
- If a unit has just completed service, the simulation proceeds in the manner shown in the flow diagram of Figure 2.2.
    - Note that the server has only two possible states: it is either busy or idle.

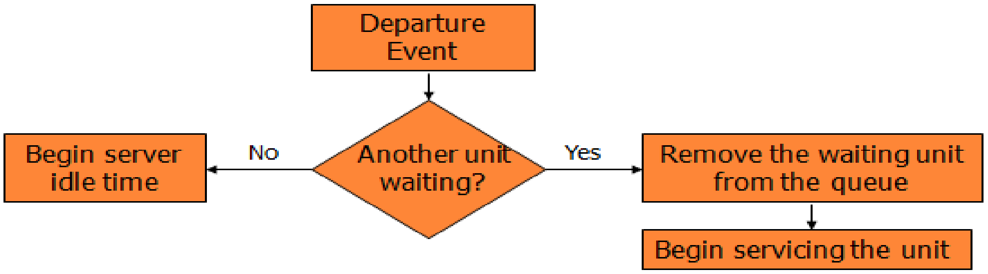


Fig. 2.2 Service-just-completed flow diagram

- The arrival event occurs when a unit enters the system.
    - The unit may find the server either idle or busy.
        - Idle: the unit begins service immediately
        - Busy: the unit enters the queue for the server.

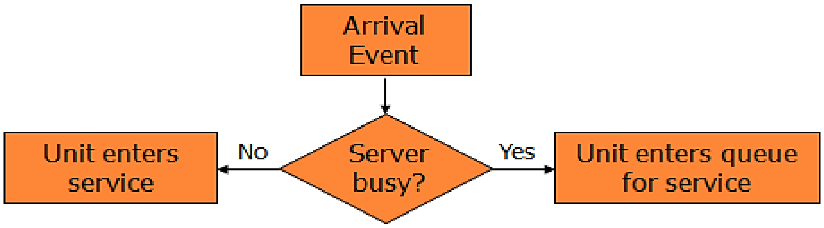


Fig. 2.3 Unit-entering-system flow diagram

		Queue status	
		Not empty	Empty
Server status	Busy	Enter queue	Enter queue
	Idle	Impossible	Enter service

Figure 2.4 Potential unit actions upon arrival.

		Queue status	
		Not empty	Empty
Server outcomes	Busy		Impossible
	Idle	Impossible	

Figure 2.5 Server outcomes after the completion of service.

Problems:

Single channel queuing system problem formulas:

1. Time Customer wait in queue= Time service begin – Arrival Time
2. Time Service End= Service time + Time service begin
3. Time customer Spend In system= Time service end-Arrival Time
4. Idle Time of Server=Time service Begin (N)-Time Service end (N-1)

Standard Formulas:

1. **Average waiting time (i.e. customer wait)**=total time customer wait in queue / Total number of customer
2. **Probability (Wait i.e. customer wait) =**Number of Customer who wait / Total number of customer
3. **Probability of idle server (idle time of server) =**total idle time of server **/** total run time of simulation
4. **Average service time**=total service time/total number of customer
5. **Average times between arrivals**=sum of all times between arrival/number of arrivals-1
6. **Average waiting time those who wait in queue**=total time customer wait in queue/total number of customer who wait
7. **Average time customer spend In the system**=Total time customer spend in system/total number of customer

① Single channel and Multi channel (Able Baker prob.) Queuing system problem :

UNIT-1

$$\text{Average Waiting time (of customer)} = \frac{\text{Total time customer waiting in queue}}{\text{Total no. of customers}}$$

$$\text{Probability wait (of customer)} = \frac{\text{No. of customers to wait}}{\text{Total no. of customers}}$$

$$\text{Probability of idle server (Idle time of server)} = \frac{\text{Total idle time of server}}{\text{Total run time of simulation (TSE last value)}}$$

$$\text{Average Service Time} = \frac{\text{Total service time}}{\text{Total no. of customers}}$$

$$\text{Average time b/w arrival} = \frac{\text{sum of all time b/w arrival}}{\text{No. of arrival} - 1}$$

$$\text{Average waiting time those who wait in queue} = \frac{\text{Total time customer wait in queu}}{\text{Total no. of customer who wait}}$$

$$\text{Average time customer spends in the system} = \frac{\text{Total time customer spends in S/m}}{\text{Total no. of customers}}$$

AT → Arrival Time

IAT → Inter Arrival Time

ST → Service Time

C.No → Customer Number

RD → Random Digit

RDA → Random Digit Assessment

TSB → Time Service Begin

TSE → Time Service End

CP → Cumulative Probability

P → Probability

1. A small grocery store has only one check out counter. The customer arrives at this check out counter at random from 1 to 8 min apart. Each possible value of service time has same probability of occurance. The service time varies from 1 to 6 mins apart. Each possible value of service time has same probability of occurance. Develop simulation distribution table for 8 customers.

Random digit for arrival time :

913 727 015 948 309 922 753 235 302

Service Time : (Random digit)

84 10 74 53 17 79 91 67 89 38

i) Determine Inter Arrival Time distribution table

C.No	P(1/8)	Cumulative Probability	Random Digit Assessment
1	0.125	0.125	001 – 125
2	0.125	0.250	126 – 250
3	0.125	0.375	251 – 375
4	0.125	0.500	376 – 500
5	0.125	0.625	501 – 625
6	0.125	0.750	626 – 750
7	0.125	0.875	751 – 875
8	0.125	1.000	876 – 000

(2). ii] Compute Arrival time from IAT

C.No	Random Digit	IAT	Arrival Time
1	–	–	0
2	913	8	8
3	727	6	14
4	015	1	15
5	948	8	23
6	309	3	26
7	922	8	34
8	753	7	41
9	235	2	43
10	302	3	46

iii] Determine Service Time Distribution Table

S.No	P (1/6)	Cumulative probability	Random Digit Assess
1	0.16	0.16	01 – 16
2	0.16	0.32	17 – 32
3	0.16	0.48	33 – 48
4	0.16	0.64	49 – 64
5	0.16	0.80	65 – 80
6	0.16	0.96	81 – 96

iv] Compute Service Time

S.No	Random digit	Service Time
1	84	6
2	10	1
3	74	5
4	53	4
5	17	2
6	79	5
7	91	6
8	67	5
9	89	6
10	38	3

v] Simulation Table for 10 customers

					TSE − AT	TSB(n) − TSE(n−1)	TSB − AT
C.No	Arrival Time	Service Time	Time Service Begin	Time Service End	Customer Time spent in System	Idle Server Time	Customer Time wait in queue
1	0	6	0	6	6 (6−0)	0	0
2	8	1	8	9	1 (9−8)	2(8−6)	0 (8−8)
3.	14	5	14	19	5(19−14)	5(14−9)	0 (14−14
4	15	4	19	23	8	0	4 (19−15
5	23	2	23	25	2	0	0
6	26	5	26	31	5	1	0
7	34	6	34	40	6	3	0
8	41	5	41	46	5	1	0
9	43	6	46	52	9	0	3
10	46	3	52	55	9	0	6

$$\text{Average waiting Time} = \frac{\text{Total time customer waiting in queue}}{\text{Total no. of customers}} = \frac{4+3+6}{10} = \frac{13}{10} = 1.3 \text{ min}$$

$$\text{Probability wait} = \frac{\text{No. of customers to wait}}{\text{Total no. of customers}} = \frac{3}{10} = 0.3 \text{ min}$$

$$\text{Probability of idle server} = \frac{\text{Total idle time of server}}{\text{Total run time of simulat}^{n}} = \frac{2+5+1+3+1}{55} = \frac{12}{55} = 0.218 \text{ min}$$

(TSE last value)

3) Average Service Time $= \frac{\text{Total Service Time}}{\text{Total no. of customers}}$

$= \frac{6+1+5+4+2+5+6+5+6+3}{10} = \frac{43}{10} = 4.3\text{m}$

Average Time b/w arrival $= \frac{\text{Sum of all time b/w arrival}}{\text{No. of arrival} - 1}$

$= \frac{46}{10-1} = \frac{46}{9} = 5.11 \text{ min}$

Average waiting time those who wait in queue $= \frac{\text{Total time customer wait in queue}}{\text{Total no. of customer who wait}}$

$= \frac{4+3+6}{3} = \frac{13}{3} = 4.3 \text{ min}$

Average time customer spent in system $= \frac{\text{Total time customer spends in s/r}}{\text{Total no. of customers}}$

$= \frac{6+1+5+8+2+5+6+5+9+9}{10}$

$= \frac{56}{10} = 5.6 \text{ min}$

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2. A small grocery store has only one check out counte at random from 1-6 min apart. Each possible value of IAT has the same probability of occurance. The service time vary from 1 to 6 mins with probability shown below.

ST :	1	2	3	4	5	6
P :	0.10	0.20	0.25	0.30	0.10	0.05

Develop a simulation Table for 10 customers.

Take a random digit for arrival :

13 27 15 48 9 22 53 35 2

Random digit for Service Time

84 10 74 53 17 91 79 67 38 89 sequentially.

Calc. Avg Service Time, probability of idle Service time, average b/w arrivals & avg. time customer spent in System.

i] Determine IAT Distribution Table

C.No	Probability (1/6)	Cumulative probability	Random digit assessment
1	0.16	0.16	01−16
2	0.16	0.32	17−32
3	0.16	0.48	33−48
4	0.16	0.64	49−64
5	0.16	0.80	65−80
6	0.16	0.96	81−96

ii] Compute Arrival time from IAT

C.No	RD	IAT	AT
1	−	−	0
2	13	1	1
3	27	2	3
4	15	1	4
5	48	3	7
6	9	1	8
7	22	2	10
8	53	4	14
9	35	3	17
10	2	1	18

iii] Determine ST Dist. Table

S.No	Probability	CP	RDA
1	0.10	0.10	01−10
2	0.20	0.30	11−30
3	0.25	0.55	31−55
4	0.30	0.85	56−85
5	[illegible]	0.95	86−95

iv) Compute Service Time from Distribution Table

S.No	RD	ST
1	84	4
2	10	1
3	74	4
4	53	3
5	17	2
6	91	5
7	79	4
8	67	4
9	38	3
10	89	5

imulation Table for 10 customers

o	Arrival Time	Service Time	Time Service Begin	Time Service End	Cust. time spent in s/m	Idle Server Time	Cust. time wait in queue
	0	4	0	4	4	0	0
	1	1	4	5	4	0	3
	3	4	5	9	6	0	2
	4	3	9	12	8	0	5
	7	2	12	14	7	0	5
	8	5	14	19	11	0	6
	10	4	19	23	13	0	9
	14	4	23	27	13	0	9
	17	3	27	30	13	0	10
	18	5	30	35	17	0	12

erage Service Time $= \frac{35}{10} = \underline{\underline{3.5 \text{ min}}}$

bability of Idle Server Time $= \frac{0}{35} = \underline{\underline{0}}$

erage Time b/w arrivals $= \frac{18}{9} = \underline{\underline{2 \text{ min}}}$

erage time Customer spent in system $= \frac{61}{10} = \underline{\underline{6.1 \text{ min}}}$

3. Consider a store with one checkout counter. Prepare simulation table & find out average waiting time of customer in waiting queue, probability of idle server, average service time

* IAT : 3 2 6 4 4 5 8 7

ST : 4 5 5 8 4 6 2 3 4

Assume 1^{st} customer arrives at $t=0$.

i] Inter arrival distribution Table

C.No	IAT	AT
1	–	0
2	3	3
3	2	5
4	6	11
5	4	15
6	4	19
7	5	24
8	8	32
9	7	39

ii] Simulation table for 9 customers

C.No	AT	ST	TSB	TSE	Cust. spent in s/m.	Idle time of server	Cust. time wait in que
1	0	4	0	4	4	0	0
2	3	5	4	9	6	0	1
3	5	5	9	14	9	0	4
4	11	8	14	22	11	0	3
5	15	4	22	26	11	0	7
6	19	6	26	32	13	0	7
7	24	2	32	34	10	0	8
8	32	3	34	37	5	0	2
9	39	4	39	43	4	2	0

(5) Average waiting time of customer in waiting queue $= \frac{32}{7} = 4.57$ min

Probability of Idle Server $= \frac{2}{10} = 0.2$ min

Average Service Time $= \frac{41}{9} = 4.56$ min

Multichannel Problems :

Type I

4 Consider a simulation with a restaurant system where car ~~$~~ hope takes order & brings an item to the car. The car arrives in the manner :

Time b/w arrival :	1	2	3	4
Probability :	0.25	0.40	0.20	0.15

Consider 2 persons Able & Baker. Able is better & bit faster than Baker.

Able Service Time :

ST :	2	3	4	5
Prob :	0.30	0.28	0.25	0.17

Baker Service Time :

ST :	3	4	5	6
Prob :	0.35	0.25	0.20	0.20

Take a random digit for arrival :

26 98 90 26 42 74 80 68 22

Random digit for service time :

95 21 51 92 89 38 13 61 50 49

i) Determine Inter Arrival distribution Table

C.No	Probability	Cumulative Probability	Random Digit Assessment
1	0.25	0.25	01-25
2	0.40	0.65	26-65
3	0.20	0.85	66-85
4	0.15	1.00	86-00

ii) Compute AT from IAT distribution Table

C.No	RD	IAT	AT
1	-	-	0
2	26	2	2
3	98	4	6
4	90	4	10
5	26	2	12
6	42	2	14
7	74	3	17
8	80	3	20
9	68	3	23
10	22	1	24

iii) Able Service Time Distribution Table

S.No	Probability	CP	RDA
2	0.30	0.30	01-30
3	0.28	0.58	31-58
4	0.25	0.83	59-83
5	0.17	1.00	84-00

iv) Baker Service Time Distribution Table

S.No	Probability	CP	RDA
3	0.35	0.35	01-35
4	0.25	0.60	36-60
5	0.20	0.80	61-80
6	0.20	1.00	81-00

6) v) Simulation table for 10 customers

C.No	AT	RD for service	ST	When Able is Available	When Baker is Available	Server Choosen	Able TSB	Able TSE	Baker TSB	Baker TSE	cust in s/m	Idle Server time	wa in que
1	0	95	5	0	0	A	0	5	—	—	5	0	0
2	2	21	3	5	0	B	—	—	2	5	3	2	0
3	6	51	3	5	5	A	6	9	—	—	3	1	0
4	10	92	5	9	5	A	10	15	—	—	5	1	0
5	12	89	6	15	5	B	—	—	12	18	6	7	0
6	14	38	3	15	18	A	15	18	—	—	4	0	1
7	17	13	2	18	18	A	18	20	—	—	3	0	1
8	20	61	4	20	18	A	20	24	—	—	4	0	0
9	23	50	4	24	18	B	—	—	23	27	4	5	0
10	24	49	3	24	27	B	24	27	—	—	3	0	0

5. Consider Simulation table for Able & Baker problem where time b/w arrival are :

Time b/w arrival : 1 2 3 4 5

Probability : 0.20 0.15 0.05 0.20 0.40

Baker is faster than Able.

Service Time for Able :

ST : 3 4 5 6 2

Prob : 0.20 0.05 0.15 0.20 0.40

Service Time for Baker :

ST : 2 3 5 6 1

Prob : 0.15 0.20 0.05 0.20 0.40

Random Digit for Arrival :

98 90 42 80 22 26 74 26 68

Random Digit for Service Time :

49 50 61 13 38 89 92 51 21 95

i) Determine IAT distribution Table

C.NO	Probability	CP	RDA
1	0.20	0.20	01-20
2	0.15	0.35	21-35
3	0.05	0.40	36-40
4	0.20	0.60	41-60
5	0.40	1.00	61-00

ii) Compute Arrival Time from IAT dist. table.

C.NO	RD	IAT	AT
1	-	-	0
2	98	5	5
3	90	5	10
4	42	4	14
5	80	5	19
6	22	2	21
7	26	2	23
8	74	5	28
9	26	2	30
10	68	5	35

iii) Able Service Time Distribution Table

S.No	P	CP	RDA
3	0.20	0.20	01-20
4	0.05	0.25	21-25
5	0.15	0.40	26-40
6	0.20	0.60	41-60
2	0.40	1.00	61-00

iv) Baker Service Time Distribution Table

S.No	P	CP	RDA
2	0.15	0.15	01-15
3	0.20	0.35	16-35
5	0.05	0.40	36-40
6	0.20	0.60	41-60
1	0.40	1.00	61-00

(RD - Random Digit)

v) Simulation table for 10 customers

C.No	Arrival Time	RD of Service	Service Time	When Able is Available	When Baker is Available	Server Choosen	Able TSB	Able TSE	Baker TSB	Baker TSE	Cust. time spent in s/m	Idle time of Server	Cust. t wait queu
1	0	49	6	0	0	B	—	—	0	6	6	0	0
2	5	50	6	0	6	A	5	11	—	—	6	5	0
3	10	61	1	11	6	B	—	—	10	11	1	4	0
4	14	13	2	11	11	B	—	—	14	16	2	3	0
5	19	38	5	11	16	B	—	—	19	24	5	3	0
6	21	89	2	11	24	A	21	23	—	—	2	10	0
7	23	92	2	23	24	A	23	25	—	—	2	0	0
8	28	51	6	25	24	B	—	—	28	34	6	4	0
9	30	21	4	25	34	B	30	34	—	—	4	5	0
10	35	95	1	34	34	A	—	—	35	36	1	1	0

Customer time spent in s/m = TSE − AT

Idle time of Server = TSB(n) − TSE(n−1)

Customer time waiting in queue = TSB − AT

6. Consider Able Baker Dist. table, IAT having equal probability ratio 1 to 6 min apart.

* Able Service Time have equal probability ratio 1-5 min
Baker Service Time have equal probability ratio 1-4 min.
1-6

Random Digit for time b/w arrival :

62 89 9 62 24 47 8 86 22

Random Digit for Service Time :

59 12 15 29 28 83 13 16 5 94

Able is faster than Baker.

i) Determine IAT Distribution Table

C.No	P	CP	RDA
1	0.16	0.16	01-16
2	0.16	0.32	17-32
3	0.16	0.48	33-48
4	0.16	0.64	49-64
5	0.16	0.80	65-80
6	0.16	0.96	81-96

ii) Compute Arrival Time from IAT Dist. Table

C.No	RD	IAT	AT
1	-	-	0
2	62	4	4
3	89	6	10
4	9	1	11
5	62	4	15
6	24	2	17
7	47	3	20
8	8	1	21
9	86	6	27
10	22	2	29

Distribution Table

S.No	P	CP	RDA
1	0.20	0.20	01-20
2	0.20	0.40	21-40
3	0.20	0.60	41-60
4	0.20	0.80	61-80
5	0.20	1.00	81-00

Distribution Table

S.No	P	CP	RDA
1	0.25	0.25	01-25
2	0.25	0.50	26-50
3	0.25	0.75	51-75
4	0.25	1.00	76-00

v) Simulation Table for 10 customers

C.No	AT	RD of service	ST	When Able is Available	When Baker is Available	Server Choosen	Able		Baker		Cust. spent in s/m	Idle time of server	Cust. time wait in queue
							TSB	TSE	TSB	TSE			
1	0	59	3	0	0	A	0	3	-	-	3	0	0
2	4	12	1	3	0	A	4	5	-	-	1	1	0
3	10	15	1	5	0	A	10	11	-	-	1	5	0
4	11	29	2	11	0	A	11	13	-	-	2	0	0
5	15	98	15	13	0	A	15	20	-	-	2	2	0
6	17	83	4	20	0	B	-	-	17	21	2	17	0
7	20	13	1	20	21	A	20	21	-	-	1	0	0
8	21	16	1	21	21	A	21	22	-	-	1	0	0
9	27	5	1	22	21	A	27	28	-	-	2	5	0
10	29	94	5	28	21	A	29	34	-	-	5	1	0

7. Develop a simulation table for Able & Baker problem.

Inter Arrival Times : 3 2 1 5 4 6

Service Time (Able) : 3 5 2 6 2 1 7

Service Time (Baker): 2 1 4 5 7 6 2

Able is much faster than Baker

i) Compute AT from IAT

C. No	IAT	AT
1	–	0
2	3	3
3	2	5
4	1	6
5	5	11
6	4	15
7	6	21

ii) Simulation Table for 7 customers

C.No	AT	ST	When Able is Available	When Baker is Available	Server choosen	Able TSB	Able TSE	Baker TSB	Baker TSE	Cust. spent in s/m	Idle time of Server	Cust. time wait in Queue
1	0	3	0	0	A	0	3	–	–	0	0	3
2	3	5	3	0	A	3	8	–	–	0	0	5
3	5	4	8	0	B	–	–	5	9	0	5	4
4	6	6	8	9	A	8	14	–	–	2	0	8
5	11	7	14	9	B	–	–	11	18	0	2	7
6	15	1	14	18	A	15	16	–	–	0	1	1
7	21	7	16	18	A	21	28	–	–	0	5	7

(9) having equal probability ratio 1 to 7 min apart.

Service Time for Able :

ST :	1	2	3	4	5
Prob :	0.20	0.10	0.30	0.20	0.20

Service Time for Baker :

ST :	1	2	3	4	5
Prob. :	0.10	0.20	0.20	0.30	0.20

Random Visit for Arrival : 95 60 35 40 52 54 10

Random visit for Service : 60 95 35 40 24 54 10 25

Baker is faster than Able.

i) Determine Inter Arrival Distribution Table.

C.No	P	CP	RDA
1	0.14	0.14	01-14
2	0.14	0.28	15-28
3	0.14	0.42	29-42
4	0.14	0.56	43-56
5	0.14	0.70	57-70
6	0.14	0.84	71-84
7	0.14	0.98	85-98

ii) Compute AT from IADT.

C.No	RD	IAT	AT
1	-	-	0
2	95	7	7
3	60	5	12
4	35	3	15
5	40	3	18
6	52	4	22
7	54	4	26
8	10	1	27

iii) Determine Able Service Time Distribution Table

S.No	P	CP	RDA
1	0.20	0.20	01-20
2	0.10	0.30	21-30
3	0.30	0.60	31-60
4	0.20	0.80	61-80
5	0.20	1.00	81-00

iv) Determine Baker Service Time Distribution table

S.No	P	CP	RDA
1	0.10	0.10	1-10
2	0.20	0.30	11-30
3	0.20	0.50	31-50
4	0.30	0.80	51-80
5	0.20	1.00	81-00

v] Simulation Table for 8 customers.

C.No	Arrival Time	RD of Service	Service Time	When Able is Available	When Baker is Available	Server Choosen	Able		Baker		Cust. spent in s/m	Idle Server Time	Cust.W in que
							TSB	TSE	TSB	TSE			
1	0	60	4	0	0	B	–	–	0	4	4	0	0
2	7	95	5	0	4	B	–	–	7	12	5	3	0
3	12	35	3	0	12	B	–	–	12	15	3	0	0
4	15	40	3	0	15	B	–	–	15	18	3	0	0
5	18	24	2	0	18	B	–	–	18	20	2	0	0
6	22	54	4	0	20	B	–	–	22	26	4	2	0
7	26	10	1	0	26	B	–	–	26	27	1	0	0
8	27	25	2	0	27	B	–	–	27	29	2	0	0

General Principles

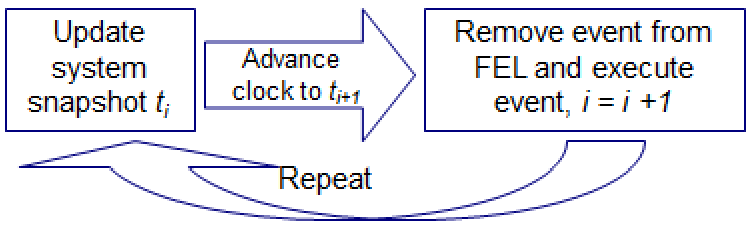
- Develops a common framework for the modeling of complex systems.
- Covers the basic blocks for all discrete-event simulation models.
- Introduces and explains the fundamental concepts and methodologies underlying all discrete-event simulation packages.
    - These concepts and methodologies are not tied to any particular simulation package.
- Deals exclusively with dynamic, stochastic systems.
- Discrete-event models are appropriate for those systems for which changes in system state occur only at discrete points in time.
    - Covers general principles and concepts:
        - Event scheduling/time advance algorithm.
        - The three prevalent world views.
    - Introduces some of the notions of list processing.
- System: a collection of entities that interact together over time, e.g., people and machines.
- Model: an abstract representation of a system.
- System state: a collection of variables that contain all the info necessary to describe the system at any time.
- Entity: any object or component in the system, e.g., a server, a customer, a machine.
- Attributes: the properties of a given entity.
- Lists: a collection of associated entities, ordered in some logical fashion such as sets, queues and chains.
- Event: an instantaneous occurrence that changes the state of a system, e.g., an arrival of a new customer.
- Event list: a list of event notices for future events, ordered by time of occurrence such as the future event list (FEL)
- Activity: duration of time of specified length which is known when it begins, e.g., a service time.
- Simulation Clock: a variable representing simulated time.

Note: different simulation packages use different terminology for the same or similar concepts.

Event Scheduling/Time Advance Algorithm

- The mechanism for advancing simulation time and guaranteeing that all events occur in correct chronological order is based on the future event list (FEL).
- At any given time *t*, the FEL contains all previously scheduled future events and their associated event times ($t_1, t_2, \ldots$)
  - FEL is ordered by event time, and the event time satisfy:

$t \leq t_1 \leq t_2 \leq t_3 \leq \ldots \leq t_n$ where t is the value of CLOCK



Old system snapshot at time t

CLOCK	*System State*	. . .	*Future Event List*	. . .
t	(5, 1, 6)		$(3, t_1)$ — Type 3 event to occur at time t_1 $(1, t_2)$ — Type 1 event to occur at time t_2 $(1, t_3)$ — Type 1 event to occur at time t_3 . . . $(2, t_n)$ — Type 2 event to occur at time t_n	

Event-scheduling/time-advance algorithm

Step 1. Remove the event notice for the imminent event (event 3, time t_1) from FEL

Step 2. Advance CLOCK to imminent event time (i.e., advance CLOCK from t to t_1).

Step 3. Execute imminent event: update system state, change entity attributes, and set membership as needed.

Step 4. Generate future events (if necessary) and place their event notices on FEL ranked by event time. (*Example:* Event 4 to occur at time t^*, where $t_2 < t^* < t_3$.)

Step 5. Update cumulative statistics and counters.

New system snapshot at time t_1

CLOCK	*System State*	. . .	*Future Event List*	. . .
t_1	(5, 1, 5)		$(1, t_2)$ — Type 1 event to occur at time t_2 $(4, t^*)$ — Type 4 event to occur at time t^* $(1, t_3)$ — Type 1 event to occur at time t_3 . . . $(2, t_n)$ — Type 2 event to occur at time t_n	

Figure 3.2 Advancing simulation time and updating system image.

Manual Simulation Using Event Scheduling

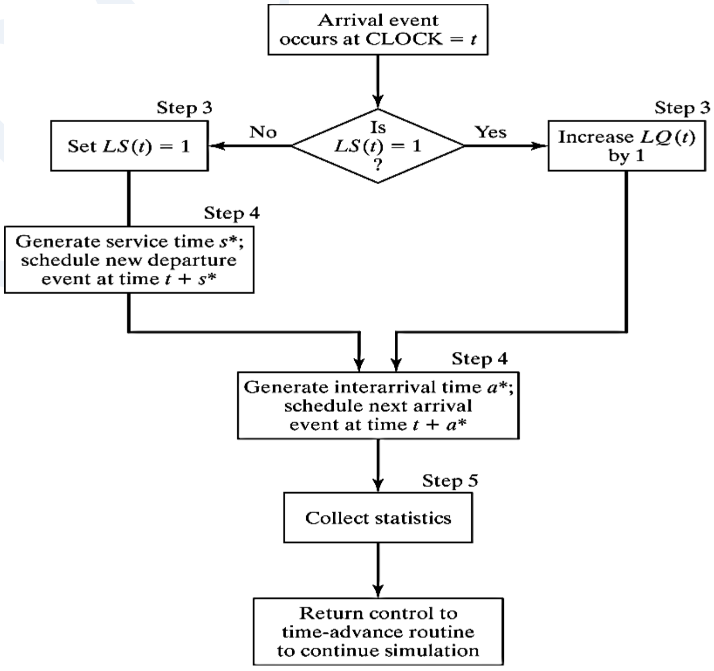
In an event-scheduling simulation, a simulation table is used to record the successive system snapshots as time advances.

Let us consider the example of a grocery shop which has only one checkout counter. **(Single-Channel Queue)** The system consists of those customers in the waiting line plus the one (if any) checking out. The model has the following components:

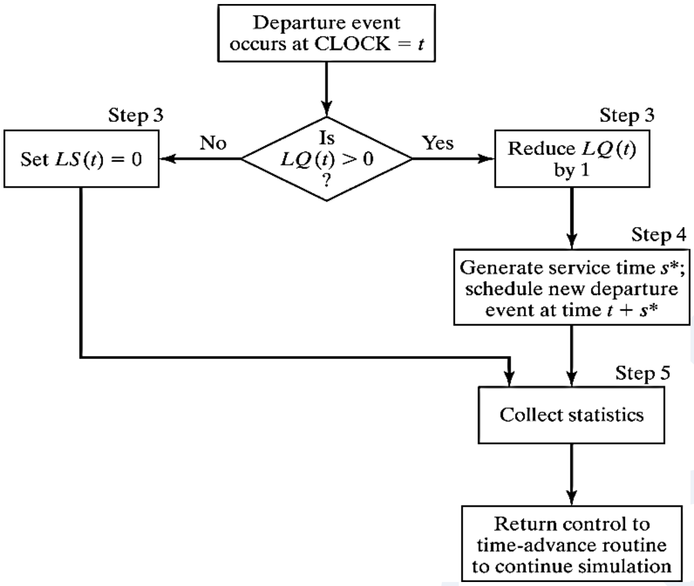
- System state (LQ(t), LS(t)) :
  - LQ(t) is the number of customers in the waiting line
  - LS(t) is the number being served (0 or 1) at time t
- Entities: The server and customers are not explicitly modeled, except in terms of the state variables above.
- Events :
  - Arrival (A)
  - Departure (D)
  - Stopping event (E), scheduled to occur at time 60.
- Event notices (event type, event time) :
  - (A, t), representing an arrival event to occur at future time t
  - (D, t), representing a customer departure at future time t
  - (E, 60), representing the simulation-stop event at future time 60.
- Activities:
  - Interarrival time
  - Service time
- Delay: Customer time spent in waiting line.

❖ FEL will always contain two or three event notices.

❖ Event logic – execution of arrival event.



❖ Event logic – execution of departure event



(1) UNIT-2 [NOTE : Check out time column should be included in prob. No 1 to 5. It is included in prob. 6 to 8.]

Problems on Event Scheduling / Time advanced Algorithm

1. Prepare a simulation table for 1 channel queuing s/m using ES/TA algorithm. Stopping event is at 30.

IAT : 8 6 1 8 2 8

ST : 4 1 4 3 2 4

LQ → Load Queue

LS → Load Service

Step 1 : Compute Departure Time

C.No	Inter Arrival Time	Arrival Time	Service Time	Departure Time
1	–	0	4	4
2	8	8	1	9 (8+1)
3	6	14	4	18 (14+4)
4	1	15	3	21 (18+3)
5	8	23	2	25 (23+2)
6	2	25	4	29 (25+4)
7	8	33	–	–

Step 2 : Simulation Table for 6 customers

Event Type	Clock	System State		Future Event List	Cumulative Statistics	
		LQ(E)	LS(E)		Busy	Maximum Queue
A_1	0	0	1	(D, 4) (A, 8) (E, 30)	0	0
D_1	4	0	0	(A, 8) (D, 9) (E, 30)	4	0
A_2	8	0	1	(D, 9) (A, 14) (E, 30)	4	0
D_2	9	0	0	(A, 14) (D, 18) (E, 30)	5	0
A_3	14	0	1	(A, 15) (D, 18) (E, 30)	5	0
A_4	15	1	1	(D, 18) (A, 23) (E, 30)	6	1
D_3	18	0	1	(D, 21) (A, 23) (E, 30)	9	1
D_4	21	0	0	(A, 23) (D, 25) (E, 30)	12	1
A_5	23	0	1	(D, 25) (A, 25) (E, 30)	12	1

A_6/D_5	25	0	1	(A, 33) (D, 29) (E, 30)	14	1
D_6	29	0	0	(E, 30) (A, 33)	14	1
~~D_6~~	~~29~~	~~0~~	~~0~~	~~(E, 30)~~	~~18~~	~~1~~

2. Stopping Time = 30

IAT : 2 4 3 1 2 5 6

ST : 4 3 2 5 2 1

i] Compute Departure Time

C. No	IAT	AT	ST	DT
1	–	0	4	4
2	2	2	3	7
3	4	6	2	9
4	3	9	5	14
5	1	10	2	16
6	2	12	1	17
7	5	17	–	–
8	6	23	–	–

ii] Simulation Table for 6 customers

Event Type	Clock	System State		Future Event List	CS	
		$LQ(t)$	$LS(t)$		B	MQ
A_1	0	0	1	(A, 2) (D, 4) (E, 30)	0	0
A_2	2	1	1	(D, 4) (A, 6) (E, 30)	2	1
D_1	4	0	1	(A, 6) (D, 7) (E, 30)	4	1
A_3	6	1	1	(D, 7) (A, 9) (E, 30)	6	1
D_2	7	0	1	(A, 9) (D, 9) (E, 30)	7	1
A_4/D_3	9	0	1	(A, 10) (D, 14) (E, 30)	9	1
~~A_4~~	[illegible]	[illegible]	[illegible]	~~(A, 10) (D, 14) (E, 30)~~	[illegible]	[illegible]

(2) A_6	12	2	1	(D, 14) (A, 17) (E, 30)	12	2
D_4	14	1	1	(A, 17) (D, 16) (E, 30)	14	2
D_5	16	1	1	(A, 17) (D, 17) (E, 30)	16	2
A_7/D_6	17	0	1	(A, 23) (E, 30)	17	2

3. Stopping Time = 60

IAT : 1 1 6 3 7 5 2 4 1

ST : 4 2 5 4 1 5 4 1 4

i) Compute Departure Time

C. No	IAT	AT	ST	DT
1	–	0	4	4
2	1	1	2	6
3	1	2	5	11
4	6	8	4	15
5	3	11	1	16
6	7	18	5	23
7	5	23	4	27
8	2	25	1	28
9	4	29	4	33
10	1	–	–	–

ii) Simulation Table for 9 customers

Event Type	Clock	S/m State $LQ(t)$	$LS(t)$	Future Event List	CS B	MQ
A_1	0	0	1	(A, 1) (D, 4) (E, 60)	0	0
A_2	1	1	1	(A, 2) (D, 4) (E, 60)	1	1
A_3	2	2	1	(D, 4) (A, 8) (E, 60)	2	2
D_1	4	1	1	(D, 6) (A, 8) (E, 60)	4	2
D_2	6	0	1	(A, 8) (D, 11) (E, 60)	6	2
A_4	8	1	1	(A, 11) (D, 11) (E, 60)	8	2

D_3/A_5	11	1	1	(D, 15) (A, 18) (E, 60)	11	2
				~~(D, 15) (A, 18) (E, 60)~~		
D_4	15	0	1	(D, 16) (A, 18) (E, 60)	15	2
D_5	16	0	0	(A, 18) (D, 23) (E, 60)	16	2
A_6	18	0	1	(D, 23) (A, 23) (E, 60)	16	2
D_6	23	0	1	(A, 23) (D, 27) (E, 60)	21	2
A_7	23	0	1	(D, 27) (A, 25) (E, 60)	21	2
A_8	25	1	1	(D, 27) (A, 29) (E, 60)	23	2
D_7	27	0	1	(A, 29) (A, 28) (E, 60)	25	2
D_8	28	0	0	(A, 29) (D, 33) (E, 60)	26	2
A_9	29	0	1	(D, 33) (E, 60)	26	2
D_9	33	0	0	(E, 60)	30	2

4. Prepare a simulation table using ES/TA algorithm until the clock reaches time 23 using IAT & ST given :
Stopping Time is 30.

IAT : 5 1 2 3 4 9 5 8 6 1

ST : 4 7 8 1 4 2 5 3 1 4

i) Compute Departure Time

C. No	IAT	AT	ST	DT
1	-	0	4	4
2	5	5	7	12
3	1	6	8	20
4	2	8	1	21
5	3	11	4	25
6	4	15	2	27
7	9	24	5	32
8	5	29	3	35

C. No	IAT	AT	ST	DT
9	8	37	1	38
10	6	43	4	47
11	1	-	-	-

(3) ii] Simulation Table for 9 customers.

Event Type	Clock	S/m state		Future Event List	CS	
		LQ(t)	LS(t)		B	MQ
A_1	0	0	1	(A, 1) (D, 4)	0	0
A_2	1	1	1	(D, 4) (A, 7)	1	1
D_1	4	0	1	(D, 6) (A, 7)	4	1
D_2	6	0	0	(A, 7) (D, 12)	6	1
A_3	7	0	1	(A, 8) (D, 12)	6	1
A_4	8	1	1	(A, 11) (D, 12)	7	1
A_5	11	2	1	(D, 12) (A, 16)	10	2
D_3	12	1	1	(A, 16) (D, 17)	11	2
A_6	16	2	1	(D, 17) (A, 23)	15	2
D_4	17	1	1	(D, 18) (A, 23)	16	2
D_5	18	0	1	(D, 22) (A, 23)	17	2
D_6	22	0	0	(A, 23) (D, 27)	21	2
A_7	23	0	1	(A, 24) (D, 27)	21	2
A_8	24	1	1	(D, 27) (A, 28)	22	2
D_7	27	0	1	(A, 28) (D, 31)	25	2
A_9	28	1	1	(D, 31) (D, 32)	26	2
D_8	31	0	1	(D, 32)	29	2
D_9	32	1	0	–	30	2

S → Customer who taken response time from system

N_D → No. of customer departured

F → customer who spent more than given time (Eg: 4 min)

S = Response Time + Current Departure

Response Time = Clock − Current Arrival

ii) Simulation Table

Event Type	Clock	S/m state LQ(t)	LS(t)	Future Event List	CS B	MQ
A_1	0	0	1	(D, 4) (A, 5) (E, 30)	0	0
D_1	4	0	0	(A, 5) (D, 12) (E, 30)	4	0
A_2	5	0	1	(A, 6) (D, 12) (E, 30)	4	0
A_3	6	1	1	(A, 8) (D, 12) (E, 30)	5	1
A_4	8	2	1	(A, 11) (D, 12) (E, 30)	7	2
A_5	11	3	1	(D, 12) (A, 15) (E, 30)	10	3
D_2	12	2	1	(A, 15) (D, 20) (E, 30)	11	3
A_6	15	3	1	(D, 20) (A, 24) (E, 30)	14	3
D_3	20	2	1	(D, 21) (A, 24) (E, 30)	19	3
D_4	21	1	1	(A, 24) (D, 25) (E, 30)	20	3

5. Prepare a simulation table using Time Advanced Algorithm:

IAT : 1 6 1 3 5 7 1 4

ST : 4 2 5 5 1 4 4 4 1

i) Compute Departure Time

C. No	IAT	AT	ST	DT
1	–	0	4	4
2	1	1	2	6
3	6	7	5	12
4	1	8	5	17
5	3	11	1	18
6	5	16	4	22
7	7	23	4	27
8	1	24	4	31
9	4	28	1	32

6. Prepare a simulation table using Time advanced algorithm
(4) with IAT : 1 1 8 6 8

ST : 4 2 3 4 1 2

Find customers who spent more than 4 min.

i) Compute Departure Time

C. No	IAT	AT	ST	DT
1	–	0	4	4
2	1	1	2	6
3	1	2	3	9
4	8	10	4	14
5	6	16	1	17
6	8	24	2	26

ii) Simulation table of 6 customers.

Event type	Clock	S/m state		Check out time	Future Event List	CS				
		LQ(t)	LS(t)			S	N_D	F	B	MQ
A_1	0	0	1	$(C_1, 0)$	(A, 1) (D, 4)	0	0	0	0	0
A_2	1	1	1	$(C_1, 0)$ $(C_2, 1)$	(A, 2) (D, 4)	0	0	0	1	1
A_3	2	2	1	$(C_1, 0)$ $(C_2, 1)$ $(C_3, 2)$	(D, 4) (A, 10)	0	0	0	2	2
D_1	4	1	1	$(C_2, 1)$ $(C_3, 2)$	(D, 6) (A, 10)	4	1	0	4	2
D_2	6	0	1	$(C_3, 2)$	(D, 9) (A, 10)	9	2	1	6	2
D_3	9	0	0	–	(A, 10) (D, 14)	16	3	2	9	2
A_4	10	0	1	$(C_4, 10)$	(D, 14) (A, 16)	16	3	2	9	2
D_4	14	0	0	–	(A, 16) (D, 17)	20	4	2	13	2
A_5	16	0	1	$(C_5, 16)$	(D, 17) (A, 24)	20	4	2	13	2
D_5	17	0	0	–	(A, 24) (D, 26)	21	5	2	14	2
A_6	24	0	1	$(C_6, 24)$	(D, 26)	21	5	2	14	2
D_6	26	0	0	–	–	23	6	2	16	2

7. Consider single server queue with one checkout counter using ES/TA algorithm

IAT : 4 2 8 1 8 3 6 8

ST : 4 6 5 2 3 4 4 1

Find the no. of customers who spent 4 or more ~~min~~ in the system. Stopping time = 32

i) Compute Arrival & Departure time

C·NO	IAT	AT	ST	DT
1	–	0	4	4
2	4	4	6	10
3	2	6	5	15
4	8	14	2	17
5	1	15	3	20
6	8	23	4	27
7	3	26	4	31
8	6	32	1	33
9	8	40	–	–

ii) Simulation table

Event type	clock	S/m state $LQ(t)$	$LS(t)$	check out time	Future Event List	CS: S	N_D	F	B	MQ
A_1	0	0	1	$(C_1,1)$	(A,4) (D,4) (E,32)	0	0	0	0	0
D_1/A_2	4	0	1	$(C_2,4)$	(A,6) (D,10) (E,32)	4	1	1	4	0
A_3	6	1	1	$(C_2,4)$ $(C_3,6)$	(D,10) (A,14) (E,32)	4	1	1	6	1
D_2	10	0	1	$(C_3,6)$	(A,14) (D,15) (E,32)	10	2	2	10	1
A_4	14	1	1	$(C_3,6)$ $(C_4,14)$	(A,15) (D,15) (E,32)	10	2	2	14	1
A_5/D_3	15	1	1	$(C_4,14)$ $(C_5,15)$	(D,17) (A,23) (E,32)	19	3	3	15	1
D_4	17	0	1	$(C_5,15)$	(D,20) (A,23) (E,32)	22	4	3	17	1
D_5	20	0	0	–	(A,23) (D,27) (E,32)	27	5	4	20	1
A_6	23	0	1	$(C_6,23)$	(A,26) (D,27) (E,32)	27	5	4	20	1

5	D_6	27	0	1	$(C_7, 26)$	(D, 31) (A, 32) (E, 32)	31	6	5	24	1
	D_7	31	0	0	–	(A, 32) (D, 33) (E, 32)	36	7	6	28	1
	A_8	32	0	1	$(C_8, 32)$	(A, 40) (D, 33) (E, 32)	36	7	6	28	1

8. Develop a simulation table for single server queue with one check out counter using TA algorithm. Find busy time of server, maximum queue length, Total no. of customer who spent 3 min or more in system, Total number of departure.

IAT : 1 6 8 8 3 8 4 2 8

ST : 4 1 4 4 2 3 5 6 4

i) Compute Arrival and Departure time

C.No	IAT	AT	ST	DT
1	–	0	4	4
2	1	1	1	5
3	6	7	4	11
4	8	15	4	19
5	8	23	2	25
6	3	26	3	29
7	8	34	5	39
8	4	38	6	45
9	2	40	4	49
10	8	48	–	–

Event type	Clock	S/m state $LQ(t)$	$LS(t)$	Check Out Time	Future Event List	CS: S	N_D	F	B	MQ
A_1	0	0	1	$(C_1, 0)$	(A, 1) (D, 4)	0	0	0	0	0
A_2	1	1	1	$(C_1, 0)(C_2, 1)$	(A, 7) (D, 4)	0	0	0	1	1
D_1	4	0	1	$(C_2, 1)$	(A, 7) (D, 5)	4	1	1	4	1
D_2	5	0	0	–	(A, 7) (D, 11)	8	2	2	5	1
A_3	7	0	1	$(C_3, 7)$	(A, 15) (D, 11)	8	2	2	5	1
D_3	11	0	0	–	(A, 15) (D, 19)	12	3	3	9	1
A_4	15	0	1	$(C_4, 15)$	(D, 19) (A, 23)	12	3	3	9	1
D_4	19	0	0	–	(A, 23) (D, 25)	16	4	4	13	1
A_5	23	0	1	$(C_5, 23)$	(D, 25) (A, 26)	16	4	4	13	1
D_5	25	0	0	–	(A, 26) (D, 29)	18	5	4	15	1
A_6	26	0	1	$(C_6, 26)$	(D, 29) (A, 34)	18	5	4	15	1
D_6	29	0	0	–	(A, 34) (D, 39)	21	6	5	18	1
A_7	34	0	1	$(C_7, 34)$	(A, 38) (D, 39)	21	6	5	18	1
A_8	38	1	1	$(C_7, 34)(C_8, 38)$	(D, 39) (A, 40)	21	6	5	22	1
D_7	39	0	1	$(C_8, 38)$	(A, 40) (D, 45)	26	7	6	23	1
A_9	40	1	1	$(C_8, 38)(C_9, 40)$	~~(A, 48)~~ (D, 45)	26	7	6	24	1
D_8	45	0	1	$(C_9, 40)$	~~(A, 48)~~ (D, 49)	33	8	7	29	1
[illegible]	[illegible]	[illegible]	[illegible]	~~$(C_9, 40)(C_{10}, 48)$~~	~~(D, 49)~~	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
D_9	49	0	0	$(C_{10}, 48)$	–	42	9	8	33	1

Busy time of server = 33 min

Maximum Queue Length = 1

Total no. of customer who spent 3 or more in System = 8

Total no. of departure = 9

Travel Times as given below: Consider stopping time 32 clock cycle.
(B) The stopping event will be completion of 2 weighings (2 A_{QL})

Loading Times	5	15	10	15	15	10	5
Weighing Times	17	15	15	15	15	12	
Travel Times	40	60	60	60	80		

Solution:
Simulation table for Dump-truck Operation.

Clock t	System State				Lists		Future Event List	Cumulative Statistics	
	LQ(t)	L(t)	WQ(t)	W(t)	Loader Queue	Weigh Queue		B_L	B_S
0	4	2	0	1	DT_4 DT_5 DT_6 DT_7	–	(E_L, 5, DT_2)* (E_L, 15, DT_3) (E_W, 17, DT_1)	0	0
5	3	2	1	1	DT_5 DT_6 DT_7	DT_2	(E_L, 15, DT_3)* (E_W, 17, DT_1) (E_L, $\underset{(5+10)}{15}$, DT_4)	10 (5-0)×2+0	5 (5-0)+0
15	2	2	2	1	DT_6 DT_7	DT_2 DT_3	(E_W, 17, DT_1) (E_L, 15, DT_4)* (E_L, $\underset{(15+15)}{30}$, DT_5)	30 (15-5)×2+10	15 (15-5)+5
15	1	2	3	1	DT_7	DT_2 DT_3 DT_4	(E_W, 17, DT_1)* (E_L, 30, DT_5) (E_L, 30, DT_6)	30	15
17	1	2	2	1	DT_7	DT_3 DT_4	(E_L, 30, DT_5)* (E_L, 30, DT_6) (A_{QL}, 57, DT_1) (17+40) (E_W, 32, DT_2) (17+15)	34	17
30	0	2	3	1	–	DT_3 DT_4 DT_5	(E_L, 30, DT_6)* (A_{QL}, 57, DT_1) (E_W, 32, DT_2) (E_L, 40, DT_7)	60	30
30	0	1	4	1	–	DT_3 DT_4 DT_5 DT_6	(A_{QL}, 57, DT_1) (E_W, 32, DT_2)* (E_L, 40, DT_7)	60	30
32	0	1	3	1	–	DT_4 DT_5 DT_6	(A_{QL}, 57, DT_1) (E_L, 40, DT_7) (A_{QL}, 92, DT_2) (E_W, 47, DT_3)	62	32

2. Consider 6 Dump-trucks with loading times, weighing time & Traveling times are given below,

Loading Times	5	5	10	15	20	5	5
Weighing Times	12	15	20	12	15	15	
Travel Time	40	60	20	80			

- Until clock cycle 52
- Calculate
  i) Avg loader utilization
  ii) Avg scale utilization.

Solution:

Simulation table for Dump truck operations.

Clock	System State				Lists		Future Event List	Cumulative Statistics	
t	LQ(t)	L(t)	WQ(t)	W(t)	Loader queue	Weigh queue		B_L	B_S
0	3	2	0	1	DT4 DT5 DT6	–	(E_L, 5, DT_2)* (E_L, 5, DT_3) (E_W, 12, DT_1)	0	0
5	2	2	1	1	DT5 DT6	DT_2	(E_L, 5, DT_3)* (E_W, 12, DT_1) (E_L, 15, DT_4)	10	5
5	1	2	2	1	DT6	DT_2 DT_3	(E_W, 12, DT_1)* (E_L, 15, DT_4) (E_L, 20, DT_5)	10	5
12	1	2	1	1	DT6	DT_3	(E_L, 15, DT_4)* (E_L, 20, DT_5) (AQL, 52, DT_1) (E_W, 27, DT_2)	24	12
15	0	2	2	1	–	DT_3 DT4	(E_L, 20, DT_5)* (AQL, 52, DT_1) (E_W, 27, DT_2) (E_L, 35, DT_6)	30	15
20	0	1	3	1	–	DT_3 DT4 DT_5	(AQL, 52, DT_1) (E_W, 27, DT_2)* (E_L, 35, DT_6)	40	20
27	0	1	2	1	–	DT4 DT_5	(AQL, 52, DT_1) (E_L, 35, DT_6)* (AQL, 87, DT_2) (E_W, 47, DT_3)	47	27
35	0	0	3	1	–	DT4 DT_5 DT_6	(AQL, 52, DT_1) (AQL, 87, DT_2) (E_W, 47, DT_3)*	55	35

Clock t	System State				Lists		Future event List	Cumulative statistics	
	LQ(t)	L(t)	WQ(t)	W(t)	Loader queue	weigh queue		B_L	B_S
47	0	0	2	1	–	DT_5 DT_6	$(AQL, 52, DT_1)$ * $(AQL, 87, DT_2)$ $(AQL, 67, DT_3)$ $(E_W, 59, DT_4)$	55	47
52	0	1	2	1	–	DT_5 DT_6	$(AQL, 87, DT_2)$ $(AQL, 67, DT_3)$ $(E_L, 57, DT_1)$ [52+5] $(E_W, 59, DT_4)$	$\boxed{55}$	$\boxed{52}$

i) Average loader utilization $= \frac{55/2}{52} = \underline{0.529}$

ii) Average scale utilization $= \frac{52}{52} = \underline{1.00}$

3. Consider Loading Times, weighing Times & Travel times of 6 Dump trucks. Loading Times & weighing Times are based on FIFO, Until clock cycle 60 $\boxed{\text{or}}$ entire system runs for 1 hours $\boxed{\text{or}}$ estimation time is 60 mins.

Loading Times	10	10	15	5	10	5
Weighing Times	10	15	5	10	10	
Travel Times	40	60	80	100		

* if all the trucks have equal value use FIFO method.

Solution:

Simulation table for Dump-truck operation.

Clock t	System State				Lists		Future event List	Cumulative Statistics	
	LQ(t)	L(t)	WQ(t)	W(t)	loaders queue	weigh queue		B_L	B_S
0	3	2	0	1	DT_4 DT_5 DT_6	–	$(E_L, 10, DT_2)$ $(E_L, 10, DT_3)$ $(E_W, 10, DT_1)$*	0	0
10	2	2	0	1	DT_5 DT_6	–	$(E_L, 10, DT_3)$* $(AQL, 50, DT_1)$ [10+40] $(E_W, 25, DT_2)$ [10+15] $(E_L, 25, DT_4)$ [10+15]	20	10
10	1	2	1	1	DT_6	DT_3	$(AQL, 50, DT_1)$ $(E_W, 25, DT_2)$ $(E_L, 25, DT_4)$ $(E_L, 15, DT_5)$* [10+5]	20	10

Clock t	System State				Lists		Future Event List	Cumulative Statistics	
	LQ(t)	L(t)	WQ(t)	W(t)	Loader Queue	Weigh Queue		B_L	B_S
15	0	2	2	1	–	DT_3 DT_5	(AQL, 50, DT_1) (E_W, 25, DT_2)* (E_L, 25, DT_4) (E_L, 25, DT_6)	30	15
25	0	2	1	1	–	DT_5	(AQL, 50, DT_1) (E_L, 25, DT_4)* (E_L, 25, DT_6) (AQL, 85, DT_2) (E_W, 30, DT_3)	50	25
25	0	1	2	1	–	DT_5 DT_4	(AQL, 50, DT_1) (E_L, 25, DT_6)* (AQL, 85, DT_2) (E_W, 30, DT_3)	50	25
25	0	0	3	1	–	DT_5 DT_4 DT_6	(AQL, 50, DT_1) (AQL, 85, DT_2) (E_W, 30, DT_3)*	50	25
30	0	0	2	1	–	DT_4 DT_6	(AQL, 50, DT_1) (AQL, 85, DT_2) (AQL, 110, DT_3) (E_W, 40, DT_5)*	50	30
40	0	0	1	1	–	DT_6	(AQL, 50, DT_1)* (AQL, 85, DT_2) (AQL, 110, DT_3) (AQL, 140, DT_5) (E_W, 50, DT_4)	50	40
50	0	1	1	1	–	DT_6	(AQL, 85, DT_2) (AQL, 110, DT_3) (E_L, 55, DT_1) (E_W, 50, DT_4)* (AQL, 140, DT_5)	50	50
50	0	1	0	1	–	–	(AQL, 85, DT_2) (AQL, 110, DT_3) (AQL, 140, DT_5) (AQL, 90, DT_4) (E_W, 60, DT_6) (E_L, 55, DT_1)*	50	50
55	0	0	1	1	–	DT_1	(AQL, 85, DT_2) (AQL, 110, DT_3) (AQL, 140, DT_5) (AQL, 90, DT_4) (E_W, 60, DT_6)*	55	55

Clock t	System State				Lists		Future Event List	Cumulative Statistics	
	$LQ(t)$	$L(t)$	$WQ(t)$	$W(t)$	Loader Queue	Weigh queue		B_L	B_S
60	0	0	0	1	—	—	$(ALQ, 85, DT_2)$ $(ALQ, 110, DT_3)$ $(ALQ, 140, DT_2)$ $(ALQ, 90, DT_4)$ $(ALQ, 120, DT_6)$ $(EW, 70, DT_1)$	55	60

SYSTEM MODELLING AND SIMULATION

Module 2

Statistical Models in Simulation: Review of terminology and concepts, Useful statistical models, discrete distributions, Continuous distributions, Poisson process, Empirical distributions. Queuing Models: Characteristics of queuing systems, Queuing notation, Long-run measures of performance of queuing systems, Long-run measures of performance of queuing systems, Steady-state behavior of M/G/1 queue, Networks of queues

Statistical Models in Simulation

- The world the model-builder sees is probabilistic rather than deterministic.
  - Some statistical model might well describe the variations.
- An appropriate model can be developed by sampling the phenomenon of interest:
  - Select a known distribution through educated guesses
  - Make estimate of the parameter(s)
  - Test for goodness of fit
- In this chapter:
  - Review several important probability distributions
  - Present some typical application of these models

Review of Terminology and Concepts

- In this section, we will review the following concepts:
  - Discrete random variables
  - Continuous random variables
  - Cumulative distribution function
  - Expectation

- **Discrete Random Variables**
  - X is a discrete random variable if the number of possible values of X is finite, or countably infinite.
  - Example: Consider jobs arriving at a job shop.
    - Let X be the number of jobs arriving each week at a job shop.
    - R_x = possible values of X (range space of X) = $\{0,1,2,\ldots\}$
    - $p(x_i)$ = probability the random variable is $x_i = P(X = x_i)$

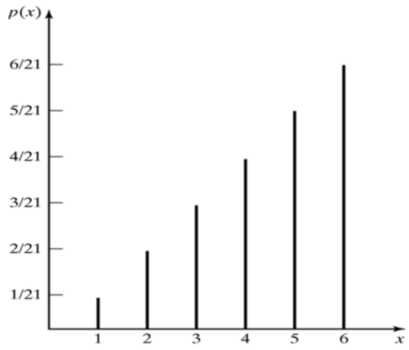
- $p(x_i)$, $i = 1,2, \ldots$ must satisfy:

$$1.\ p(x_i) \geq 0, \text{ for all } i$$

$$2.\ \sum_{i=1}^{\infty} p(x_i) = 1$$

- The collection of pairs $[x_i, p(x_i)]$, $i = 1,2,\ldots,$ is called the probability distribution of X, and $p(x_i)$ is called the probability mass function (pmf) of X.
    - Example: Assume the die is loaded so that the probability that a given face lands up is proportional to the number of spots showing.

x_i	1	2	3	4	5	6
$P(x_i)$	1/21	2/21	3/21	4/21	5/21	6/21



- $p(x_i)$, $i = 1,2,\ \ldots$ must satisfy:

$$1.\ p(x_i) \geq 0,\ \text{for all } i$$

$$2.\ \sum_{i=1}^{\infty} p(x_i) = 1$$

❖ Continuous Random Variables

- X is a continuous random variable if its range space R_x is an interval or a collection of intervals.
- The probability that X lies in the interval $[a, b]$ is given by:

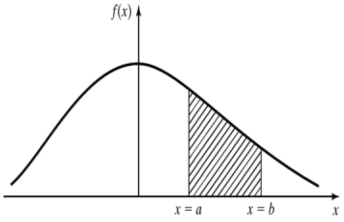
$$P(a \leq X \leq b) = \int_a^b f(x)dx$$

- $f(x)$, denoted as the pdf of X, satisfies:

$$1.\ f(x) \geq 0,\ \text{for all } x \text{ in } R_X$$

$$2.\ \int_{R_X} f(x)dx = 1$$

$$3.\ f(x) = 0,\ \text{ if } x \text{ is not in } R_X$$



- Properties

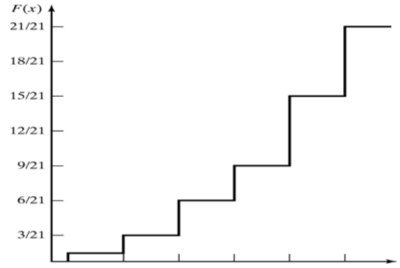
$$1.\ P(X = x_0) = 0,\ \text{because} \int_{x_0}^{x_0} f(x)dx = 0$$

$$2.\ P(a \leq X \leq b) = P(a \prec X \leq b) = P(a \leq X \prec b) = P(a \prec X \prec b)$$

- Example: The die-tossing experiment described in last example has a cdf given as follows:

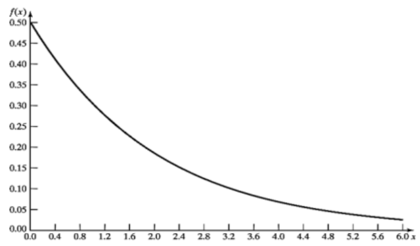
x	(-∞,1)	[1,2)	[2,3)	[3,4)	[4,5)	[5,6)	[6,∞)
F(x)	0	1/21	3/21	6/21	10/21	15/21	21/21

- [a, b) = {a ≤ x ≤ b}



- Example: Life of an inspection device is given by X, a continuous random variable with pdf:

$$f(x)=\begin{cases}\frac{1}{2}e^{-x/2}, & x\geq 0\\ 0, & \text{otherwise}\end{cases}$$



- X has an exponential distribution with mean 2 years
- Probability that the device's life is between 2 and 3 years is:

$$P(2\leq x\leq 3)=\frac{1}{2}\int_{2}^{3}e^{-x/2}dx=0.14$$

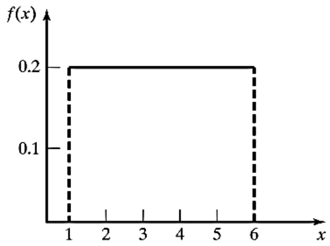
- A random variable X *is* uniformly distributed on the interval *(a, b)* if its PDF is given by

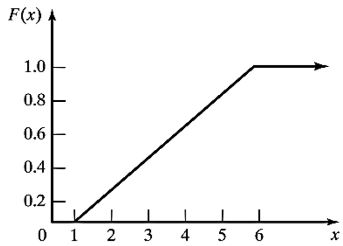
$$f(x)=\begin{cases}\frac{1}{b-a}, & a\leq x\leq b\\ 0, & \text{otherwise}\end{cases}$$

- The CDF is given by

$$F(x)=\begin{cases}0, & x<a\\ \frac{x-a}{b-a}, & a\leq x<b\\ 1, & x\geq b\end{cases}$$

- The PDF and CDF when *a=1* and *b=6:*

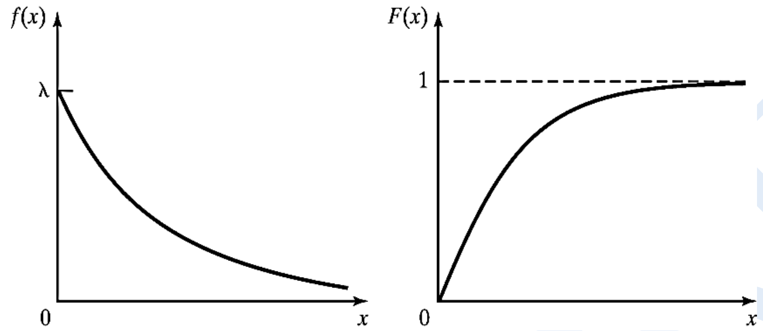




❖ Exponential Distribution

- A random variable X is said to be exponentially distributed with parameter if its PDF is given by

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & \text{otherwise} \end{cases}$$



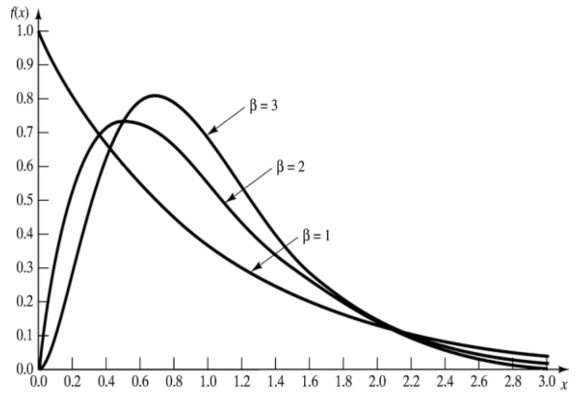
❖ Gamma Distribution

- A function used in defining the gamma distribution is the gamma function, which is defined for all $\beta > 0$ as

$$\Gamma(\beta) = \int_0^\infty x^{\beta-1} e^{-x} dx$$

- A random variable X is gamma distributed with parameters and if its PDF is given by

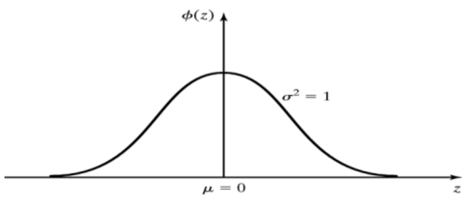
$$f(x) = \begin{cases} \dfrac{\beta\theta}{\Gamma(\beta)} (\beta\theta x)^{\beta-1} e^{-\beta\theta x}, & x > 0 \\ 0, & \text{otherwise} \end{cases}$$



❖ Normal Distribution

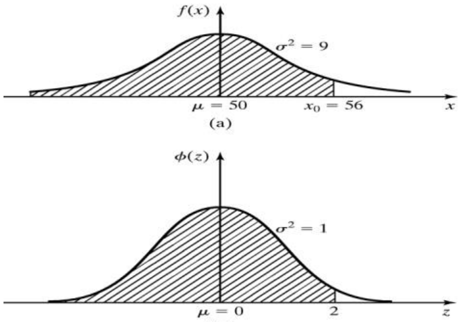
- A random variable X with mean $-\infty < x < \infty$ and variance $\sigma^2 > 0$ has a normal distribution if it has the PDF

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right], -\infty < x < \infty$$

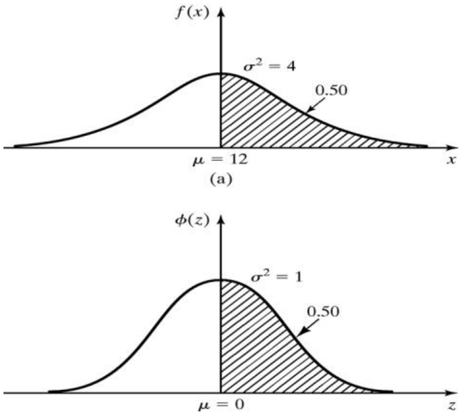


- Example: Suppose that $X \sim N\ (50, 9)$.

  $$F(56) = \Phi\left(\frac{56-50}{3}\right) = \Phi(2) = 0.9772$$



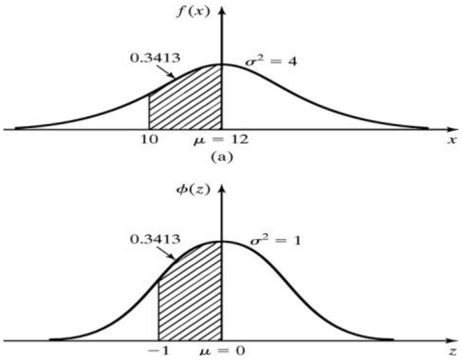
❖ Example: The time in hours required to load a ship, X, is distributed as $N(12, 4)$. The probability that 12 or more hours will be required to load the ship is:



$$P(X > 12) = 1 - F\ (12) = 1 - 0.50 = 0.50$$

(The shaded portions in both figures)

❖ The probability that between 10 and 12 hours will be required to load a ship is given by



$$P\ (10 \le X \le 12) = F\ (12) - F\ (10) = 0.5000 - 0.1587 = 0.3413$$

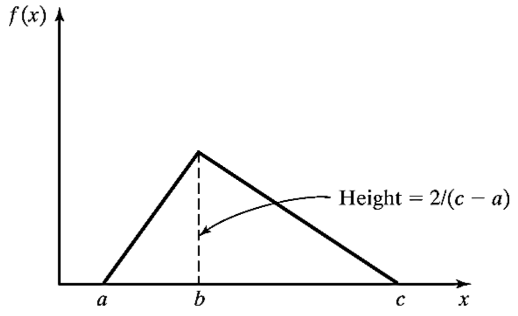
The area is shown in shaded portions of the figure

❖ **Triangular Distribution**

- A random variable X has a triangular distribution if its PDF is given by

$$f(x) = \begin{cases} \dfrac{2(x-a)}{(b-a)(c-a)}, & a \le x \le b \\ \dfrac{2(c-x)}{(c-b)(c-a)}, & b < x \le c \\ 0, & elsewhere \end{cases}$$

Where $\mathbf{a \le b \le c.}$

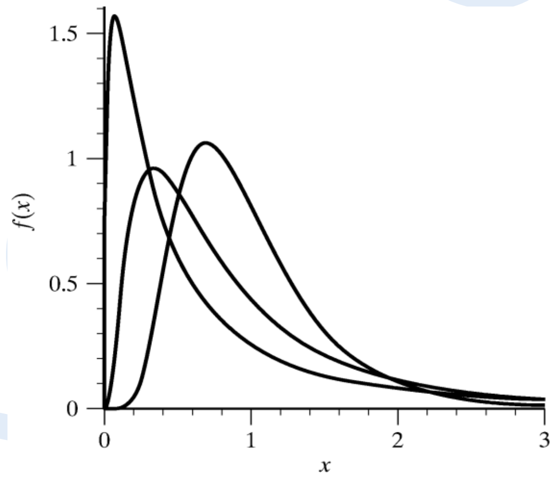


❖ Lognormal Distribution

- A random variable X has a lognormal distribution if its PDF is given by

$$f(x) = \begin{cases} \frac{1}{\sqrt{2\pi}\sigma x} \exp[-\frac{(\ln x - \mu)^2}{2\sigma^2}], & x > 0 \\ 0, & \text{otherwise} \end{cases}$$

where $\sigma^2 > 0$



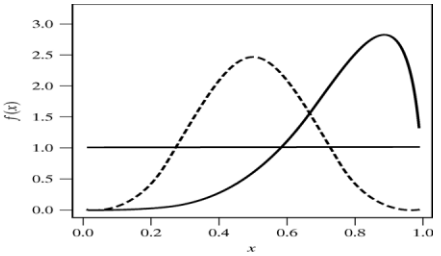
❖ Beta Distribution

- A random variable X is beta-distributed with parameters $\beta_1 > 0$ and $\beta_2 > 0$ if its PDF is given by

$$f(x) = \begin{cases} \frac{x^{\beta_1 - 1}(1-x)^{\beta_2 - 1}}{B(\beta_1, \beta_2)}, & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$$

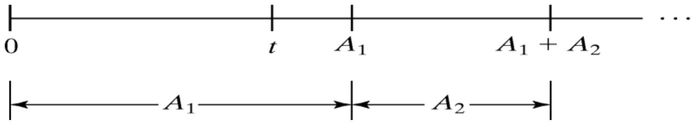
where

$$B(\beta_1, \beta_2) = \Gamma(\beta_1)\Gamma(\beta_2) / \Gamma(\beta_1 + \beta_2)$$



❖ Poisson Process

- Consider the time at which arrivals occur.
- Let the first arrival occur at time A_1, the second occur at time A_1+A_2, and so on.



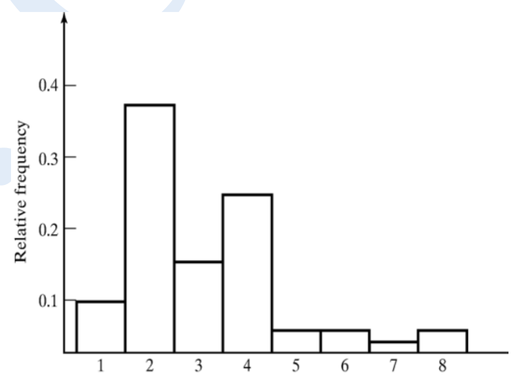
- The probability that the first arrival will occur in *[0, t]* is given by

$$P(A_1 \le t) = 1 - e^{-\lambda t}$$

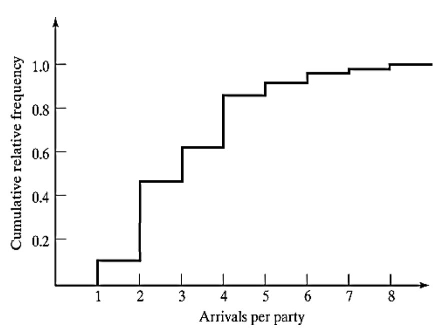
❖ Empirical Distributions

- Example:
- Customers arrive at lunchtime in groups of from one to eight persons.
- The number of persons per party in the last 300 groups has been observed.
- The results are summarized in a table.
- The histogram of the data is also included.

Arrivals per Party	*Frequency*	*Relative Frequency*	*Cumulative Relative Frequency*
1	30	0.10	0.10
2	110	0.37	0.47
3	45	0.15	0.62
4	71	0.24	0.86
5	12	0.04	0.90
6	13	0.04	0.94
7	7	0.02	0.96
8	12	0.04	1.00



- The CDF in the figure is called the empirical distribution of the given data.



❖ Cumulative Distribution Function

- Cumulative Distribution Function (cdf) is denoted by *F(x)*, where *F(x) = P(X <= x)*
    - If *X* is discrete, then $F(x) = \sum_{\substack{\text{all} \\ x_i \le x}} p(x_i)$
    - If *X* is continuous, then $F(x) = \int_{-\infty}^{x} f(t)dt$
- Properties
    1. F is nondecreasing function. If $a \prec b$, then $F(a) \le F(b)$
    2. $\lim_{x \to \infty} F(x) = 1$
    3. $\lim_{x \to -\infty} F(x) = 0$
- All probability question about *X* can be answered in terms of the cdf, e.g.:

$$P(a \prec X \le b) = F(b) - F(a), \text{ for all } a \prec b$$

- Example: An inspection device has cdf:

$$F(x) = \frac{1}{2}\int_{0}^{x} e^{-t/2} dt = 1 - e^{-x/2}$$

- The probability that the device lasts for less than 2 years:

$$P(0 \le X \le 2) = F(2) - F(0) = F(2) = 1 - e^{-1} = 0.632$$

- The probability that it lasts between 2 and 3 years:

$$P(2 \le X \le 3) = F(3) - F(2) = (1 - e^{-(3/2)}) - (1 - e^{-1}) = 0.145$$

❖ Expectation

- The expected value of *X* is denoted by *E(X)*
    - If *X* is discrete $E(x) = \sum_{\text{all } i} x_i p(x_i)$
    - If *X* is continuous $E(x) = \int_{-\infty}^{\infty} x f(x) dx$
    - a.k.a the mean, m, or the 1st moment of *X*
    - A measure of the central tendency
- The variance of *X* is denoted by *V(X)* or *var (X)* or σ^2
    - Definition: $V(X) = E[(X - E[X]^2]$
    - Also, $V(X) = E(X^2) - [E(x)]^2$
    - A measure of the spread or variation of the possible values of X around the mean

- The standard deviation of X is denoted by σ
  - Definition: square root of $V(X)$
  - Expressed in the same units as the mean

- Example: The mean of life of the previous inspection device is:

$$E(X)=\frac{1}{2}\int_0^\infty xe^{-x/2}dx=-xe^{-x/2}\Big|_0^\infty+\int_0^\infty e^{-x/2}dx=2$$

- To compute variance of X, we first compute $E(X^2)$:

$$E(X^2)=\frac{1}{2}\int_0^\infty x^2e^{-x/2}dx=-x^2e^{-x/2}\Big|_0^\infty+\int_0^\infty e^{-x/2}dx=8$$

- Hence, the variance and standard deviation of the device's life are:

$$V(X)=8-2^2=4$$
$$\sigma=\sqrt{V(X)}=2$$

Useful Statistical Models

- In this section, statistical models appropriate to some application areas are presented. The areas include:
  - Queueing systems
  - Inventory and supply-chain systems
  - Reliability and maintainability
  - Limited data

- **Queueing Systems**

- In a queueing system, interarrival and service-time patterns can be probabilistic (for more queueing examples, see Chapter 2).
- Sample statistical models for interarrival or service time distribution:
  - Exponential distribution: if service times are completely random
  - Normal distribution: fairly constant but with some random variability (either positive or negative)
  - Truncated normal distribution: similar to normal distribution but with restricted value.
  - Gamma and Weibull distribution: more general than exponential (involving location of the modes of pdf's and the shapes of tails.)

- ❖ **Inventory and supply chain**

- In realistic inventory and supply-chain systems, there are at least three random variables:
    - The number of units demanded per order or per time period
    - The time between demands
    - The lead time
- Sample statistical models for lead time distribution:
    - Gamma
- Sample statistical models for demand distribution:
    - Poisson: simple and extensively tabulated.
    - Negative binomial distribution: longer tail than Poisson (more large demands).
    - Geometric: special case of negative binomial given at least one demand has occurred.

- ❖ **Reliability and maintainability**

- Time to failure (TTF)
    - Exponential: failures are random
    - Gamma: for standby redundancy where each component has an exponential TTF
    - Weibull: failure is due to the most serious of a large number of defects in a system of components
    - Normal: failures are due to wear

- ❖ **Other areas**

- For cases with limited data, some useful distributions are:
    - Uniform, triangular and beta
- Other distribution: Bernoulli, binomial and hyperexponential.

Discrete Distributions

- Discrete random variables are used to describe random phenomena in which only integer values can occur.
- In this section, we will learn about:
    - Bernoulli trials and Bernoulli distribution
    - Binomial distribution
    - Geometric and negative binomial distribution
    - Poisson distribution

- ❖ **Bernoulli Trials and Bernoulli distribution**

- Bernoulli Trials:
    - Consider an experiment consisting of n trials, each can be a success or a failure.
        - ➢ Let $X_j = 1$ if the jth experiment is a success
        - ➢ and $X_j = 0$ if the jth experiment is a failure

- The Bernoulli distribution (one trial):

$$p_j(x_j) = p(x_j) = \begin{cases} p, & x_j = 1,\ j = 1,2,\ldots,n \\ 1 - p = q, & x_j = 0, j = 1,2,\ldots,n \\ 0, & \text{otherwise} \end{cases}$$

- where $E(X_j) = p$ *and* $V(X_j) = p\,(1\text{-}p) = p\,q$

▪ Bernoulli process:

- The n Bernoulli trials where trails are independent:

$$p(x_1,x_2,\ldots,x_n) = p_1(x_1)\,p_2(x_2)\,\ldots\,p_n(x_n)$$

❖ **Binomial Distribution**

▪ The number of successes in n Bernoulli trials, X, has a binomial distribution.

$$p(x) = \begin{cases} \binom{n}{x} \underbrace{p^x q^{n-x}}, & x = 0,1,2,\ldots,n \\ 0, & \text{otherwise} \end{cases}$$

The number of outcomes having the required number of successes and failures

Probability that there are x successes and (n-x) failures

- The mean, $E(x) = p + p + \ldots + p = n*p$
- The variance, $V(X) = pq + pq + \ldots + pq = n*pq$

❖ **Geometric & Negative Binomial Distribution**

▪ Geometric distribution

- The number of Bernoulli trials, X, to achieve the 1st success:

$$p(x) = \begin{cases} q^{x-1}p, & x = 0,1,2,\ldots,n \\ 0, & \text{otherwise} \end{cases}$$

- $E(x) = 1/p$, *and* $V(X) = q/p^2$

▪ Negative binomial distribution

- The number of Bernoulli trials, X, until the k^{th} success
- If Y is a negative binomial distribution with parameters p and k, then:

$$p(x) = \begin{cases} \binom{y-1}{k-1} q^{y-k} p^k, & y = k, k+1, k+2, \ldots \\ 0, & \text{otherwise} \end{cases}$$

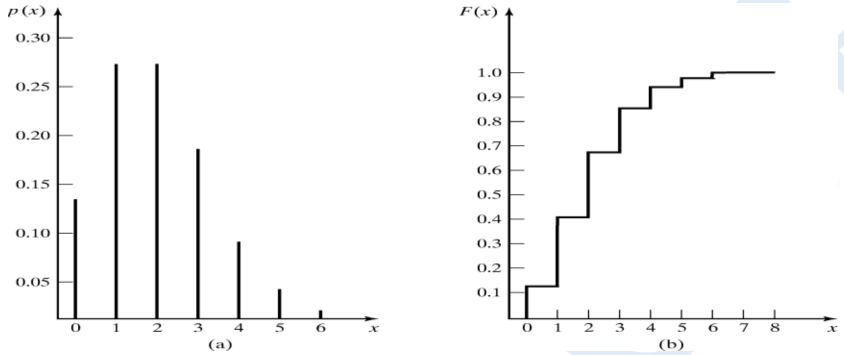
- $E(Y) = k/p$, *and* $V(X) = kq/p^2$

❖ **Poisson Distribution**

- Poisson distribution describes many random processes quite well and is mathematically quite simple.
    - where a > 0, pdf and cdf are:

$$p(x) = \begin{cases} \dfrac{e^{-\alpha}\alpha^x}{x!}, & x = 0,1,\ldots \\ 0, & \text{otherwise} \end{cases} \qquad\qquad F(x) = \sum_{i=0}^{x} \frac{e^{-\alpha}\alpha^i}{i!}$$

    - *E(X) = a = V(X)*



- Example: A computer repair person is "beeped" each time there is a call for service. The number of beeps per hour ~ Poisson(a = 2 per hour).
    - The probability of three beeps in the next hour:

$$p(3) = e^{-2}2^3/3! = 0.18$$

also, $p(3) = F(3) - F(2) = 0.857 - 0.677 = 0.18$

    - The probability of two or more beeps in a 1-hour period:

$$\begin{aligned} p(2 \text{ or more}) &= 1 - p(0) - p(1) \\ &= 1 - F(1) \\ &= 0.594 \end{aligned}$$

Continuous Distributions

- Continuous random variables can be used to describe random phenomena in which the variable can take on any value in some interval.
- In this section, the distributions studied are:
    - Uniform
    - Exponential
    - Normal
    - Weibull
    - Lognormal

❖ **Uniform Distribution**

- A random variable X is uniformly distributed on the interval (a,b), $U(a,b)$, if its pdf and cdf are:

$$f(x)=\begin{cases}\dfrac{1}{b-a}, & a\le x\le b\\ 0, & \text{otherwise}\end{cases}\qquad F(x)=\begin{cases}0, & x<a\\ \dfrac{x-a}{b-a}, & a\le x<b\\ 1, & x\ge b\end{cases}$$

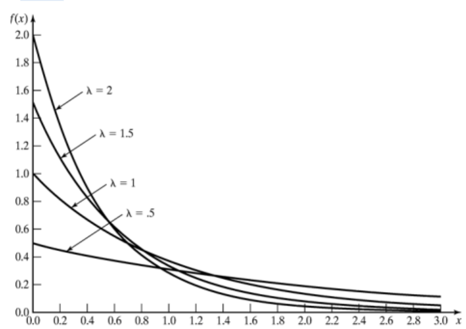
- Properties
  - $P(x_1 < X < x_2)$ is proportional to the length of the interval
    $[F(x_2) - F(x_1) = (x_2-x_1)/(b-a)]$
  - $E(X) = (a+b)/2$ $\qquad\qquad V(X) = (b-a)^2/12$
- U(0,1) provides the means to generate random numbers, from which random variates can be generated.

❖ **Exponential Distribution**

- A random variable X is exponentially distributed with parameter l > 0 if its pdf and cdf are:

$$f(x)=\begin{cases}\lambda e^{-\lambda x}, & x\ge 0\\ 0, & \text{elsewhere}\end{cases}\qquad F(x)=\begin{cases}0, & x<0\\ \int_0^x \lambda e^{-\lambda t}dt = 1-e^{-\lambda x}, & x\ge 0\end{cases}$$

- $E(X) = 1/l$ $\qquad V(X) = 1/l^2$
- Used to model interarrival times when arrivals are completely random, and to model service times that are highly variable
- For several different exponential pdf's (see figure), the value of intercept on the vertical axis is l, and all pdf's eventually intersect.



- Memoryless property
  - For all s and t greater or equal to 0:
    $$P(X > s+t \mid X > s) = P(X > t)$$
  - Example: A lamp ~ exp(l = 1/3 per hour), hence, on average, 1 failure per 3 hours.
    - ➢ The probability that the lamp lasts longer than its mean life is:
      $P(X > 3) = 1-(1-e^{-3/3}) = e^{-1} = 0.368$
    - ➢ The probability that the lamp lasts between 2 to 3 hours is:
      $P(2 <= X <= 3) = F(3) – F(2) = 0.145$
    - ➢ The probability that it lasts for another hour given it is operating for 2.5 hours:

      $P(X > 3.5 \mid X > 2.5) = P(X > 1) = e^{-1/3} = 0.717$

❖ Normal Distribution

- A random variable X is normally distributed has the pdf:

$$f(x)=\frac{1}{\sigma\sqrt{2\pi}}\exp\left[-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right],\quad -\infty<x<\infty$$

  - Mean: $-\infty<\mu<\infty$
  - Variance: $\sigma^2>0$
  - Denoted as $X \sim N(\mu,\sigma^2)$
- Special properties:
  - $\lim_{x\to-\infty} f(x)=0$, and $\lim_{x\to\infty} f(x)=0$
  - $f(\mu-x)=f(\mu+x)$; the pdf is symmetric about μ.
  - The maximum value of the pdf occurs at $x=\mu$; the mean and mode are equal.
- Evaluating the distribution:
  - Use numerical methods (no closed form)
  - Independent of μ and σ, using the standard normal distribution:

$$Z \sim N(0,1)$$

  - Transformation of variables: let $Z = (X - \mu) / \sigma$,

$$F(x)=P(X\le x)=P\left(Z\le\frac{x-\mu}{\sigma}\right)$$

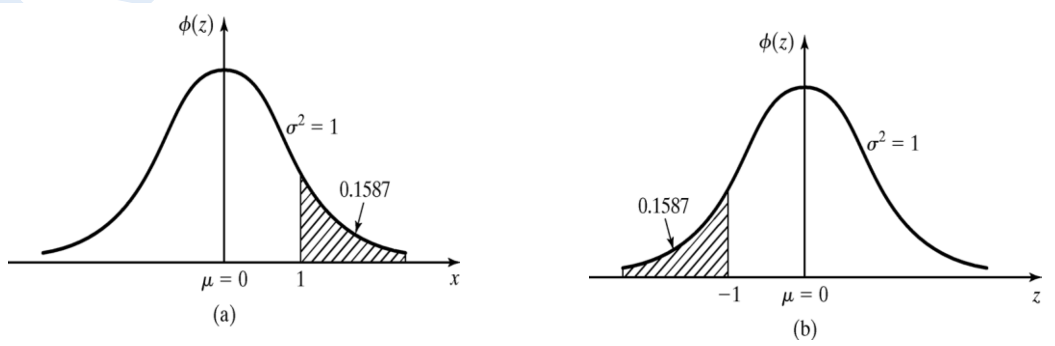
$$=\int_{-\infty}^{(x-\mu)/\sigma}\frac{1}{\sqrt{2\pi}}e^{-z^2/2}dz$$

$$=\int_{-\infty}^{(x-\mu)/\sigma}\phi(z)dz=\Phi\left(\tfrac{x-\mu}{\sigma}\right)\qquad\text{, where }\quad \Phi(z)=\int_{-\infty}^{z}\frac{1}{\sqrt{2\pi}}e^{-t^2/2}dt$$

- Example: The time required to load an oceangoing vessel, X, is distributed as N(12,4)
  - The probability that the vessel is loaded in less than 10 hours:

$$F(10)=\Phi\left(\frac{10-12}{2}\right)=\Phi(-1)=0.1587$$

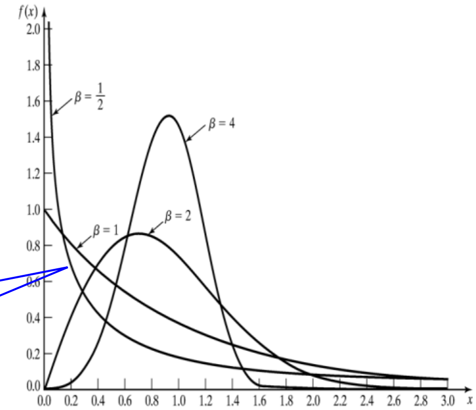
  - ➢ Using the symmetry property, $\Phi(1)$ is the complement of $\Phi(-1)$



❖ Weibull Distribution

- A random variable X has a Weibull distribution if its pdf has the form:

$$f(x)=\begin{cases}\dfrac{\beta}{\alpha}\left(\dfrac{x-\nu}{\alpha}\right)^{\beta-1}\exp\left[-\left(\dfrac{x-\nu}{\alpha}\right)^{\beta}\right], & x\geq \nu\\ 0, & \text{otherwise}\end{cases}$$



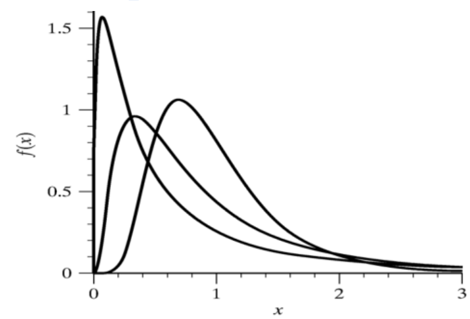
- 3 parameters:
  - Location parameter: υ, $(-\infty<\nu<\infty)$
  - Scale parameter: β, $(\beta > 0)$
  - Shape parameter. α, (> 0)
- Example: $\upsilon = 0$ and $\alpha = 1$:

When $\beta = 1$,
$X \sim exp(\lambda = 1/\alpha)$

❖ Lognormal Distribution

- A random variable X has a lognormal distribution if its pdf has the form:

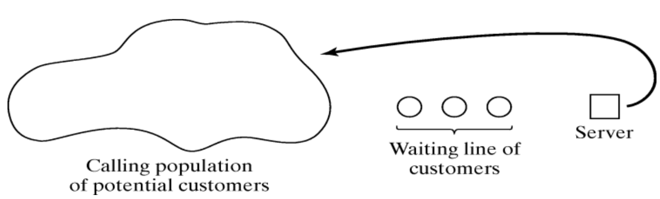
$$f(x)=\begin{cases}\dfrac{1}{\sqrt{2\pi}\sigma x}\exp\left[-\dfrac{(\ln x-\mu)^2}{2\sigma^2}\right], & x>0\\ 0, & \text{otherwise}\end{cases}$$



  - Mean $E(X) = e^{\mu+\sigma^2/2}$
  - Variance $V(X) = e^{2\mu+\sigma^2/2}\,(e^{\sigma^2} - 1)$
- Relationship with normal distribution
  - When $Y \sim N(\mu, \sigma^2)$, then $X = e^Y \sim$ lognormal(μ, σ^2)
  - Parameters μ and σ^2 are not the mean and variance of the lognormal

Queuing Models

- Simulation is often used in the analysis of queueing models.
- A simple but typical queueing model:



- Queueing models provide the analyst with a powerful tool for designing and evaluating the performance of queueing systems.

- Typical measures of system performance:
  - Server utilization, length of waiting lines, and delays of customers
  - For relatively simple systems, compute mathematically
  - For realistic models of complex systems, simulation is usually required.

Characteristics of Queueing Systems

- Key elements of queueing systems:
  - Customer: refers to anything that arrives at a facility and requires service, e.g., people, machines, trucks, emails.
  - Server: refers to any resource that provides the requested service, e.g., repairpersons, retrieval machines, runways at airport.

Calling Population

- Calling population: the population of potential customers, may be assumed to be finite or infinite.
  - Finite population model: if arrival rate depends on the number of customers being served and waiting, e.g., model of one corporate jet, if it is being repaired, the repair arrival rate becomes zero.
  - Infinite population model: if arrival rate is not affected by the number of customers being served and waiting, e.g., systems with large population of potential customers.

System Capacity

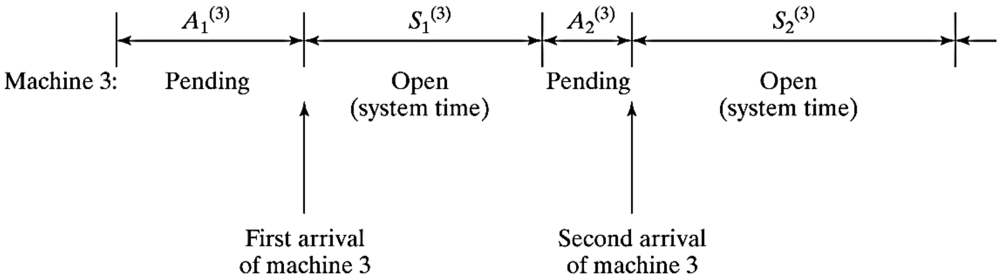
- System Capacity: a limit on the number of customers that may be in the waiting line or system.
  - Limited capacity, e.g., an automatic car wash only has room for *10* cars to wait in line to enter the mechanism.
  - Unlimited capacity, e.g., concert ticket sales with no limit on the number of people allowed waiting to purchase tickets.

Arrival Process

- For infinite-population models:
  - In terms of interarrival times of successive customers.
  - Random arrivals: interarrival times usually characterized by a probability distribution.
    - Most important model: Poisson arrival process (with rate l), where A_n represents the interarrival time between customer $n-1$ and customer n, and is exponentially distributed (with mean $1/l$).
  - Scheduled arrivals: interarrival times can be constant or constant plus or minus a small random amount to represent early or late arrivals.

- e.g., patients to a physician or scheduled airline flight arrivals to an airport.
- At least one customer is assumed to always be present, so the server is never idle, e.g., sufficient raw material for a machine.

- For finite-population models:
  - Customer is pending when the customer is outside the queueing system, e.g., machine-repair problem: a machine is "pending" when it is operating, it becomes "not pending" the instant it demands service form the repairman.
  - Runtime of a customer is the length of time from departure from the queueing system until that customer's next arrival to the queue, e.g., machine-repair problem, machines are customers and a runtime is time to failure.
  - Let $A_1^{(i)}$, $A_2^{(i)}$, … be the successive runtimes of customer i, and $S_1^{(i)}$, $S_2^{(i)}$ be the corresponding successive system times:



Queue Behavior and Queue Discipline

- Queue behavior: the actions of customers while in a queue waiting for service to begin, for example:
  - Balk: leave when they see that the line is too long,
  - Renege: leave after being in the line when its moving too slowly,
  - Jockey: move from one line to a shorter line.
- Queue discipline: the logical ordering of customers in a queue that determines which customer is chosen for service when a server becomes free, for example:
  - First-in-first-out (FIFO)
  - Last-in-first-out (LIFO)
  - Service in random order (SIRO)
  - Shortest processing time first (SPT)
  - Service according to priority (PR).

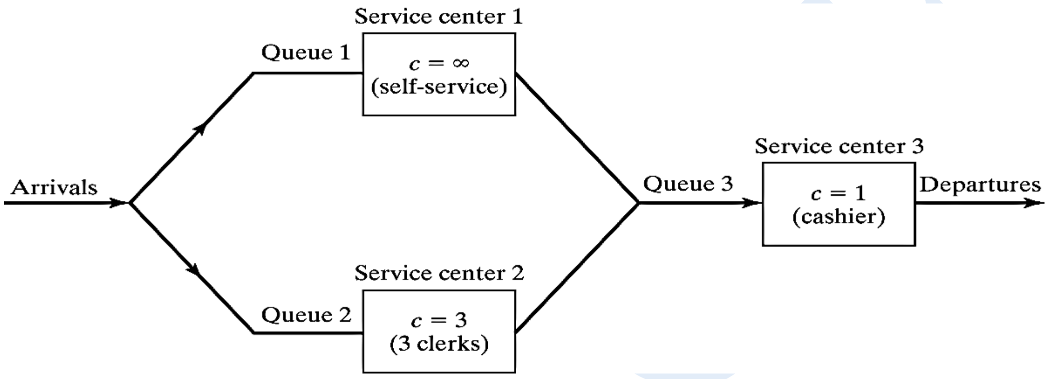
Service Times and Service Mechanism

- Service times of successive arrivals are denoted by S_1, S_2, S_3.
  - May be constant or random.

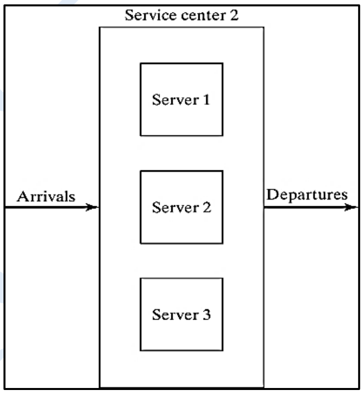
- $\{S_1, S_2, S_3, \ldots\}$ is usually characterized as a sequence of independent and identically distributed random variables, e.g., exponential, Weibull, gamma, lognormal, and truncated normal distribution.

▪ A queueing system consists of a number of service centers and interconnected queues.
  - Each service center consists of some number of servers, *c*, working in parallel, upon getting to the head of the line, a customer takes the 1^{st} available server.

▪ Example: consider a discount warehouse where customers may:
  - Serve themselves before paying at the cashier:



  - Wait for one of the three clerks:



  - Batch service (a server serving several customers simultaneously), or customer requires several servers simultaneously.

Queueing Notation

▪ A notation system for parallel server queues: *A/B/c/N/K*
  - *A* represents the interarrival-time distribution,
  - *B* represents the service-time distribution,
  - *c* represents the number of parallel servers,
  - *N* represents the system capacity,
  - *K* represents the size of the calling population.

- Primary performance measures of queueing systems:
    - P_n: steady-state probability of having n customers in system,
    - $P_n(t)$: probability of n customers in system at time t,
    - λ: arrival rate,
    - λ_e: effective arrival rate,
    - μ: service rate of one server,
    - ρ: server utilization,
    - A_n: interarrival time between customers n-1 and n,
    - S_n: service time of the nth arriving customer,
    - W_n: total time spent in system by the nth arriving customer,
    - W_n^Q: total time spent in the waiting line by customer n,
    - $L(t)$: the number of customers in system at time t,
    - $L_Q(t)$: the number of customers in queue at time t,
    - L: long-run time-average number of customers in system,
    - L_Q: long-run time-average number of customers in queue,
    - w : long-run average time spent in system per customer,
    - w_Q: long-run average time spent in queue per customer.

Long-run Measures of performance of queueing systems

- The primary long run measures of performance of queueing system are the long run time average number of customer in s/m(L) & queue(L_Q)
- The long run average time spent in s/m(w) & in the queue(w_Q) per customer
- Server utilization or population of time that a server is busy (p).

Time-Average Number in System *L*

- Consider a queueing system over a period of time T,
    - Let T_i denote the total time during $[0,T]$ in which the system contained exactly i customers, the time-weighted-average number in a system is defined by:

$$\hat{L} = \frac{1}{T}\sum_{i=0}^{\infty} iT_i = \sum_{i=0}^{\infty} i\left(\frac{T_i}{T}\right)$$

    - Consider the total area under the function is $L(t)$, then,

$$\hat{L} = \frac{1}{T}\sum_{i=0}^{\infty} iT_i = \frac{1}{T}\int_0^T L(t)dt$$

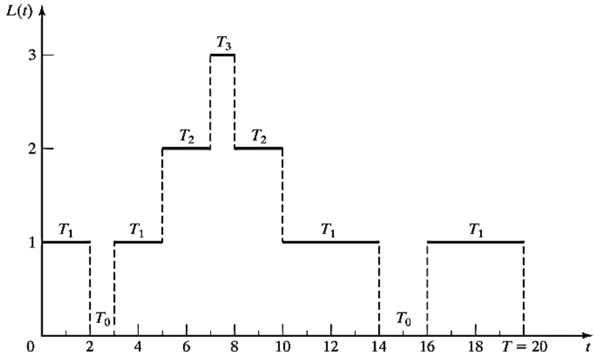
    - The long-run time-average # in system, with probability 1:

$$\hat{L} = \frac{1}{T}\int_0^T L(t)dt \rightarrow L \quad \text{as} \quad T \rightarrow \infty$$

- The time-weighted-average number in queue is:

$$\hat{L}_Q = \frac{1}{T}\sum_{i=0}^{\infty} iT_i^Q = \frac{1}{T}\int_0^T L_Q(t)dt \to L_Q \quad \text{as} \quad T \to \infty$$

- *G/G/1/N/K* example: consider the results from the queuing system (*N > 4, K > 3*).



$$\hat{L} = [0(3) + 1(12) + 2(4) + 3(1)]/20$$
$$= 23/20 = 1.15 \text{ cusomters}$$

$$L_Q(t) = \begin{cases} 0, & \text{if } L(t) = 0 \\ L(t) - 1, & \text{if } L(t) \ge 1 \end{cases}$$

$$\hat{L}_Q = \frac{0(15) + 1(4) + 2(1)}{20} = 0.3 \text{ customers}$$

Average Time Spent in System per Customer *w*

- The average time spent in system per customer, called the average system time, is:

$$\hat{w} = \frac{1}{N}\sum_{i=1}^{N} W_i$$

Where $W_1, W_2, \ldots, W_N$ are the individual times that each of the N customers spend in the system during $[0,T]$.

- For stable systems: $\hat{w} \to w$ as $N \to \infty$
- If the system under consideration is the queue alone:

$$\hat{w}_Q = \frac{1}{N}\sum_{i=1}^{N} W_i^Q \to w_Q \quad \text{as} \quad N \to \infty$$

- *G/G/1/N/K* example (cont.): the average system time is

$$\hat{w} = \frac{W_1 + W_2 + \ldots + W_5}{5} = \frac{2 + (8-3) + \ldots + (20-16)}{5} = 4.6 \text{ time units}$$

Server Utilization

- Definition: the proportion of time that a server is busy.
  - Observed server utilization, $\hat{\rho}$ is defined over a specified time interval [0,T].
  - Long-run server utilization is ρ.
  - For systems with long-run stability: $\hat{\rho} \to \rho \quad \text{as} \quad T \to \infty$
- For *G/G/1/∞/∞* queues:

- Any single-server queueing system with average arrival rate *l* customers per time unit, where average service time $E(S) = 1/\mu$ time units, infinite queue capacity and calling population.
- Conservation equation, $L = \lambda w$, can be applied.
- For a stable system, the average arrival rate to the server, λ_s, must be identical to λ.
- The average number of customers in the server is:
- In general, for a single-server queue:

$$\hat{L}_s = \hat{\rho} \rightarrow L_s = \rho \quad \text{as } T \rightarrow \infty$$

$$\text{and} \qquad \rho = \lambda E(s) = \frac{\lambda}{\mu}$$

  - For a single-server stable queue: $\rho = \dfrac{\lambda}{\mu} < 1$
  - For an unstable queue ($\lambda > m$), long-run server utilization is *1*

- **For *G/G/c/∞/∞* queues:**
  - A system with c identical servers in parallel.
  - If an arriving customer finds more than one server idle, the customer chooses a server without favoring any particular server.
  - For systems in statistical equilibrium, the average number of busy servers, L_s, is: L_s, $= \lambda\, E(s) = \lambda / m$.
  - The long-run average server utilization is:

$$\rho = \frac{L_s}{c} = \frac{\lambda}{c\mu}, \quad \text{where } \lambda < c\mu \text{ for stable systems}$$

Server Utilization and System Performance

- System performance varies widely for a given utilization ρ.
  - For example, a *D/D/1* queue where $E(A) = 1/\lambda$ and $E(S) = 1/\mu$, where:
    $L = \rho = \lambda/\mu, \quad w = E(S) = 1/\mu, \quad L_Q = W_Q = 0.$
    - By varying λ and μ, server utilization can assume any value between *0* and *1*.
    - Yet there is never any line.
  - In general, variability of interarrival and service times causes lines to fluctuate in length.

- Example: A physician who schedules patients every *10* minutes and spends S_i minutes with the i^{th} patient:

$$S_i = \begin{cases} 9 \text{ minutes with probability } 0.9 \\ 12 \text{ minutes with probability } 0.1 \end{cases}$$

  - Arrivals are deterministic, $A_1 = A_2 = \ldots = \lambda^{-1} = 10$.
  - Services are stochastic, $E(S_i) = 9.3$ min and $V(S_0) = 0.81$ min^2.

- On average, the physician's utilization = $\rho = \lambda/\mu = 0.93 < 1$.
- Consider the system is simulated with service times: $S_1 = 9, S_2 = 12, S_3 = 9, S_4 = 9, S_5 = 9$, …. The system becomes:
- The occurrence of a relatively long service time ($S_2 = 12$) causes a waiting line to form temporarily.

Costs in Queueing Problems

- Costs can be associated with various aspects of the waiting line or servers:
  - System incurs a cost for each customer in the queue, say at a rate of *$10* per hour per customer.
    - ➢ The average cost per customer is:

$$\sum_{j=1}^{N} \frac{\$10 * W_j^Q}{N} = \$10 * \hat{w}_Q$$

W_j^Q is the time customer j spends in queue

    - ➢ If $\hat{\lambda}$ customers per hour arrive (on average), the average cost per hour is:

$$\left(\hat{\lambda}\,\frac{\text{customer}}{\text{hour}}\right)\left(\frac{\$10 * \hat{w}_Q}{\text{customer}}\right) = \$10 * \hat{\lambda}\hat{w}_Q = \$10 * \hat{L}_Q \text{ / hour}$$

  - Server may also impose costs on the system, if a group of c parallel servers ($1 \le c \le \infty$) have utilization r, each server imposes a cost of *$5* per hour while busy.
    - ➢ The total server cost is: $\$5 * c\rho$.

Steady-State Behavior of Infinite-Population Markovian Models

- Markovian models: exponential-distribution arrival process (mean arrival rate = λ).
- Service times may be exponentially distributed as well (M) or arbitrary (G).
- A queueing system is in statistical equilibrium if the probability that the system is in a given state is not time dependent:

$$P(\,L(t) = n\,) = P_n(t) = P_n.$$

- Mathematical models in this chapter can be used to obtain approximate results even when the model assumptions do not strictly hold (as a rough guide).
- Simulation can be used for more refined analysis (more faithful representation for complex systems).
- For the simple model studied in this chapter, the steady-state parameter, L, the time-average number of customers in the system is:

$$L = \sum_{n=0}^{\infty} nP_n$$

  - Apply Little's equation to the whole system and to the queue alone:

$$w = \frac{L}{\lambda}, \quad w_Q = w - \frac{1}{\mu}$$

$$L_Q = \lambda w_Q$$

- $G/G/c/\infty/\infty$ example: to have a statistical equilibrium, a necessary and sufficient condition is $\lambda / (c\mu) < 1$.

M/G/1 Queues

- Single-server queues with Poisson arrivals & unlimited capacity.
- Suppose service times have mean $1/\mu$ and variance σ^2 and $r = \lambda/\mu < 1$, the steady-state parameters of $M/G/1$ queue:

$$\boxed{\begin{array}{l} \rho = \lambda/\mu, \quad P_0 = 1-\rho \\ L = \rho + \dfrac{\rho^2(1+\sigma^2\mu^2)}{2(1-\rho)}, \quad L_Q = \dfrac{\rho^2(1+\sigma^2\mu^2)}{2(1-\rho)} \\ w = \dfrac{1}{\mu} + \dfrac{\lambda(1/\mu^2+\sigma^2)}{2(1-\rho)}, \quad w_Q = \dfrac{\lambda(1/\mu^2+\sigma^2)}{2(1-\rho)} \end{array}}$$

  - No simple expression for the steady-state probabilities $P_0, P_1, \ldots$
  - $L - L_Q = \rho$ is the time-average number of customers being served.
  - Average length of queue, L_Q, can be rewritten as:

$$L_Q = \frac{\rho^2}{2(1-\rho)} + \frac{\lambda^2\sigma^2}{2(1-\rho)}$$

- If λ and μ are held constant, L_Q depends on the variability, σ^2, of the service times.
- Example: Two workers competing for a job, Able claims to be faster than Baker on average, but Baker claims to be more consistent,
  - Poisson arrivals at rate $\lambda = 2$ per hour ($1/30$ per minute).
  - Able: $1/\mu = 24$ minutes and $\sigma^2 = 20^2 = 400$ minutes^2:

$$L_Q = \frac{(1/30)^2[24^2+400]}{2(1-4/5)} = 2.711 \text{customers}$$

    - The proportion of arrivals who find Able idle and thus experience no delay is $P_0 = 1-\rho = 1/5 = 20\%$.
  - Baker: $1/\mu = 25$ *minutes* and $\sigma^2 = 2^2 = 4$ minutes^2:

$$L_Q = \frac{(1/30)^2[25^2+4]}{2(1-5/6)} = 2.097 \text{customers}$$

    - The proportion of arrivals who find Baker idle and thus experience no delay is $P_0 = 1-\rho = 1/6 = 16.7\%$.
  - Although working faster on average, Able's greater service variability results in an average queue length about *30%* greater than Baker's.

- Suppose the service times in an $M/G/1$ queue are exponentially distributed with mean $1/\mu$, then the variance is $\sigma^2 = 1/\mu^2$.

- *M/M/1* queue is a useful approximate model when service times have standard deviation approximately equal to their means.
- The steady-state parameters:

$$\rho = \lambda / \mu, \quad P_n = (1-\rho)\rho^n$$

$$L = \frac{\lambda}{\mu-\lambda} = \frac{\rho}{1-\rho}, \quad L_Q = \frac{\lambda^2}{\mu(\mu-\lambda)} = \frac{\rho^2}{1-\rho}$$

$$w = \frac{1}{\mu-\lambda} = \frac{1}{\mu(1-\rho)}, \quad w_Q = \frac{\lambda}{\mu(\mu-\lambda)} = \frac{\rho}{\mu(1-\rho)}$$

- Example: *M/M/1* queue with service rate $\mu=10$ customers per hour.
  - Consider how L and w increase as arrival rate, λ, increases from 5 to 8.64 by increments of 20%:
  - If $\lambda/\mu \geq 1$, waiting lines tend to continually grow in length.

λ	5.0	6.0	7.2	8.64	10.0
ρ	0.500	0.600	0.720	0.864	1.000
L	1.00	1.50	2.57	6.35	∞
w	0.20	0.25	0.36	0.73	∞

  - Increase in average system time (w) and average number in system (L) is highly nonlinear as a function of ρ.

Effect of Utilization and Service Variability

- For almost all queues, if lines are too long, they can be reduced by decreasing server utilization (ρ) or by decreasing the service time variability (σ^2).
- A measure of the variability of a distribution, coefficient of variation (cv):

$$(cv)^2 = \frac{V(X)}{[E(X)]^2}$$

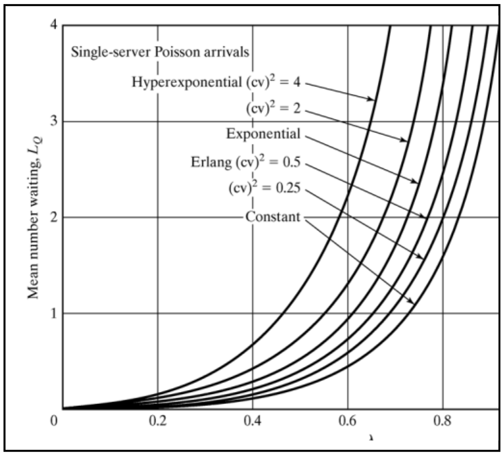
  - The larger cv is, the more variable is the distribution relative to its expected value

- Consider L_Q for any *M/G/1* queue:

$$L_Q = \frac{\rho^2(1+\sigma^2\mu^2)}{2(1-\rho)} = \left(\frac{\rho^2}{1-\rho}\right)\left(\frac{1+(cv)^2}{2}\right)$$

L_Q for *M/M/1* queue

Corrects the *M/M/1* formula to account for a non-exponential service time dist'n



Multiserver Queue

- $M/M/c/\infty/\infty$ queue: c channels operating in parallel.
  - Each channel has an independent and identical exponential service-time distribution, with mean $1/\mu$.
  - To achieve statistical equilibrium, the offered load (λ/μ) must satisfy $\lambda/\mu < c$, where $\lambda/(c\mu) = \rho$ is the server utilization.
  - Some of the steady-state probabilities:

$$\rho = \lambda / c\mu$$

$$P_0 = \left\{ \left[\sum_{n=0}^{c-1} \frac{(\lambda/\mu)^n}{n!} \right] + \left[\left(\frac{\lambda}{\mu}\right)^c \left(\frac{1}{c!}\right)\left(\frac{c\mu}{c\mu - \lambda}\right) \right] \right\}^{-1}$$

$$L = c\rho + \frac{(c\rho)^{c+1} P_0}{c(c!)(1-\rho)^2} = c\rho + \frac{\rho\, P(L(\infty) \geq c)}{1-\rho}$$

$$w = \frac{L}{\lambda}$$

- Other common multiserver queueing models:
  - $M/G/c/\infty$: general service times and c parallel server. The parameters can be approximated from those of the $M/M/c/\infty/\infty$ model.
  - $M/G/\infty$: general service times and infinite number of servers, e.g., customer is its own system, service capacity far exceeds service demand.
  - $M/M/C/N/\infty$: service times are exponentially distributed at rate m and c servers where the total system capacity is $N \geq c$ customer (when an arrival occurs and the system is full, that arrival is turned away).

Steady-State Behavior of Finite-Population Models

- When the calling population is small, the presence of one or more customers in the system has a strong effect on the distribution of future arrivals.
- Consider a finite-calling population model with K customers ($M/M/c/K/K$):
  - The time between the end of one service visit and the next call for service is exponentially distributed, (mean = $1/\lambda$).
  - Service times are also exponentially distributed.
  - c parallel servers and system capacity is K.
- Some of the steady-state probabilities:

$$P_0 = \left\{ \sum_{n=0}^{c-1} \binom{K}{n} \left(\frac{\lambda}{\mu}\right)^n + \sum_{n=c}^{K} \frac{K!}{(K-n)!\,c!\,c^{n-c}} \left(\frac{\lambda}{\mu}\right)^n \right\}^{-1}$$

$$P_n = \begin{cases} \binom{K}{n} \left(\frac{\lambda}{\mu}\right)^n P_0, & n = 0,1,\ldots,c-1 \\ \frac{K!}{(K-n)!\,c!\,c^{n-c}} \left(\frac{\lambda}{\mu}\right)^n, & n = c, c+1, \ldots K \end{cases}$$

$$L = \sum_{n=0}^{K} n P_n, \qquad w = L/\lambda_e, \qquad \rho = \lambda_e / c\mu$$

where λ_e is the long run effective arrival rate of customers to queue (or entering/exiting service)

$$\lambda_e = \sum_{n=0}^{K} (K-n)\lambda P_n$$

- Example: two workers who are responsible for 10 milling machines.
  - ➢ Machines run on the average for 20 minutes, then require an average 5-minute service period, both times exponentially distributed: $\lambda = 1/20$ and $\mu = 1/5$.
  - ➢ All of the performance measures depend on P_0:

$$P_0 = \left\{ \sum_{n=0}^{2-1} \binom{10}{n} \left(\frac{5}{20}\right)^n + \sum_{n=2}^{10} \frac{10!}{(10-n)!2!2^{n-2}} \left(\frac{5}{20}\right)^n \right\}^{-1} = 0.065$$

  - ○ Then, we can obtain the other P_n.
  - ○ Expected number of machines in system:

$$L = \sum_{n=0}^{10} nP_n = 3.17 \text{ machines}$$

  - ○ The average number of running machines:

$$K - L = 10 - 3.17 = 6.83 \text{ machines}$$

Networks of Queues

- ▪ Many systems are naturally modeled as networks of single queues: customers departing from one queue may be routed to another.
- ▪ The following results assume a stable system with infinite calling population and no limit on system capacity:
  - Provided that no customers are created or destroyed in the queue, then the departure rate out of a queue is the same as the arrival rate into the queue (over the long run).
  - If customers arrive to queue i at rate λ_i, and a fraction $0 \le p_{ij} \le 1$ of them are routed to queue j upon departure, then the arrival rate form queue i to queue j is $\lambda_i p_{ij}$ (over the long run).

    The overall arrival rate into queue j:

$$\lambda_j = a_j + \sum_{\text{all } i} \lambda_i p_{ij}$$

Arrival rate from outside the network

Sum of arrival rates from other queues in network

- If queue j has $c_i < \infty$ parallel servers, each working at rate μ_i, then the long-run utilization of each server is $\rho_i = \lambda_j / (c\mu_i)$ (where $\rho_i < 1$ for stable queue).
- If arrivals from outside the network form a Poisson process with rate a_j for each queue j, and if there are c_i identical servers delivering exponentially distributed service times with mean $1/\mu_i$, then, in steady state, queue j behaves likes an $M/M/c_j$ queue with arrival rate

$$\lambda_j = a_j + \sum_{\text{all } i} \lambda_i p_{ij}$$

- Discount store example:
- Suppose customers arrive at the rate 80 per hour and 40% choose self-service. Hence:
  - Arrival rate to service center 1 is $\lambda_1 = 80(0.4) = 32$ per hour
  - Arrival rate to service center 2 is $\lambda_2 = 80(0.6) = 48$ per hour.
  - c2 = 3 clerks and $\mu_2 = 20$ customers per hour.
  - The long-run utilization of the clerks is:

$$\rho_2 = 48/(3*20) = 0.8$$

  - All customers must see the cashier at service center 3, the overall rate to service center 3 is $\lambda_3 = \lambda_1 + \lambda_2 = 80$ per hour.
    - If $\mu_3 = 90$ per hour, then the utilization of the cashier is:

$$\rho_3 = 80/90 = 0.89$$

SYSTEM MODELLING AND SIMULATION

Module 3

Random-Number Generation: Properties of random numbers, Generation of pseudo-random numbers, Techniques for generating random numbers, Techniques for generating random numbers, Tests for Random Numbers, Random - Variate Generation, Inverse transform technique Acceptance-Rejection technique, Inverse transform technique Acceptance-Rejection technique.

RANDOM-NUMBER GENERATION

Random numbers are a necessary basic ingredient in the simulation of almost all discrete systems. Most computer languages have a subroutine, object, or function that will generate a random number. Similarly simulation languages generate random numbers that are used to generate event times and other random variables.

Properties of Random Numbers

- Two important statistical properties:
  - Uniformity
  - Independence.
- Random Number, R_i, must be independently drawn from a uniform distribution with pdf:

$$f(x) = \begin{cases} 1, & 0 \le x \le 1 \\ 0, & \text{otherwise} \end{cases}$$

$$E(R) = \int_0^1 x\,dx = \left.\frac{x^2}{2}\right|_0^1 = \frac{1}{2}$$

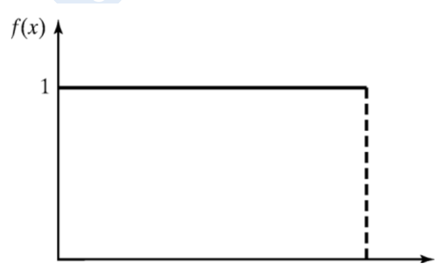


Figure: pdf for random numbers

Generation of Pseudo-Random Numbers

- "Pseudo", because generating numbers using a known method removes the potential for true randomness.
- Goal: To produce a sequence of numbers in [*0,1*] that simulates, or imitates, the ideal properties of random numbers (RN).
- Important considerations in RN routines:
  - Fast
  - Portable to different computers
  - Have sufficiently long cycle

- Replicable
- Closely approximate the ideal statistical properties of uniformity and independence.

Techniques for Generating Random Numbers

- Linear Congruential Method (LCM).
- Combined Linear Congruential Generators (CLCG).
- Random-Number Streams.

❖ **Linear Congruential Method**

- To produce a sequence of integers, $X_1, X_2, \ldots$ between 0 and $m-1$ by following a recursive relationship:

$$X_{i+1} = (\underbrace{a}_{\text{The multiplie}} X_i + \underbrace{c}_{\text{The increment}}) \bmod \underbrace{m}_{\text{The modulus}}, \quad i = 0,1,2,\ldots$$

- The selection of the values for a, c, m, and X_0 drastically affects the statistical properties and the cycle length.
- The random integers are being generated $[0,m-1]$, and to convert the integers to random numbers:

$$R_i = \frac{X_i}{m}, \quad i = 1,2,\ldots$$

Example

- Use $X_0 = 27$, $a = 17$, $c = 43$, and $m = 100$.
- The X_i and R_i values are:

$X_1 = (17*27+43) \bmod 100 = 502 \bmod 100 = 2$, $R_1 = 0.02$;
$X_2 = (17*2+32) \bmod 100 = 77$, $R_2 = 0.77$;
$X_3 = (17*77+32) \bmod 100 = 52$, $R_3 = 0.52$;
…

Characteristics of a Good Generator

- Maximum Density
  - ➢ Such that he values assumed by R_i, $i = 1,2,\ldots$, leave no large gaps on $[0,1]$
  - ➢ Problem: Instead of continuous, each R_i is discrete
  - ➢ Solution: a very large integer for modulus m
    - Approximation appears to be of little consequence
- Maximum Period
  - ➢ To achieve maximum density and avoid cycling.
  - ➢ Achieve by: proper choice of a, c, m, and X_0.
- Most digital computers use a binary representation of numbers
  - ➢ Speed and efficiency are aided by a modulus, m, to be (or close to) a power of 2.

❖ Combined Linear Congruential Generators

- Reason: Longer period generator is needed because of the increasing complexity of stimulated systems.
- Approach: Combine two or more multiplicative congruential generators.
- Let $X_{i,1}$, $X_{i,2}$, …, $X_{i,k}$, be the i^{th} output from k different multiplicative congruential generators.
  - ➢ The j^th generator:
    - Has prime modulus m_j and multiplier a_j and period is m_{j-1}
    - Produces integers $X_{i,j}$ is approx ~ Uniform on integers in $[1, m-1]$
    - $W_{i,j} = X_{i,j} -1$ is approx ~ Uniform on integers in $[1, m-2]$
  - ○ Suggested form:

$$X_i = \left(\sum_{j=1}^{k} (-1)^{j-1} X_{i,j} \right) \bmod m_1 - 1 \qquad \text{Hence, } R_i = \begin{cases} \dfrac{X_i}{m_1}, & X_i \succ 0 \\ \dfrac{m_1 - 1}{m_1}, & X_i = 0 \end{cases}$$

The coefficient: Performs the subtraction $X_{i\,1-1}$

- The maximum possible period is:

$$P = \frac{(m_1 - 1)(m_2 - 1)\ldots(m_k - 1)}{2^{k-1}}$$

- Example: For 32-bit computers, L'Ecuyer [1988] suggests combining $k = 2$ generators with $m_1 = 2,147,483,563$, $a_1 = 40,014$, $m_2 = 2,147,483,399$ and $a_2 = 20,692$. The algorithm becomes:

Step 1: Select seeds

- $X_{1,0}$ in the range $[1, 2,147,483,562]$ for the 1^st generator
- $X_{2,0}$ in the range $[1, 2,147,483,398]$ for the 2^nd generator.

Step 2: For each individual generator,

$X_{1,j+1} = 40,014\ X_{1,j}$ mod $2,147,483,563$

$X_{2,j+1} = 40,692\ X_{1,j}$ mod $2,147,483,399$.

Step 3: $X_{j+1} = (X_{1,j+1} - X_{2,j+1})$ mod $2,147,483,562$.

Step 4: Return

$$R_{j+1} = \begin{cases} \dfrac{X_{j+1}}{2,147,483563}, & X_{j+1} > 0 \\ \dfrac{2,147,483562}{2,147,483563}, & X_{j+1} = 0 \end{cases}$$

Step 5: Set $j = j+1$, go back to step 2.

- ➢ Combined generator has period: $(m_1 - 1)(m_2 - 1)/2 \sim 2 \times 10^{18}$

❖ **Random-Numbers Streams**

- The seed for a linear congruential random-number generator:
  - Is the integer value X_0 that initializes the random-number sequence.
  - Any value in the sequence can be used to “seed” the generator.
- A random-number stream:
  - Refers to a starting seed taken from the sequence $X_0, X_1, \ldots, X_P$.
  - If the streams are b values apart, then stream i could defined by starting seed:
  - Older generators: $b = 10^5$; Newer generators: $b = 10^{37}$.
- A single random-number generator with k streams can act like k distinct virtual random-number generators
- To compare two or more alternative systems.
  - Advantageous to dedicate portions of the pseudo-random number sequence to the same purpose in each of the simulated systems.

Tests for Random Numbers

- Two categories:
  - Testing for uniformity:

    H_0: $R_i \sim U[0,1]$

    H_1: $R_i \nsim U[0,1]$

    ➢ Failure to reject the null hypothesis, H_0, means that evidence of non-uniformity has not been detected.
  - Testing for independence:

    H_0: $R_i \sim$ independently

    H_1: $R_i \nsim$ independently

    ➢ Failure to reject the null hypothesis, H_0, means that evidence of dependence has not been detected.
- Level of significance a, the probability of rejecting H_0 when it is true:

  $$\alpha = P(\text{reject } H_0 | H_0 \text{ is true})$$
- When to use these tests:
  - If a well-known simulation languages or random-number generators is used, it is probably unnecessary to test
  - If the generator is not explicitly known or documented, e.g., spreadsheet programs, symbolic/numerical calculators, tests should be applied to many sample numbers.
- Types of tests:
  - Theoretical tests: evaluate the choices of m, a, and c without actually generating any numbers
  - Empirical tests: applied to actual sequences of numbers produced. Our emphasis.

Frequency Tests

- Test of uniformity
- Two different methods:
  - Kolmogorov-Smirnov test
  - Chi-square test

❖ **Kolmogorov-Smirnov Test**

- Compares the continuous cdf, $F(x)$, of the uniform distribution with the empirical cdf, $S_N(x)$, of the N sample observations.
  - We know: $F(x) = x, \quad 0 \le x \le 1$
  - If the sample from the RN generator is $R_1, R_2, \ldots, R_N$, then the empirical cdf, $S_N(x)$ is:

$$S_N(x) = \frac{\text{number of } R_1, R_2, \ldots, R_n \text{ which are} \le x}{N}$$

- Based on the statistic: $D = max| F(x) - S_N(x)|$
  - Sampling distribution of D is known (a function of N, tabulated in Table)
- A more powerful test, recommended.
- Example: Suppose 5 generated numbers are *0.44, 0.81, 0.14, 0.05, 0.93*.

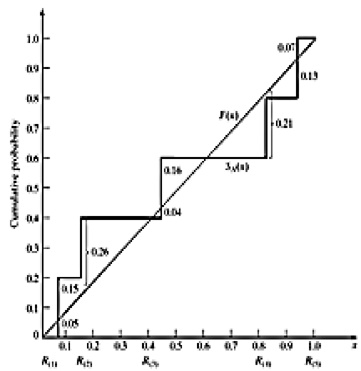
Step 1:	$R_{(i)}$	0.05	0.14	0.44	0.81	0.93	Arrange $R_{(i)}$ from smallest to largest
	i/N	0.20	0.40	0.60	0.80	1.00	
Step 2:	$i/N - R_{(i)}$	0.15	0.26	0.16	-	0.07	$D^+ = max\ \{i/N - R_{(i)}\}$
	$R_{(i)} - (i-1)/N$	0.05	-	0.04	0.21	0.13	$D^- = max\ \{R_{(i)} - (i-1)/N\}$

Step 3: $D = max(D^+, D^-) = 0.26$

Step 4: For $\alpha = 0.05$,

$D_\alpha = 0.565 > D$

Hence, H_0 is not rejected.



❖ **Chi-square test**

- Chi-square test uses the sample statistic:

$$X_0^2 = \sum_{i=1}^{n} \frac{(O_i - E_i)^2}{E_i}$$

n is the # of classes

E_i is the expected # in the i^{th} class

O_i is the observed # in the i^{th} class

- Approximately the chi-square distribution with *n-1* degrees of freedom (where the critical values are tabulated in Table A.6)
- For the uniform distribution, E_i, the expected number in the each class is:

$$E_i = \frac{N}{n}, \quad \text{where N is the total \# of observation}$$

- Valid only for large samples, e.g. N >= 50

Tests for Autocorrelation

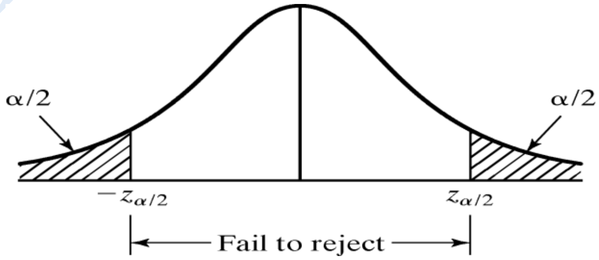
- Testing the autocorrelation between every m numbers (m is a.k.a. the lag), starting with the i^{th} number
  - The autocorrelation ρ_{im} between numbers: $R_i, R_{i+m}, R_{i+2m}, R_{i+(M+1)m}$
  - M is the largest integer such that $i+(M+1)m \leq N$
- Hypothesis: $$H_0: \rho_{im} = 0, \quad \text{if numbers are independent}$$ $$H_1: \rho_{im} \neq 0, \quad \text{if numbers are dependent}$$
- If the values are uncorrelated:
  - For large values of M, the distribution of the estimator of ρ_{im}, denoted is approximately normal.
- Test statistics is: $$Z_0 = \frac{\hat{\rho}_{im}}{\hat{\sigma}_{\hat{\rho}_{im}}}$$
  - Z_0 is distributed normally with mean = *0* and variance = *1*, and:

$$\hat{\rho}_{im} = \frac{1}{M+1}\left[\sum_{k=0}^{M} R_{i+km} R_{i+(k+1)m}\right] - 0.25$$

$$\hat{\sigma}_{\rho_{im}} = \frac{\sqrt{13M+7}}{12(M+1)}$$

- If $\rho_{im} > 0$, the subsequence has positive autocorrelation
  - High random numbers tend to be followed by high ones, and vice versa.
- If $\rho_{im} < 0$, the subsequence has negative autocorrelation
  - Low random numbers tend to be followed by high ones, and vice versa.

Normal Hypothesis Test



Example

- Test whether the 3rd, 8th, 13th, and so on, for the following output on P. 265.
  - Hence, $\alpha = 0.05$, $i = 3$, $m = 5$, $N = 30$, and M = 4

$$\hat{\rho}_{35} = \frac{1}{4+1}\begin{bmatrix}(0.23)(0.28)+(0.25)(0.33)+(0.33)(0.27)\\ +(0.28)(0.05)+(0.05)(0.36)\end{bmatrix} - 0.25$$

$$= -0.1945$$

$$\hat{\sigma}_{\rho_{35}} = \frac{\sqrt{13(4)+7}}{12(4+1)} = 0.128$$

$$Z_0 = -\frac{0.1945}{0.1280} = -1.516$$

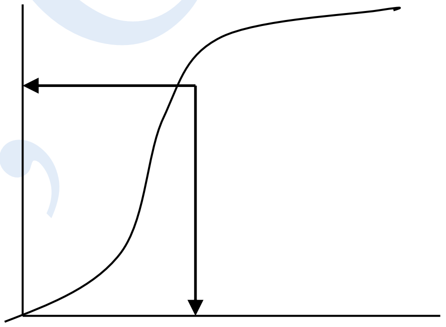
  - From Table A.3, $z_{0.025} = 1.96$. Hence, the hypothesis is not rejected.

Random-Variate Generation

- Illustrate some widely-used techniques for generating random variates.
  - Inverse-transform technique
  - Acceptance-rejection technique

❖ **Inverse-transform Technique**

- The concept:
  - For cdf function: r = F(x)
  - Generate r from uniform (0,1)
  - Find x:



r

Steps in inverse-transform technique

Step 1. Compute the cdf of the desired random variable X: $F(x) = 1 - e^{-\lambda x}$

$x \geq 0$

Step 2. Set F(X) = R on the range of X

Step 3. Solve the equation F(x) = R for X in terms of R.

$$1 - e^{-\lambda X} = R$$

$$e^{-\lambda X} = 1 - R$$

$$-\lambda X = \ln(1-R)$$

$$X = -\frac{1}{\lambda}\ln(1-R)$$

Step 4. Generate (as needed) uniform random numbers R1, R2, R3, . . . and compute the desired random variates

- Examples of other distributions for which inverse cdf works are:
    - Uniform distribution
      $X = a + (b - a)R$
    - Weibull distribution – time to failure – see steps on p278
      $X = \alpha\,[-\ln(1 - R)]^{1/\beta}$
    - Triangular distribution

$$X = \begin{cases} \sqrt{2R}, & 0 \le R \le 1/2 \\ 2 - \sqrt{2(1-R)}, & 1/2 < R \le 1 \end{cases}$$

❖ **Acceptance-Rejection technique**

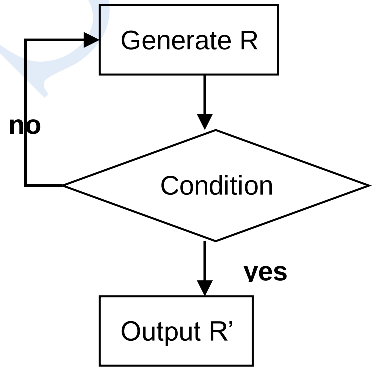
- Useful particularly when inverse cdf does not exist in closed form, a.k.a. thinning
- Illustration: To generate random variates, $X \sim U(1/4, 1)$

Procedures:

Step 1. Generate R ~ U[0,1]

Step 2a. If R >= ¼, accept X=R.

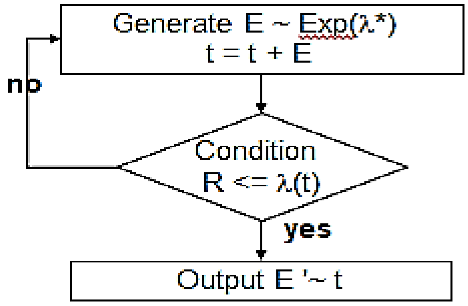
Step 2b. If R < ¼, reject R, return to Step 1



- *R* does not have the desired distribution, but *R* conditioned (*R'*) on the event $\{R \ge 1/4\}$ does
- Efficiency: Depends heavily on the ability to minimize the number of rejections.

NSPP

- Non-stationary Poisson Process (NSPP): a Possion arrival process with an arrival rate that varies with time
- Idea behind thinning:
    - Generate a stationary Poisson arrival process at the fastest rate, $\lambda^* = \max \lambda(t)$
    - But "accept" only a portion of arrivals, thinning out just enough to get the desired time-varying rate



- Poisson Distribution
    - Step 1 set n = 0, P =1
    - Step 2 generate a random number R_{n+1}

And replace P by P * R_{n+1}

- Step 3 if $P < e^{-\lambda}$, then accept, otherwise, reject the current n, increase n by 1 and return to step 2

- Example: Generate a random variate for a NSPP

Data: Arrival Rates

t (min)	*Mean Time Between Arrivals (min)*	*Arrival Rate $\lambda(t)$ (#/min)*
0	15	1/15
60	12	1/12
120	7	1/7
180	5	1/5
240	8	1/8
300	10	1/10
360	15	1/15
420	20	1/20
480	20	1/20

Procedures:

Step 1. $\lambda^* = \max \lambda(t) = 1/5$, $t = 0$ and $i = 1$.

Step 2. For random number $R = 0.2130$,

$E = -5\ln(0.213) = 13.13$

$t = 13.13$

Step 3. Generate $R = 0.8830$

$\lambda(13.13)/\lambda^*=(1/15)/(1/5)=1/3$

Since $R>1/3$, do not generate the arrival

Step 2. For random number $R = 0.5530$,

$E = -5\ln(0.553) = 2.96$

$t = 13.13 + 2.96 = 16.09$

Step 3. Generate $R = 0.0240$

$\lambda(16.09)/\lambda^*=(1/15)/(1/5)=1/3$

Since $R<1/3$, $T_1 = t = 16.09$,

and $i = i + 1 = 2$

Unit-5

Random Number Generation & Random Variate Generation

Problems :

(1) Generate a sequence of 5 integer Random number with $a=19$, $m=100$, $x_0=63$, $\boxed{c=1}$ (mixed Linear Congruential Method)

Solution : $\boxed{X_{i+1} = (a x_i + c) \bmod m,}$ $i = 0 \ldots m-1$

→ $i=0$ $X_1 = (19 \times 63 + 1) \bmod 100$

$= 1198 \bmod 100 = 98$

$i=1$ $\boxed{R_i = \frac{x_i}{m}, \quad i = 1 \ldots m}$

$R_1 = \frac{98}{100} = 0.98$

→ $i=1$ $X_2 = (19 \times 98 + 1) \bmod 100 = 63$

$i=2$ $R_2 = \frac{63}{100} = 0.63$

→ $i=2$ $X_3 = (19 \times 63 + 1) \bmod 100 = 98$

$i=3$ $R_3 = 98/100 = 0.98$

→ $i=3$ $X_4 = (19 \times 98 + 1) \bmod 100 = 63$

$i=4$ $R_4 = 63/100 = 0.63$

→ $i=4$ $X_5 = (19 \times 63 + 1) \bmod 100 = 98$

$i=5$ $R_5 = 98/100 = 0.98$

∴ Random numbers are 0.98, 0.63, 0.98, 0.63, 0.98

(2) Using Multiplicative Congruential Method, Generate sequence of 5 integer number where $a = 24$, $m = 100$, seed value $= 64$.

Solution:

Given
$a = 24$
$m = 100$
$x_0 = 64$
$c = 0$

$X_{i+1} = (aX_i + c) \bmod m, \quad i = 0 \dots m-1$

$R_i = \frac{X_i}{m}, \quad i = 1 \dots m$

$X_1 = (24 \times 64 + 0) \bmod 100 = 36$

$R_1 = 36/100 = \underline{\underline{0.36}}$

$X_2 = (24 \times 36 + 0) \bmod 100 = 64$

$R_2 = 64/100 = \underline{\underline{0.64}}$

$X_3 = (24 \times 64 + 0) \bmod 100 = 36$

$R_3 = 36/100 = \underline{\underline{0.36}}$

$X_4 = (24 \times 36 + 0) \bmod 100 = 64$

$R_4 = 64/100 = \underline{\underline{0.64}}$

$X_5 = (24 \times 64 + 0) \bmod 100 = 36$

$R_5 = 36/100 = \underline{\underline{0.36}}$

$\therefore$ Random numbers are 0.36, 0.64, 0.36, 0.64, 0.36

(3) Using **K.S Test** with $\alpha = 0.05$, to Test wether the data shown below are uniformly distributed 0.44, 0.81, 0.14, 0.05, 0.93 & also plot graph for $f(x)$.

Solution:

given, $\alpha = 0.05$
$N = 5$ (No. of random number given)

i	$R_{(i)}$	i/N	$\frac{i-1}{N}$	$D^+ = \max_{1\le i\le N}\{\frac{i}{N} - R_i\}$	$D^- = \max_{1\le i\le N}\{R_i - \frac{i-1}{N}\}$
1	0.05	0.20	0	0.15	0.05
2	0.14	0.40	0.20	$\boxed{0.26}$ max.	~
3	0.44	0.60	0.40	0.16	0.04
4	0.81	0.80	0.60	~	$\boxed{0.21}$ max.
5	0.93	1.00	0.80	0.07	0.13

$$D = \max(D^+, D^-)$$
$$= \max(0.26, 0.21) = \underline{0.26}$$

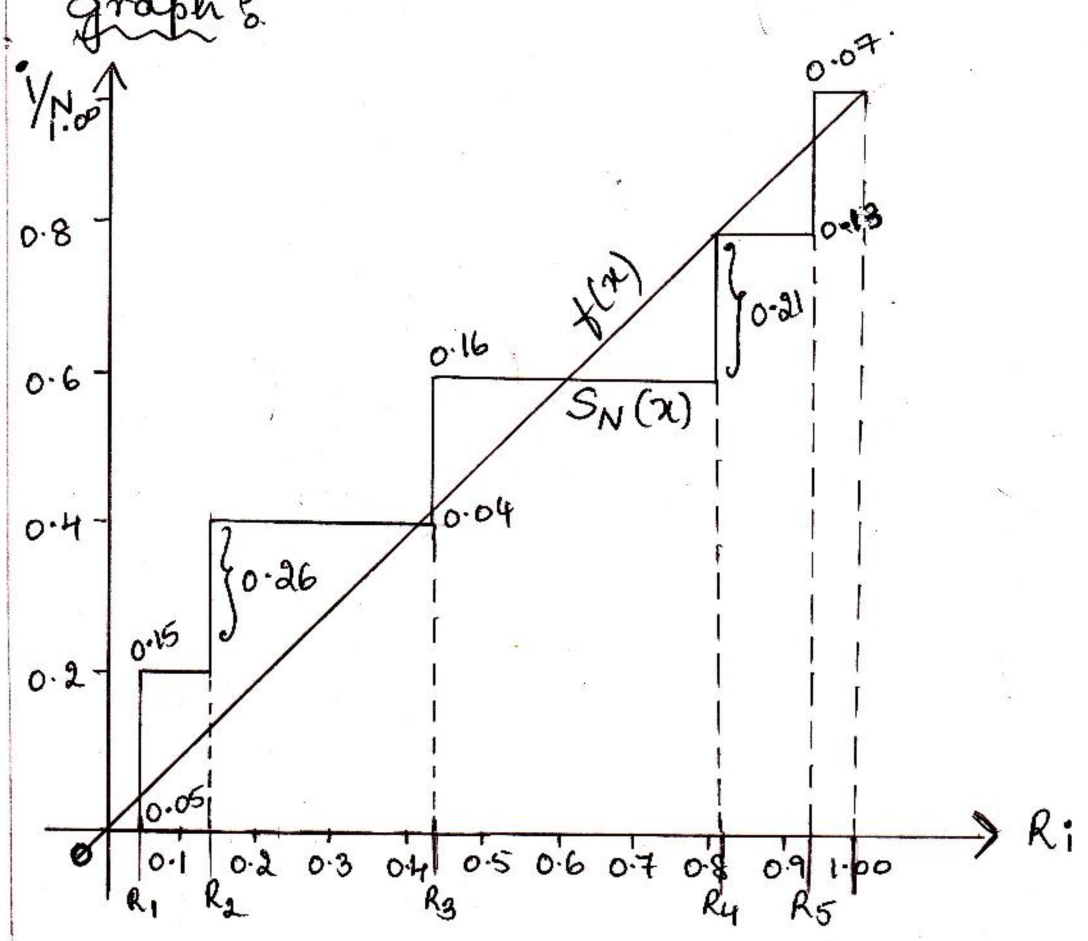
$D_{\alpha, N} \Rightarrow D_{0.05, 5} = 0.565$, from table A8

$\therefore D \le D_\alpha \Rightarrow 0.26 \le 0.565$

$\therefore$ Accepted Null hypothesis

hence Random Numbers are uniformly distributed.

graph :



(4) Using K.S Test with $\alpha = 0.05$, & sample data are given below also plot graph for $f(x)$.
0.46, 0.18, 0.23, 0.64, 0.36, 0.44

Solution : given $\alpha = 0.05$, $N = 6$.

i	R_i	i/N	$\frac{i-1}{N}$	$D^+ = \max\limits_{1\le i\le N} \{ i/N - R_i \}$	$D^- = \max\limits_{1\le i\le N} \{ R_i - \frac{i-1}{N} \}$
1	0.18	0.16	0	~	$\boxed{0.18}$ max
2	0.23	0.33	0.16	0.1	0.07
3	0.36	0.5	0.33	0.14	0.03
4	0.44	0.66	0.5	0.22	~
5	0.46	0.83	0.66	$\boxed{0.37}$ max	~
6	0.64	1.00	0.83	0.36	~

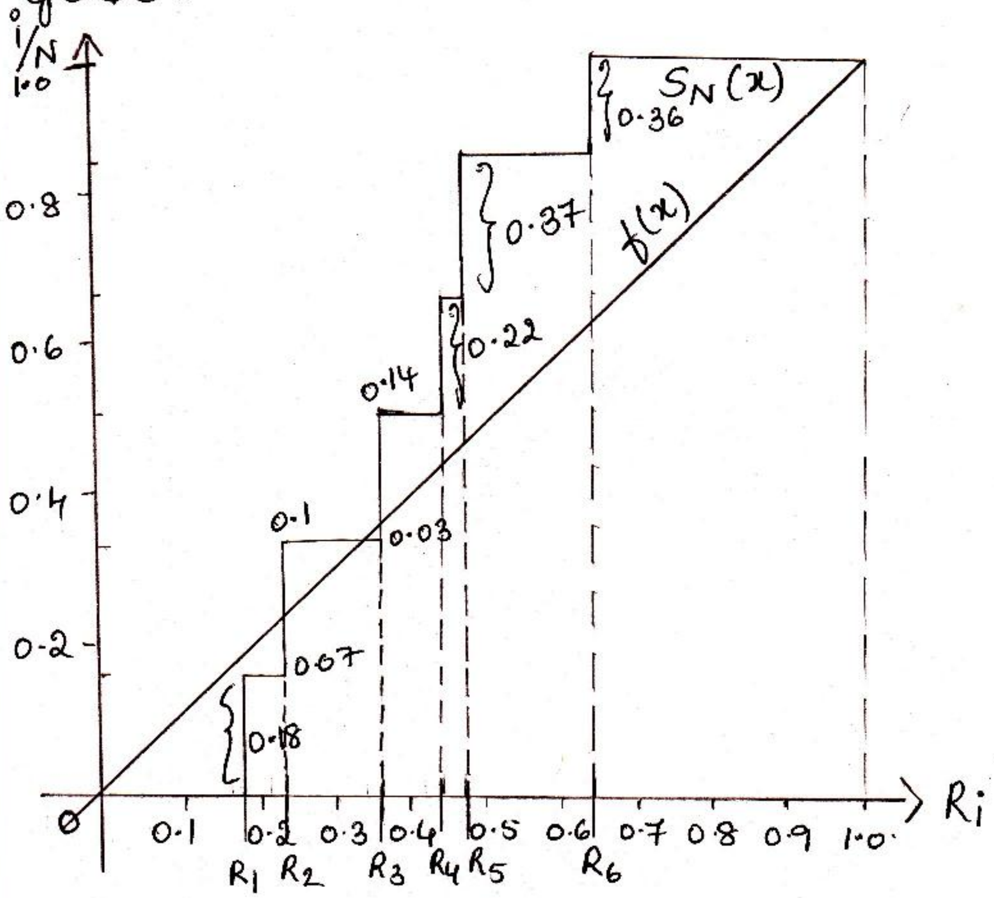
$\therefore D = \max(D^+, D^-) \Rightarrow \max(0.37, 0.18) = 0.37$

from table A8, $D_{\alpha}, N = D_{0.05}, 6 = 0.521$

$\therefore D \le D_{\alpha} = 0.37 \le 0.521$

$\therefore$ Accepted Null hypothesis
hence random numbers are uniformly distributed.

Graph :



⑤ Using chi-square Test with level of significance $\alpha = 0.05$, Test wether given data are uniformly distributed. with test uses 10 interval of equal length. total number of sample data are 100.

$\therefore$ $\therefore$ VVI

Interval	1	2	3	4	5	6	7	8	9	10
Observation	8	8	10	9	12	8	10	14	10	11

Solution :

given, $\alpha = 0.05$, $n = 10$, $N = 100$

$$E_i = \frac{N}{n} = \frac{100}{10} = 10$$

interval	O_i	$O_i - E_i$	$(O_i - E_i)^2$	E_i	$X_0^2 = \sum \frac{(O_i - E_i)^2}{E_i}$
1	8	-2	4	10	0.4
2	8	-2	4	10	0.4
3	10	0	0	10	0
4	9	-1	1	10	0.1
5	12	2	4	10	0.4
6	8	-2	4	10	0.4
7	10	0	0	10	0
8	14	4	16	10	1.6
9	10	0	0	10	0
10	11	1	1	10	0.1
Sum	100				3.4

$\therefore X_0^2 = 3.4$

from table A6, $X^2_{\alpha, n-1} = X^2_{0.05, 9} = 16.9$

$\therefore X_0^2 \leq X^2_{\alpha, n-1} = 3.4 \leq 16.9$

$\therefore$ Accepted null hypothesis

⑥ use chi-square Test with $\alpha = 0.05$ where $n = 10$, intervals of equal length. sample data are given below :

0.34, 0.90, 0.25, 0.89, 0.87, 0.44, 0.12, 0.21, 0.46, 0.67,
0.83, 0.76, 0.79, 0.64, 0.70, 0.81, 0.94, 0.74, 0.22, 0.74,
0.96, 0.99, 0.77, 0.67, 0.56, 0.41, 0.52, 0.73, 0.99, 0.02
0.47, 0.30, 0.17, 0.82, 0.56, 0.05, 0.45, 0.31, 0.78, 0.05
0.79, 0.71, 0.23, 0.19, 0.82, 0.93, 0.65, 0.37, 0.39, 0.4
0.10, 0.17, 0.10, 0.46, 0.05, 0.66, 0.10, 0.42, 0.18, 0.49,
0.37, 0.51, 0.54, 0.01, 0.81, 0.28, 0.69, 0.34, 0.75, 0.49,
0.72, 0.43, 0.56, 0.97, 0.30, 0.94, 0.96, 0.58, 0.73, 0.05,
0.06, 0.39, 0.84, 0.24, 0.40, 0.64, 0.40, 0.19, 0.79, 0.62,
0.18, 0.26, 0.97, 0.88, 0.64, 0.47, 0.60, 0.11, 0.29, 0.78

Solution : given, $\alpha = 0.05$, $n = 10$, $N = 100$

$E_i = N/n = 100/10 = 10$

interval	O_i	E_i	$O_i - E_i$	$(O_i - E_i)^2$	$X_0^2 = \sum \frac{(O_i - E_i)^2}{E_i}$
0.01-0.10	10	10	0	0	0
0.11-20	8	10	-2	4	0.4
0.21-30	10	10	0	0	0
0.31-40	9	10	-1	1	0.1
0.41-50	12	10	2	4	0.4
0.51-60	8	10	-2	4	0.4
0.61-70	10	10	0	0	0
0.71-80	14	10	4	16	1.6
0.81-90	10	10	0	0	0
0.91-1.00	9	10	-1	1	0.1

$X_0^2 = 3$

from Table A6, $X^2_{\alpha, n-1} = X^2_{0.05, 9} = 16.9$

$\therefore X_0^2 \leq X^2_{\alpha, n-1}$

$3 \leq 16.9$

$\therefore$ Accepted Null hypothesis

---

(7) Using Auto corelation Test to test wether numbers are uniformely distributed with starting period 3rd, 8th, 13th & so on and largest integer number is 4. $Z_{\alpha/2} = 1.96$. the sample data are given below:

0.12, 0.01, 0.23, 0.28, 0.89, 0.31, 0.64, 0.28, 0.83, 0.93, 0.99, 0.15, 0.33, 0.35, 0.91, 0.41, 0.60, 0.27, 0.75, 0.88, 0.68, 0.49, 0.05, 0.43, 0.95, 0.58, 0.19, 0.36, 0.69, 0.87.

Solution: given, $i = 3$ (period starts from 3rd)

$m = 5$ (difference b/w periods i.e 8-3, 13-8 ...)

$M = 4$ (largest number)

$Z_{\alpha/2} = 1.96$

$$\boxed{\hat{P}_{im} = \frac{1}{M+1}\left[\sum_{K=0}^{M} R_{i+Km} R_{i+(K+1)m}\right] - 0.25}$$

$$= \frac{1}{4+1}\left[\begin{matrix}(0.23)(0.28) + (0.28)(0.33) + (0.33)(0.27) + \\ (0.27)(0.05) + (0.05)(0.36)\end{matrix}\right] - 0.25$$

$$= -0.1945$$

$$\boxed{\sigma_{\hat{P}_{iM}} = \frac{\sqrt{13M+7}}{12(M+1)}}$$

$$= \frac{\sqrt{13\times4+7}}{12(\cancel{4}+1)} = 0.1280$$

$$\boxed{Z_0 = \frac{\hat{P}_{iM}}{\sigma_{\hat{P}_{iM}}}} = \frac{-0.1945}{0.1280} = -1.5196$$

$\therefore \quad -Z_{\alpha/2} \leq Z_0 \leq Z_{\alpha/2}$

$-1.96 \leq -1.5196 \leq 1.96$

$\therefore$ Accepted Null Hypothesis

(8). Generate 5 Random Variation from sample data 0.30, 0.20, 0.10, 0.50, 0.60 using Exponential distribution with mean = 1

% Imp.

Solution : given $\lambda = 1$

i	1	2	3	4	5
R_i	0.30	0.20	0.10	0.50	0.60
X_i	0.3566	0.2231	0.1053	0.6931	0.9162

$$\boxed{X_i = -\frac{1}{\lambda} \ln(1-R_i)}$$

$$X_1 = -\frac{1}{1} \ln(1-0.30) = 0.3566$$

$$X_2 = \frac{-1}{} \ln(1-0.20) = 0.2231$$

$$X_3 = \frac{-1}{1} \ln(1-0.10) = \underline{0.1053}$$

$$X_4 = \frac{-1}{1} \ln(1-0.50) = \underline{0.6931}$$

$$X_5 = \frac{-1}{1} \ln(1-0.60) = \underline{0.9162}$$

---

(9) Generate 5 Random Variation using Uniform Distribution technique with interval $0.3 \le x \le 2$.
consider sample data, 0.30, 0.25, 0.80, 0.75, 2.5

Solution: given, $a = 0.3$ $b = 2$

$$\boxed{X_i = a + (b-a) R_i \quad , i = 1 \ldots n}$$

i	1	2	3	4	5
R_i	0.30	0.25	0.80	0.75	2.5
X_i	0.81	0	1.66	1.575	1

$X_1 = 0.3 + (2-0.3)\,0.3 = \underline{0.81}$

$X_2 = 0.25 < 0.3 = \underline{0}$

$X_3 = 0.3 \le 0.8 \le 2 \Rightarrow 0.3 + (2-0.3)\,0.80 = \underline{1.66}$

$X_4 = 0.3 \le 0.75 \le 2 \Rightarrow 0.3 + (2-0.3)\,0.75 = \underline{1.575}$

$X_5 = 2.5 > 2 \Rightarrow \underline{1}$

(10) Generate 6 random variation using Weibull Distribution with slope = 10 & shape = 2. the sample data are 0.10, 0.60, 0.50, 0.80, 0.20, 0.45

Solution: given $\alpha = 10$ (slope)
$\beta = 2$ (shape)

$$\boxed{X_i = \alpha\left[-\ln(1-R_i)\right]^{1/\beta}}$$

i	1	2	3	4	5	6.
R_i	0.10	0.60	0.50	0.80	0.20	0.45
X_i	3.24	9.57	8.32	12.68	4.72	7.73

$$X_1 = 10 \times \left[-\ln(1-0.10)\right]^{1/2} = 3.24$$

$$X_2 = 10\left[-\ln(1-0.60)\right]^{1/2} = 9.57$$

$$X_3 = 10\left[-\ln(1-0.50)\right]^{1/2} = 8.32$$

$$X_4 = 10\left[-\ln(1-0.80)\right]^{1/2} = 12.68$$

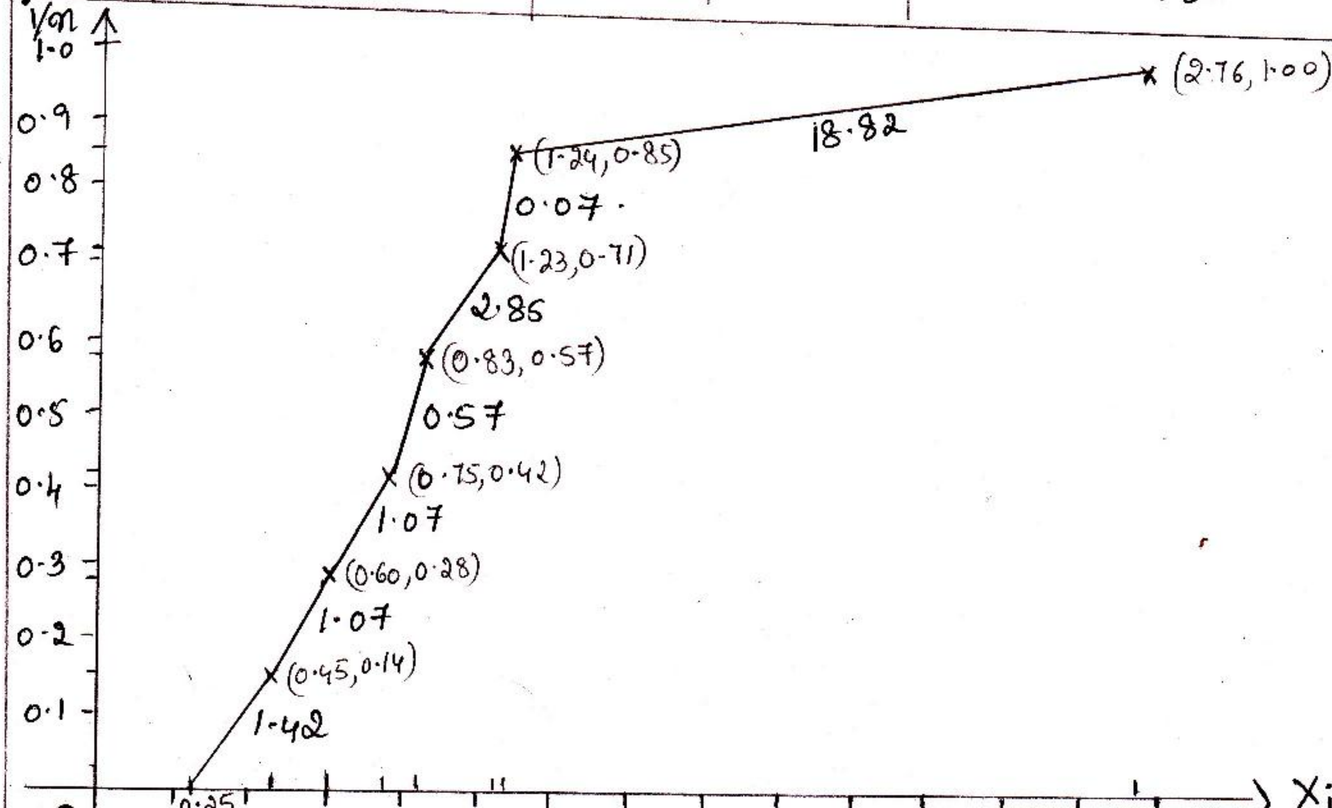
$$X_5 = 10\left[-\ln(1-0.20)\right]^{1/2} = 4.72$$

$$X_6 = 10\left[-\ln(1-0.45)\right]^{1/2} = 7.73$$

(11) consider the data : 0.83, 0.45, 1.23, 0.60, 0.75, 2.76, 1.24 with the probability = $1/n$ where $x_0 = 0.25$. Find slope of ith line segment using Emphirical continuous distribution method. (without frequency)

Solution:

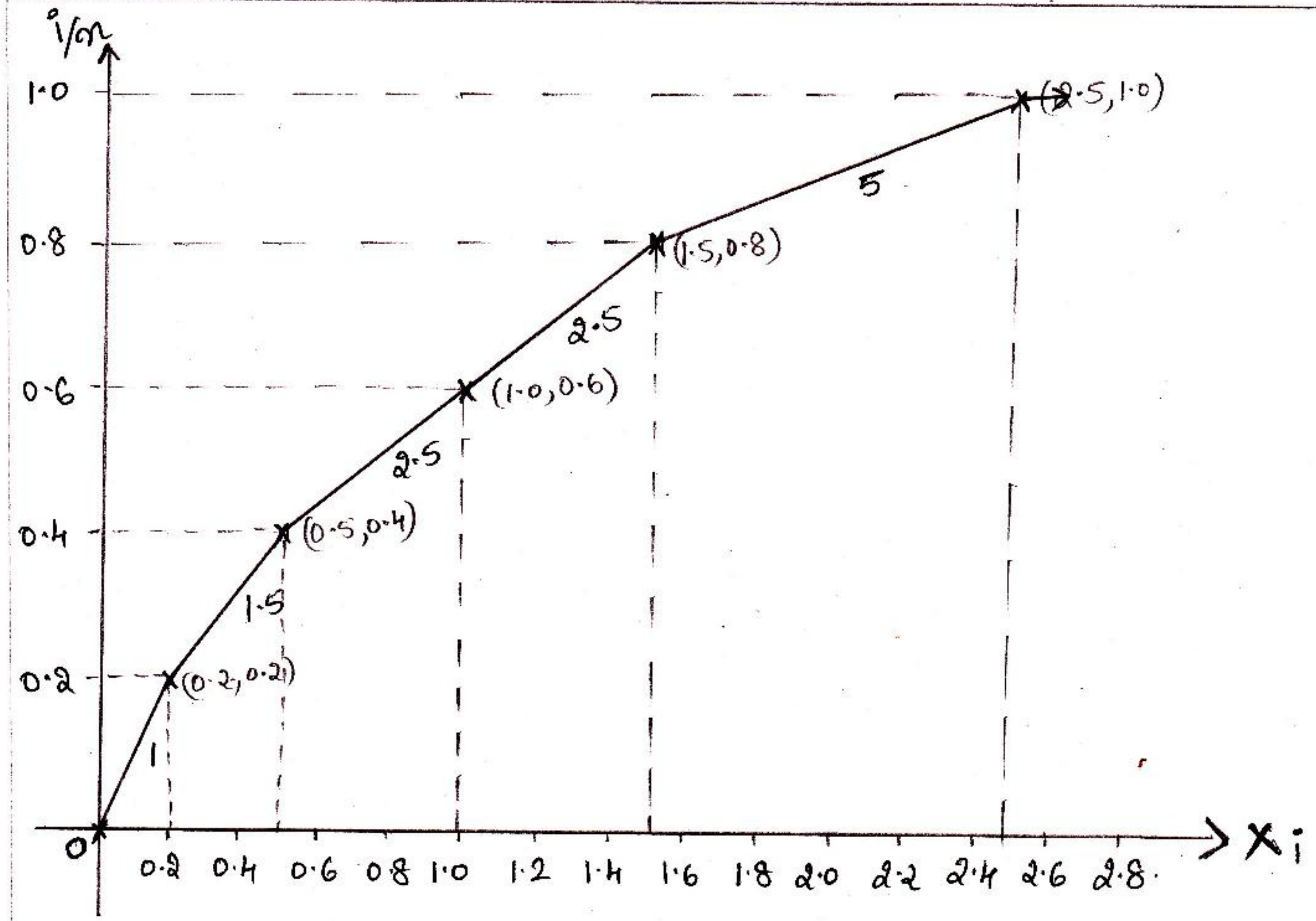
i	$x_{i-1} \le x \le x_i$	$1/n$	i/n	$a_i = \frac{x_i - x_{i-1}}{1/n}$
1	$0.25 \le x \le 0.45$	0.14	0.14	$a_i = \frac{0.45-0.25}{0.14} = 1.42$
2	$0.45 \le x \le 0.60$	0.14	0.28	1.07
3	$0.60 \le x \le 0.75$	0.14	0.42	1.07
4	$0.75 \le x \le 0.83$	0.14	0.57	0.57
5	$0.83 \le x \le 1.23$	0.14	0.71	2.85
6	$1.23 \le x \le 1.24$	0.14	0.85	0.07
7	$1.24 \le x \le 2.76$	0.14	1.00	18.82



(12) Consider the data 1.0, 0.5, 0.20, 1.5, 2.5 & Frequency are 31, 10, 25, 24, 30. Find the slope of ith line segment using Empirical Continuous Distribution (with frequency)

Solution: Consider $x_0 = 0.0$ & $e_0 = 0.0$

i	$x_{i-1} \leq x \leq x_i$	Frequency	Relative Frequency	Cumulative Probability (i/n)	$a_i = \frac{x_i - x_{i-1}}{e_i - e_{i-1}}$
1	$0.0 \leq x \leq 0.2$	25	0.25	0.2	$a_i = \frac{0.2-0.0}{0.2-0.0} = 1$
2	$0.2 \leq x \leq 0.5$	10	0.10	0.4	$\frac{0.5-0.2}{0.4-0.2} = 1.5$
3	$0.5 \leq x \leq 1.0$	31	0.31	0.6	$\frac{1.0-0.5}{0.6-0.4} = 2.5$
4	$1.0 \leq x \leq 1.5$	24	0.24	0.8	$\frac{1.5-1.0}{0.8-0.6} = 2.5$
5	$1.5 \leq x \leq 2.5$	30	0.30	1.00	$\frac{2.5-1.5}{1.0-0.8} = 5$



13 10M * Generate 3 poisson variate with mean 0.2. consider random numbers 0.4357, 0.4146, 0.8353, 0.9952, 0.8004.
vvimp

Solution : Steps : (2marks)

Step 0: Set $n=0$, $p=1$

2: $R_1 = 0.4357$ $P = P \cdot R_1 = 1 \times 0.4357 = 0.4357$

$e^{-\alpha} = e^{-0.2} = 0.8187$

3: $P < e^{-\alpha} = 0.4357 < 0.8187$ $\boxed{\therefore \text{ Accept } N=0}$

Step 1: $n=0$, $p=1$

2: $R_2 = 0.4146$ $P = P \cdot R_2 = 1 \times 0.4146 = 0.4146$

3: $0.4146 < 0.8187$ $\boxed{\therefore \text{ Accept } N=0}$

Step 1: $n=0$, $p=1$

2: $R_3 = 0.8353$ $p = 1 \times 0.8353 = 0.8353$

3: $0.8353 < 0.8187$ $\boxed{\therefore \text{ Reject } n=1}$ $p = 0.8353$.

$n=1$, $p=0.8353$

Step 2: $R_4 = 0.9952$ $p = P \cdot R_4 = 0.8353 \times 0.9952 = 0.8312$

3: $0.8312 < 0.8187$ $\boxed{\therefore \text{ Reject } n=2}$

$n=2$, $p=0.8312$

Step 2: $R_5 = 0.8004$ $p = 0.8312 \times 0.8004 = 0.6652$

3: $0.6652 < 0.8187$ $\boxed{\therefore \text{ Accept } N=2}$

* table: (8 marks)

n	P	R_{n+1}	$P = P \cdot R_{n+1}$	$P < e^{-\alpha}$ A/R	$N = n$.
0	1	0.4357	0.4357	$0.4357 < 0.8187$ Accept	$N=0$
0	1	0.4146	0.4146	$0.4146 < 0.8187$ Accept	$N=0$
0	1	0.8353	0.8353	$0.8353 < 0.8187$ Reject	–
1	0.8353	0.9952	0.8312	$0.8312 < 0.8187$ Reject	–
2	0.8312	0.8004	0.6652	$0.6652 < 0.8187$ Accept	$N=2$

∴ the 3 poisson variates are $N=0, N=0, N=2$.

(14) Generate 5 poisson variate with mean $= 0.2$

Solution: Steps:

$e^{-\alpha} = e^{-0.2} = 0.8187.$

$n = 5$

*) If random numbers are not given, then the random numbers are selected so that it satisfies following conditions: If x is random number then,

$1/4 \leq x \leq 1$ & $x \leq e^{-\alpha}$ then accepted, else reject.

Step 1: $n=0$, $p=1$

2: $R_1 = 0.3568$ $\quad p = p \times R_1 = 1 \times 0.3568 = 0.3568$

3: $0.3568 \leq 0.8187$ & $0.25 \leq 0.3568 \leq 1$ $\boxed{\therefore \text{Accept } N=0}$

Step 1: $n=0$ $\quad p=1$

2: $R_2 = 0.4123$ $\quad p = 1 \times 0.4123 = 0.4123$

3: $0.4123 \leq 0.8187$ & $0.25 \leq 0.4123 \leq 1$ $\boxed{\therefore \text{Accept } N=0}$

Step 1: $n=0$, $p=1$

2: $R_3 = 0.5067$ $\quad p = 1 \times 0.5067 = 0.5067$.

3: $0.5067 \leq 0.8187$ & $0.25 \leq 0.5067 \leq 1$ $\boxed{\therefore \text{Accept } N=0}$

$R_4 = 0.6818$ $\quad p = 1 \times 0.6818 = 0.6818$

$0.6818 \leq 0.8187$ & $0.25 \leq 0.6818 \leq 1$ $\boxed{\therefore \text{Accept } N=0}$

Step 1: $n=0$, $p=1$

2: $R_5 = 0.7293$ $\quad p = 1 \times 0.7293 = 0.7293$

3: $0.7293 \leq 0.8187$ & $0.25 \leq 0.7293 \leq 1$ $\boxed{\therefore \text{Accept } N=0}$

Table :

n	P	R_{n+1}	$P = P \cdot R_{n+1}$	$P \leq e^{-\alpha}$ A / R	$N = n$.
0	1	0.3568	0.3568	$0.3568 \leq 0.8187$ Accept	N=0
0	1	0.4123	0.4123	$0.4123 \leq 0.8187$ Accept	N=0
0	1	0.5067	0.5067	$0.5067 \leq 0.8187$ Accept	N=0
0	1	0.6818	0.6818	$0.6818 \leq 0.8187$ Accept	N=0
0	1	0.7293	0.7293	$0.7293 \leq 0.8187$ Accept	N=0

$\therefore$ The 5 poisson variates are N=0, N=0, N=0, N=0, N=0

(15) Generate 5 gamma variate using Gamma distribution with slope, $\beta = 2.30$ & mean, $\theta = 0.4545$.
Consider the random numbers : 0.4357, 0.1806, 0.1508, 0.8353, 0.1202, 0.8004, 0.9550, 0.1460, 0.196, 0.234.

Solution :

given $\beta = 2.30$, $\theta = 0.4545$

Formula :

$$a = (2\beta - 1)^{1/2}$$

$$b = 2\beta - \ln 4 + 1/a$$

$$X = \beta\left[R_1/(1-R_1)\right]^a$$

$X \le b - \ln(R_1^2 \cdot R_2)$ then Accept & $X = X/\beta\theta$ else reject.

Steps :

1: $a = (2 \times 2.3 - 1)^{1/2} = \underline{\underline{1.8973}}$

$b = 2 \times 2.3 - \ln 4 + \frac{1}{1.8973} = \underline{\underline{3.7407}}$

2: $R_1 = 0.4357 \quad R_2 = 0.1806$

3: $X = 2.30\left[0.4357/(1-0.4357)\right]^{1.8973} = \underline{\underline{1.4080}}$

4: $1.4080 \le 3.7407 - \ln(0.4357^2 \times 0.1806) = \underline{\underline{7.1137}}$

$\boxed{\therefore \text{Accept}}$ hence $X = 1.4080/2.30 \times 0.4545 = \boxed{1.3469}$

1: $a = 1.8973 \quad b = 3.7407$

2: $R_1 = 0.1508 \quad R_2 = 0.8353$

3: $X = 2.30\left[0.1508/(1-0.1508)\right]^{1.8973} = \underline{\underline{0.0866}}$

4: $0.0866 \le 3.7407 - \ln(0.1508^2 \times 0.8353)$

$0.0866 \le 7.7042$ $\boxed{\therefore \text{Accept}}$

$X = 0.0866/2.3 \times 0.4545 = \boxed{0.0828}$

1: $a = 1.8973$, $b = 3.7407$

2: $R_1 = 0.1202$ $R_2 = 0.8004$

3: $X = 2.3\left[0.1202/(1-0.1202)\right]^{1.8973} = 0.0526$

4: $0.0526 \leq 8.2005$ $\boxed{\therefore \text{Accept}}$

$X = 0.0526/2.3\times0.4545 = \boxed{0.0503}$

Step 1: $a = 1.8973$ $b = 3.7407$.

2: $R_1 = 0.9550$ $R_2 = 0.1460$

3: $X = 756.9167$

4: $756.9167 \nleq 5.7569$ $\boxed{\therefore \text{Reject}}$

Step 2: $R_1 = 0.1960$ $R_2 = 0.2340$

3: $X = 0.1580$

4: $0.1580 \leq 8.4524$ $\boxed{\therefore \text{Accept}}$

$X = 0.1580/2.3\times0.4545 = \boxed{0.1511}$

table :

a	b	R_1 & R_2	X	$X < b - \ln(R_1^2 R_2)$ A / R	$X = X/\beta\theta$
1.8973	3.7407	$0.4357 = R_1$ $0.1806 = R_2$	1.4080	$1.4080 \leq 7.1137$ Accept	1.3469.
1.8973	3.7407	$R_1 = 0.1508$ $R_2 = 0.8353$	0.0866	$0.0866 \leq 7.7042$ Accept	0.0828
1.8973	3.7407	$R_1 = 0.1202$ $R_2 = 0.8004$	0.0526	$0.0526 \leq 8.2005$ Accept	0.0503
1.8973	3.7407	$R_1 = 0.9550$ $R_2 = 0.1460$	756.916	$756.916 \nleq 5.7569$ Reject	-
-	-	$R_1 = 0.196$ $R_2 = 0.234$	0.1580	$0.1580 \leq 8.4524$ Accept	0.1511

$\therefore$ The Gamma Variates are 1.34, 0.082, 0.0503, 0.1511

SYSTEM MODELLING AND SIMULATION

Module 4

Input Modeling: Data Collection; Identifying the distribution with data, Parameter estimation. Goodness of Fit Tests. Goodness of Fit Tests. Fitting a non-stationary Poisson process, selecting input models without data. Fitting a non-stationary Poisson process, selecting input models without data. Multivariate and Time-Series input models. Estimation of Absolute Performance: Types of simulations with respect to output analysis. Stochastic nature of output data, Measures of performance and their estimation.

Input Modeling

- Input models provide the driving force for a simulation model.
- The quality of the output is no better than the quality of inputs.
- In this chapter, we will discuss the 4 steps of input model development:
  - Collect data from the real system
  - Identify a probability distribution to represent the input process
  - Choose parameters for the distribution
  - Evaluate the chosen distribution and parameters for goodness of fit.

Data Collection

- One of the biggest tasks in solving a real problem. GIGO – garbage-in-garbage-out
- Suggestions that may enhance and facilitate data collection:
  - Plan ahead: begin by a practice or pre-observing session, watch for unusual circumstances
  - Analyze the data as it is being collected: check adequacy
  - Combine homogeneous data sets, e.g. successive time periods, during the same time period on successive days
  - Be aware of data censoring: the quantity is not observed in its entirety, danger of leaving out long process times
  - Check for relationship between variables, e.g. build scatter diagram
  - Check for autocorrelation
  - Collect input data, not performance data

Input Data Examples

- Queueing Systems
  - Interarrival time
  - Service time
- Inventory Systems

- Demand
- Lead time

▪ Reliability Systems
- Time to failure

Identifying the Distribution

1. Histograms
2. Selecting families of distribution
3. Parameter estimation
4. Goodness-of-fit tests

❖ Histograms

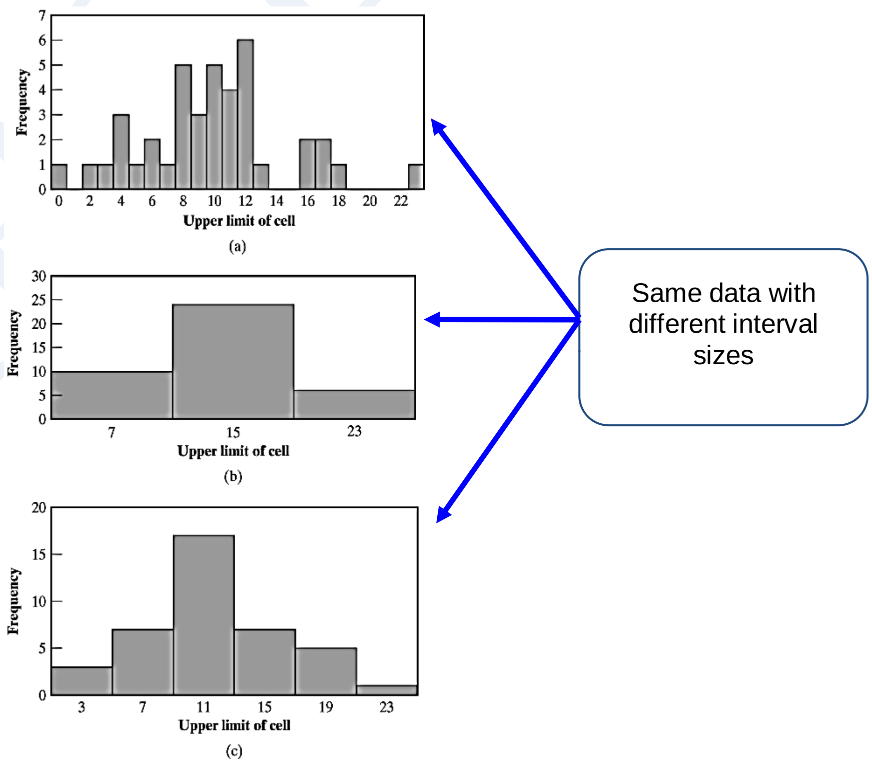
▪ A frequency distribution or histogram is useful in determining the shape of a distribution

▪ The number of class intervals depends on:
- The number of observations
- The dispersion of the data
- Suggested: the square root of the sample size

▪ For continuous data:
- Corresponds to the probability density function of a theoretical distribution

▪ For discrete data:
- Corresponds to the probability mass function

▪ If few data points are available: combine adjacent cells to eliminate the ragged appearance of the histogram



❖ Selecting the Family of Distributions

- A family of distributions is selected based on:
  - The context of the input variable
  - Shape of the histogram
- Frequently encountered distributions:
  - Easier to analyze: exponential, normal and Poisson
  - Harder to analyze: beta, gamma and Weibull
- Use the physical basis of the distribution as a guide, for example:
  - Binomial: # of successes in *n* trials
  - Poisson: # of independent events that occur in a fixed amount of time or space
  - Normal: distribution of a process that is the sum of a number of component processes
  - Exponential: time between independent events, or a process time that is memoryless
  - Weibull: time to failure for components
  - Discrete or continuous uniform: models complete uncertainty
  - Triangular: a process for which only the minimum, most likely, and maximum values are known
  - Empirical: resamples from the actual data collected
- Remember the physical characteristics of the process
  - Is the process naturally discrete or continuous valued?
  - Is it bounded?
- No "true" distribution for any stochastic input process
- Goal: obtain a good approximation

❖ Quantile-Quantile Plots

- Q-Q plot is a useful tool for evaluating distribution fit
  - a subjective method
- If X is a random variable with cdf F, then the q-quantile of X is the γ such that

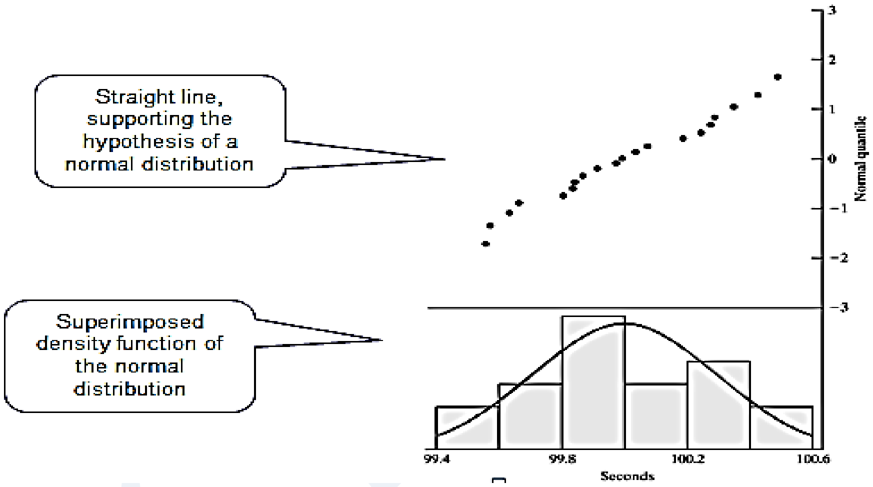
$$F(\gamma) = P(X \le \gamma) = q, \qquad \text{for } 0 < q < 1$$

  - When F has an inverse, $\gamma = F^{-1}(q)$
- Let $\{y_j, j = 1,2, \ldots, n\}$ be the observations in ascending order
- The plot of y_j versus $F^{-1}((j-0.5)/n)$ is
  - Approximately a straight line if F is a member of an appropriate family of distributions
  - The line has slope 1 if F is a member of an appropriate family of distributions with appropriate parameter values

- Example: Check whether the door installation times follow a normal distribution.
  - The observations are now ordered from smallest to largest:

j	Value	j	Value	j	Value
1	99.55	6	99.98	11	100.26
2	99.56	7	100.02	12	100.27
3	99.62	8	100.06	13	100.33
4	99.65	9	100.17	14	100.41
5	99.79	10	100.23	15	100.47

  - y_j are plotted versus $F^{-1}((j-0.5)/n)$ where F has a normal distribution with the sample mean (*99.99 sec*) and sample variance (0.2832^2 sec^2)
- Example (continued): Check whether the door installation times follow a normal distribution.



- Consider the following while evaluating the linearity of a *q-q* plot:
  - The observed values never fall exactly on a straight line
  - The ordered values are ranked and hence not independent, unlikely for the points to be scattered about the line
  - Variance of the extremes is higher than the middle. Linearity of the points in the middle of the plot is more important.
- *Q-Q* plot can also be used to check homogeneity
  - Check whether a single distribution can represent both sample sets
  - Plotting the order values of the two data samples against each other

Parameter Estimation

- Next step after selecting a family of distributions
- If observations in a sample of size n are $X_1, X_2, \ldots, X_n$ (discrete or continuous), the sample mean and variance are:

$$\bar{X} = \frac{\sum_{i=1}^{n} X_i}{n} \qquad S^2 = \frac{\sum_{i=1}^{n} X_i^2 - n\bar{X}^2}{n-1}$$

- If the data are discrete and have been grouped in a frequency distribution:

$$\bar{X} = \frac{\sum_{j=1}^{n} f_j X_j}{n} \qquad S^2 = \frac{\sum_{j=1}^{n} f_j X_j^2 - n\bar{X}^2}{n-1}$$

where f_j is the observed frequency of value X_j

- When raw data are unavailable (data are grouped into class intervals), the approximate sample mean and variance are:

$$\bar{X} = \frac{\sum_{j=1}^{c} f_j X_j}{n} \qquad S^2 = \frac{\sum_{j=1}^{n} f_j m_j^2 - n\bar{X}^2}{n-1}$$

where f_j is the observed frequency of in the jth class interval

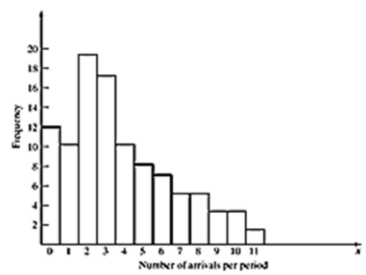
m_j is the midpoint of the jth interval, and c is the number of class intervals

- A parameter is an unknown constant, but an estimator is a statistic.
- Vehicle Arrival Example: Table 9.1 in book can be analyzed to obtain:

$$n = 100,\ f_1 = 12,\ X_1 = 0,\ f_2 = 10,\ X_2 = 1, \ldots,$$
$$\text{and } \sum_{j=1}^{k} f_j X_j = 364, \text{ and } \sum_{j=1}^{k} f_j X_j^2 = 2080$$

  - The sample mean and variance are

$$\bar{X} = \frac{364}{100} = 3.64$$
$$S^2 = \frac{2080 - 100*(3.64)^2}{99} = 7.63$$



  - The histogram suggests X to have a Possion distribution
    - ➢ However, note that sample mean is not equal to sample variance.
    - ➢ Reason: each estimator is a random variable, is not perfect.

Suggested Estimators

- Poisson Distribution
  - Estimate mean $\quad \bar{X} = \frac{\sum_{i=1}^{n} X_i}{n}$
- Exponential Distribution
  - Estimate rate $\quad \lambda = \frac{n}{\sum_{i=1}^{n} X_i}$

- Normal Distribution
  - Estimate mean and variance

$$\overline{X} = \frac{\sum_{i=1}^{n} X_i}{n} \qquad S^2 = \frac{\sum_{i=1}^{n} X_i^2 - n\overline{X}^2}{n-1}$$

❖ **Goodness-of-Fit Tests**

- Conduct hypothesis testing on input data distribution using:
  - Kolmogorov-Smirnov test
  - Chi-square test
- No single correct distribution in a real application exists.
  - If very little data are available, it is unlikely to reject any candidate distributions
  - If a lot of data are available, it is likely to reject all candidate distributions

Chi-Square test

- Intuition: comparing the histogram of the data to the shape of the candidate density or mass function
- Valid for **large** sample sizes when parameters are estimated by maximum likelihood
- By arranging the *n* observations into a set of *k* class intervals or cells, the test statistics is:

$$\chi_0^2 = \sum_{i=1}^{k} \frac{(O_i - E_i)^2}{E_i}$$

Observed Frequency

Expected Frequency
$E_i = n * p_i$
where p_i is the theoretical prob. of the *i*th interval.

which **approximately** follows the chi-square distribution with *k-s-1* degrees of freedom, where s = # of parameters of the hypothesized distribution estimated by the sample statistics.

- The hypothesis of a chi-square test is:

  H_0: The random variable, X, conforms to the distributional assumption with the parameter(s) given by the estimate(s).

  H_1: The random variable X does not conform.
- If the distribution tested is discrete and combining adjacent cell is not required (so that E_i > minimum requirement):
  - Each value of the random variable should be a class interval, unless combining is necessary, and

$$p_i = p(x_i) = P(X = x_i)$$

- If the distribution tested is continuous:

$$p_i = \int_{a_{i-1}}^{a_i} f(x)\,dx = F(a_i) - F(a_{i-1})$$

where a_i-1 and a_i are the endpoints of the i^{th} class interval and $f(x)$ is the assumed pdf, $F(x)$ is the assumed cdf.

  - Recommended number of class intervals (k):

Sample Size, n	Number of Class Intervals, k
20	Do not use the chi-square test
50	5 to 10
100	10 to 20
> 100	$n^{1/2}$ to n/5

  - Caution: Different grouping of data (i.e., k) can affect the hypothesis testing result.

- Vehicle Arrival Example (continued):
  H_0: the random variable is Poisson distributed.
  H_1: the random variable is not Poisson distributed.

x_i	Observed Frequency, O_i	Expected Frequency, E_i	$(O_i - E_i)^2/E_i$
0	12	2.6	7.87
1	10	9.6	
2	19	17.4	0.15
3	17	21.1	0.8
4	19	19.2	4.41
5	6	14.0	2.57
6	7	8.5	0.26
7	5	4.4	11.62
8	5	2.0	
9	3	0.8	
10	3	0.3	
> 11	1	0.1	
	100	100.0	27.68

$$E_i = np(x) = n\frac{e^{-\alpha}\alpha^x}{x!}$$

Combined because of min E_i

  - Degree of freedom is k-s-1 = 7-1-1 = 5, hence, the hypothesis is rejected at the 0.05 level of significance.

$$\chi_0^2 = 27.68 > \chi_{0.05,5}^2 = 11.1$$

Kolmogorov-Smirnov Test

- Intuition: formalize the idea behind examining a q-q plot
- The test compares the continuous cdf, $F(x)$, of the hypothesized distribution with the discrete empirical cdf, $S_N(x)$, of the N sample observations.
  - Based on the maximum difference statistics (Tabulated in A.8):

$$D = \max| F(x) - S_N(x)|$$

- A more powerful test, particularly useful when:

- Sample sizes are small,
- No parameters have been estimated from the data.

- No need to group the data
  - No information is lost
  - Eliminates the problem of interval specification

The Kolmogorov-Smirnov Test for Uniformity

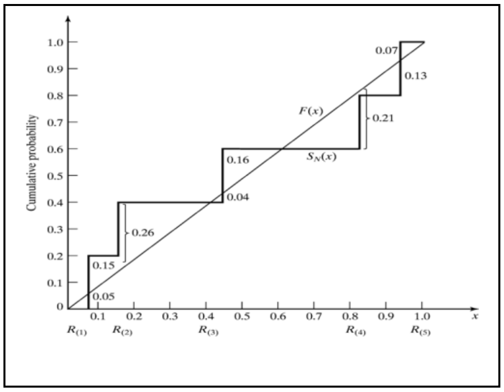
- Intuition: formalize the idea behind examining a *q-q* plot
- The test compares the continuous cdf, $F(x)$, of the hypothesized distribution with the discrete empirical cdf, $S_N(x)$, of the N sample observations.
  - Based on the maximum difference statistics (Tabulated in A.8):

$$D = max|\ F(x) - S_N(x)|$$

- A more powerful test, particularly useful when:
  - Sample sizes are small,
  - No parameters have been estimated from the data.
- No need to group the data
  - No information is lost
  - Eliminates the problem of interval specification

- STEP 1: Rank the data from smallest to largest. ($R_{(i)}$ denotes the i th smallest observation => $R_{(1)} <= R_{(2)} <= \ldots <= R_{(N)}$)
- STEP 2: Compute $D^+ = \max \{i/N - R_{(i)}\}$ (over i)

  $D^- = \max \{R_{(i)} - (i-1)/N\}$ (over i)
- STEP 3: Compute $D = \max (D^+, D^-)$
- STEP 4: Determine the critical value, D_α, from Table A.8 for the specified significance level, α, and the given sample size N
- STEP 5: If the sample statistic D is greater than the critical value, D_α, the null hypothesis that the data are sampled from uniform distribution is rejected. Otherwise, we cannot reject H_0



Example

- 5 numbers generated:
- 0.44, 0.81, 0.14, 0.05, 0.93
- We want to test uniformity using the K-S test with $\alpha = 0.05$ ($D_\alpha = 0.565$)

$R_{(i)}$	0.05	0.14	0.44	0.81	0.93
i/N	0.20	0.40	0.60	0.80	1.00
i/N- $R_{(i)}$	0.15	0.26	0.16	-	0.07
$R_{(i)}$ – (i-1)/N	0.05	-	0.04	0.21	0.13

- D= max (0.26, 0.21) = 0.26 => The uniformity of the underlying distribution for our samples is not rejected

Selecting input models without data

- If data is not available, some possible sources to obtain information about the process are:
  - Engineering data: often product or process has performance ratings provided by the manufacturer or company rules specify time or production standards.
  - Expert option: people who are experienced with the process or similar processes, often, they can provide optimistic, pessimistic and most-likely times, and they may know the variability as well.
  - Physical or conventional limitations: physical limits on performance, limits or bounds that narrow the range of the input process.
  - The nature of the process.
- The uniform, triangular distributions are often used as input models.
- Sensitivity to input data must be tested.
- Example: Production planning simulation.
  - Input of sales volume of various products is required, salesperson of product XYZ says that:
    - No fewer than *1,000* units and no more than *5,000* units will be sold.
    - Given her experience, she believes there is a *90%* chance of selling more than *2,000* units, a *25%* chance of selling more than *2,500* units, and only a *1%* chance of selling more than *4,500* units.
  - Translating these information into a cumulative probability of being less than or equal to those goals for simulation input:

i	Interval (Sales)	Cumulative Frequency, c_i
1	$1000 \le x \le 2000$	0.10
2	$2000 < x \le 3000$	0.75
3	$3000 < x \le 4000$	0.99
4	$4000 < x \le 5000$	1.00

Multivariate and Time-Series Input Models

- Multivariate:
  - For example, lead time and annual demand for an inventory model, increase in demand results in lead time increase, hence variables are dependent.
- Time-series:
  - For example, time between arrivals of orders to buy and sell stocks, buy and sell orders tend to arrive in bursts, hence, times between arrivals are dependent.
- Consider the model that describes relationship between X_1 and X_2:

$$(X_1 - \mu_1) = \beta(X_2 - \mu_2) + \varepsilon$$

ε is a random variable with mean *0* and is independent of X_2

  - $\beta = 0$, X_1 and X_2 are statistically independent
  - $\beta > 0$, X_1 and X_2 tend to be above or below their means together
  - $\beta < 0$, X_1 and X_2 tend to be on opposite sides of their means
- Covariance between X_1 and X_2 :
  - where $cov(X_1, X_2) \begin{cases} = 0, \\ < 0, \\ > 0, \end{cases}$ then $\beta \begin{cases} = 0 \\ < 0 \\ > 0 \end{cases}$
- Correlation between X_1 and X_2 (values between *-1* and *1*):

$$\rho = \text{corr}(X_1, X_2) = \frac{\text{cov}(X_1, X_2)}{\sigma_1 \sigma_2}$$

  - where $corr(X_1, X_2) \begin{cases} = 0, \\ < 0, \\ > 0, \end{cases}$ then $\beta \begin{cases} = 0 \\ < 0 \\ > 0 \end{cases}$
  - The closer ρ is to *-1* or *1*, the stronger the linear relationship is between X_1 and X_2.

- A time series is a sequence of random variables $X_1, X_2, X_3, \ldots$, are identically distributed (same mean and variance) but dependent.
  - $\text{cov}(X_t, X_{t+h})$ is the *lag-h autocovariance*
  - $\text{corr}(X_t, X_{t+h})$ is the *lag-h autocorrelation*
  - If the autocovariance value depends only on *h* and not on *t*, the time series is covariance stationary

Multivariate Input Models

- If X1 and X2 are normally distributed, dependence between them can be modeled by the bivariate normal distribution with μ_1, μ_2, σ_1^2, σ_2^2 and correlation ρ
  - To Estimate μ_1, μ_2, σ_1^2, σ_2^2, see "Parameter Estimation"
  - To Estimate r, suppose we have n independent and identically distributed pairs (X_{11}, X_{21}), (X_{12}, X_{22}), … (X_{1n}, X_{2n}), then:

$$\hat{\text{cov}}(X_1, X_2) = \frac{1}{n-1}\sum_{j=1}^{n}(X_{1j} - \hat{X}_1)(X_{2j} - \hat{X}_2)$$

$$= \frac{1}{n-1}\left(\sum_{j=1}^{n} X_{1j}X_{2j} - n\hat{X}_1\hat{X}_2\right)$$

$$\hat{\rho} = \frac{\hat{\text{cov}}(X_1, X_2)}{\hat{\sigma}_1\hat{\sigma}_2}$$

Sample

Time-Series Input Models

- If X_1, X_2, X_3, … is a sequence of identically distributed, but dependent and covariance-stationary random variables, then we can represent the process as follows:
  - Autoregressive order-1 model, AR(1)
  - Exponential autoregressive order-1 model, EAR(1)
    - Both have the characteristics that:

$$\rho_h = corr(X_t, X_{t+h}) = \rho^h, \quad \text{for } h = 1,2,...$$

    - *Lag-h* autocorrelation decreases geometrically as the lag increases, hence, observations far apart in time are nearly independent

AR (1) Time-Series Input Models

- Consider the time-series model:

$$X_t = \mu + \phi(X_{t-1} - \mu) + \varepsilon_t, \quad \text{for } t = 2,3,...$$

where $\varepsilon_2, \varepsilon_3, \ldots$ are i.i.d. normally distributed with $\mu_\varepsilon = 0$ and variance σ_ε^2

- If X_1 is chosen appropriately, then
  - X_1, X_2, … are normally distributed with mean = μ, and variance = $\sigma^2/(1-\phi^2)$
  - Autocorrelation $\rho_h = \phi^h$
- To estimate ϕ, μ, σ_ε^2 :

$$\hat{\mu} = \bar{X}\,, \qquad \hat{\sigma}_{\varepsilon}^{2} = \hat{\sigma}^{2}(1-\hat{\phi}^{2})\,, \qquad \hat{\phi} = \frac{\hat{\text{cov}}(X_t, X_{t+1})}{\hat{\sigma}^{2}}$$

where $\hat{\text{cov}}(X_t, X_{t+1})$ is the *lag*-1 autocovariance

EAR (1) Time-Series Input Models

- Consider the time-series model:

$$X_t = \begin{cases} \phi X_{t-1}, & \text{with probability } \phi \\ \phi X_{t-1} + \varepsilon_t, & \text{with probability } 1\text{-}\varphi \end{cases} \qquad \text{for } t = 2,3,\ldots$$

where $\varepsilon_2, \varepsilon_3, \ldots$ are i.i.d. exponentially distributed with $\mu_{\varepsilon} = 1/\lambda$, and $0 \le \phi < 1$

- If X1 is chosen appropriately, then
  - $X_1, X_2, \ldots$ are exponentially distributed with mean = $1/\lambda$
  - Autocorrelation $\rho_h = \phi^h$, and only positive correlation is allowed.
- To estimate ϕ, λ :

$$\hat{\lambda} = 1/\bar{X}\,, \qquad \hat{\phi} = \hat{\rho} = \frac{\hat{\text{cov}}(X_t, X_{t+1})}{\hat{\sigma}^{2}}$$

where $\hat{\text{cov}}(X_t, X_{t+1})$ is the *lag*-1 autocovariance

Type of Simulations

- Terminating verses non-terminating simulations
- Terminating simulation:
  - Runs for some duration of time T_E, where E is a specified event that stops the simulation.
  - Starts at time *0* under well-specified initial conditions.
  - Ends at the stopping time T_E.
  - Bank example: Opens at 8:30 am (time *0*) with no customers present and *8* of the *11* teller working (initial conditions), and closes at 4:30 pm (Time T_E = *480* minutes).
  - The simulation analyst chooses to consider it a terminating system because the object of interest is one day's operation.
- Non-terminating simulation:
  - Runs continuously or at least over a very long period of time.
  - Examples: assembly lines that shut down infrequently, telephone systems, hospital emergency rooms.
  - Initial conditions defined by the analyst.

- Runs for some analyst-specified period of time T_E.
- Study the steady-state (long-run) properties of the system, properties that are not influenced by the initial conditions of the model.

- Whether a simulation is considered to be terminating or non-terminating depends on both
  - The objectives of the simulation study and
  - The nature of the system.

Stochastic Nature of Output Data

- Model output consist of one or more random variables (r. v.) because the model is an input-output transformation and the input variables are r.v.'s.
- M/G/1 queueing example:
- Poisson arrival rate = 0.1 per minute;
  service time ~ $N(\mu = 9.5, \sigma = 1.75)$.
  - System performance: long-run mean queue length, $L_Q(t)$.
  - Suppose we run a single simulation for a total of 5,000 minutes
    - Divide the time interval [0, 5000) into 5 equal subintervals of 1000 minutes.
    - Average number of customers in queue from time (j-1)1000 to j(1000) is Y_j
- M/G/1 queueing example (cont.):
  - Batched average queue length for 3 independent replications:

Batching Interval (minutes)	Batch, j	Replication 1, Y_{1j}	Replication 2, Y_{2j}	Replication 3, Y_{3j}
[0, 1000)	1	3.61	2.91	7.67
[1000, 2000)	2	3.21	9.00	19.53
[2000, 3000)	3	2.18	16.15	20.36
[3000, 4000)	4	6.92	24.53	8.11
[4000, 5000)	5	2.82	25.19	12.62
[0, 5000)		3.75	15.56	13.66

  - Inherent variability in stochastic simulation both within a single replication and across different replications.
  - The average across 3 replications, $\bar{Y}_{1.}, \bar{Y}_{2.}, \bar{Y}_{3.}$, can be regarded as independent observations, but averages within a replication, $Y_{11}, \ldots, Y_{15}$, are not.

Measures of performance

- Consider the estimation of a performance parameter, θ (or ϕ), of a simulated system.
  - Discrete time "tally" data: $[Y_1, Y_2, \ldots, Y_n]$, with ordinary mean: θ
    - Average System Time
    - Average Waiting Time
  - Continuous-time "time-persistent" data: $\{Y(t), 0 \le t \le T_E\}$ with time-weighted mean: ϕ

- ➢ Average Queue Length
- ➢ Average Utilization

Point Estimator

- Point estimation for discrete-time data.
  - The point estimator
$$\hat{\theta} = \frac{1}{n}\sum_{i=1}^{n} Y_i$$
  - Is unbiased if $E(\hat{\theta}) = \theta$ Desired
- Point estimation for continuous-time data.
  - The point estimator:
$$\hat{\phi} = \frac{1}{T_E}\int_0^{T_E} Y(t)dt$$
    - ➢ Is biased if $E(\hat{\phi}) \neq \phi$
    - ➢ An unbiased or low-bias estimator is desired.

Confidence-Interval Estimation

- Suppose the model is the normal distribution with mean θ, variance σ^2 (both unknown).
  - Let Y_i be the average cycle time for parts produced on the i^{th} replication of the simulation (its mathematical expectation is θ).
  - Average cycle time will vary from day to day, but over the long-run the average of the averages will be close to θ.
  - Sample variance across R replications:
$$S^2 = \frac{1}{R-1}\sum_{i=1}^{R}(Y_{i.} - Y_{..})^2$$
- Confidence Interval (CI):
  - A measure of error.
  - Where $Y_{i\cdot}$ are normally distributed.
$$\bar{Y}_{..} \pm t_{\alpha/2,R-1}\frac{S}{\sqrt{R}}$$
  - We cannot know for certain how far $\bar{Y}_{..}$ is from θ but CI attempts to bound that error.
  - A CI, such as 95%, tells us how much we can trust the interval to actually bound the error between $\bar{Y}_{..}$ and θ.
  - The more replications we make, the less error there is in $\bar{Y}_{..}$ (converging to 0 as R goes to infinity).

(1) UNIT-6 : INPUT MODELLING

I. Chi Square Test using Poisson Assumption

1. Using goodness of fit test, test whether random Nos are uniformly distributed based on poisson assumption with level of significance $\alpha = 0.05$. $\hat{\alpha} = 3.64$. Sample data are :

Interval :	0	1	2	3	4	5	6	7	8	9	10	11
observed Frequency :	12	10	19	17	10	8	7	5	5	3	3	1

⇒ Given : $\alpha = 0.05$

$\hat{\alpha} = 3.64$

$n = 12 + 10 + 19 + 17 + 10 + 8 + 7 + 5 + 5 + 3 + 3 + 1 = 100$

Step 1 : Compute Poisson Distribution

$$P(x) = \frac{e^{-\alpha} \alpha^{x}}{x!} \quad \text{where } x = 0, 1, \ldots, 11$$

$$P(0) = \frac{e^{-3.64} * (3.64)^0}{0!} = 0.026$$

$P(1) = 0.096$

$P(2) = 0.174$

$P(3) = 0.211$

$P(4) = 0.192$

$P(5) = 0.140$

$P(6) = 0.085$

$P(7) = 0.044$

$P(8) = 0.020$

$P(9) = 0.008$

$P(10) = 0.003$

$P(11) = 0.001$

Step 2 : Apply Chi Square with poisson assumption.

X_i	O_i	$E_i = n \cdot P_i$	$O_i - E_i$	$(O_i - E_i)^2$	$X_0^2 = \sum_{i=1}^{n} \frac{(O_i - E_i)^2}{E_i}$
0	12 } 22	2.6 } 12.2	9.8	96.04	7.87
1	10	9.6			
2	19	17.4	1.6	2.56	0.15
3	17	21.1	−4.1	16.81	0.79
4	10	19.2	−9.2	84.64	4.41
5	8	14.0	−6	36	2.57
6	7	8.5	−1.5	2.25	0.27
7	5 } 17	4.4 } 7.6	9.4	88.36	11.63
8	5	2.0			
9	3	0.8			
10	3	0.3			
11	1	0.1			

We have $k = 7$, $s = 1$ $X_0^2 = 27.69$

Step 3 : Compute level of Significance from Table A6

$$X^2_{0,\alpha, k-s-1} = X^2_{0\ 0.05,\ 7-1-1}$$

$$= X^2_{0\ 0.05,\ 5} = 11.1$$

Step 4 : Check whether Random No's are uniformly distributed.

Compare X_0^2 & $X^2_{0\ 0.05,\ 5}$

$\because 27.69 > 11.1 \Rightarrow$ Random No's are not uniformly distributed.

2. Using goodness of fit test, check whether Random No.s are uniformly distributed over interval [0, 1] using poisson assumption with level of significance = 0.05. Simulation table for critical values is given:

Interval (X_i) :	0	1	2	3	4	5	6	7
Frequency (f_i) :	5	10	5	8	12	10	8	12

⇒ Given: $\alpha = 0.05$

$n = 5+10+5+8+12+10+8+12 = 70$

$\hat{\alpha} = ?$

$$\hat{\alpha} = \bar{X} = \frac{\sum_{i=1}^{n} f_i X_i}{n} = \frac{0+10+10+24+48+50+48+84}{70}$$

$$\hat{\alpha} = \frac{274}{70} = 3.91$$

Step 1: Compute Poisson Distribution

$$P(x) = \frac{e^{-\alpha}\alpha^{x}}{x!}, \quad x = 0, 1, \ldots, 7. \quad \& \quad \alpha = 3.91$$

$P(0) = 0.020$

$P(1) = 0.078$

$P(2) = 0.153$

$P(3) = 0.199$

$P(4) = 0.195$

$P(5) = 0.153$

$P(6) = 0.099$

$P(7) = 0.056$

Step 2 : Apply Chi Square test with poisson assumption

X_i	O_i	$E_i = n \cdot P_i$	$O_i - E_i$	$(O_i - E_i)^2$	$X_0^2 = \sum_{i=1}^{n} \frac{(O_i - E_i)^2}{E_i}$
0	5 } 15	1.4 } 6.86	8.14	66.26	9.66
1	10	5.46			
2	5	10.71	−5.71	32.60	3.04
3	8	13.93	−5.93	35.16	2.52
4	12	13.65	−1.65	2.72	0.19
5	10	10.71	−0.71	0.50	0.05
6	8 } 20	6.93 } 10.85	9.15	83.72	7.72
7	12	3.92			

$X_0^2 = 23.18$

Here $k = 6$, $s = 1$

Step 3 : Compute level of Significance from Table A6

$X_{0\,\alpha,\,k-s-1}^2 = X_{0\,0.05,\,6-1-1}^2 = 9.49$

Step 4 : Check whether Random No.s are uniformly distributed.

Compare X_0^2 & $X_{0\,0.05,\,4}^2$

$\because 23.18 > 9.49 \Rightarrow$ Random No.s are not uniformly distributed

3. Apply goodness of fittest, check whether Random Nos are uniformly distributed over Interval [0, 1] with given size of data 100. Assume $\alpha = 0.01$. Simulation table to check critical values using poisson assumption is given below :

Interval :	1	2	3	4	5	6	7	8	9	10
Frequency :	8	6	10	11	12	8	10	12	12	11

⇒ Given : $\alpha = 0.01$ $\hat{\alpha} = ?$

$n = 100$

$$\alpha = \bar{x} = \frac{\sum_{i=1}^{n} f_i x_i}{n} = \frac{586}{100} = 5.86$$

Step 1 : Compute Poisson Distribution

$$P(x) = \frac{e^{-\alpha} \cdot \alpha^{x}}{x!} \quad \text{where } x = 1, 2 \ldots 10 \quad \& \quad \alpha = 5.86$$

$P(1) = 0.017$

$P(2) = 0.049$

$P(3) = 0.096$

$P(4) = 0.140$

$P(5) = 0.164$

$P(6) = 0.160$

$P(7) = 0.134$

$P(8) = 0.098$

$P(9) = 0.064$

$P(10) = 0.038$

Step 2 : Apply Chi square with poisson assumption

X_i	O_i	$E_i = n \cdot P_i$	$O_i - E_i$	$(O_i - E_i)^2$	$X_0^2 = \sum_{i=1} \frac{(O_i - E_i)^2}{E_i}$
1	8 } 14	1.7 } 6.6	7.4	54.76	8.29
2	6	4.9			
3	10	9.6	0.4	0.16	0.02
4	11	14.0	−3	9	0.64
5	12	16.4	−4.4	19.36	1.18
6	8	16.0	−8	64	4
7	10	13.4	−3.4	11.56	0.86
8	12	9.8	2.2	4.84	0.49
9	12 } 23	6.4 } 10.2	12.8	163.84	16.06
10	11	3.8			

$k = 8, \ s = 1$ $\qquad X_0^2 = 31.54$

Step 3 : Compute level of Significance from Table A6

$X_{0\ \alpha,\ k-s-1}^2 = X_{0\ 0.01,\ 8-1-1}^2 = 20.1$

Step 4 : Check whether Random No.s are uniformly distributed

Compare X_0^2 & $X_{0\ 0.05,\ 6}^2$

$31.54 > 20.1 \Rightarrow$ Random No.s are not uniformly distributed

(A)

II. Chi Square Test with Equal Probability (Exponential Dist.)

1. Apply goodness of fit test to check whether random No.s are uniformly distributed over [0, 1] using equal probability. Use $\alpha = 0.05$, interval $k = 8$ to check whether given sample datas are accepted or rejected.

79.918	3.081	0.062	1.961	5.845	3.027	6.505
0.021	0.013	0.123	6.769	59.899	1.192	34.760
5.009	18.387	0.141	43.565	24.420	0.433	144.695
2.663	17.967	0.091	9.003	0.941	0.878	3.371
2.157	7.579	0.624	5.380	3.148	7.078	23.960
0.590	1.928	0.300	0.002	0.543	7.004	31.764
1.005	1.147	0.219	3.217	14.382	1.008	2.336
4.562						

$\Rightarrow$ Given : $k = 8$, $\alpha = 0.05$

Step 1 : Compute mean

$$\bar{\lambda} = \frac{1}{\bar{X}} \quad \text{where} \quad \bar{X} = \frac{\Sigma X_i}{n}$$

$$\bar{X} = \frac{594.674}{50} = 11.894$$

$$\bar{\lambda} = \frac{1}{11.894}$$

$$\boxed{\bar{\lambda} = 0.084}$$

Step 2 : Compute class intervals

$$P = \frac{1}{k} = \frac{1}{8} = 0.125$$

$$a_i = -\frac{1}{\lambda} \ln [1 - i * p] \quad \text{where } i = 0 \ldots 8$$

$\lambda = 0.084$

$P = 0.125$

$a_0 = 0$

$a_1 = 1.589$ $\qquad a_5 = 11.677$

$a_2 = 3.425$ $\qquad a_6 = 16.504$

$a_3 = 5.595$ $\qquad a_7 = 24.755$

$a_4 = 8.252$ $\qquad a_8 = \infty$

Step 3 : Compute Chi square with equal probability

Class Interval	O_i	$E_i = \frac{N}{k}$ (50/8)	$O_i - E_i$	$(O_i - E_i)^2$	$X_0^2 = \sum_{i=1} \frac{(O_i - E_i)^2}{E_i}$
0 – 1.589	19	6.25	12.75	162.563	26.01
1.589 – 3.425	10	6.25	3.75	14.063	2.25
3.425 – 5.595	3	6.25	−3.25	10.563	1.69
5.595 – 8.252	6	6.25	−0.25	0.0625	0.01
8.252 – 11.677	1	6.25	−5.25	27.563	4.41
11.677 – 16.504	1	6.25	−5.25	27.563	4.41
16.504 – 24.755	4	6.25	−2.25	5.0623	0.81
24.755 – ∞	6	6.25	−0.25	0.063	0.01
	50				$X_0^2 = 39.60$

Step 3 : Compute level of significance from table A6.

$X_{0\,\alpha,\,k-s-1}^2 = X_{0\,0.05,\,8-1-1}^2 = 12.6$

Step 4 : Check whether random No.s are uniformly distributed.

$39.60 > 12.6 \Rightarrow$ Random No.s are rejected.

3. Consider goodness of fit test using Chi Square test with equal probability. Given $k=6$, $\alpha=0.05$. Sample data:

(5)

0.34	0.90	1.88	1.90	0.74	2.62	2.67	3.53	4.91
5.50	1.10	1.03	1.73	1.00	2.03	1.49	2.16	0.80
0.48	5.60	0.45	0.26	0.24	0.63	0.36	1.28	0.82
2.16	0.05	0.04	0.39	0.21	0.79	0.53	3.53	2.62
0.53	1.50	2.81						

Given: $k=6$ $\alpha=0.05$ $N=39$

Step 1: Compute Mean

$\lambda = \frac{1}{\bar{X}}$ where $\bar{X} = \frac{\sum X_i}{N} = \frac{61.61}{39} = 1.579$

$\lambda = \frac{1}{1.579} = \underline{\underline{0.63}}$

Step 2: Compute class intervals

$P = \frac{1}{k} = \frac{1}{6} = \underline{\underline{0.17}}$

$a_i = \frac{-1}{\lambda} \ln\left[1 - i * P\right]$ where $i = 0, 1 \dots 6$

$\lambda = 0.63$

$P = 0.17$

$a_0 = 0$

$a_1 = 0.29$

$a_2 = 0.66$

$a_3 = 1.13$

$a_4 = 1.81$

$a_5 = 3.01$

$a_6 = \infty$

Class Interval	O_i	$E_i = \frac{N}{K}$	$O_i - E_i$	$(O_i - E_i)^2$	$X_0^2 = \sum_{i=1}^{n} \frac{(O_i - E_i)^2}{E_i}$
0 - 0.29	5	6.5	-1.5	2.25	0.35
0.29 - 0.66	8	6.5	1.5	2.25	0.35
0.66 - 1.13	8	6.5	1.5	2.25	0.35
1.13 - 1.81	4	6.5	-2.5	6.25	0.96
1.81 - 3.01	9	6.5	2.5	6.25	0.96
3.01 - ∞	5	6.5	-1.5	2.25	0.35

$X_0^2 = 3.32$

Step 3 : Compute level of Significance from table A6

$X_0^2{}_{\alpha,\ k-s-1} = X_0^2{}_{0.05,\ 6-1-1} = 9.49$

Step 4 : Check whether Random Nos are uniformly distributed.

$\because 3.32 < 9.49 \Rightarrow$ Random Nos are accepted

III K-S Test

1. Apply goodness of fit test to check whether Random Nos are uniformly distributed over $(0, T)$ for an interval 100. Take $\alpha = 0.05$

Simulation table for Critical values :

0.44 0.53 2.04 2.74 2.00 0.30 2.54 0.52 2.02 1.89

Given : $\alpha = 0.05$, $n = 10$, $T = 100$

$$R_{(i)} = \left\{ \frac{T_1}{T}, \frac{T_1 + T_2}{T}, \ldots\ldots, \frac{T_1 + T_2 + \ldots T_n}{T} \right\}$$

(6) Step 1 :

$R_{(i)} = \{0.0044, 0.0097, 0.0301, 0.0575, 0.0775, 0.0805, 0.1059, 0.1111, 0.1313, 0.1502\}$

Step 2 :

i	$R_{(i)}$	i/n	$i-1/n$	$D^+ = \max\{\frac{i}{n} - R_{(i)}\}$	$D^- = \max\{R_{(i)} - \frac{i-1}{n}\}$
1	0.0044	0.1	0	0.0956	0.0044
2	0.0097	0.2	0.1	0.1903	~
3	0.0301	0.3	0.2	0.2699	~
4	0.0575	0.4	0.3	0.3425	~
5	0.0775	0.5	0.4	0.4225	~
6	0.0805	0.6	0.5	0.5195	~
7	0.1059	0.7	0.6	0.5941	~
8	0.1111	0.8	0.7	0.6889	~
9	0.1313	0.9	0.8	0.7687	~
10	0.1502	1.0	0.9	0.8498	~

Step 3 :

$D = \max\{D^+, D^-\} = \max\{0.8498, 0.0044\}$

$D = \underline{0.8498}$

Step 4 :

$D_{\alpha, n}$ from A8 table.

$D_{0.05, 10} = \underline{0.410}$

Step 5 :

$\therefore$ $0.8498 > 0.410 \Rightarrow$ Random Nos are rejected

2. Consider Sample data. Perform KS Test.

0.10 1.42 0.46 0.07 1.09 0.76 5.53 3.93 1.07

2.26 2.88 0.67 1.12 0.26

Interval (0-T) = 100 min $\alpha = 0.05$ $n = 14$

= Step 1 :

$R_{(i)} = \{$ 0.0010, 0.0152, 0.0198, 0.0205, 0.0314, 0.039, 0.0943, 0.1336, 0.1443, 0.1669, 0.1957, 0.2024, 0.2136, 0.2162 $\}$

Step 2 :

i	$R_{(i)}$	i/n	$i-1/n$	$D^+ = \max\{\frac{i}{n} - R_{(i)}\}$	$D^- = \max\{R_{(i)} - \frac{i-1}{n}\}$
1	0.0010	0.0714	0	0.0704	0.0010
2	0.0152	0.1429	0.0714	0.1277	~
3	0.0198	0.2143	0.1429	0.1945	~
4	0.0205	0.2857	0.2143	0.2652	~
5	0.0314	0.3571	0.2857	0.3257	~
6	0.039	0.4286	0.3571	0.3896	~
7	0.0943	0.5	0.4286	0.4057	~
8	0.1336	0.5714	0.5	0.4378	~
9	0.1443	0.6429	0.5714	0.4986	~
10	0.1669	0.7143	0.6429	0.5474	~
11	0.1957	0.7857	0.7143	0.59	~
12	0.2024	0.8571	0.7857	0.6547	~
13	0.2136	0.9286	0.8571	0.715	~
14	0.2162	1	0.9286	0.7838	~

Step 3 : $D = \max\{0.7838, 0.0010\} = \underline{0.7838}$

Step 4 : $D_{0.05, 14} = 0.349$ (A8 table)

Step 5 : $\because$ 0.7838 > 0.349 $\Rightarrow$ <u>Random No.s are rejected</u>

SYSTEM MODELLING AND SIMULATION

Module 5

Output analysis for terminating simulations, Output analysis for steady-state simulations Verification, Calibration and Validation: Optimization, Model building, verification and validation, Verification of simulation models, Calibration and validation of models, Optimization via Simulation.

Output Analysis for Terminating Simulations

- A terminating simulation: runs over a simulated time interval $[0, T_E]$.
- A common goal is to estimate:

$$\theta = E\left(\frac{1}{n}\sum_{i=1}^{n} Y_i\right), \qquad \text{for discrete output}$$

$$\phi = E\left(\frac{1}{T_E}\int_0^{T_E} Y(t)dt\right), \quad \text{for continuous output } Y(t), 0 \le t \le T_E$$

- In general, independent replications are used, each run using a different random number stream and independently chosen initial conditions.

Statistical Background

- Important to distinguish within-replication data from across-replication data.
- For example, simulation of a manufacturing system
    - Two performance measures of that system: cycle time for parts and work in process (WIP).
    - Let Y_{ij} be the cycle time for the j^{th} part produced in the i^{th} replication.
    - Across-replication data are formed by summarizing within-replication data.
- Across Replication:
    - For example: the daily cycle time averages (discrete time data)
        - The average:
        $$\bar{Y}_{..} = \frac{1}{R}\sum_{i=1}^{R} Y_{i.}$$
        - The sample variance:
        $$S^2 = \frac{1}{R-1}\sum_{i=1}^{R}(Y_{i.} - \bar{Y}_{..})^2$$
        - The confidence-interval half-width:
        $$H = t_{\alpha/2,R-1}\frac{S}{\sqrt{R}}$$

- Within replication:
  - For example: the WIP (a continuous time data)
    - The average:

$$\bar{Y}_{i.} = \frac{1}{T_{Ei}} \int_0^{T_{Ei}} Y_i(t)dt$$

    - The sample variance:

$$S_i^2 = \frac{1}{T_{Ei}} \int_0^{T_{Ei}} \left(Y_i(t) - \bar{Y}_{i.}\right)^2 dt$$

- Overall sample average, $\bar{Y}_{..}$, and the interval replication sample averages, $\bar{Y}_{..}$, are always unbiased estimators of the expected daily average cycle time or daily average WIP.

- Across-replication data are independent (different random numbers) and identically distributed (same model), but within-replication data do not have these properties.

C.I. with Specified Precision

- The half-length H of a 100*(1 – α)%* confidence interval for a mean θ, based on the *t* distribution, is given by:

$$H = t_{\alpha/2,R-1} \frac{S}{\sqrt{R}}$$

R is the # of replications

S^2 is the sample variance

- Suppose that an error criterion e is specified with probability *1 - α*, a sufficiently large sample size should satisfy:

$$P\left(\left|\bar{Y}_{..} - \theta\right| < \varepsilon\right) \geq 1 - \alpha$$

- Assume that an initial sample of size R_0 (independent) replications have been observed.

- Obtain an initial estimate S_0^2 of the population variance σ^2.
- Then, choose sample size R such that $R \geq R_0$:
  - Since $t_{\alpha/2,\, R-1} \geq z_{\alpha/2}$, an initial estimate of R:

$$R \geq \left(\frac{z_{\alpha/2} S_0}{\varepsilon}\right)^2, \quad z_{\alpha/2} \text{ is the standard normal distribution.}$$

  - R is the smallest integer satisfying $R \geq R_0$ and $R \geq \left(\frac{t_{\alpha/2,R-1} S_0}{\varepsilon}\right)^2$
- Collect $R - R_0$ additional observations.
- The *100(1-α)%* C.I. for θ:

$$\bar{Y}_{..} \pm t_{\alpha/2,R-1} \frac{S}{\sqrt{R}}$$

- Call Center Example: estimate the agent's utilization ρ over the first 2 hours of the workday.
  - Initial sample of size $R_0 = 4$ is taken and an initial estimate of the population variance is $S_0^2 = (0.072)^2 = 0.00518$.
  - The error criterion is $\varepsilon = 0.04$ and confidence coefficient is $1\text{-}\alpha = 0.95$, hence, the final sample size must be at least:

$$\left(\frac{z_{0.025}S_0}{\varepsilon}\right)^2 = \frac{1.96^2 * 0.00518}{0.04^2} = 12.14$$

  - For the final sample size:

R	13	14	15
$t_{0.025,\,R-1}$	2.18	2.16	2.14
$(t_{\alpha/2,R-1}S_0/\varepsilon)^2$	15.39	15.1	14.83

  - R = 15 is the smallest integer satisfying the error criterion, so R - R0 = 11 additional replications are needed.
  - After obtaining additional outputs, half-width should be checked.

Output Analysis for Steady-State Simulation

- Consider a single run of a simulation model to estimate a steady-state or long-run characteristics of the system.
  - The single run produces observations $Y_1, Y_2, \ldots$ (generally the samples of an autocorrelated time series).
  - Performance measure:

$$\theta = \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} Y_i, \quad \text{for discrete measure} \quad \text{(with probability 1)}$$

$$\phi = \lim_{T_E\to\infty} \frac{1}{T_E}\int_0^{T_E} Y(t)\,dt, \quad \text{for continuous measure} \quad \text{(with probability 1)}$$

    - Independent of the initial conditions.
- The sample size is a design choice, with several considerations in mind:
  - Any bias in the point estimator that is due to artificial or arbitrary initial conditions (bias can be severe if run length is too short).
  - Desired precision of the point estimator.
  - Budget constraints on computer resources.
- Notation: the estimation of q from a discrete-time output process.
  - One replication (or run), the output data: $Y_1, Y_2, Y_3, \ldots$
  - With several replications, the output data for replication r: $Y_{r1}, Y_{r2}, Y_{r3}, \ldots$

Initialization Bias

- Methods to reduce the point-estimator bias caused by using artificial and unrealistic initial conditions:
  - Intelligent initialization.
  - Divide simulation into an initialization phase and data-collection phase.
- Intelligent initialization
  - Initialize the simulation in a state that is more representative of long-run conditions.
  - If the system exists, collect data on it and use these data to specify more nearly typical initial conditions.
  - If the system can be simplified enough to make it mathematically solvable, e.g. queueing models, solve the simplified model to find long-run expected or most likely conditions, use that to initialize the simulation.
- Divide each simulation into two phases:
  - An initialization phase, from time *0* to time T_0.
  - A data-collection phase, from T_0 to the stopping time T_0+T_E.
  - The choice of T_0 is important:
    - After T_0, system should be more nearly representative of steady-state behavior.
  - System has reached steady state: the probability distribution of the system state is close to the steady-state probability distribution (bias of response variable is negligible).
- M/G/1 queueing example: A total of 10 independent replications were made.
  - Each replication beginning in the empty and idle state.
  - Simulation run length on each replication was T_0+T_E = 15,000 minutes.
  - Response variable: queue length, $L_Q(t,r)$ (at time t of the rth replication).
  - Batching intervals of 1,000 minutes, batch means
- Ensemble averages:
  - To identify trend in the data due to initialization bias
  - The average corresponding batch means *across* replications:

$$\bar{Y}_{.j} = \frac{1}{R}\sum_{r=1}^{R} Y_{rj}$$

R replications

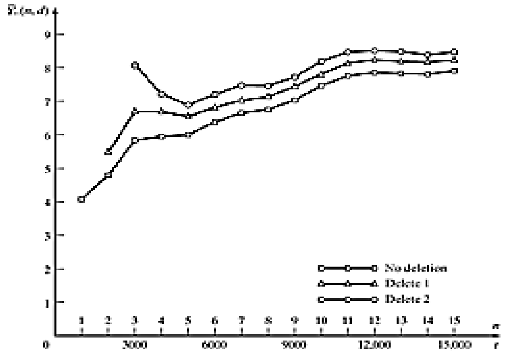
  - The preferred method to determine deletion point.

- A plot of the ensemble averages, $\bar{Y}_{..}(n,d)$, versus *1000j*, for *j = 1,2, …,15*.

  - Illustrates the downward bias of the initial observations.
- Cumulative average sample mean (after deleting *d* observations):

$$\bar{Y}_{..}(n,d) = \frac{1}{n-d}\sum_{j=d+1}^{n}\bar{Y}_{.j}$$

  - Not recommended to determine the initialization phase.



  - It is apparent that downward bias is present and this bias can be reduced by deletion of one or more observations.
- No widely accepted, objective and proven technique to guide how much data to delete to reduce initialization bias to a negligible level.
- Plots can, at times, be misleading but they are still recommended.
  - Ensemble averages reveal a smoother and more precise trend as the # of replications, R, increases.
  - Ensemble averages can be smoothed further by plotting a moving average.
  - Cumulative average becomes less variable as more data are averaged.
  - The more correlation present, the longer it takes for $\bar{Y}_{.j}$ to approach steady state.
  - Different performance measures could approach steady state at different rates.

Replication Method

- Use to estimate point-estimator variability and to construct a confidence interval.
- Approach: make R replications, initializing and deleting from each one the same way.
- Important to do a thorough job of investigating the initial-condition bias:
  - Bias is not affected by the number of replications, instead, it is affected only by deleting more data (i.e., increasing T_0) or extending the length of each run (i.e. increasing T_E).
- Basic raw output data $\{Y_{rj},\ r = 1, \ldots, R;\ j = 1, \ldots, n\}$ is derived by:
  - Individual observation from within replication *r*.
  - Batch mean from within replication *r* of some number of discrete-time observations.
  - Batch mean of a continuous-time process over time interval *j*.
- Each replication is regarded as a single sample for estimating θ. For replication *r:*

$$\bar{Y}_{r.}(n,d) = \frac{1}{n-d}\sum_{j=d+1}^{n} Y_{rj}$$

- The overall point estimator:

$$\bar{Y}_{..}(n,d) = \frac{1}{R}\sum_{r=1}^{R}\bar{Y}_{r.}(n,d) \quad \text{and} \quad E[\bar{Y}_{..}(n,d)] = \theta_{n,d}$$

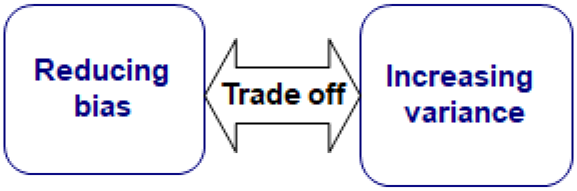
- If d and n are chosen sufficiently large:
  - $\theta_{n,d} \sim \theta$.
  - $\bar{Y}_{..}(n,d)$ is an approximately unbiased estimator of θ.
- To estimate standard error of $\bar{Y}_{..}$, the sample variance and standard error:

$$S^2 = \frac{1}{R-1}\sum_{r=1}^{R}(\bar{Y}_{r.} - \bar{Y}_{..})^2 = \frac{1}{R-1}\left(\sum_{r=1}^{R}\bar{Y}_{r.}^{\,2} - R\bar{Y}_{..}^{\,2}\right) \quad \text{and} \quad s.e.(\bar{Y}_{..}) = \frac{S}{\sqrt{R}}$$

- Length of each replication (*n*) beyond deletion point (*d*):

$$(n - d) > 10d$$

- Number of replications (*R*) should be as many as time permits, up to about *25* replications.
- For a fixed total sample size (*n*), as fewer data are deleted (↓*d*):
  - C.I. shifts: greater bias.
  - Standard error of $\bar{Y}_{..}(n,d)$ decreases: decrease variance.



- M/G/1 queueing example:
  - Suppose $R = 10$, each of length $T_E = 15{,}000$ minutes, starting at time 0 in the empty and idle state, initialized for $T_0 = 2{,}000$ minutes before data collection begins.
  - Each batch means is the average number of customers in queue for a *1,000*-minute interval.
  - The 1st two batch means are deleted ($d = 2$).
  - The point estimator and standard error are:

$$\bar{Y}_{..}(15{,}2) = 8.43 \qquad \text{and} \qquad s.e.\left(\bar{Y}_{..}(15{,}2)\right) = 1.59$$

  - The *95%* C.I. for long-run mean queue length is:

$$\bar{Y}_{..} - t_{\alpha/2,R-1} S/\sqrt{R} \le \theta \le \bar{Y}_{..} + t_{\alpha/2,R-1} S/\sqrt{R}$$
$$8.43 - 2.26(1.59) \le L_Q \le 8.42 + 2.26(1.59)$$

  - A high degree of confidence that the long-run mean queue length is between *4.84* and *12.02* (if d and n are "large" enough).

Sample Size

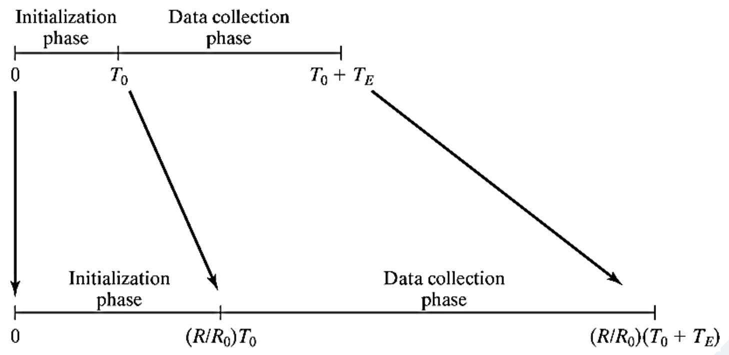
- To estimate a long-run performance measure, θ, within with confidence $100(1-\alpha)\%$.
- M/G/1 queueing example (cont.):
  - We know: $R_0 = 10$, $d = 2$ and $S_0^2 = 25.30$.
  - To estimate the long-run mean queue length, L_Q, within $\varepsilon = 2$ customers with *90%* confidence ($\alpha = 10\%$).
  - Initial estimate:

$$R \ge \left(\frac{z_{0.05} S_0}{\varepsilon}\right)^2 = \frac{1.645^2(25.30)}{2^2} = 17.1$$

  - Hence, at least 18 replications are needed, next try $R = 18, 19, \ldots$ using . We found that:

$$R = 19 \ge \left(t_{0.05,19-1} S_0/\varepsilon\right)^2 = (1.74 * 25.3/2)^2 = 18.93$$

  - Additional replications needed is $R - R_0 = 19 - 10 = 9$.
- An alternative to increasing R is to increase total run length T_0+T_E within each replication.
  - Approach:
    - ➢ Increase run length from (T_0+T_E) to $(R/R_0)(T_0+T_E)$, and
    - ➢ Delete additional amount of data, from time 0 to time $(R/R_0)T_0$.
  - Advantage: any residual bias in the point estimator should be further reduced.
  - However, it is necessary to have saved the state of the model at time T_0+T_E and to be able to restart the model.



Batch Means for Interval Estimation

- Using a single, long replication:
  - Problem: data are dependent so the usual estimator is biased.
  - Solution: batch means.
- Batch means: divide the output data from *1* replication (after appropriate deletion) into a few large batches and then treat the means of these batches as if they were independent.
- A continuous-time process, $\{Y(t), T_0 \le t \le T_0+T_E\}$:
  - k batches of size $m = T_E/k$, batch means:

$$\bar{Y}_j = \frac{1}{m}\int_{(j-1)m}^{jm} Y(t+T_0)\,dt$$

  - A discrete-time process, $\{Y_i,\ i = d+1, d+2, \ldots, n\}$:

k batches of size $m = (n-d)/k$, batch means:

$$\bar{Y}_j = \frac{1}{m}\sum_{i=(j-1)m+1}^{jm} Y_{i+d}$$

$$\underbrace{Y_1,\ldots,Y_d}_{\text{deleted}},\ \underbrace{Y_{d+1},\ldots,Y_{d+m}}_{\bar{Y}_1},\ \underbrace{Y_{d+m+1},\ldots,Y_{d+2m}}_{\bar{Y}_2},\ \cdots,\ \underbrace{Y_{d+(k-1)m+1},\ldots,Y_{d+km}}_{\bar{Y}_k}$$

- Starting either with continuous-time or discrete-time data, the variance of the sample mean is estimated by:

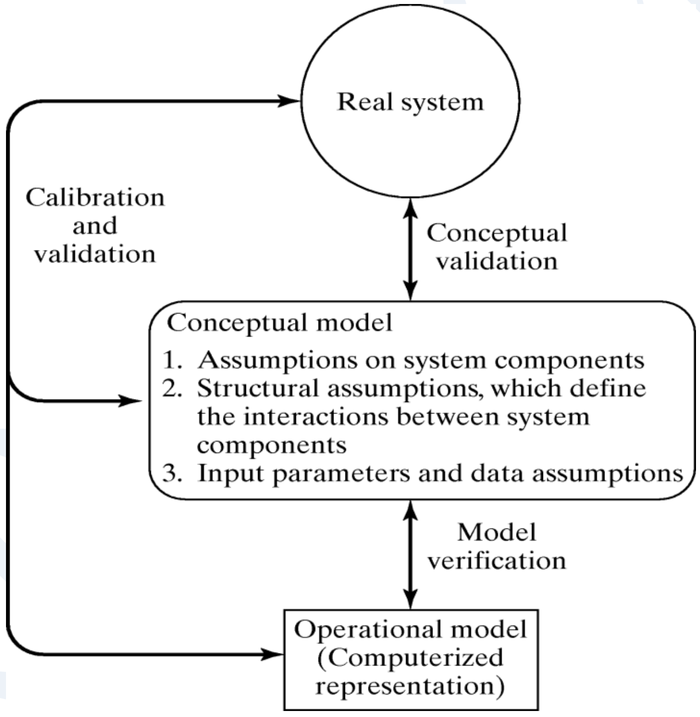
$$\frac{S^2}{k} = \frac{1}{k}\sum_{j=1}^{k}\frac{(\bar{Y}_j-\bar{Y})^2}{k-1} = \sum_{j=1}^{k}\frac{\bar{Y}_j^2 - k\bar{Y}^2}{k(k-1)}$$

- If the batch size is sufficiently large, successive batch means will be approximately independent, and the variance estimator will be approximately unbiased.
- No widely accepted and relatively simple method for choosing an acceptable batch size m (see text for a suggested approach). Some simulation software does it automatically.

Verification, Calibration and Validation

- The goal of the validation process is:
  - To produce a model that represents true behavior closely enough for decision-making purposes
  - To increase the model's credibility to an acceptable level
- Validation is an integral part of model development
  - Verification – building the model correctly (correctly implemented with good input and structure)
  - Validation – building the correct model (an accurate representation of the real system)
- Most methods are informal subjective comparisons while a few are formal statistical procedures

Modeling-Building, Verification & Validation



Verification

- Purpose: ensure the conceptual model is reflected accurately in the computerized representation.
- Many common-sense suggestions, for example:
  - Have someone else check the model.
  - Make a flow diagram that includes each logically possible action a system can take when an event occurs.

- Closely examine the model output for reasonableness under a variety of input parameter settings. (Often overlooked!)
- Print the input parameters at the end of the simulation make sure they have not been changed inadvertently.

Examination of Model Output for Reasonableness

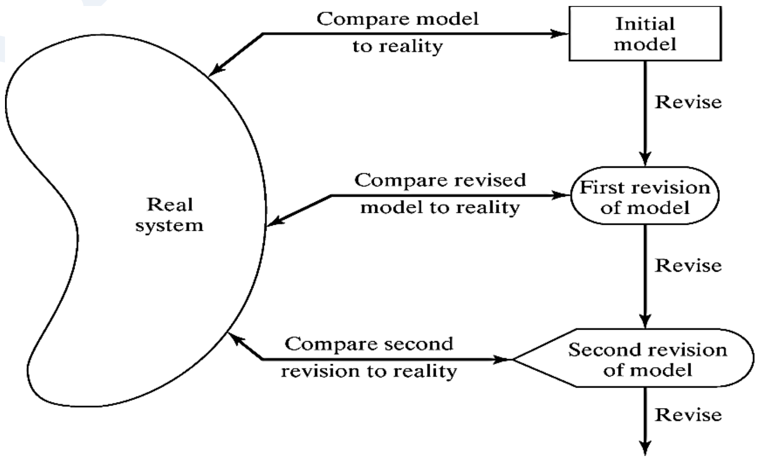
- Example: A model of a complex network of queues consisting many service centers.
  - Response time is the primary interest, however, it is important to collect and print out many statistics in addition to response time.
    - Two statistics that give a quick indication of model reasonableness are current contents and total counts, for example:
      - If the current content grows in a more or less linear fashion as the simulation run time increases, it is likely that a queue is unstable
      - If the total count for some subsystem is zero, indicates no items entered that subsystem, a highly suspect occurrence
      - If the total and current count are equal to one, can indicate that an entity has captured a resource but never freed that resource.
    - Compute certain long-run measures of performance, e.g. compute the long-run server utilization and compare to simulation results

Other Important Tools

- Documentation
  - A means of clarifying the logic of a model and verifying its completeness
- Use of a trace
  - A detailed printout **of the state of the simulation model over time.**

Calibration and Validation

- Validation: the overall process of comparing the model and its behavior to the real system.
- Calibration: the iterative process of comparing the model to the real system and making adjustments.



- No model is ever a perfect representation of the system
  - The modeler must weigh the possible, but not guaranteed, increase in model accuracy versus the cost of increased validation effort.
- Three-step approach:
  - Build a model that has high face validity.
  - Validate model assumptions.
  - Compare the model input-output transformations with the real system's data.

High Face Validity

- Ensure a high degree of realism: Potential users should be involved in model construction (from its conceptualization to its implementation).
- Sensitivity analysis can also be used to check a model's face validity.
  - Example: In most queueing systems, if the arrival rate of customers were to increase, it would be expected that server utilization, queue length and delays would tend to increase

Validate Model Assumptions

- General classes of model assumptions:
  - Structural assumptions: how the system operates.
  - Data assumptions: reliability of data and its statistical analysis.
- Bank example: customer queueing and service facility in a bank.
  - Structural assumptions, e.g., customer waiting in one line versus many lines, served FCFS versus priority.
  - Data assumptions, e.g., interarrival time of customers, service times for commercial accounts.
    - Verify data reliability with bank managers.
    - Test correlation and goodness of fit for data (see Chapter 9 for more details).

Validate Input-Output Transformation

- Goal: Validate the model's ability to predict future behavior
  - The only objective test of the model.
  - The structure of the model should be accurate enough to make good predictions for the range of input data sets of interest.
- One possible approach: use historical data that have been reserved for validation purposes only.
- Criteria: use the main responses of interest.

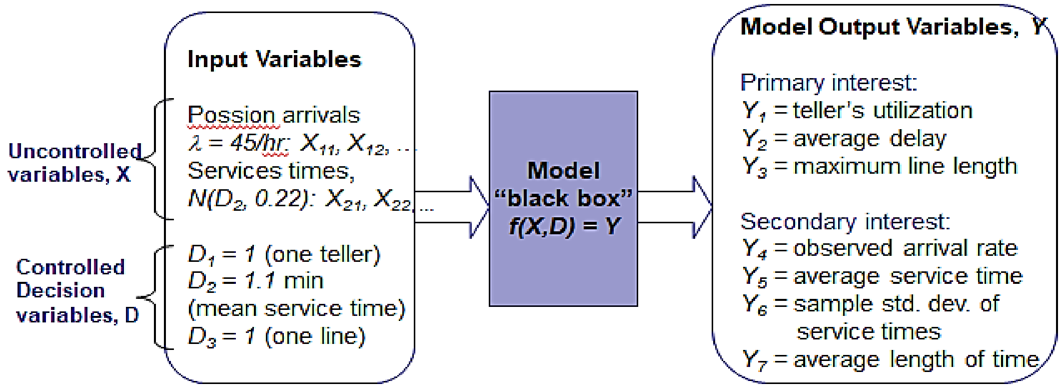
Bank Example

- Example: One drive-in window serviced by one teller, only one or two transactions are allowed.
  - Data collection: 90 customers during 11 am to 1 pm.

    - ➢ Observed service times $\{S_i,\ i = 1,2,\ \dots,\ 90\}$.
    - ➢ Observed interarrival times $\{A_i,\ i = 1,2,\ \dots,\ 90\}$.
- Data analysis let to the conclusion that:
    - ➢ Interarrival times: exponentially distributed with rate $\lambda = 45$
    - ➢ Service times: $N(1.1,\ 0.2^2)$

The Black Box [Bank Example: Validate I-O Transformation]

- A model was developed in close consultation with bank management and employees
- Model assumptions were validated
- Resulting model is now viewed as a "black box":



Comparison with Real System Data [Bank Example: Validate I-O Transformation]

- Real system data are necessary for validation.
    - System responses should have been collected during the same time period (from *11*am to *1*pm on the same Friday.)
- Compare the average delay from the model Y_2 with the actual delay Z_2:
    - Average delay observed, $Z_2 = 4.3$ minutes, consider this to be the true mean value $m_0 = 4.3$.
    - When the model is run with generated random variates X_{1n} and X_{2n}, Y_2 should be close to Z_2.
    - Six statistically independent replications of the model, each of *2*-hour duration, are run.

Hypothesis Testing [Bank Example: Validate I-O Transformation]

- Compare the average delay from the model Y_2 with the actual delay Z_2 (continued):
    - Null hypothesis testing: evaluate whether the simulation and the real system are *the same* (w.r.t. output measures):

$$H_0\text{: } E(Y_2) = 4.3 \text{ minutes}$$
$$H_1\text{: } E(Y_2) \neq 4.3 \text{ minutes}$$

    - ➢ If H_0 is not rejected, then, there is no reason to consider the model invalid
    - ➢ If H_0 is rejected, the current version of the model is rejected, and the modeler needs to improve the model

- Conduct the *t* test:
  - Chose level of significance ($a = 0.5$) and sample size ($n = 6$), see result in Table 10.2.
  - Compute the same mean and sample standard deviation over the n replications:

$$\bar{Y}_2 = \frac{1}{n}\sum_{i=1}^{n} Y_{2i} = 2.51 \text{ minutes} \qquad S = \frac{\sum_{i=1}^{n}(Y_{2i} - \bar{Y}_2)^2}{n-1} = 0.81 \text{ minutes}$$

  - Compute test statistics:

$$|t_0| = \left|\frac{\bar{Y}_2 - \mu_0}{S/\sqrt{n}}\right| = \left|\frac{2.51 - 4.3}{0.82/\sqrt{6}}\right| = 5.24 \;>\; t_{critical} = 2.571 \;\text{(for a 2-sided test)}$$

  - Hence, reject H_0. Conclude that the model is inadequate.
  - Check: the assumptions justifying a *t* test, that the observations (Y_{2i}) are normally and independently distributed.
- Similarly, compare the model output with the observed output for other measures: $Y_4 \leftrightarrow Z_4$, $Y_5 \leftrightarrow Z_5$, and $Y_6 \leftrightarrow Z_6$

Type II Error [Validate I-O Transformation]

- For validation, the power of the test is:
  - Probability[detecting an invalid model] = $1 - \beta$
  - β = P(Type II error) = P(failing to reject $H_0 | H_1$ is true)
  - Consider failure to reject H_0 as a strong conclusion, the modeler would want β to be small.
  - Value of β depends on:
  - Sample size, n
  - The true difference, δ, between E(Y) and μ: $\delta = \frac{|E(Y) - \mu|}{\sigma}$
- In general, the best approach to control b error is:
  - Specify the critical difference, δ.
  - Choose a sample size, n, by making use of the operating characteristics curve (OC curve).

Type I and II Error [Validate I-O Transformation]

- Type I error (α):
  - Error of rejecting a valid model.
  - Controlled by specifying a small level of significance α.
- Type II error (β):
  - Error of accepting a model as valid when it is invalid.
  - Controlled by specifying critical difference and find the n.
- For a fixed sample size n, increasing α will decrease β.

Confidence Interval Testing [Validate I-O Transformation]

- Confidence interval testing: evaluate whether the simulation and the real system are close enough.
- If Y is the simulation output, and $\mu = E(Y)$, the confidence interval (C.I.) for μ is:
- Validating the model:
  - Suppose the C.I. does not contain μ_0: $\bar{Y} \pm t_{\alpha/2,n-1} S/\sqrt{n}$
    - If the best-case error is > ε, model needs to be refined.
    - If the worst-case error is ≤ ε, accept the model.
    - If best-case error is ≤ ε, additional replications are necessary.
    - Suppose the C.I. contains μ_0:
    - If either the best-case or worst-case error is > ε, additional replications are necessary.
    - If the worst-case error is ≤ ε, accept the model.
- Bank example: $\mu_0 = 4.3$, and "close enough" is $\varepsilon = 1$ minute of expected customer delay.
  - A 95% confidence interval, based on the 6 replications is [1.65, 3.37] because:

$$\bar{Y} \pm t_{0.025,5} S/\sqrt{n}$$

$$4.3 \pm 2.51(0.82/\sqrt{6})$$

  - Falls outside the confidence interval, the best case |3.37 – 4.3| = 0.93 < 1, but the worst case |1.65 – 4.3| = 2.65 > 1, additional replications are needed to reach a decision.

Using Historical Output Data

- An alternative to generating input data:
  - Use the actual historical record.
  - Drive the simulation model with the historical record and then compare model output to system data.
  - In the bank example, use the recorded interarrival and service times for the customers $\{A_n, S_n, n = 1,2,\ldots\}$.
- Procedure and validation process: similar to the approach used for system generated input data.

Using a Turing Test

- Use in addition to statistical test, or when no statistical test is readily applicable.
- Utilize persons' knowledge about the system.
- For example:
  - Present 10 system performance reports to a manager of the system. Five of them are from the real system and the rest are "fake" reports based on simulation output data.

- If the person identifies a substantial number of the fake reports, interview the person to get information for model improvement.
- If the person cannot distinguish between fake and real reports with consistency, conclude that the test gives no evidence of model inadequacy.

Optimization via Simulation

- Optimization usually deals with problems with certainty, but in stochastic discrete-event simulation, the result of any simulation run is a random variable.
- Let $x_1, x_2, \ldots, x_m$ be the m controllable design variables & $Y(x_1, x_2, \ldots, x_m)$ be the observed simulation output performance on one run:
  - To optimize $Y(x_1, x_2, \ldots, x_m)$ with respect to $x_1, x_2, \ldots, x_m$ is to maximize or minimize the mathematical expectation (long-run average) of performance, $E[Y(x_1, x_2, \ldots, x_m)]$.
- Example: select the material handling system that has the best chance of costing less than *$D* to purchase and operate.
  - Objective: maximize $Pr(Y(x_1, x_2, \ldots, x_m) \leq D)$.
  - Define a new performance measure:

$$Y'(x_1, x_2, \ldots x_m) = \begin{cases} 1, & \text{if } Y(x_1, x_2, \ldots x_m) \leq D \\ 0, & \text{otherwise} \end{cases}$$

  - Maximize $E(Y'(x_1, x_2, \ldots, x_m))$ *instead.*

Robust Heuristics [Optimization via Simulation]

- The most common algorithms found in commercial optimization via simulation software.
- Effective on difficult, practical problems.
- However, do not guarantee finding the optimal solution.
- Example: genetic algorithms and tabu search.
- It is important to control the sampling variability.

Control sampling variability [Optimization via Simulation]

- To determine how much sampling (replications or run length) to undertaken at each potential solution.
  - Ideally, sampling should increase as heuristic closes in on the better solutions.
  - If specific and fixed number of replications per solution is required, analyst should:
    - Conduct preliminary experiment.
    - Simulate several designs (some at extremes of the solution space and some nearer the center).
    - Compare the apparent best and apparent worst of these designs.
    - Find the minimum for the number of replications required to declare these designs to be statistically significantly different.
    - After completion of optimization run, perform a 2^{nd} set of experiments on the top *5* to *10* designs identified by the heuristic, rigorously evaluate which are the best or near-best of these designs.